S-asymptotically w-periodic mild solutions for non-instantaneous impulsive integro-differential equations with state-dependent delay

Yinuo Wang[1], Chuandong Li[1], Hao Deng[1], and Hangjun Wu[1]

[1]Southwest University

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Abstract

A kind of nonlinear non-instantaneous impulsive equation with state-dependent delay is studied here. By ultilizing suitable fixed point theorem and the theory of semigroup in Banach space, the uniqueness and existence results of S-asymptotically w-periodic mild solutions are obtained, respectively. In the end, two examples are presented to demonstrate the validity of the obtained results.

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$\mathcal{S}$-asymptotically ω-periodic mild solutions for non-instantaneous impulsive integro-differential equations with state-dependent delay

Yinuo Wang | Chuandong Li* | Hao Deng | Hongjuan Wu

Chongqing Key Laboratory of Nonlinear Circuits and Intelligent Information Processing, College of Electronic and Information Engineering, Southwest University, Chongqing 400715, PR China

Correspondence
*Chuandong Li. Email: cdli@swu.edu.cn

Summary

A kind of nonlinear non-instantaneous impulsive equation with state-dependent delay is studied here. By ultilizing suitable fixed point theorem and the theory of semigroup in Banach space, the uniqueness and existence results of $\mathcal{S}$-asymptotically ω-periodic mild solutions are obtained, respectively. In the end, two examples are presented to demonstrate the validity of the obtained results.

KEYWORDS:
non-instantaneous impulse, integro-differential equations, state-dependent delay, fixed point theorem, $\mathcal{S}$-asymptotically ω-periodic mild solution

1 | INTRODUCTION

In nature and human social activities, impulse is a common phenomenon. According to the duration of the changing process, the impulse can be divided into instantaneous impulse and non-instantaneous impulse. Just as the name implies, the instantaneous impulse means that the time of the sudden change process is very short relative to the whole development process and can be ignored. A non-instantaneous impulse means that the process of change is dependent on the state and lasts for a period of time that cannot be ignored. Over the past years, instantaneous impulsive equations have received great attention, which are often used to describe abrupt change, for instance, harvesting, disasters and so on. Detailed information and applications, see e.g. [1,2,3,4,5,6,7,8,9,10] and the cited references. However, some phenomena in real life can not be described by the action of instantaneous impulses, for instance, earthquakes and tsunamis. Thus, more and more scholars began to pay attention to the study of non-instantaneous impulse. In the context of a person injecting drugs, Hernández and O'Regan [11] firstly introduced the non-instantaneous impulsive equations. In Banach space, by utilizing the theory of semigroup, they obtained the existence and uniqueness results. Along this line, non-instantaneous impulse differential equations have received a significant amount of attention, see for example [11,12,13,14,15,16,17].

Since the speed limitation and connection between the system internal subsystem takes time, which leads to almost all of the sports system delay is inevitable, so there has been extensive integro-differential equations with delay in the natural sciences and engineering technology. This kind of problem in the theory study of mathematics and engineering technology has been paid more attention. In the early 1960s, J.J. Levin and J. Nohel studied the integro-differential equations encountered in the theory of the nuclear reactor fuel cycle,

$$z'(t) = -\int_{t-\tau}^{t} a(t-u)g(z(u))du,$$

where $z(t)$ represents the number of neutrons at time t. Since then, this kind of problem also appeared in a large number of biological engineering, electrical and electronic fields. Differential equations with state-dependent delay has gained more and more attention because of its wide applications and its qualitative theory is different from equations with discrete and time-dependent delays. In recent years, the research on semilinear differential or integro-differential systems with delay are becoming more and more active, see for instance [18,19,20,21,22,23,24,25,26]. In [18], Suganya et.al took an impulsive fractional integro-differential equation in neutral form with non-instantaneous impulses and state-dependent delay into consideration, they got the existence results through the fixed piont theorem and the measure of non-compactness. Mesmouli et.al [25] studied the existence of periodic solutions of the nonlinear integro-differential equations with delay. They used the Krasnoselski's and Banach's fixed point theorem to get the desired results.

Recently, the existence of $\mathcal{S}$-asymptotically w-periodic solutions of differential equations or inclusions were studied in [27,28,29,30,31,32]. In [27], Hernández et.al gave the concept of $\mathcal{S}$-asymptotically w-periodic functions and introduced the relations between the $\mathcal{S}$-asymptotically w-periodic functions and asymptotically w-periodic functions. Besides, the existence of $\mathcal{S}$-asymptotically w-periodic mild solutions for a class of abstract Cauchy problem was studied. Wang [29] considered a kind of differential equations with alomost sectorial operator in a complete normed vector space which is of infinite demensional. And they presented the uniqueness and existence results under sufficeint conditions. The authors in [30] studied the non-instantaneous impulsive differential inclusion of order $\alpha \in (1,2)$ and proved the existence of $\mathcal{S}$-asymptotically w-periodic mild solutions via a fixed point theorem for contraction multivalues function and a compactness criterion in the space of bounded piecewise continuous functions defined on the bounded interval. In [31], Andrade et.al studied the systems determined by partial differential equations with infinite and state-dependent delay. The existence of $\mathcal{S}$-asymptotically periodic solutions and asymptotically periodic solutions were presented via the local Lipschitz conditions of the function concerned. What's more, Cueves et.al [32] studied the abstract fractional equations with infinite delay in complete normed vector space and they got the existence results of $\mathcal{S}$-asymptotically w-periodic mild solutions.

As far as we know, there has been limiting lierature concerning on the existence of $\mathcal{S}$-asymptotically w-periodic mild solutions for non-instantaneous impulsive integro-differential equations with state-dependent delay. Therefore, inspired by the above existing papers, we mainly consider

$$\begin{cases} z'(t) = Az(t) + f\Big(t, z_{\alpha(t,z_t)}, \displaystyle\int_0^w h(t,s,z_{\alpha(s,z_s)})ds\Big), & t \in \displaystyle\bigcup_{j=0}^{m}(s_j, t_{j+1}], \\ z(t) = g_j(t, z_{\alpha(t,z_t)}), & t \in \displaystyle\bigcup_{j=1}^{m}(t_j, s_j], \\ z(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{1}$$

in which $A : D(A) \subset G \to G$ is a linear operator and is also closed, $\{U(t), t \geq 0\}$ denotes the C_0-semigroup and the infinitesimal generator is denoted by A on Banach space G with a norm $\|\cdot\|$. Let $0 = s_0 \leq t_1 \leq s_1 \leq t_2 \leq \cdots \leq t_m \leq s_m = w \leq t_{m+1} \leq \cdots$ and $\lim_{j\to\infty} t_j = \infty$, $t_{j+m} = t_j + w$, $s_{j+m} = s_j + w$. $f \in C([0,w] \times G \times \mathcal{D}, G)$, where $\mathcal{D}$ is a phase space and $g_j \in C([t_j, s_j] \times G, G)$, $j = 1, 2, \cdots, m$. $\alpha : [0,w] \times \mathcal{D} \to (-\infty, w]$ is a suitable function. For any z defined on $(-\infty, w]$ and for any $t \geq 0$, we define $z_t(\theta) = z(t+\theta)$, $\theta \in (-\infty, 0]$, where $z_t(\cdot)$ is the element of $\mathcal{D}$ and it denotes the history of the state from each time θ up to the present time t.

In the following, necessary notations and important Lemmas are provided in part 2. In part 3, the uniqueness and existence results of $\mathcal{S}$-asymptotically w-periodic solutions are given, respectively. In part 4, examples are presented to illustrate the applications of the results obtained.

