On a nonlinear transmission eigenvalue problem with a Neumann-Robin boundary condition

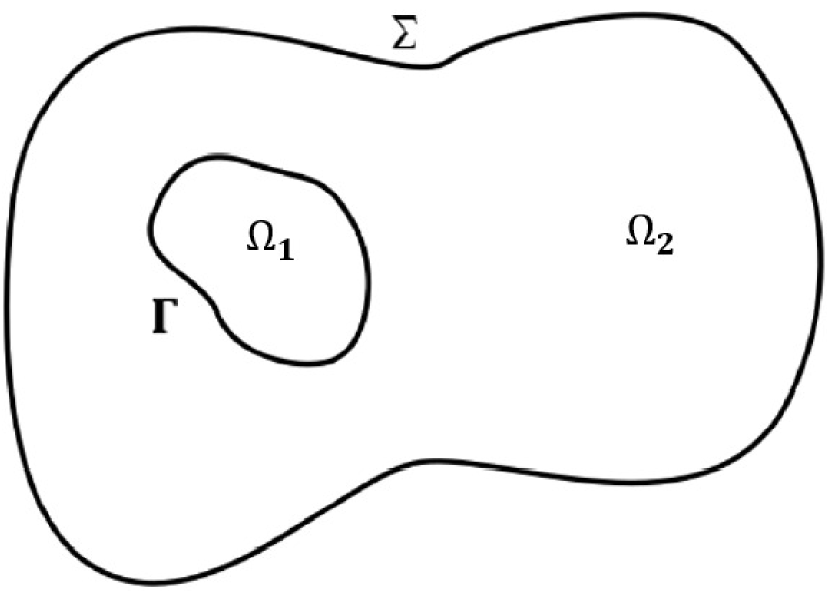
Luminita Barbu[1], Andreea Burlacu[1], and Gheorghe Morosanu[2]

[1]Ovidius University of Constanta
[2]Babes-Bolyai University

February 22, 2024

Abstract

Let Ω be a bounded domain in $\mathbb{R}^N, N\geq 2,$ with smooth boundary Σ and let Ω_1 be a subdomain of Ω with smooth boundary $\Gamma,$ such that $\overline{\Omega}_1\subset \Omega$. Denote $\Omega_2 = \Omega \setminus \overline{\Omega}_1.$ Consider the transmission eigenvalue problem \begin{equation*} \left\{\begin{array}{l} -\Delta_p u_1+\gamma_1(x)\mid u_1\mid ^{r-2}u_1=\lambda \mid u_1\mid ^{p-2}u_1\ \ \mbox{in} ~ \Omega_1,\\[1mm] -\Delta_q u_2+\gamma_2(x)\mid u_2\mid ^{s-2}u_2=\lambda \mid u_2\mid ^{q-2}u_2\ \ \mbox{in} ~ \Omega_2,\\[1mm] u_1=u_2,~~\frac{\partial u_1}{\partial\nu_{p}}=\frac{\partial u_2}{\partial\nu_{q}} ~~ \mbox{on} ~ \Gamma,\\[1mm] \frac{\partial u_2}{\partial\nu_{q}}+\beta (x) \mid u_2\mid^{\zeta-2} u_2=0 ~~ \mbox{on} ~ \Sigma, \end{array}\right. \end{equation*} where λ is a real parameter $p, q, r, s, \zeta \in (1, \infty)$ and $\gamma_i\in L^{\infty}(\Omega_i), ~i=1, 2, \beta\in L^{\infty}(\Sigma),$ $\beta\geq 0$ a.e. on $\Sigma.$ Under additional suitable assumptions on p, q, r, s, ζ we prove the existence of a sequence of eigenvalues $\big(\lambda_n\big)_n, \lambda_n\rightarrow \infty.$ The proof is based on the Lusternik-Schnirelmann theory on C^1- manifolds.



Received ...; | Revised ...; | Accepted ...

DOI: xxx/xxxx

ARTICLE TYPE

On a nonlinear transmission eigenvalue problem with a Neumann-Robin boundary condition†

Luminiţa Barbu*[1] | Andreea Burlacu[1] | Gheorghe Moroşanu[2,3]

[1]Faculty of Mathematics and Informatics, Ovidius University, Constanţa, Romania
[2]Academy of Romanian Scientists,, Bucharest, Romania
[3]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania

Correspondence
*Luminiţa Barbu, 124 Mamaia Blvd, 900527 Constanţa, Romania. Email: lbarbu@univ-ovidius.ro

