Differential inclusion obstacle problems with variable exponents and convection terms

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Abstract

The elliptic obstacle problems with variable exponents and multivalued reaction terms, depending on the gradient, are considered in this paper. Under general assumptions on the convection term, we prove two existence theorems of a (weak) nontrivial solution by using the surjective theorem and the Leray-Schauder fixed point theorem for multivalued mappings, respectively. Our assumptions are suitable and different from those required in the previous literature.

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1. Introduction and the main results

The $p(\cdot)$-Laplacian operator is motivated by numerous models arising in electrorheological fluids ([33, 49, 50]), elastic mechanics ([59]) and image restoration ([11, 52, 61]). For almost twenty years, there have been some existence and multiplicity results for elliptic equations with this operator. The interested readers may refer to [1, 2, 14, 23, 27, 28, 32, 34, 39, 40, 45, 51, 54, 55, 57, 58, 60] and the references therein. For the study the regularity of solutions of $p(\cdot)$-Laplacian equation, many results have been obtained [3, 6–9, 13, 15, 26, 35, 46, 56].

In this paper, we consider the following differential inclusion problem involving the $p(\cdot)$-Laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in f(x,u,\nabla u), & \text{in } \Omega,\\ u \le \Phi, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto 2^{\mathbb{R}}$ is a multivalued function depending on the gradient of the solution, $p(\cdot)$ is a logarithmic Hölder continuous with $1 < \min_{x\in\overline{\Omega}} p(x) \le \max_{x\in\overline{\Omega}} p(x) < N$, and the following condition holds:

(H_p) *there exists* $\xi_0 \in \mathbb{R}^N\backslash\{0\}$ *such that for any* $x \in \Omega$, *the map* $t \mapsto p(x + t\xi_0)$ *is monotone on* $t \in \{t : x + t\xi_0 \in \Omega\}$.

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Especially, when $\Phi \equiv +\infty$ and f is a single-valued function, the problem (P) reduces to

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

which was recently studied by Hammou-Azroul-Lahmi [57] and Yin-Li-Ke [1]. It should be noted that the single-valued function f depends on the gradient of the solution. Such functions are usually called *convection terms.* This means that problem (P) has a non-variational structure. Thus the well developed critical point theory can not be directly applied. To overcome this difficulty, Yin, Li and Ke in [57] obtained the existence of positive solutions for problem (P_1) by using Krasnoselskii fixed point theorem on the cone. In [1], by using the topological degree approach, Ait Hammou, Azroul and Lahmi obtained the existence of at least one solution of problem (P_1) under some suitable assumptions on $f = f(x,u,\nabla u)$. Moreover, there is a rich literature concerning the multiplicity of solutions of problem (P) without convection term, see e.g., [5, 14, 16, 19, 21, 24, 36, 42, 43, 53, 62] and the references therein.

In the case when $\Phi \equiv +\infty$, existence results for the differential inclusion problem (P), without convection term, have been obtained by several authors, see, for example, Ge [22], Ge-Zhou [31], Ge-Xue-Zhou [30], Ge-Xue [29] and Qian-Shen [48].

Unlike the aforementioned works, in this article, we study (P) in the two cases when the nonlinearity f satisfies subcritical growth and sub-$p(\cdot)$ linear growth, respectively. It has come to our knowledge that our setting is more general than those of [1, 57] and our approach contrast with other treatments of (P). We should also point out that our technique is based on the surjective result of Le [41] and the Leray-Schauder alternative theorem of Bader [4] for multivalued mappings, respectively.

To this end, we first assume that the multivalued mapping $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto 2^{\mathbb{R}}$ *is nonempty, compact and convex values and satisfies the following assumptions:*

(f_1) *for all* $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ *the map* $x \mapsto f(x,t,\xi)$ *is measurable;*

(f_2) *for almost all* $x \in \Omega$, $(t,\xi) \mapsto f(x,t,\xi)$ *is upper semi-continuous;*

(f_3) *there exist* $k \in L^{\frac{r(x)}{r(x)-1}}(\Omega)$, $1 < r(x) < p^*(x) := \frac{Np(x)}{N-p(x)}$ *and* $C > 0$ *such that for all* $w \in f(x,t,\xi)$

$$|w| \le C\big(k(x) + |t|^{r(x)-1} + |\xi|^{p(x)\frac{r(x)-1}{r(x)}}\big)$$

for a.a. $x \in \Omega$, *for all* $t \in \mathbb{R}$ *and for all* $\xi \in \mathbb{R}^N$*;*

(f_4) *there exist* $\delta \in L^1(\Omega)$ *and* $b_1, b_2 \ge 0$, *with* $b_1 + b_2\lambda_*^{-1} < 1$, *such that for all* $w \in f(x,t,\xi)$

$$wt \le b_1|\xi|^{p(x)} + b_2|t|^{p(x)} + \delta(x)$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where λ_* is infimum of all eigenvalues of the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

Remark 1.1. Thanks to Theorem 3.4 of [20], it is well known that condition (H_p) yields that

$$\lambda_* := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx} > 0.$$

The first result reads as follows.

Theorem 1.1. *Assume the validity of (H_p), (f_1)-(f_4) and that the following condition holds*

$$0 \notin f(x,0,0) \ \text{ for all } \ x \in \Omega.$$

Then problem (P) admits at least one nontrivial (weak) solution.