2 | PRELIMINARIES

Set $\mathrm{K} := [0, w]$. $C(\mathrm{K}, G)$ is the set of mapping $z : \mathrm{K} \to G$ whose components are continuous functions. It forms a Banach space and $\|z\|_C$ denotes the norm. $C_b(\mathrm{K}, G)$ is the space of mapping $z : \mathrm{K} \to G$ whose components are continuous and bounded functions, and $\|\cdot\|_\infty$ denotes the norm. $C_\phi(\mathrm{K}, G)$, $C_w(\mathrm{K}, G)$ for $w > 0$ are the subspaces of $C_b(\mathrm{K}, G)$ defined as

$$C_\phi(\mathrm{K}, G) = \{z \in C_b(\mathrm{K}, G) : z(0) = \phi(0)\},$$

$$C_w(\mathrm{K}, G) = \{z \in C_b(\mathrm{K}, G) : z \text{ is } w \text{ periodic}\}.$$

Let $PC(\mathrm{K}, G) = \{z : \mathrm{K} \to G : z \in C((t_j, t_{j+1}], G)$ and there exist $z(t_j^-)$ and $z(t_j^+)$ with $z(t_j^-) = z(t_j)\}$, its norm is denoted by $\|z\|_{PC}$. And $SAP_wPC(\mathrm{K}, G) := \{z : \mathrm{K} \to G, z$ is bounded and $z \in PC(\mathrm{K}, G)$, $\lim_{t\to\infty} \|z(t+w) - z(t)\| = 0\}$, it is a complete normed vector space with the norm $\|z\| := \max_{t\in \mathrm{K}} \|z(t)\|$. The noncompact Kuratowski measure is represented by $\mu(\cdot), \mu_C(\cdot), \mu_{PC}(\cdot)$ on the bounded set of $G, C(\mathrm{K}, G), PC(\mathrm{K}, G)$, we refer readers to [17,33] and the reference therein for more details.

Lemma 2.1. ([34]). *$\mathcal{D}$ is a function mapping $(-\infty, 0]$ into G, which is seminormed linear endowed with the norm $\|\cdot\|_{\mathcal{D}}$ and satisfies:*

(i) *If $z : (-\infty, w] \to G$, where $w > 0$, is continuous on* K *and $z_0 \in \mathcal{D}$, then for every $t \in \mathrm{K}$, there holds*

- (a) $z_t \in \mathcal{D}$.
- (b) *There is $C_0 > 0$ such that $\|z(t)\| \leq C_0\|z_t\|_{\mathcal{D}}$, where C_0 is a constant.*
- (c) *There exist $C_1, C_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|z_t\|_{\mathcal{D}} \leq C_1(t) \sup_{s\in[0,w]} \|z(s)\| + C_2(t)\|z_0\|_{\mathcal{D}}, \tag{2}$$

where C_1, C_2 are both independent of $z(\cdot)$ with C_1 continuous and C_2 locally bounded.

(ii) *For function $z(\cdot)$ defined in (i), z_t is a $\mathcal{D}$-valued continuous function on* K.

(iii) *$\mathcal{D}$ is a complete space.*

Definition 2.1. ([35]) The semigroup $(U(t))_{t\geq 0}$ is strongly continuous bounded linear operator, if there exist constants $M \geq 1$ and $\gamma > 0$ such that

$$\|U(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

then, $(U(t))_{t\geq 0}$ is called uniformly exponentially stable.

Lemma 2.2. ([33]). *Suppose the semigroup $\{U(t), t \geq 0\}$ is uniformly exponentially stable. Set $q \in C([0, \infty), G)$ and vanishes at infinity. Let*

$$v(t) = \int_0^t U(t-s)q(s)ds, \quad t \geq 0, \tag{3}$$

then, it also vanishes at infinity.

Lemma 2.3. ([33]). *Assume $v : [0, \infty) \to G$ is defined by* (3) *and $\{U(t), t \geq 0\}$ is uniformly exponentially stable. Let $q \in SAP_w(G)$, then the function $v(\cdot) \in SAP_w(G)$.*

Definition 2.2. If $f \in C_b(\mathrm{K}, G)$ and there is $w > 0$ such that $f(t + w) - f(t) \to 0$ as $t \to \infty$. Then, f is said to be S-asymptotically w-periodic.

Definition 2.3. $f \in C([0, \infty) \times G, G)$, if for every $\Omega \subset G$ (Ω is bounded), the set $\{f(t, z) : t \geq 0, z \in \Omega\}$ is bounded and $f(t, z) - f(t + w, z) \to 0$ as $t \to \infty$ uniformly in $z \in \Omega$. Then, f is called uniformly S-asymptotically w-periodic on bounded sets.

Lemma 2.4. ([36]). *G is a complete normed vector space, Λ is a subset of G and it is bounded, then there is $\Lambda_0 \subset \Lambda$ which is countable satisfying $\mu(\Lambda) \leq 2\mu(\Lambda_0)$.*

Lemma 2.5. ([37]). *G is a complete normed vector space, $\Lambda = \{z_n\}$ is a subset of $PC(\mathrm{K}, G)$ and it is bounded and countable, therefore $\mu(\Lambda(t))$ satisfies*

$$\mu\Big(\Big\{\int_{\mathrm{K}} z_n(t)ds | n \in \mathbb{N}\Big\}\Big) \leq 2\int_{\mathrm{K}} \mu(\Lambda(t))dt.$$

Besides, it is Lebesgue integral on K.

Lemma 2.6. ([38]). *G is a complete normed vector space and for each $[t_j, t_{j+1}]$, $j = 0, 1, \cdots, m$, $\Lambda \subset PC(\mathrm{K}, G)$ is bounded and equicontinuous, thus $\mu(\Lambda(t)) \in PC(\mathrm{K}, \mathbb{R}^+)$ and $\mu_{PC}(\Lambda) = \sup\limits_{t\in \mathrm{K}} \mu(\Lambda(t))$.*

Lemma 2.7. ([33]). *G is a complete normed vector space. $S \subset G$ and S is nonempty. $Q : S \to G$ is continuous, which called the strict μ-set-contraction operator if for every $\Gamma \subset S$ (S is bounded), there is a constant $0 \le \delta < 1$ such that $\mu(Q(\Gamma)) \le \delta\mu(\Gamma)$.*

Lemma 2.8. ([33]). *G is a complete normed vector space. Suppose Γ is a bounded subset on G and is also closed and convex, $\Phi : \Gamma \to \Gamma$ is μ-set-contraction operator, therefore, Φ has at least one fixed point in Γ.*

Definition 2.4. If $z \in SAP_w PC(\mathrm{K}, G)$ satisfies

$$z(t) = \begin{cases} U(t)\phi(0) + \int\limits_0^t U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int\limits_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds, & t \in [0, t_1], \\ g_j(t, z_{\alpha(t,z_t)}), & t \in \bigcup\limits_{j=1}^m (t_j, s_j], \\ g_j(s_j, z_{\alpha(s_j, z_{s_j})}) + \int\limits_{s_j}^t U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int\limits_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds, & t \in \bigcup\limits_{j=0}^m (s_j, t_{j+1}], \end{cases} \tag{4}$$

then it is called an S-asymptotically w-periodic mild solution of problem (1).