Summary

Let Ω be a bounded domain in $\mathbb{R}^N,\ N \geq 2$, with smooth boundary Σ and let Ω_1 be a subdomain of Ω with smooth boundary Γ, such that $\overline{\Omega}_1 \subset \Omega$. Denote $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Consider the transmission eigenvalue problem

$$\begin{cases} -\Delta_p u_1 + \gamma_1(x) \mid u_1 \mid^{r-2} u_1 = \lambda \mid u_1 \mid^{p-2} u_1 \ \text{ in } \Omega_1, \\ -\Delta_q u_2 + \gamma_2(x) \mid u_2 \mid^{s-2} u_2 = \lambda \mid u_2 \mid^{q-2} u_2 \ \text{ in } \Omega_2, \\ u_1 = u_2, \ \frac{\partial u_1}{\partial \nu_p} = \frac{\partial u_2}{\partial \nu_q} \ \text{ on } \Gamma, \\ \frac{\partial u_2}{\partial \nu_q} + \beta(x) \mid u_2 \mid^{\zeta-2} u_2 = 0 \ \text{ on } \Sigma, \end{cases}$$

where λ is a real parameter, $p, q, r, s, \zeta \in (1, \infty)$, and $\gamma_i \in L^\infty(\Omega_i),\ i = 1, 2,\ \beta \in L^\infty(\Sigma),\ \beta \geq 0$ a.e. on Σ. Under additional suitable assumptions on p, q, r, s, ζ we prove the existence of a sequence of eigenvalues $\left(\lambda_n\right)_n,\ \lambda_n \to \infty$. The proof is based on the Lusternik-Schnirelmann theory on C^1- manifolds.

KEYWORDS:
Nonlinear transmission problem, $p-$Laplacian, Sobolev spaces, Krasnosel'skiĭ genus, Lusternik–Schnirelmann theory, C^1-manifold

MSC CLASSIFICATION
35J50; 35J55; 35P30

1 | INTRODUCTION

Consider a bounded domain $\Omega \subset \mathbb{R}^N,\ N \geq 2$, with smooth boundary Σ, and a subdomain Ω_1 with smooth boundary Γ, such that $\overline{\Omega}_1 \subset \Omega$, as in Fig. 1 below, where $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.

Consider the following transmission eigenvalue problem

$$\begin{cases} -\Delta_p u_1 + \gamma_1(x) \mid u_1 \mid^{r-2} u_1 = \lambda \mid u_1 \mid^{p-2} u_1 \ \text{ in } \Omega_1, \\ -\Delta_q u_2 + \gamma_2(x) \mid u_2 \mid^{s-2} u_2 = \lambda \mid u_2 \mid^{q-2} u_2 \ \text{ in } \Omega_2, \\ u_1 = u_2, \ \frac{\partial u_1}{\partial \nu_p} = \frac{\partial u_2}{\partial \nu_q} \ \text{ on } \Gamma, \\ \frac{\partial u_2}{\partial \nu_q} + \beta(x) \mid u_2 \mid^{\zeta-2} u_2 = 0 \ \text{ on } \Sigma, \end{cases} \tag{1}$$

where λ is a real parameter.

†On a nonlinear transmission eigenvalue problem.

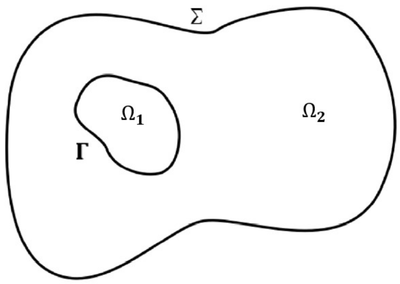


FIGURE 1

As usual, for $\theta \in (1, \infty)$, we denote by Δ_θ the θ-Laplace operator, i.e., $\Delta_\theta u = \operatorname{div}(|\nabla u|^{\theta-2} \nabla u)$.

In the second transmission condition on Γ, $\partial/\partial \nu_\theta$, $\theta \in \{p, q\}$, denote the conormal derivatives corresponding to the differential operators of the problem, i.e.,

$$\frac{\partial v}{\partial \nu_\theta} := |\nabla v|^{\theta-2} \nabla v \cdot \nu_\theta ,$$

with ν_p being the outward unit normal on the boundary Γ of Ω_1 pointing outward and $\nu_q = -\nu_p$.

Throughout the paper we will assume that the following conditions are satisfied

$(h)_1$ $p, q, r, s, \zeta \in (1, \infty)$, $\zeta < q_*$,

$$\begin{aligned} r < p\Big(1 + \frac{p}{N}\Big) \ \text{ in case } \ (r > p \text{ and } p < N), \\ s < q\Big(1 + \frac{q}{N}\Big) \ \text{ in case } \ (s > q \text{ and } q < N) \end{aligned} \tag{2}$$

(here q_* denotes the critical Sobolev exponent for the boundary trace embedding defined in Remark 1 below);

$(h)_2$ $\gamma_i \in L^\infty(\Omega_i)$, $i = 1, 2$, $\beta \in L^\infty(\Sigma)$, $\beta \geq 0$ a.e. on Σ.