The next theorem concerns problems where the multivalued nonlinearity is $p(\cdot)$-sublinear. To this end, we assume that the multivalued mapping $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto 2^{\mathbb{R}}$ *has nonempty, compact and convex values and satisfies the following assumptions:*

(f_5) *$(x,t,\xi) \mapsto f(x,t,\xi)$ is graph measurable;*

(f_6) *for almost all $x \in \Omega$, $(t,\xi) \mapsto f(x,t,\xi)$ has a closed graph;*

(f_7) *there exist $k \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$ and $q \in C(\Omega)$ with $1 < q^- \leq q(x) \leq q^+ < p^-$, and a number $C > 0$ such that for all $w \in f(x,t,\xi)$*

$$|w| \leq C\big(k(x) + |t|^{q(x)-1} + |\xi|^{q(x)-1}\big)$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

The second result reads as follows.

Theorem 1.2. *Assume the validity of (H_p), (f_5)-(f_7) and that the following condition holds*

$$0 \notin f(x,0,0) \ \text{ for all } \ x \in \Omega.$$

Then problem (P) admits at least one nontrivial (weak) solution.

The article is organized as follows. First, we briefly introduce the definitions and collect some preliminary results for the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$ and we recall the surjective result and the Leray-Schauder alternative theorem for pseudomonotone operators. Finally, we complete the proofs of main Theorems 1.1-1.2 of the paper.

2. Preliminaries

In section, we start with the definition of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$, and the presentation of some properties of them. For more details we refer to [12, 17, 18, 38].

Let Ω and p be as stated in the Introduction. Let us put

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for every } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we write

$$h^- := \min_{x\in\overline{\Omega}} h(x) \text{ and } h^+ := \max_{x\in\overline{\Omega}} h(x).$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(\Omega)$, is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u|^{p(x)}dx$ is finite, that is,

$$L^{p(\cdot)}(\Omega) = \Big\{u \text{ is a measurable real valued function with } \int_\Omega |u|^{p(x)}dx < \infty\Big\}.$$

We endow this space with the so-called Luxemburg norm

$$|u|_{p(\cdot)} = \inf\Big\{\lambda > 0 \,:\, \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\Big\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

and it is equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}. \tag{2.1}$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. With these norms, the spaces $L^{p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable reflexive Banach spaces; see [18, 38] for the details.

Proposition 2.1 ([18]). *Set $\rho(u) = \int_\Omega |u|^{p(x)}dx$. Let $u \in L^{p(\cdot)}(\Omega)$. Then,*

(i) $|u|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$.

(ii) $|u|_{p(\cdot)\ge 1} \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$.

(iii) $|u|_{p(\cdot)\le 1} \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$.

Proposition 2.2 ([18, 38]). *(1) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for every $x \in \Omega$, then the embedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is continuous and compact.*

(2) In $W_0^{1,p(\cdot)}(\Omega)$, the well-known Poincaré inequality is still valid, namely there is a constant $C_0 > 0$ such that

$$|u|_{p(\cdot)} \le C_0 |\nabla u|_{p(\cdot)} \quad \textit{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

Thanks to Proposition 2.2(1), there is a constant $c_\vartheta > 0$ such that

$$|u|_{q(\,\cdot\,)} \leq c_\vartheta \|u\| \quad \text{for all } u \in W_0^{1,p(\,\cdot\,)}(\Omega).$$

In view of Proposition 2.2(2), $|\nabla u|_{p(\,\cdot\,)}$ is an equivalent norm in $W_0^{1,p(\,\cdot\,)}(\Omega)$. Thus, we can consider the equivalent norm $\|u\| = |\nabla u|_{p(\,\cdot\,)}$ for any $u \in W_0^{1,p(\,\cdot\,)}(\Omega)$.

Since the surjective theorem is a key tool to prove Theorem 1.1, it is useful to recall the next definition and results.

Definition 2.3. ([25, Definition 1.4.8]) Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot, \cdot \rangle$ the corresponding duality pairing. Let $F : X \to 2^{X^*}$ be a multivalued mapping, then

(i) F is called *pseudomonotone* if the following conditions hold:

(a) the set $F(u)$ is nonempty, bounded, closed, and convex in X^* for all $u \in X$;

(b) F is upper semi-continuous from each finite-dimensional subspace of X to X^* with respect to the weak topology;

(c) if $u_n \rightharpoonup u$ in X, $u_n^* \in F(u_n)$ and $\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0$, then for each element $v \in X$ there exists $u_v^* \in F(u)$ such that $\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle \geq \langle u_v^*, u - v\rangle$.

(ii) F is called *generalized pseudomonotone* if $u_n \rightharpoonup u$ in X, $u_n^* \rightharpoonup u^*$ in X^* and

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \leq 0 \quad \text{for all } u_n^* \in F(u_n),$$

then $u^* \in F(u)$ and $\lim_{n\to\infty} \langle u_n^*, u_n\rangle = \langle u^*, u\rangle$.

With these definitions, we have that every pseudomonotone mapping is generalized pseudomonotone, see [25, Proposition 1.4.11]. On the contrary, the generalized pseudomonotone mapping is pseudomonotone, cf. [25, Proposition 1.4.12], under the additional assumption of boundedness. The latter statement is perfectly described by the next result:

Lemma 2.4. *Let X be a real, reflexive Banach space, and let $F : X \to 2^{X^*}$ be a generalized pseudomonotone mapping. Assume that*

(a) the set $F(u)$ is nonempty, closed and convex in X^* for all $u \in X$;

(b) $F : X \to 2^{X^*}$ is bounded.

Then $F : X \to 2^{X^*}$ is pseudomonotone.

The main tool is based on the surjective result for multivalued mappings which is formulated by the sum of a maximal monotone multivalued operator and a bounded multivalued pseudomonotone mapping, we refer to [41, Theorem 2.2]) for more details. In the sequel, we use the notation $D(F)$ for the domain of F, and $B_R := \{u \in X \,:\, \|u\|_X < R\}$.