We assume $f,\ g_j,\ h$ and α satisfy the conditions:

$(H1)$ $f \in C(\mathrm{K} \times \mathcal{D} \times G, G)$ and satisfies

- (a) for every $\zeta > 0$, there is $L_f(\cdot) > 0$ satisfying
$$\|f(t, z_{t_2}, u) - f(t, z_{t_1}, u)\| \le L_f(\zeta)|t_2 - t_1|, \quad t, t_1, t_2 \ge 0,$$
for all $z : (-\infty, w] \to G$ such that $z_0 = \phi \in \mathcal{D},\ z : [0, w] \to G$ is continuous and $\max\limits_{s\in[0,w]} \|z(s)\| \le \zeta$;
- (b) for $z_1, z_2 \in \mathcal{D},\ u, v \in G$ and each $t \in \bigcup\limits_{j=0}^m [s_j, t_{j+1}]$, there exists a constant $L'_f > 0$ satisfying
$$\|f(t, z_1, u) - f(t, z_2, v)\| \le L'_f[\|z_1 - z_2\|_{\mathcal{D}} + \|u - v\|];$$
- (c) there exist constants $L_0, L_1, L_2 > 0$ such that $\|f(t, z_1, z_2)\| \le L_0 + L_1\|z_1\| + L_2\|z_2\|$ for each $t \in [s_j, t_{j+1}]$ and all $z_1, z_2 \in G,\ j = 0, 1, 2, \cdots, m$;
- (d) there is a positive function $L_j(t) \in L^1(\mathrm{K}, \mathbb{R}^+)$ such that for any bounded subset $B_1 \subset \mathcal{D},\ B_2 \subset G$,
$$\mu(f(t, B_1, B_2)) \le L_j(t)\Big(\sup_{\theta\in(-\infty,0]} \mu(B_1(\theta)) + \mu(B_2)\Big),\ t \in \bigcup_{j=0}^m (s_j, t_{j+1}].$$

$(H2)$ The function $\alpha \in C(\mathrm{K} \times \mathcal{D}, \mathbb{R}^+)$ satisfies

- (a) $-\infty < \alpha(t, z) \le t$, for $z \in \mathcal{D}$. And
$$\alpha(t + w, \phi) - \alpha(t, \phi) \to 0, \quad t \to \infty,$$
uniformly for ϕ in bounded sets;
- (b) there exists a constant $L_\alpha > 0$ such that
$$\|\alpha(t, \psi_2) - \alpha(t, \psi_1)\| \le L_\alpha\|\psi_2 - \psi_1\|_{\mathcal{D}},\ \ \psi_1, \psi_2 \in \mathcal{D}.$$

$(H3)$ $h : \{(t, s) \in \mathrm{K} \times \mathrm{K} : s \le t\} \times \mathcal{D} \to G$ is continuous and satisfies

(a) there exists a constant $L_h > 0$ such that for $z_1, z_2 \in \mathcal{D}$,

$$\Big\| \int_0^w [h(t,s,z_1) - h(t,s,z_2)]ds \Big\| \le L_h \|z_1 - z_2\|_{\mathcal{D}};$$

(b) there exists a constant $L_3 > 0$ such that $\|h(t,s,z)\| \le L_3(1+\|z\|)$ for each $t \in [s_j, t_{j+1}]$ and all $z \in G$, $j = 0, 1, 2, \cdots, m$.

$(H4)$ There exists $\rho_j \in L^1(\mathrm{K}\times\mathrm{K}, \mathbb{R}^+)$ such that for each bounded set $\psi \in \mathcal{D}$,

$$\mu(h(t,s,\psi)) \le \rho_j(t,s)\Big(\sup_{\theta\in(-\infty,0]} \mu(\psi(\theta))\Big).$$

$(H5)$ For any $j \in \mathbb{N}$, $g_j : [t_j, s_j] \times \mathcal{D} \to G$ such that for any $z \in G$, the function $t \to g_j(t,z)$ is differentiable at s_j and

(a) for all $z \in G$, there is

$$\lim_{t\to\infty, j\to\infty} \|g_{j+m}(t+w, z) - g_j(t,z)\| = 0; \tag{5}$$

(b) there exists $L'_{g_j} > 0$ such that

$$\|g_j(t, z_{t_1}) - g_j(t, z_{t_2})\| \le L'_{g_j}(r)\|t_1 - t_2\|,\ j \in \mathbb{N}, \tag{6}$$

for all $z : (-\infty, w] \to G$ such that $z_0 = \phi \in \mathcal{D}$, $z : [0,w] \to G$ is continuous and $\max_{s\in[0,w]} \|z(s)\| \le r$;

(c) for $z_1, z_2 \in \mathcal{D}$ and each $t \in [t_j, s_j]$, there is $\|g_j(t,z_1) - g_j(t,z_2)\| \le L_{g_j}\|z_1 - z_2\|_{\mathcal{D}}$, where $L_{g_j} > 0$ is a constant, $j = 1, 2, \cdots, m$;

(d) there is a constant $L_4 > 0$ such that $\|g_j(t,z)\| \le L_4(1+\|z\|)$ for each $t \in [t_j, s_j]$ and all $z \in G$, $j = 1, 2, \cdots, m$;

(e) there exist constants $\beta_j > 0$ such that for each bounded set $B \subset \mathcal{D}$,

$$\mu(g_j(t,B)) \le \beta_j\Big(\sup_{\theta\in(\infty,0]} \mu(B(\theta))\Big).$$

3 | MAIN RESULTS

Theorem 3.1. *Suppose $(H1)(a)(b)(c)$, $(H2)(a)(b)$, $(H3)(a)(b)$ and $(H5)(a)(b)(c)(d)$ hold, f, h are uniformly $\mathcal{S}$-asymptotically w-periodic on bounded sets, and $\{U(t), t \ge 0\}$ is uniformly exponentially stable. If*

$$\rho := \max\Big\{\delta\Big[\tfrac{M(1+L_h)(t_{j+1}-s_j)}{\gamma} \cdot L'_f + L_{g_j}\Big]\Big\} < 1,$$

then a unique $\mathcal{S}$-asymptotically w-periodic mild solution of problem (1) *can be obtained.*

Proof. We define the operator H on the space $SAP_wPC(\mathrm{K}, G)$ by

$$(Hz)(t) = \begin{cases} U(t)\phi(0) + \int_0^t U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau,z_{\alpha(\tau,z_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ g_j(t, z_{\alpha(t,z_t)}), & t \in \bigcup_{j=1}^m (t_j, s_j], \\ g_j(s_j, z_{\alpha(s_j, z_{s_j})}) + \int_{s_j}^t U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau,z_{\alpha(\tau,z_\tau)})d\tau\Big)ds, & t \in \bigcup_{j=0}^m (s_j, t_{j+1}]. \end{cases} \tag{7}$$

Obviously, H is well defined and the fixed points of H is actually the mild solutions of the problem (1). Firstly, we claim for $z \in SAP_wPC(\mathrm{K}, G)$, then $Hz \in SAP_wPC(\mathrm{K}, G)$.

For $t \in (s_j, t_{j+1}]$, then $t + w \in (s_j + w, t_{j+1} + w] = (s_{j+m}, t_{j+1+m}]$. Hence

$$
\begin{aligned}
&\Big\| g_{j+m}(s_{j+m}, z_{\alpha(s_{j+m}, z_{s_{j+m}})}) + \int_{s_{j+m}}^{t+w} U(t+w-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds \\
&\quad - g_j(s_j, z_{\alpha(s_j, z_{s_j})}) - \int_{s_j}^{t} U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds \Big\| \\
&\le \Big\| \int_{s_j}^{t} U(t+w-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds \\
&\quad - \int_{s_j}^{t} U(t-s) f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) ds \Big\| + \Big\| g_{j+m}(s_{j+m}, z_{\alpha(s_{j+m}, z_{s_{j+m}})}) - g_j(s_j, z_{\alpha(s_j, z_{s_j})}) \Big\| \\
&\le \Big\| \int_{s_j}^{t} U(t-s) \Big[f\Big(s+w, z_{\alpha(s+w, z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \\
&\quad - f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big] ds \Big\| + \Big\| g_{j+m}(s_{j+m}, z_{\alpha(s_{j+m}, z_{s_{j+m}})}) - g_j(s_j, z_{\alpha(s_j, z_{s_j})}) \Big\|.
\end{aligned} \tag{8}
$$

For the first term in (8), we define

$$
u(s) = f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}\Big) d\tau, \quad v(s) = \int_{s_j}^{t} U(t-s)\Big(u(t+w) - u(t)\Big) ds,
$$

then,

$$
\begin{aligned}
&\|u(s+w) - u(s)\| \\
&= \Big\| f\Big(s+w, z_{\alpha(s+w, z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) - f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big\| \\
&\le \Big\| f\Big(s+w, z_{\alpha(s+w, z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) - f\Big(s, z_{\alpha(s+w,z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big\| \\
&\quad + \Big\| f\Big(s, z_{\alpha(s+w, z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) - f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big\| \\
&\quad + \Big\| f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) - f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big\|, \quad s \ge 0.
\end{aligned} \tag{9}
$$