Lemma 2.5. *Let X be a real, reflexive Banach space, and let $\mathcal{G} : D(\mathcal{G}) \subset X \to 2^{X^*}$ be a maximal monotone operator. Let $\mathcal{H} : D(\mathcal{H}) \subset X \to 2^{X^*}$ be* *a bounded multivalued pseudomonotone operator and let $T \in X^*$. Assume that there exist $u_0 \in X$ and $R \geq \|u_0\|_X$ such that $D(\mathcal{G}) \cap B_R \neq \emptyset$ and $\langle \xi + \eta - T, u - u_0 \rangle > 0$ for all $u \in D(\mathcal{G})$ with $\|u_0\|_X = R$, for all $\xi \in \mathcal{G}(u)$ and for all $\eta \in \mathcal{H}(u)$. Then the inclusion $T \in \mathcal{G}(u) + \mathcal{H}(u)$ has a solution in $D(\mathcal{G})$.*

In the sequel, we present the multivalued generalization of the Leray-Schauder alternative theorem that we use in the proof of Theorem 1.2. More information on this subject may be found in the reference [4].

Lemma 2.6. *Let X, X_1 be two Banach spaces, and let $F : X \to \mathcal{P}_{wkc}(X_1)$ be upper semi-continuous from X into X_1 endowed with weak topology, where $\mathcal{P}_{wkc}(X_1) := \{M \subset X_1 : M \text{ is nonempty weakly compact and convex}\}$. Assume that $\Psi : X_1 \to X$ is a completely continuous and $\Phi = \Psi \circ F$ maps bounded sets into relatively compact sets. Then one of the following statements holds:*

(a) the set $B = \{u \in X : u \in \lambda\Phi(u), \lambda \in (0, 1)\}$ is unbounded, or

(b) Φ has a fixed point, i.e. there exists a $u \in X$, such that $u \in \Phi(u)$.

Now let us consider the $p(\,\cdot\,)$-Laplacian operator $\mathcal{L} : W_0^{1,p(\,\cdot\,)}(\Omega) \to (W_0^{1,p(\,\cdot\,)}(\Omega))^*$ defined by

$$\langle \mathcal{L}(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx, \ \forall u, v \in W_0^{1,p(\,\cdot\,)}(\Omega),$$

where $(W_0^{1,p(\,\cdot\,)}(\Omega))^*$ denotes the dual space of $W_0^{1,p(\,\cdot\,)}(\Omega)$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $W_0^{1,p(\,\cdot\,)}(\Omega)$ and $(W_0^{1,p(\,\cdot\,)}(\Omega))^*$. The properties of operator $\mathcal{L}$ are summarized in the following proposition, see Fan-Zhang [17, Theorem 3.1].

Proposition 2.7. *Set $E = W_0^{1,p(\,\cdot\,)}(\Omega)$, $\mathcal{L}$ is as above, then*

(1) $\mathcal{L} : E \to E^*$ is a continuous, bounded and strictly monotone operator.

(2) $\mathcal{L} : E \to E^*$ is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in E and $\limsup\limits_{n\to+\infty}\langle \mathcal{L}(u_n) - \mathcal{L}(u), u_n - u\rangle \leq 0$, implies $u_n \to u$ in E.

(3) $\mathcal{L} : E \to E^*$ is a homeomorphism.

3. Proof of Theorem 1.1

This section is devoted to prove the existence of nontrivial solutions of problem (P) by applying the Lemma 2.4. To this end, we need to define a subset $\mathcal{K}$ in E, where for simplicity from here $E = W_0^{1,p(\,\cdot\,)}(\Omega)$,

$$\mathcal{K} := \{u \in E : u(x) \leq \Phi(x) \text{ for a.e. } x \in \Omega\}.$$

It is easy to check that $0 \in \mathcal{K}$ and so the set $\mathcal{K}$ is a nonempty, closed and convex subset of E.

A function $u \in \mathcal{K}$ is said to be a (weak) *solution* of (P) if there exists $w \in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)$ such that $w(x) \in f(x, u(x), \nabla u(x))$ for a.a. $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(v-u) dx = \int_\Omega w(x)(v-u) dx, \tag{3.1}$$

is satisfied for all test functions $v \in \mathcal{K}$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Define the Nemytskii operator $\overline{N}_f : E \subseteq L^{r(\cdot)}(\Omega) \to L^{\frac{r(x)}{r(x)-1}}(\Omega)$ defined by

$$\overline{N}_f(u) =: \Big\{ w \in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega) : w(x) \in f\big(x, u(x), \nabla u(x)\big) \Big\}.$$

Moreover, let $i^* : L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega) \to E^*$ be the adjoint operator for the embedding $i : E \to L^{r(\cdot)}(\Omega)$. Then we define

$$N_f = i^* \circ \overline{N}_f : E \to E^*,$$

which is well-defined by condition (f_3). Also, let us consider the indicator function $I_{\mathcal{K}} : E \to \mathbb{R} \cup \{+\infty\}$ of $\mathcal{K}$ defined by

$$I_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{K}, \\ +\infty, & \text{if } u \notin \mathcal{K}. \end{cases}$$

It is easy to check that $I_{\mathcal{K}}$ is a convex, lower semi-continuous, proper functional from E to $[0, +\infty]$ with $D(I_{\mathcal{K}}) = \mathcal{K}$. With the above definitions, it is easy to see that $u \in \mathcal{K}$ is a (weak) *solution* of problem (P), if and only if u solves the following problem:

Find $u \in \mathcal{K}$ and $w \in N_f(u)$ such that

$$\langle \mathcal{L} - w, v - u \rangle + I_{\mathcal{K}}(v) - I_{\mathcal{K}}(u) \geq 0, \ \forall v \in E, \tag{3.2}$$

where $\mathcal{L}$ is given in Proposition 2.7.