Obviously, the first term in (9) tends to 0 as $t \to \infty$. For the second term in (9) , combining $(H1)(a)$ and $(H2)(b)$, we have

$$
\begin{aligned}
&\Big\| f\Big(s, z_{\alpha(s+w, z_{s+w})}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) - f\Big(s, z_{\alpha(s,z_s)}, \int_0^w h(s+w,\tau, z_{\alpha(\tau,z_\tau)}) d\tau\Big) \Big\| \\
&\le L_f(\zeta) |\alpha(s+w, z_{s+w}) - \alpha(s, z_s)| \\
&\le L_f(\zeta) |\alpha(s+w, z_{s+w}) - \alpha(s, z_{s+w})| + L_f(\zeta) |\alpha(s, z_{s+w}) - \alpha(s, z_s)| \\
&\le L_f(\zeta) |\alpha(s+w, z_{s+w}) - \alpha(s, z_{s+w})| + L_f(\zeta) L_\alpha |z_{s+w} - z_s| \\
&\to 0, \quad s \to \infty.
\end{aligned}
$$

For the last term in (9), combining $(H1)(b)$, one gets

$$
\begin{aligned}
&\left\| f\left(s, z_{\alpha(s,z_s)}, \int_0^w h(s+w,\tau,z_{\alpha(\tau,z_\tau)})d\tau\right) - f\left(s, z_{\alpha(s,z_s)}, \int_0^w h(s+w,\tau,z_{\alpha(\tau,z_\tau)})d\tau\right)\right\| \\
&\leq L'_f \int_0^w \left\| h(s+w,\tau,z_{\alpha(\tau,z_\tau)})d\tau - h(s,\tau,z_{\alpha(\tau,z_\tau)})\right\| d\tau \\
&\to 0, \quad s \to \infty.
\end{aligned}
$$

Therefore, there holds $\|u(s+w) - u(s)\| \to 0,\ s \to \infty$, that is $u \in SAP_w PC(\mathrm{K}, G)$. Combining $\{U(t)\}_{t\geq 0}$ is uniformly exponentially stable, then from Lemma 2.3, $v(s) \in SAP_w PC(\mathrm{K}, G)$ as $s \to \infty$.

For the second term in (8), combining (6), we have

$$
\begin{aligned}
&\left\| g_{j+m}(s_{j+m}, z_{\alpha(s_{j+m},z_{s_{j+m}})}) - g_j(s_j, z_{\alpha(s_j,z_{s_j})})\right\| \\
&\leq \left\| g_{j+m}(s_j+w, z_{\alpha(s_{j+m},z_{s_{j+m}})}) - g_{j+m}(s_j+w, z_{\alpha(s_j,z_{s_j})})\right\| + \left\| g_{j+m}(s_j+w, z_{\alpha(s_j,z_{s_j})}) - g_j(s_j, z_{\alpha(s_j,z_{s_j})})\right\| \\
&\leq L'_{g_j}(r)\left\| \alpha(s_{j+m}, z_{s_{j+m}}) - \alpha(s_j, z_{s_j})\right\| + \left\| g_{j+m}(s_j+w, z_{\alpha(s_j,z_{s_j})}) - g_j(s_j, z_{\alpha(s_j,z_{s_j})})\right\|
\end{aligned}
$$

From (5), one can get that $\left\| g_{j+m}(s_j+w, z_{\alpha(s_j,z_{s_j})}) - g_j(s_j, z_{\alpha(s_j,z_{s_j})})\right\| \to 0$ as $s_j \to \infty$. From $(H2)(a)$ and $(H2)(b)$, there is

$$
\begin{aligned}
\left\| \alpha(s_{j+m}, z_{s_{j+m}}) - \alpha(s_j, z_{s_j})\right\| &\leq \left\| \alpha(s_{j+m}, z_{s_{j+m}}) - \alpha(s_j, z_{s_{j+m}})\right\| + \left\| \alpha(s_j, z_{s_{j+m}}) - \alpha(s_j, z_{s_j})\right\| \\
&\leq \left\| \alpha(s_{j+m}, z_{s_{j+m}}) - \alpha(s_j, z_{s_{j+m}})\right\| + L_\alpha \|z_{s_j+w} - z_{s_j}\| \\
&\to 0, \quad s_j \to \infty.
\end{aligned}
$$

Then $\left\| g_{j+m}(s_{j+m}, z_{\alpha(s_{j+m},z_{s_{j+m}})}) - g_j(s_j, z_{\alpha(s_j,z_{s_j})})\right\| \to 0$ as $s_j \to \infty$, which shows $Hz \in SAP_w PC(\mathrm{K}, G)$ for $t \in (s_j, t_{j+1}],\ j = 0, 1, 2, \cdots, m$.

For $t \in (t_j, s_j]$, then $t + w \in (t_j + w, s_j + w] = (t_{j+m}, s_{j+m}]$, combining $(H2)(a)(b)$ and $(H5)(a)(b)$, one has

$$
\begin{aligned}
&\|g_{j+m}(t+w, z_{\alpha(t+w,z_{t+w})}) - g_j(t, z_{\alpha(t,z_t)})\| \\
&\leq \|g_{j+m}(t+w, z_{\alpha(t+w,z_{t+w})}) - g_{j+m}(t+w, z_{\alpha(t,z_t)})\| + \|g_{j+m}(t+w, z_{\alpha(t,z_t)}) - g_j(t, z_{\alpha(t,z_t)})\| \\
&\leq L'_{g_j}(r)\Big[\|\alpha(t+w, z_{t+w}) - \alpha(t, z_{t+w})\| + L_\alpha \|z_{t+w} - z_t\|\Big] + \|g_{j+m}(t+w, z_{\alpha(t+w,z_{t+w})}) - g_{j+m}(t, z_{\alpha(t+w,z_{t+w})})\| \\
&\to 0, \quad t \to \infty,
\end{aligned}
$$

which implies that $Hz \in SAP_w PC(\mathrm{K}, G)$ for $t \in (t_j, s_j]$.

For $t \in [0, t_1]$, since $U(t)\phi(0) \to 0$ as $t \to \infty$, then $U(\cdot)\phi \in SAP_w PC(\mathrm{K}, G)$. Therefore the problem is reduced to verify that $\int_0^t U(t-s) f\left(s, z_{\alpha(s,z_s)}, \int_0^w h(s,\tau,z_{\alpha(\tau,z_\tau)})d\tau\right) ds \in SAP_w PC(\mathrm{K}, G)$, which can be viewed as the special case when $t \in (s_j, t_{j+1}]$. Thus, $Hz \in SAP_w PC(\mathrm{K}, G)$ for $t \in [0, t_1]$ is obtained.

Secondly, set $B_\eta = \{z \in SAP_w PC(\mathrm{K}, G) : \|z\| \leq \eta\}$, it is obvious that B_η is a closed and convex subset of $SAP_w PC(\mathrm{K}, G)$, then we show that for any $\eta > 0$, there is a constant $b > 0$ such that for each $z \in B_\eta$, there holds $\|Hz\| \leq b$. Define ψ,which can be regarded as the extension of $\phi \in \mathcal{D}$, as

$$
\psi(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ U(t)\phi(0), & t \in [0, t_1], \\ 0, & t \in (t_1, \infty). \end{cases}
$$

Therefore, $z_0 = \phi$. Let $z(t) = y(t) + \psi(t),\ t \in (-\infty, w]$, if $z(\cdot)$ satisfies (4), then $y_0 = 0$ and $z_t = y_t + \psi_t$, where $y(t)$ is defined by

$$y(t) = \begin{cases} \int_0^t U(t-s) f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ g_j(t, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}), & t \in \bigcup_{j=1}^{m} (t_j, s_j], \\ \int_{s_j}^t U(t-s) f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds & \\ +g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})} + \psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})}), & t \in \bigcup_{j=0}^{m} (s_j, t_{j+1}]. \end{cases}$$

Set $SAP_{w,0}PC(\mathrm{K}, G) = \{y \in SAP_w PC(\mathrm{K}, G) : y_0 = 0\}$. For simplicity, define $\bar{D}_w = SAP_{w,0}PC(\mathrm{K}, G)$. Then for any $y \in \bar{D}_w$, one has

$$\|y\|_{\bar{D}_w} = \sup_{t\in[0,\infty)} \|y\|.$$

The space $(\bar{D}_w, \|\cdot\|_{\bar{D}_w})$ is a Banach space. Define $N : \bar{D}_w \to \bar{D}_w$ by

$$(Ny)(t) = \begin{cases} \int_0^t U(t-s) f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds, & t \in [0, t_1], \\ g_j(t, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}), & t \in \bigcup_{j=1}^{m} (t_j, s_j], \\ \int_{s_j}^t U(t-s) f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds & \\ \qquad +g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})} + \psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})}), & t \in \bigcup_{j=0}^{m} (s_j, t_{j+1}]. \end{cases} \tag{10}$$

Since $\{U(t), t \geq 0\}$ is uniformly exponentially stable, then from Definition 2.1, there yields $\|U(t)\| \leq Me^{-\gamma t} < M$, thus, $U(t)\phi(0)$ is bounded. The problem is reduced to prove that N maps any closed ball $B_{\mathcal{L}}$ of $\bar{D}_w$ into bounded sets in $\bar{D}_w$. We only need to show that for any $y \in B_{\mathcal{L}} = \{y \in \bar{D}_w : \|y\|_{\bar{D}_w} \leq \mathcal{L}\}$, one gets $\|Ny\|$ is also bounded.

For any $y \in B_{\mathcal{L}}$ and for $t \in [0, t_1]$, we have

$$\begin{aligned} \|(Ny)(t)\|_{\bar{D}_w} &\leq M \int_0^t e^{-\gamma(t-s)} \Big\| f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|_D ds \\ &\leq M \int_0^t \Big[L_0 + L_1 \|y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}\|_D + L_2 \int_0^w \|h(s,\tau, y_{\alpha(\tau,y_{\tau+\psi_\tau})} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}\|_D d\tau\Big] ds \\ &\leq ML_0t_1 + ML_1t_1 \|y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}\|_D + ML_2 \int_0^t \int_0^w \|h(s,\tau, y_{\alpha(\tau,y_{\tau+\psi_\tau})} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)})\|_D d\tau ds \\ &\leq ML_0t_1 + ML_1t_1 \|y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}\|_D + ML_2L_3 \int_0^t \int_0^w (1 + \|y_{\alpha(\tau,y_{\tau+\psi_\tau})} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}\|_D) d\tau ds \\ &\leq ML_0t_1 + ML_1t_1 \|y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}\|_D + wMt_1L_2L_3 + ML_2L_3 \int_0^t \int_0^w \|y_{\alpha(\tau,y_{\tau+\psi_\tau})} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}\|_D d\tau ds. \end{aligned}$$

From (2) and the properties of the norm, one has

$$\begin{aligned}\|y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}\|_{\mathcal{D}} &\le \|y_{\alpha(s,y_s+\psi_s)}\|_{\mathcal{D}}+\|\psi_{\alpha(s,y_s+\psi_s)}\|_{\mathcal{D}}\\ &\le C_1(t)\sup_{s\in[0,w]}\|y(s)\|+C_2(t)\|y_0\|_{\mathcal{D}}+C_1(t)\sup_{t\in[0,t]}\|\psi(s)\|+C_2(t)\|\phi\|_{\mathcal{D}}\\ &\le C_1(t)\sup_{s\in[0,w]}\|y(s)\|+[MC_1(t)C_0+C_2(t)]\|\phi\|_{\mathcal{D}}\\ &\le \delta\sup_{s\in[0,w]}\|y(s)\|+C',\end{aligned}\tag{11}$$

where $\delta=\sup_{t\in[0,w]}C_1(t)$ and $C'=[MC_0+C_2(t)]\|\phi\|_{\mathcal{D}}$. Therefore, combining (11), we can obtain that

$$\begin{aligned}\|(Ny)(t)\|_{\bar{\mathcal{D}}_w} &\le t_1ML_0+t_1ML_1(\delta\eta+C')+wMt_1L_2L_3+ML_2L_3(\delta\eta+C')wt_1\\ &= Mt_1[L_0+(L_1+wL_2L_3)(\delta\eta+C')+wL_2L_3].\end{aligned}$$

For $t\in(t_j,s_j]$, we have

$$\|(Ny)(t)\|_{\bar{\mathcal{D}}_w}=\|g_j(t,y_{\alpha(t,y_t+\psi_t)}+\psi_{\alpha(t,y_t+\psi_t)})\|\le L_4(1+\|y_{\alpha(t,y_t+\psi_t)}+\psi_{\alpha(t,y_t+\psi_t)}\|)\le L_4(1+\delta\eta+C').$$

For $t\in(s_j,t_{j+1}]$, one gets

$$\begin{aligned}\|(Ny)(t)\|_{\bar{\mathcal{D}}_w} \le M\int_{s_j}^{t}\Big\|f\Big(s,y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)},\int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\|ds\\ +\|g_j(s_j,y_{\alpha(s_j,y_{s_j}+\psi_{s_j})}+\psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})})\Big\|\\ \le M(t_{j+1}-s_j)\Big[L_0+(L_1+wL_2L_3)(\delta\eta+C')+wL_2L_3\Big]+L_4(1+\delta\eta+C'),\end{aligned}$$

then, one can get that $\|Ny\|_{\bar{\mathcal{D}}_w}\le q$, where $q=M(t_{j+1}-s_j)\Big[L_0+(L_1+wL_2L_3)(\delta\eta+C')+wL_2L_3\Big]+L_4(1+\delta\eta+C')$. Set $b=M\|\phi(0)\|+q$, then for any $z\in B_\eta$, there holds $\|Hz\|\le b$.

Finally, for $u,v\in\bar{\mathcal{D}}_w$ and for $t\in[0,t_1]$, there holds

$$\begin{aligned}&\|Hu(t)-Hv(t)\|\\ &\le\Big\|\int_0^t U(t-s)f\Big(s,u_{\alpha(s,u_s)},\int_0^w h(s,\tau,u_{\alpha(\tau,u_\tau)})d\tau\Big)ds-\int_0^t U(t-s)f\Big(s,v_{\alpha(s,v_s)},\int_0^w h(s,\tau,v_{\alpha(\tau,v_\tau)})d\tau\Big)ds\Big\|\\ &\le M\int_0^t e^{-\gamma(t-s)}L_f'\Big[\|u_{\alpha(s,u_s)}-v_{\alpha(s,v_s)}\|_{\mathcal{D}}+L_h\|u_{\alpha(\tau,u_\tau)}-v_{\alpha(\tau,v_\tau)}\|\Big]ds\\ &\le M(1+L_h)\|L_f'\int_0^t e^{-\gamma(t-s)}\|u_{\alpha(s,u_s)}-v_{\alpha(s,v_s)}\|_{\mathcal{D}}ds.\end{aligned}$$

From (2), we have

$$\|u_{\alpha(s,u_s)}-v_{\alpha(s,v_s)}\|_{\mathcal{D}}\le C_1(t)\sup_{t\in[0,w]}\|u(t)-v(t)\|\le\delta\|u-v\|.\tag{12}$$