Now, we define the multivalued operator $\mathcal{H} : E \to 2^{E^*}$ given as

$$\mathcal{H}(u) = \mathcal{L}(u) - N_f(u), \ \forall u \in E$$

Then problem (3.2) is equivalent to the following inclusion problem:

$$0 \in \mathcal{H}(u) + \partial I_{\mathcal{K}}(u), \tag{3.3}$$

where $\partial I_{\mathcal{K}}(u)$ is the subdifferential with respect to the u-variable in the sense of Clarke [10].

Set $\mathcal{G} = \partial I_{\mathcal{K}}$. Obviously, $\mathcal{G}$ is a maximal monotone operator. In view of Lemma 2.5, it suffices to show that,

(A_1) *$\mathcal{H}$ is bounded;*

(A_2) *$\mathcal{H}$ is pseudomonotone;*

(A_3) *there exists a constant $R > 0$ such that*

$$\langle u^* + w, u \rangle > 0, \ \forall u^* \in \mathcal{H}(u), \ \forall w \in \mathcal{G}(u), \tag{3.4}$$

for all $u \in E$ with $\|u\| = R$.

Verification of (A_1). By virtue of the growth condition (f_3), for any $u\in E$ and $w\in N_f(u)$, it follows from Proposition 2.1 that

$$\begin{aligned}
\|w\|_{E^*} =&\|i^*\eta\|_{E^*}\leq C_1|\eta|_{\frac{r(\cdot)}{r(\cdot)-1}}\\
\leq& C_1\Big(\int_\Omega |\eta|^{\frac{r(x)}{r(x)-1}}dx\Big)^{\frac{r^--1}{r^+}}+C_1\Big(\int_\Omega |\eta|^{\frac{r(x)}{r(x)-1}}dx\Big)^{\frac{r^+-1}{r^-}}\\
\leq& C_2\Big[\int_\Omega\Big(k(x)+|u|^{r(x)-1}+|\nabla u|^{p(x)\frac{r(x)-1}{r(x)}}\Big)^{\frac{r(x)}{r(x)-1}}dx\Big]^{\frac{r^--1}{r^+}}\\
&+C_2\Big[\int_\Omega\Big(k(x)+|u|^{r(x)-1}+|\nabla u|^{p(x)\frac{r(x)-1}{r(x)}}\Big)^{\frac{r(x)}{r(x)-1}}dx\Big]^{\frac{r^+-1}{r^-}}\\
\leq& C_3\Big(\int_\Omega(|k|^{\frac{r(x)}{r(x)-1}}+|u|^{r(x)}+|\nabla u|^{p(x)})dx\Big)^{\frac{r^--1}{r^+}}\\
&+C_3\Big(\int_\Omega(|k|^{\frac{r(x)}{r(x)-1}}+|u|^{r(x)}+|\nabla u|^{p(x)})dx\Big)^{\frac{r^+-1}{r^-}}\\
\leq& C_3\Big(|k|^{\frac{r^+}{r^--1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|k|^{\frac{r^-}{r^+-1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|u|^{r^+}_{r(\cdot)}+|u|^{r^-}_{r(\cdot)}\\
&+|\nabla u|^{p^+}_{p(\cdot)}+|\nabla u|^{p^-}_{p(\cdot)}\Big)^{\frac{r^--1}{r^+}}+C_3\Big(|k|^{\frac{r^+}{r^--1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|k|^{\frac{r^-}{r^+-1}}_{\frac{r(\cdot)}{r(\cdot)-1}}\\
&+|u|^{r^+}_{r(\cdot)}+|u|^{r^-}_{r(\cdot)}+|\nabla u|^{p^+}_{p(\cdot)}+|\nabla u|^{p^-}_{p(\cdot)}\Big)^{\frac{r^+-1}{r^-}},
\end{aligned}\tag{3.5}$$

for some constants $C_1,C_2,C_3>0$, where $\eta\in\overline{N}_f(u)$ is such that $i^*\eta=w$.

According to Proposition 2.2, the embedding $E\hookrightarrow L^{r(x)}(\Omega)$ is continuous. With this in mind, we have that there exists $C_r>0$ such that

$$|u|_{r(\cdot)}\leq C_r|\nabla u|_{p(\cdot)},\ \forall u\in E.\tag{3.6}$$

Hence, by (3.5) and (3.6), we can calculate that

$$\begin{aligned}
\|w\|_{E^*}\leq& C_1|\eta|_{\frac{r(\cdot)}{r(\cdot)-1}}\\
\leq& C_4\Big(|k|^{\frac{r^+}{r^--1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|k|^{\frac{r^-}{r^+-1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|\nabla u|^{r^+}_{p(\cdot)}+|\nabla u|^{r^-}_{p(\cdot)}\\
&+|\nabla u|^{p^+}_{p(\cdot)}+|\nabla u|^{p^-}_{p(\cdot)}\Big)^{\frac{r^--1}{r^+}}+C_4\Big(|k|^{\frac{r^+}{r^--1}}_{\frac{r(\cdot)}{r(\cdot)-1}}+|k|^{\frac{r^-}{r^+-1}}_{\frac{r(\cdot)}{r(\cdot)-1}}\\
&+|\nabla u|^{r^+}_{r(x)}+|\nabla u|^{r^-}_{r(x)}+|\nabla u|^{p^+}_{p(\cdot)}+|\nabla u|^{p^-}_{p(\cdot)}\Big)^{\frac{r^+-1}{r^-}},
\end{aligned}\tag{3.7}$$

for some $C_4>0$. This combined with Proposition 2.7(1) implies that $\mathcal{H}$ maps bounded sets into bounded sets. Thus (A_1) is satisfied.