Therefore,

$$\|(Hu)(t)-(Hv)(t)\|\le\frac{\delta Mt_1(1+L_h)}{\gamma}L_f'\|u-v\|.$$

For $u,v\in\bar{\mathcal{D}}_w$ and for $t\in(t_j,s_j]$, combining (12), we have

$$\|(Hu)(t)-(Hv)(t)\|\le L_{g_j}\|u_{\alpha(t,u_t)}-v_{\alpha(t,u_t)}\|_{\mathcal{D}}\le\delta L_{g_j}\|u-v\|.$$

For $u, v \in \bar{\mathcal{D}}_w$ and $t \in (s_j, t_{j+1}]$, one can obtain

$$
\begin{aligned}
&\|(Hu)(t) - (Hv)(t)\| \\
&\leq M L_f \int_{s_j}^{t} e^{-\gamma(t-s)} \Big[\|u_{\alpha(s,u_s)} - v_{\alpha(s,v_s)}\|_{\mathcal{D}} + L_h \|u_{\alpha(\tau,u_\tau)} - v_{\alpha(\tau,v_\tau)}\|_{\mathcal{D}} \Big] ds + L_{g_j} \|u_{\alpha(s_j,u_{s_j})} - v_{\alpha(s_j,v_{s_j})}\|_{\mathcal{D}} \\
&\leq \delta \Big[\frac{M(1+L_h)(t_{j+1}-s_j)}{\gamma} \cdot L'_f + L_{g_j} \Big] \|u - v\|.
\end{aligned}
$$

Thus, we get that $\|(Hu)(t) - (Hv)(t)\| \leq \rho\|u - v\|$, from which one can get that H is contractive. From the Banach's fixed point theorem, H has a unique solution which is the mild solution of problem (1). Since for any $z \in SAP_wPC(\mathrm{K}, G)$, there is $Hz \in SAP_wPC(\mathrm{K}, G)$, then, the uniqueness of $\mathcal{S}$-asymptotically w-periodic mild solution for problem (1) is obtained. □

Theorem 3.2. *Suppose $(H1)(c)(d), (H3)(b), (H4)$ and $(H5)(a)(d)(e)$ hold, $\{U(t)\}_{t\geq 0}$ is uniformly exponentially stable and equicontinuous, therefore, at least one $\mathcal{S}$-asymptotically w-periodic mild solution of problem* (1) *can be obtained provided*

$$
\iota := \max\Big\{ 4M(t_{j+1} - s_j)(1 + 2\tilde{\rho}) \int_0^t L_j(s) ds + 2\beta_j \Big\} < 1.
$$

Proof. Consider the operator defined by (10). It is obvious that the fixed pionts of (10) are actually the mild solutions of (1). Therefore, we mainly prove the operator N has at least one fixed piont.

(a) First, we are going to prove N is continuous.

Let $y^n \to y$ as $n \to \infty$ in $\mathcal{D}$. Then for $t \in [0, t_1]$, there is

$$
\begin{aligned}
&\|(Ny^n)(t) - (Ny)(t)\|_G \\
&= \Big\| \int_0^t U(t-s) f\Big(s, y^n_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y^n_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) \\
&\quad - \int_0^t U(t-s) f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) ds \Big\| \\
&\leq M \int_0^t e^{-\gamma(t-s)} \Big\| f\Big(s, y^n_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y^n_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) \\
&\quad - f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) \Big\| ds.
\end{aligned}
$$

Since $f \in C(\mathrm{K}\times\mathcal{D}\times G, G)$, and $h : \{(t,s) \in \mathrm{K}\times K : s \leq t\}\times\mathcal{D} \to G$ is continuous, then, $\|(Ny^n)(t) - (Ny)(t)\| \to 0$ as $n \to \infty$.

For $t \in (t_j, s_j]$,

$$
\|(Ny^n)(t) - (Ny)(t)\|_G = \|g_j(t, y^n_{\alpha(t,y_t+\psi_t)} + \psi_{\alpha(t,y_t+\psi_t)}) - g_j(t, y_{\alpha(t,y_t+\psi_t)} + \psi_{\alpha(t,y_t+\psi_t)})\|,
$$

from the continuity of g_j, one can easily obtain that $\|(Ny^n)(t) - (Ny)(t)\|_G \to 0$ as $n \to \infty$.

For $t \in (s_j, t_{j+1}]$, we get

$$
\begin{aligned}
\|(Ny^n)(t) - (Ny)(t)\|_G \leq M \int_{s_j}^t & e^{-\gamma(t-s_j)} \Big\| f\Big(s, y^n_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y^n_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) \\
& - f\Big(s, y_{\alpha(s,y_s+\psi_s)} + \psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau, y_{\alpha(\tau,y_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y_\tau+\psi_\tau)}) d\tau\Big) \Big\| ds \\
& + \Big\| g_j(s_j, y^n_{\alpha(s_j,y_{s_j}+\psi_{s_j})} + \psi_{\alpha(s_j,y^n_{s_j}+\psi_{s_j})}) - g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})} + \psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})}) \Big\|.
\end{aligned}
$$

Therefore, $\|(Ny^n)(t)-(Ny)(t)\|_G \to 0$ as $n\to\infty$, since f, h and g_j is continuous.

(b) We will prove that N maps bounded sets into bounded sets in $\bar{\mathcal{D}}_w$, which can be directly obtained from the proof of Theorem 3.1, we omit it here.

(c) We will prove that N maps bounded sets B_η into equicontinuous sets of $\bar{\mathcal{D}}_w$.

For each $t\in[0,t_1], 0\le \mu_2<\mu_1\le t_1, z\in B_\eta$, we have

$$\begin{aligned}
&\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\\
\le &\int_0^{\mu_2}\Big\|[U(\mu_1-s)-U(\mu_2-s)]f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|ds\\
&+\int_{\mu_2}^{\mu_1}\Big\|U(\mu_1-s)f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|ds\\
\le &\int_0^{\mu_2}\Big\|[U(\mu_1-s)-U(\mu_2-s)]f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|ds\\
&+M\int_{\mu_2}^{\mu_1}\Big\|f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|ds.
\end{aligned}$$

Since f is continuous function, thus it is integral. Combining $\{U(t), t\ge 0\}$ is equicontinuous, we can conclude that $\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\to 0$ as $\mu_1\to\mu_2$.

For each $t\in[t_j,s_j], t_j\le\mu_2<\mu_1\le s_j, z\in B_\eta$, one gets

$$\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\le\|g_j(\mu_1, y_{\alpha(\mu_1,y_{\mu_1}+\psi_{\mu_1})}+\psi_{\alpha(\mu_1,y_{\mu_1}+\psi_{\mu_1})})-g_j(\mu_2, y_{\alpha(\mu_2,y_{\mu_2}+\psi_{\mu_2})}+\psi_{\alpha(\mu_2,y_{\mu_2}+\psi_{\mu_2})})\|.$$

From the fact that $g_j(t,z)$ is continuous, thus $\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\to 0$ as $\mu_1\to\mu_2$.

For each $t\in[s_j,t_{j+1}], s_j\le\mu_2<\mu_1\le t_{j+1}, z\in B_\eta$, one has

$$\begin{aligned}
&\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\\
\le &\Big\|\int_{s_j}^{\mu_1}U(\mu_1-s)f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds\\
&-\int_{s_j}^{\mu_2}U(\mu_2-s)f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)ds\Big\|\\
&+\Big\|g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})}+\psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})})ds-g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})}+\psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})})\Big\|\\
\le &\int_{s_j}^{\mu_2}\Big\|[U(\mu_1-s)-U(\mu_2-s)]f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big\|ds\\
&+M\int_{\mu_2}^{\mu_1}\Big\|f\Big(s, y_{\alpha(s,y_s+\psi_s)}+\psi_{\alpha(s,y_s+\psi_s)}, \int_0^w h(s,\tau,y_{\alpha(\tau,y_\tau+\psi_\tau)}+\psi_{\alpha(\tau,y_\tau+\psi_\tau)})d\tau\Big)\Big\|ds\\
&+\Big\|g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})}+\psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})})-g_j(s_j, y_{\alpha(s_j,y_{s_j}+\psi_{s_j})}+\psi_{\alpha(s_j,y_{s_j}+\psi_{s_j})})\Big\|.
\end{aligned}$$

Combining the fact that f, g_j are continuous functions and $\{U(t), t\ge 0\}$ is equicontinuous, there holds $\|(Nz)(\mu_1)-(Nz)(\mu_2)\|\to 0$ as $\mu_1\to\mu_2$. So the operator N is equicontinuous.

For any $B\subset B_\eta$, by Lemma 2.4, there is $B_0=\{y'\}$ which is a subset of B and countable such that

$$\mu(N(B))_{PC}\le 2\mu(N(B_0))_{PC}. \tag{13}$$

From the boundedness and equicontinuity of $N(B_0) \subset N(B_\eta)$, by Lemma 2.6, one gets

$$\mu_{PC}(N(B_0)) = \max_{t\in[t_j,t_{j+1}]} \mu(N(B_0)(t)). \tag{14}$$

For $t \in [0, t_1]$, from Lemma 2.5 $(H1)(d), (H4)$ and the fact that $-\infty < \alpha(s, y_s + \psi_s) \le s$, we have

$$\begin{aligned}
\mu(N(B_0)(t)) &= \mu\Big(\int_0^t U(t-s) f\Big(s, y'_{\alpha(s,y'_s+\psi_s)} + \psi_{\alpha(s,y'_s+\psi_s)}, \int_0^w h(s,\tau,y'_{\alpha(\tau,y'_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y'_\tau+\psi_\tau)})d\tau\Big)ds\Big) \\
&\le M\mu\Big(\int_0^t f\Big(s, y'_{\alpha(s,y'_s+\psi_s)} + \psi_{\alpha(s,y'_s+\psi_s)}, \int_0^w h(s,\tau,y'_{\alpha(\tau,y'_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y'\tau+\psi_\tau)})d\tau\Big)ds\Big) \\
&\le 2M\int_0^t \mu\Big(f\Big(s, y'_{\alpha(s,y'_s+\psi_s)} + \psi_{\alpha(s,y'_s+\psi_s)}, \int_0^w h(s,\tau,y'_{\alpha(\tau,y'_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y'_\tau+\psi_\tau)})d\tau\Big)\Big)ds \\
&\le 2M\int_0^t L_j(s)\Big[\sup_{\theta\in(-\infty,0]} \mu((y'_{\alpha(s,y'_s+\psi_s)} + \psi_{\alpha(s,y'_s+\psi_s)}) + \sup_{\theta\in(-\infty,0]}\mu\Big(\int_0^w h(s,\tau,y'_{\alpha(\tau,y'_\tau+\psi_\tau)} + \psi_{\alpha(\tau,y'_\tau+\psi_\tau)})d\tau\Big)\Big]ds \\
&\le 2M\int_0^t L_j(s)\Big[\sup_{\theta\in(-\infty,0]} \mu((y'(s+\theta)+\psi(s+\theta))ds + 2\int_0^w \rho_j(s,\tau)\sup_{\theta\in(-\infty,0]}(y'(s+\theta)+\psi(s+\theta))d\tau\Big]ds \\
&\le 2M\int_0^t L_j(s)\Big[\sup_{\tau\in[0,s]}\mu(y'(\tau)) + 2\tilde{\rho}\sup_{\tau\in[0,s]}\mu(y'(\tau))\Big]ds \\
&\le 2Mt_1(1+2\tilde{\rho})\int_0^t L_j(s)\sup_{s\in[0,w]}\mu(y'(s))ds \\
&\le 2Mt_1(1+2\tilde{\rho})\mu_{PC}(B)\cdot\int_0^t L_j(s)ds,
\end{aligned}$$

where $\tilde{\rho} = \int_0^w \rho_j(s,\tau)d\tau < \infty$. Therefore,

$$\mu(N(B))_{PC} \le 4Mt_1(1+2\tilde{\rho})\mu_{PC}(B)\cdot\int_0^t L_j(s)ds. \tag{15}$$

For $t \in (t_j, s_j]$, combining $(H5)(e)$, one has

$$\begin{aligned}
\mu(N(B_0)(t)) &= \mu\Big(g_j(t, y'_{\alpha(t,y't+\psi_t)} + \psi_{\alpha(t,y'_t+\psi_t)})\Big) \\
&\le \beta_j \sup_{\theta\in(-\infty,0]}\mu(y'(t+\theta)+\psi(t+\theta)) \\
&\le \beta_j \sup_{\tau\in[0,t]}\mu(y'(\tau)) \\
&\le \beta_j \sup_{t\in[0,w]}\mu(y'(\tau)) \\
&\le \beta_j\mu_{PC}(B).
\end{aligned}$$

Thus,

$$\mu(N(B))_{PC} \le 2\beta_j\mu_{PC}(B). \tag{16}$$

In a similar way, for $t \in (s_j, t_{j+1}]$, from (13), $(H1)(d), (H4), (H5)(e)$ and Lemma 2.5, there is

$$\mu(N(B))_{PC} \le 2\mu(N(B_0))_{PC} \le \Big[4M(t_{j+1}-s_j)(1+2\tilde{\rho})\int_0^t L_j(s)ds + 2\beta_j\Big]\mu_{PC}(B).$$

Then, N is a μ-set-contraction. We can conclude from Lemma 2.8 that N has at least one fixed point $y^* \in B_0 \subset \bar{D}_w$. Let $z(t) = y^*(t) + \psi(t),\ t \in (-\infty, w]$, therefore, one can easily obtain that z is a fixed point of the operator H, which implies z is a mild solution of (1). From the proof of Theorem 3.1, for any $z \in SAP_wPC(\mathrm{K}, G)$, there is $Hz \in SAP_wPC(\mathrm{K}, G)$, from which one can conclude that the problem (1) has at least one $\mathcal{S}$-asymptotically w-periodic mild solution. $\square$

4 | EXAMPLES

Set $G = L^2([0,\pi],\mathbb{R})$ be a complete normed vector space equipped with the L^2 norm $\|\cdot\|_2$. Set $\mathrm{K} = [0,\pi],\ 0 = s_0 < t_1 = \frac{\pi}{4} < s_1 = \frac{\pi}{2} < t_2 = \frac{3\pi}{4} = s_2 < t_3 = \pi,\ m = 2$. Define $Az = -\frac{\partial^2}{\partial x^2} z$ for $z \in \mathcal{D}(A)$ with $\mathcal{D}(A) = \{z \in G : \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2} \in G, z(0) = z(\pi) = 0\}$. From [39], A generates an analytic C_0-semigroup of bounded operators $(U(t))_{t\geq 0}$ on G, which is uniformly exponentially stable with $\|U(t)\| \leq 1$.