Verification of (A_2). By the assumptions of (f_1)–(f_4), it is easy to see that $\mathcal{H}$ has nonempty, closed and convex values. Due to the Lemma 2.4 and (A_1), we only need to verify that $\mathcal{H}$ is a generalized pseudomonotone operator. Let $(u_n)_n \subset E$ are $(u_n^*)_n \subset E^*$ be two sequences such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } E \text{ and } u_n^* \rightharpoonup u^* \text{ in } E^*,\\
&\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \le 0,\\
&u_n^* \in \mathcal{H}(u_n),\ \forall n \in \mathbb{N}.
\end{aligned} \tag{3.8}$$

Recall that for each $n \in \mathbb{N}$ there exist $w_n \in N_f(u_n)$ and $\eta_n \in \overline{N}_f(u_n)$ such that $u_n^* = \mathcal{L}(u_n) - w_n = \mathcal{L}(u_n) - i^*\eta_n$. Moreover, due to the compact embedding $E \hookrightarrow L^{r(\,\cdot\,)}(\Omega)$, we can deduct that

$$u_n \rightharpoonup u \text{ in } L^{r(\,\cdot\,)}(\Omega). \tag{3.9}$$

Hence, by using (3.7) and (3.9), we deduce that the sequences $(w_n)_n$ is bounded in E^* and $(\eta_n)_n$ is bounded in $L^{\frac{r(\,\cdot\,)}{r(\,\cdot\,)-1}}(\Omega)$. Thus, from (3.8) and (3.9), we achieve that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \mathcal{L}(u_n), u_n - u\rangle\\
&\quad =\limsup_{n\to\infty}\langle \mathcal{L}(u_n), u_n - u\rangle - \limsup_{n\to\infty}\langle w_n, u_n - u\rangle\\
&\quad =\limsup_{n\to\infty}\langle \mathcal{L}(u_n), u_n - u\rangle - \limsup_{n\to\infty}\langle i^*\eta_n, u_n - u\rangle\\
&\quad =\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \le 0.
\end{aligned} \tag{3.10}$$

Since $\mathcal{L}$ is of type (S_+) by Proposition 2.7(2), we obtain $u_n \to u$ in E. This along with (3.8) implies

$$\lim_{n\to\infty}\langle u_n^*, u_n\rangle = \langle u^*, u\rangle.$$

Using the continuity of $\mathcal{L}$ again, it follows that

$$\mathcal{L}(u_n) \to \mathcal{L}(u) \text{ in } E \text{ as } n \to \infty.$$

Note that $(\eta_n)_n$ is bounded in $L^{\frac{r(\,\cdot\,)}{r(\,\cdot\,)-1}}(\Omega)$. Therefore, there is a subsequence (which we still denote by $(\eta_n)_n$) that converges weakly to a limit, say $\eta \in L^{\frac{r(\,\cdot\,)}{r(\,\cdot\,)-1}}(\Omega)$. We conclude from Mazur's theorem that there exists a sequence $(\xi_n)_n$ of convex combinations of $(\eta_n)_n$ such that

$$\begin{aligned}
&\xi_n \to \eta \text{ in } L^{\frac{r(\,\cdot\,)}{r(\,\cdot\,)-1}}(\Omega),\ \text{as } n \to \infty,\\
&\xi_n \to \eta,\ \text{ a.e. in } \ \Omega,\ \text{as } n \to \infty.
\end{aligned} \tag{3.11}$$

Since $\eta_n \in \overline{N}_f(u_n)$, it follows

$$\eta_n(x) \in f(x, u_n(x), \nabla u_n(x)) \text{ a.e. } x \in \Omega,$$

and consequently, by using condition (f_3) and (3.8), one can easily conclude that the sequence $(\eta_n)_n$ is bounded a.e. in Ω. Due to the second limit in (3.11), a subsequence of $(\eta_n)_n$ exists, still denoted by $(\eta_n)_n$, such that

$$\eta_n \to \eta, \ \text{ a.e. in } \ \Omega, \text{ as } n \to \infty.$$

Recalling that $u_n \to u$ in E. Then we have that

$$\begin{aligned} &u_n \to u, \text{ a.e. in } \ \Omega, \text{ as } n \to \infty, \\ &\nabla u_n \to \nabla u, \ \text{ a.e. in } \ \Omega, \text{ as } n \to \infty. \end{aligned}$$

On account of the above convergence properties and condition (f_3), we deduce from [44, Proposition 3.12] that

$$\eta(x) \in f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega,$$

which implies that $\eta \in \overline{N}_f(u)$, and consequently, $i^*\eta \in N_f(u)$. Hence, we conclude that $u^* = \mathcal{L}(u) - i^*\eta \in \mathcal{H}(u)$, this proves that $\mathcal{H}$ is generalized pseudomonotone. That is, (A_2) is satisfied.

Verification of (A_3). For any $u^* \in \mathcal{H}(u)$, we can find $\eta \in \overline{N}_f(u)$ such that $u^* = \mathcal{L}(u) - i^*\eta$. Then, by using $0 \in \mathcal{K}$, one has

$$\begin{aligned} \langle u^* + w, u\rangle &\geq \int_\Omega |\nabla u|^{p(x)}dx - \int_\Omega \eta(x)u(x)dx + I_{\mathcal{K}}(u) - I_{\mathcal{K}}(0) \\ &= \int_\Omega |\nabla u|^{p(x)}dx - \int_\Omega \eta(x)u(x)dx + I_{\mathcal{K}}(u). \end{aligned} \tag{3.12}$$

Note that $I_{\mathcal{K}} : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function, and so, by using Proposition 1.3.1 in [25], there exist $a_{\mathcal{K}}, b_{\mathcal{K}} > 0$ such that

$$I_{\mathcal{K}}(u) \geq -a_{\mathcal{K}}\|u\| - b_{\mathcal{K}}, \ \forall u \in E. \tag{3.13}$$

On the other hand, using the assumptions (f_4) and (H_p), we deduce from Remark 1.1 that

$$\begin{aligned} \int_\Omega \eta(x)u(x)dx \leq & b_1 \int_\Omega |\nabla u|^{p(x)}dx + b_2 \int_\Omega |u|^{p(x)}dx + |\delta|_1 \\ \leq & b_1 \int_\Omega |\nabla u|^{p(x)}dx + b_2\lambda_*^{-1} \int_\Omega |\nabla u|^{p(x)}dx + |\delta|_1 \\ = & \big(b_1 + b_2\lambda_*^{-1}\big) \int_\Omega |\nabla u|^{p(x)}dx + |\delta|_1. \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), together with (3.12), one obtains

$$\begin{aligned} \langle u^* + w, u\rangle \geq & \int_\Omega |\nabla u|^{p(x)}dx - \big(b_1 + b_2\lambda_*^{-1}\big) \int_\Omega |\nabla u|^{p(x)}dx \\ & - |\delta|_1 - a_{\mathcal{K}}\|u\| - b_{\mathcal{K}} \\ \geq & \big(1 - b_1 + b_2\lambda_*^{-1}\big) \min\{\|u\|^{p^-}, \|u\|^{p^+}\} - a_{\mathcal{K}}\|u\| \\ & - |\delta|_1 - b_{\mathcal{K}}. \end{aligned} \tag{3.15}$$

Therefore, since $p^+ \geq p^- > 1$ and $b_1 + b_2\lambda_*^{-1} < 1$, we take $R_0 > 0$ so large that for all $R \geq R_0$

$$\left(1 - b_1 + b_2\lambda_*^{-1}\right) \min\{R^{p^-}, R^{p^+}\} - a_{\mathcal{K}} R - |\delta|_1 - b_{\mathcal{K}} > 0. \tag{3.16}$$

This implies at once that (3.4) holds. Hence (A_3) is satisfied.

Therefore, all the assumptions of Lemma 2.5 are satisfied, so that, inclusion problem (3.3) has at least one solution $u_0 \in \mathcal{K}$ which is a solution of (3.2) and so, a solution from (P) in the sense of equality (3.1). Recalling that $0 \notin f(x, 0, 0)$ for all $x \in \Omega$, we conclude by the definition of $\mathcal{H}$ that $u_0 \neq 0$. Hence $u_0 \in E$ is a nontrivial (weak) solution of problem (P). The proof is complete. $\square$

Finally, we are ready to prove Theorem 1.2. Firstly, we show the next lemma.

Lemma 3.1. *Assume that (H_p) and (f_5)-(f_7) hold. Then for any $u \in E$, $\overline{N}_f(u)$ is a nonempty, closed and convex subset of $\left(L^{q(\cdot)}(\Omega)\right)^*$, and $\overline{N}_f(u)$ is upper semicontinuous from E into $\left(L^{q(\cdot)}(\Omega)\right)^*$ endowed with weak topology, and bounded on bounded sets.*

Proof. Define the Nemytskii operator $\overline{N}_f : E \subseteq L^{q(\cdot)}(\Omega) \to L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega)$ defined by

$$\overline{N}_f(u) =: \left\{ w \in L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega) : w(x) \in f\big(x, u(x), \nabla u(x)\big) \quad \text{a.e. in } \Omega \right\}.$$

The closedness and convexity of the value of $\overline{N}_f(\cdot)$ are clear. We now turn to prove that the nonemptyness of the value of $\overline{N}_f(\cdot)$. Let $u \in E$ and $(u_n)_n \subset E$ be a sequence of step function such that

$$u_n \to u \text{ in } L^{p(\cdot)}(\Omega),\ \nabla u_n \to \nabla u \text{ in } \left(L^{p(\cdot)}(\Omega)\right)^N,$$

$$|u_n| \leq |u|, \quad u_n \to u \ \text{ a.e. in } \ \Omega,$$

$$|\nabla u_n| \leq |\nabla u|, \quad \nabla u_n \to \nabla u \ \text{ a.e. in } \ \Omega.$$

Then, for every n, it follows from (f_5) that $x \mapsto f(x, u_n(x), \nabla u_n(x))$ is measurable from Ω into $\mathcal{P}_{kc}(\mathbb{R})$, where

$$\mathcal{P}_{kc}(\mathbb{R}) := \{M \subset \mathbb{R} : M \text{ is nonempty compact and convex}\}.$$

Therefore, using the Kuratowski and Ryll-Nardzewski selection theorem (see [37]), we conclude that there exists a measurable selector

$$v_n : \Omega \to \mathbb{R} \text{ such that } v_n(x) \in f(x, u_n(x), \nabla u_n(x)) \ \text{ for a.e. } \ x \in \Omega.$$

By using (f_7) again, we have

$$\begin{aligned} |v_n(x)| \leq & C(k(x) + |u_n(x)|^{q(x)-1} + |\nabla u_n(x)|^{q(x)-1}) \\ \leq & C(k(x) + |u(x)|^{q(x)-1} + |\nabla u(x)|^{q(x)-1}), \end{aligned}$$

which implies that $(v_n)_n \subset \big(L^{q(\cdot)}(\Omega)\big)^*$ is bounded and consequently, we can assume that $v_n \rightharpoonup v$ in $\big(L^{q(\cdot)}(\Omega)\big)^*$. Then, it follows from Theorem 3.1 in [47] and (f_6) that

$$\begin{aligned} v(x) \in \text{conv}\,\overline{\lim}\{v_n(x)\}_{n\geq 1} \subseteq & \text{conv}\,\overline{\lim} f(x, u_n(x), \nabla u_n(x)) \\ \subseteq & f(x, u(x), \nabla u(x)) \text{ a.e. on } \Omega. \end{aligned}$$

Consequently, $v \in \overline{N}_f(u)$. This fact, together with $v \in \big(L^{q(\cdot)}(\Omega)\big)^*$, gives that $\overline{N}_f$ has nonempty values.