Example 1. Consider

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_{-\infty}^{t} e^{s-t} \frac{z(s-\alpha_1(s)\alpha_2(\|z(s)\|,x))}{7} ds \\ \qquad + \int_0^{\pi} |\sin(t-s)| \int_{-\infty}^{s} e^{2(\tau-s)} \frac{z(\tau-\alpha_1(\tau)\alpha_2(\|z(\tau)\|,x))}{7} d\tau ds, & (t,x) \in (s_j, t_{j+1}] \times [0,\pi], j = 0,1,2,\cdots,m, \\ z(t,x) = \dfrac{\sigma z(t-\alpha_1(t)\alpha_2(\|z(s)\|,x)) \cdot \sin(tj)}{j}, & \sigma > 0,\ (t,x) \in (t_j, s_j] \times [0,\pi],\ j = 1,2,\cdots,m, \\ z(t,0) = z(t,\pi) = 0, & t \in (0,w), \\ z(t,x) = \phi(t,x), & t \in (-\infty,0],\ x \in [0,\pi]. \end{cases} \tag{17}$$

For $(t,\xi) \in [0,w] \times \mathcal{D}$, where $\xi(\theta)(x) = \xi(\theta,x),\ (\theta,x) \in (-\infty,0] \times [0,\pi]$. Let $z(t)(x) = z(t,x),\ \alpha(t,\xi) = \alpha_1(t)\alpha_2(\|\xi(0)\|)$, then one gets

$$f(t,\xi,p\xi)(x) = \int_{-\infty}^{0} e^s \cdot \frac{\xi}{7} ds + p\xi(x),$$
$$g_j(t,\xi)(x) = \frac{\sigma\xi\sin(tj)}{j},$$
$$\text{where } p\xi(x) = \int_0^{\pi} |\sin(t-s)| \int_{-\infty}^{0} e^{2\tau} \cdot \frac{\xi}{7} d\tau ds.$$

Therefore, the problem (17) is transformed into the form of (1). And it is obvious that f is an $\mathcal{S}$-asymptotically w-periodic function on the bounded set $[0,\pi]$. In the following, we assume that $\alpha_j : [0,\infty) \to [0,\infty),\ j = 1,2$ are continuous all the time. Then, for $t \in [0,\pi]$, we have

$$\begin{aligned} \|f(t,\xi,p\xi)\|_2 &\leq \Big(\int_0^{\pi} \Big(\int_{-\infty}^{0} e^s \cdot \Big\|\frac{\xi}{7}\Big\| ds + \int_0^{\pi} |\sin(t-s)| \int_{-\infty}^{0} e^{2\tau} \cdot \Big\|\frac{\xi}{7}\Big\| d\tau ds \Big)^2 dx \Big)^{\frac{1}{2}} \\ &\leq \Big(\int_0^{\pi} \Big(\frac{1}{7} \int_{-\infty}^{0} e^s \cdot \sup\|\xi\| ds + \frac{1}{7} \int_{-\infty}^{0} e^{2s} \cdot \sup\|\xi\| ds \Big)^2 dx \Big)^{\frac{1}{2}} \leq \frac{2\pi^{\frac{1}{2}}}{7} \|\xi\|_{\mathcal{D}}, \end{aligned}$$

and

$$\|f(t,\xi_1,p\xi_1)-f(t,\xi_2,p\xi_2)\|_2 \leq \Big(\int\limits_0^{\pi}\Big(\int\limits_{-\infty}^{0} e^{s}\cdot\Big\|\frac{\xi_1}{7}-\frac{\xi_2}{7}\Big\|ds+\int\limits_0^{\pi}|\sin(t-s)|\int\limits_{-\infty}^{0} e^{2\tau}\cdot\Big\|\frac{\xi_1}{7}-\frac{\xi_2}{7}\Big\|d\tau ds\Big)^2 dx\Big)^{\frac{1}{2}}$$
$$\leq \Big(\int\limits_0^{\pi}\Big(\frac{1}{7}\int\limits_{-\infty}^{0} e^{s}\cdot\sup\|\xi_1-\xi_2\|ds+\frac{1}{7}\int\limits_{-\infty}^{0} e^{2s}\cdot\sup\|\xi_1-\xi_2\|ds\Big)^2 dx\Big)^{\frac{1}{2}}$$
$$\leq \frac{2\pi^{\frac{1}{2}}}{7}\|\xi_1-\xi_2\|_{\mathcal{D}}.$$

In addition, $g_j : [t_j, s_j]\times \boldsymbol{G}\to \boldsymbol{G}$ is continuous and for any $z\in \boldsymbol{G}$, there is

$$\lim_{t\to\infty,j\to\infty}\|g_{j+m}(t+\pi,z)-g_j(t,z)\|_2=\lim_{t\to\infty,j\to\infty}\Big(\int\limits_0^{\pi}\Big\|\frac{\sigma z(s)\sin(t+w)(j+m)}{j+m}-\frac{\sigma z(s)\sin(tj)}{j}\Big\|^2 ds\Big)^{\frac{1}{2}}$$
$$\leq \lim_{j\to\infty}\frac{2\sigma\pi^{\frac{1}{2}}}{j}\|z\|$$
$$=0,$$

and for any $z_1, z_2\in \boldsymbol{G}$, we have

$$\|g_j(t,z_1)-g_j(t,z_2)\|_2=\frac{\sigma}{j}\Big\|\int\limits_0^{\pi}|z_1(s)\sin(tj)-z_2(s)\sin(tj)|^2 ds\Big\|^{\frac{1}{2}}\leq \sigma\pi^{\frac{1}{2}}\|z_1-z_2\|_{\mathcal{D}}.$$

Furthermore,

$$\|g_j(t,z)\|=\Big(\int\limits_0^{\pi}\Big|\frac{\sigma z(s)\sin(tj)}{j}\Big|^2 ds\Big)^{\frac{1}{2}}\leq \frac{\sigma\pi^{\frac{1}{2}}}{j}\|z\|\leq \sigma\pi^{\frac{1}{2}}(1+\|z\|).$$

Then, the conditions in Theorem 3.1 are satisfied. From Theorem 3.1, we get the following result:

Proposition 1. If $\rho := \max \frac{2\delta\pi^{\frac{3}{2}}}{28\gamma}+\sigma\pi^{\frac{1}{2}} < 1$, then under the above conditions, the problem (17) has a unique $\mathcal{S}$-asymptotically π-periodic mild solution.

Example 2. We discuss briefly the existence of $\mathcal{S}$-saymptotically w-periodic mild soltuions for problem (17).

For each bounded set $B_1\subset \mathcal{D}$ and $B_2\in \boldsymbol{G}$, there holds

$$\mu(f(t,B_1,B_2))\leq \frac{2\pi^{\frac{1}{2}}}{7}\Big(\sup_{\theta\in(-\infty,0]}\mu(B_1(\theta))+\mu(B_2)\Big),$$

and for any $t\in(t_j,s_j],\ j=1,2,\cdots,m$, we can directly derive from the proof of Example 1 that

$$\|g_j(t,z)\|=\Big(\int\limits_0^{\pi}\Big|\frac{\sigma z(s)\sin(tj)}{j}\Big|^2 ds\Big)^{\frac{1}{2}}\pi^{\frac{1}{2}}\leq \frac{\sigma\pi^{\frac{1}{2}}}{j}\|z\|\leq \sigma\pi^{\frac{1}{2}}(1+\|z\|).$$

Besides, for each bounded set $B\subset\mathcal{D}$, one has

$$\mu(g_j(t,B))\leq \sigma\pi^{\frac{1}{2}}\sup_{\theta\in(-\infty,0]}\mu(B(\theta)),\ j=1,2,\cdots,m.$$

Therefore, the conditions in Theorem 3.2 are satisfied, then, the following proposition holds:

Proposition 2. Under the above assumptions, if $\frac{(1+2\bar{\rho})\pi^{\frac{5}{2}}}{7}+2\sigma\pi^{\frac{1}{2}} < 1$, then the problem (17) has at least one $\mathcal{S}$-asymptotically w-periodic mild solution in $[0,\pi]$.

5 | CONCLUSIONS

We have mainly considered the nonlinear non-instantaneous impulsive integro-differential equations with state-dependent delay. First, by utilizing Banach's fixed point theory, the uniqueness of $\mathcal{S}$-asymptotically w-periodic mild solution has been obtained. And then we have considered the existence of at least one $\mathcal{S}$-asymptotically w-periodic mild solution via the non-compactness operator semigroup theorem. However, compared with the classical instantaneous impulse differential system, the theoretical development of the existing non-instantaneous impulse differential system with delay is still lagging behind and the research results on the properties of the solutions are not perfect. What's more, during this process, we find it is difficult to prove the existence of periodic solutions for non-instantaneous impulsive differential equations with state-dependent delay. Therefore, in the future work, we can consider suitable conditions to ensure the existence of periodic solutions for non-instantaneous impulsive differential equations with state-dependent delay.

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