Finally, it remains to prove that the upper semi-continuity of $\overline{N}_f$ from E into $\big(L^{q(\cdot)}(\Omega)\big)^*$ endowed with weak topology. For this we need to verify that

$$\overline{N}_f(C) := \{u \in E : \overline{N}_f(u) \cap C \neq \emptyset\}$$

is closed for any weakly closed subset C of $\big(L^{q(\cdot)}(\Omega)\big)^*$.

Let $(u_n)_n \subset \overline{N}_f(C)$ and assume that $u_n \to u$ in E. From the fact that the embedding from E to $L^{q(\cdot)}(\Omega)$ is continuous, we can find $M > 0$ such that

$$|u_n|_{q(\cdot)} \leq M \text{ for all } n \geq 1.$$

Let $v_n \in \overline{N}_f(u_n) \cap C$, then by (f_7), one has

$$|v_n(x)| \leq C\big(k(x) + |u_n(x)|^{q(x)-1} + |\nabla u_n(x)|^{q(x)-1}\big) \text{ a.e. in } \Omega,$$

and consequently $(v_n)_n \subset \big(L^{q(\cdot)}(\Omega)\big)^*$ is bounded. Thus, we can assume that $v_n \rightharpoonup v$ in $\big(L^{q(\cdot)}(\Omega)\big)^*$. As above we can easily deduce that $v \in \overline{N}_f(u)$. Also $v \in C$ and so $v \in \overline{N}_f(u) \cap C$, i.e., $u \in \overline{N}_f(C)$, which proves the desired upper semi-continuity of $\overline{N}_f$. Finally, using condition (f_7), one can easily conclude that $\overline{N}_f$ is bounded. $\square$

Proof of Theorem 1.2. In view of Lemma 3.1, $\overline{N}_f(\cdot)$ has values in $P_{wkc}\big(\big(L^{q(\cdot)}(\Omega)\big)^*\big)$ and is upper semi-continuity into $\big(L^{q(\cdot)}(\Omega)\big)^*$ endowed with weak topology. Consequently, problem (P) is equivalent to the following fixed point problem:

$$u \in \mathcal{L}^{-1}\overline{N}_f(u). \tag{3.17}$$

Using (f_7) and recalling that $\mathcal{L} : \big(L^{q(\cdot)}(\Omega)\big)^* \to E$ is completely continuous, we have that the multifunction $u \mapsto \mathcal{L}^{-1}\overline{N}_f(u)$ is compact.

To solve inclusion problem (3.17), we will apply the Leray-Schauder alternative theorem introducing in Section 2. To do this, we first claim that the set

$$B = \{u \in E : u \in \lambda\mathcal{L}^{-1}\overline{N}_f(u) \text{ for some } \lambda \in (0, 1)\}$$

is bounded.

Indeed, let $u \in B$, then we have

$$\begin{aligned}
&-\Delta_{p(x)}\big(\frac{u}{\lambda}\big) \in \overline{N}_f(u)\\
\Rightarrow &-\Delta_{p(x)}\big(\frac{u}{\lambda}\big) = v \;\text{ with } v \in \overline{N}_f(u)\\
\Rightarrow &\langle -\Delta_{p(x)}\big(\frac{u}{\lambda}\big), \frac{u}{\lambda}\rangle_{E^*E} = \langle v, \frac{u}{\lambda}\rangle_{V^*V},
\end{aligned} \tag{3.18}$$

where $V = L^{q(\cdot)}(\Omega)$, since the embedding $E \hookrightarrow V$ is continuous. By simple calculations, we conclude that

$$\langle -\Delta_{p(x)}\big(\frac{u}{\lambda}\big), \frac{u}{\lambda}\rangle_{E^*E} = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{\lambda^{p(x)}}dx \geq \frac{1}{\lambda^{p^-}}\int_\Omega |\nabla u(x)|^{p(x)}dx. \tag{3.19}$$

Combining (3.18) with (3.19) and using the Hölder inequality, it follows that

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)}dx \leq& \lambda^{p^-}\langle v, \frac{u}{\lambda}\rangle_{V^*V}\\
\leq& 2\lambda^{p^- -1}|v|_{\frac{q(\cdot)}{q(\cdot)-1}}|u|_{q(\cdot)}\\
\leq& 2\lambda^{p^-}c_q|v|_{\frac{q(\cdot)}{q(\cdot)-1}}\|u\|\\
\leq& 2c_q|v|_{\frac{q(\cdot)}{q(\cdot)-1}}\|u\|,
\end{aligned} \tag{3.20}$$

where c_q is the best constants for the continuous embeddings $E \hookrightarrow L^{q(\cdot)}(\Omega)$.

Moreover, it follows again from (f_7) that

$$\begin{aligned}
|v|_{\frac{q(\cdot)}{q(\cdot)-1}} \leq& C\big|k(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\big|_{\frac{q(\cdot)}{q(\cdot)-1}}\\
\leq& C(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + \big||u|^{q(x)-1}\big|_{\frac{q(\cdot)}{q(\cdot)-1}} + \big||\nabla u|^{q(x)-1}\big|_{\frac{q(\cdot)}{q(\cdot)-1}}).
\end{aligned} \tag{3.21}$$

Let us show that

$$||u|^{q(x)-1}|_{\frac{q(\cdot)}{q(\cdot)-1}} \leq |u|_{q(x)}^{q^+-1} + 2. \tag{3.22}$$

Indeed, one has:

(a) If $|u|_{q(\cdot)} \geq 1$, then $||u|^{q(\cdot)-1}|_{\frac{q(\cdot)}{q(\cdot)-1}} \leq |u|_{q(\cdot)}^{q^+-1}$.

This is seen as follows: According to Proposition 2.1, to prove (a), it is equivalent to prove that $|u|_{q(\cdot)} \geq 1$ implies

$$\int_\Omega \frac{|u(x)|^{q(x)}}{|u|_{q(\cdot)}^{(q^+-1)\frac{q(x)}{q(x)-1}}}dx = \int_\Omega \frac{|u(x)|^{\alpha(x)}}{|u|_{q(x)}^{(\alpha^+-1)\frac{q(x)}{q(x)-1}}}dx \leq 1.$$

This inequality is justified as follows. Since $|u|_{q(\cdot)} \geq 1$ and

$$\begin{aligned}(q^+ - 1)\frac{q(x)}{q(x)-1} - q(x) =& \alpha^+ \frac{q(x)}{q(x)-1} - (q(x) + \frac{q(x)}{q(x)-1})\\ =& q^+ \frac{q(x)}{q(x)-1} - \alpha(x)\frac{q(x)}{q(x)-1}\\ =& \frac{q(x)}{q(x)-1}(q^+ - q(x))\\ \geq& 0,\end{aligned}$$

we deduce that

$$\frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{(q^+-1)\frac{q(x)}{q(x)-1}}} = \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}} \frac{1}{|u|_{q(x)}^{(q^+-1)\frac{q(x)}{q(x)-1}-q(x)}} \leq \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}},$$

which implies that

$$\int_\Omega \frac{|u(x)|^{(q(x)-1)\frac{q(x)}{q(x)-1}}}{|u|_{q(x)}^{(q^+-1)\frac{q(x)}{q(x)-1}}} dx \leq \int_\Omega \frac{|u(x)|^{q(x)}}{|u|_{q(x)}^{q(x)}} dx = 1,$$

and the prove of (a) is complete.

(b) If $|u|_{q(\cdot)} < 1$, then $||u|^{q(\cdot)-1}|_{\frac{q(\cdot)}{q(\cdot)-1}} < 2$.

In fact, by using Proposition 2.1(iii) and noticing that

$$|u|_{q(\cdot)} < \int_\Omega |u(x)|^{q(x)} dx + 1,$$

we obtain

$$||u|^{q(x)-1}|_{\frac{q(\cdot)}{q(\cdot)-1}} < \int_\Omega |u(x)|^{q(x)} dx + 1 < 1 + 1 = 2.$$

Combining the previous consequence of (a) and (b), we complete the proof of (3.22). In the similar way as the proof of (3.22), we also have

$$||\nabla u|^{q(x)-1}|_{\frac{q(\cdot)}{q(\cdot)-1}} \leq |\nabla u|_{q(\cdot)}^{q^+-1} + 2. \tag{3.23}$$

Combining (3.21) with (3.22), together with (3.23), one obtains

$$|v|_{\frac{q(\cdot)}{q(\cdot)-1}} \leq C(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + |u|_{q(\cdot)}^{q^+-1} + |\nabla u|_{q(\cdot)}^{q^+-1} + 4). \tag{3.24}$$

Putting (3.24) into (3.20), we conclude

$$\int_\Omega |\nabla u(x)|^{p(x)} dx \leq 2c_q C(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + |u|_{q(\cdot)}^{q^+-1} + |\nabla u|_{q(\cdot)}^{q^+-1} + 4)\|u\|. \tag{3.25}$$

Recalling that the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and using Proposition 2.2(2), we can deduct the estimate

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)}dx \leq& 2c_q C\Big(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + c_1|u|_{p(\cdot)}^{q^+-1} + c_2|\nabla u|_{p(\cdot)}^{q^+-1} + 4)\|u\| \\ =& 2c_q C\Big(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + (c_1 C_0^{q^+-1} + c_2)\|u\|^{q^+-1} + 4\Big)\|u\| \\ \leq& c_3\Big(|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + \|u\|^{q^+-1} + 4)\|u\|,\end{aligned} \tag{3.26}$$

for positive constants c_1, c_2, c_3.

Without loss of generality, we may assume that $\|u\| = |\nabla u|_{p(\cdot)} > 1$, otherwise, B is bounded set. Obviously,

$$\int_\Omega |\nabla u(x)|^{p(x)}dx \geq \|u\|^{p^-}.$$

This fact combined with (3.26) implies that

$$\|u\|^{p^- -1} \leq c_3\|u\|^{q^+-1} + c_3|k|_{\frac{q(\cdot)}{q(\cdot)-1}} + 4c_3,$$

and consequently, the set B is bounded (since $q^+ < p^-$). According to Lemma 2.6, we know that there exits a $u_0 \in E$, such that

$$u_0 \in \mathcal{L}^{-1}\overline{N}_f(u_0),$$

that is, u_0 is a weak solution of problem (P). Recalling that $0 \notin f(x, 0, 0)$ for all $x \in \Omega$, we conclude by the definition of $\overline{N}_f$ that $u_0 \neq 0$. Hence $u_0 \in E$ is a nontrivial weak solution of problem (P). □

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