Since function β in (h_2) is allowed to be the null function, we call the boundary condition $(1)_4$ a *Neumann-Robin boundary condition*.

Note that a similar transmission eigenvalue problem was considered in[2], but here we have a different division of Ω into subdomains Ω_1 and Ω_2, as well as different boundary conditions.

Remark 1. Recall that, given a smooth domain $D \subset \mathbb{R}^N$ and $\theta > 1$, the critical Sobolev exponent θ^* is defined by $\theta^* := \frac{\theta N}{N-\theta}$ if $1 < \theta < N$ and $\theta^* := \infty$ otherwise. If $\theta < N$, we have $W^{1,\theta}(\Omega) \hookrightarrow L^\eta(\Omega)$ continuously if $1 \leq \eta \leq \theta^*$ and compactly if $1 \leq \eta < \theta^*$, $W^{1,N}(\Omega) \hookrightarrow L^\eta(\Omega)$ compactly if $1 \leq \eta < \infty$ and $W^{1,\theta}(\Omega) \hookrightarrow C(\overline{\Omega})$ compactly if $\theta > N$ (see, for example,[4, Section 9.3],[5, Theorem 3.9.52]).

Recall also that there is a compact boundary trace embedding $W^{1,\theta}(\Omega) \hookrightarrow L^\eta(\partial D)$ for every $\eta \in [1, \theta_*)$ and similarly as before in the other ranges of η. Here we denote by $\theta_* := \frac{\theta(N-1)}{N-\theta}$ if $\theta < N$ and $\theta_* := \infty$ otherwise (see, for example,[1]).

We assume in what follows that $p \leq q$. This does not restrict the generality, as can be seen by checking the proofs of our main result below (Theorem 1).

Definition 1. A *weak solution* of problem (1) is a pair $u = (u_1, u_2) \in W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2)$, such that u_i satisfies the equation $(1)_i$ on Ω_i in the sense of distributions, $i = 1, 2$, and u_1, u_2 satisfy the boundary and transmission conditions $(1)_{3,4}$ in the sense of traces.

Obviously, any solution $u = (u_1, u_2)$ of problem (1) can be identified with an element u of the space

$$W := \big\{u \in W^{1,p}(\Omega) : u|_{\Omega_2} \in W^{1,q}(\Omega_2)\big\},$$

where $u|_{\Omega_i} = u_i$, $i = 1, 2$.

For $1 < \theta \leq \infty$, the Lebesgue norms of the spaces $L^\theta(\Omega_i)$ and $L^\theta(\Sigma)$ will be denoted by $\|\cdot\|_{i\theta}$, $i = 1, 2$, and $\|\cdot\|_{\partial\theta}$, respectively.

We endow W with the norm

$$\| u \| := \| u_1 \|_1 + \| u_2 \|_2 \ \ \forall\, u = (u_1, u_2) \in W, \tag{3}$$

where $\| \cdot \|_i,\ i = 1, 2$, are defined by

$$\|u_1\|_1 := \|\nabla u_1\|_{1p} + \|u_1\|_{1p},\ \ \|u_2\|_2 := \|\nabla u_2\|_{2q} + \|u_2\|_{2q}. \tag{4}$$

Remark 2. The space W defined before can be identified with the space

$$\widetilde{W} := \{\widetilde{u} = (u_1, u_2) \in W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2);\ u_1 = u_2 \ \text{ on } \Gamma\}, \tag{5}$$

which shows that W is a reflexive Banach space, as $\widetilde{W}$ is a closed subspace of the reflexive product $W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2)$ with reflexive factors (see [2, Remark 1.1]).

Definition 2. The real number λ is said to be an eigenvalue of the problem (1) if (1) has a weak solution $\widetilde{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in \widetilde{W} \setminus \{(0,0)\}$. In this case $\widetilde{u}_\lambda$ is called an eigenfunction of the problem (1) corresponding to the eigenvalue λ, and the pair $(\lambda, \widetilde{u}_\lambda)$ is called an eigenpair of the problem (1).

The next result gives a characterization of the eigenvalues of problem (1).

Proposition 1. The real number λ is an eigenvalue of the problem (1) if and only if there exists $\widetilde{u}_\lambda = (u_{1\lambda}, u_{2\lambda}) \in \widetilde{W} \setminus \{(0,0)\}$, such that for all $(v_1, v_2) \in \widetilde{W}$

$$\begin{aligned}
&\int_{\Omega_1} | \nabla u_{1\lambda} |^{p-2}\, \nabla u_{1\lambda} \cdot \nabla v_1 \ dx + \int_{\Omega_2} | \nabla u_{2\lambda} |^{q-2}\, \nabla u_{2\lambda} \cdot \nabla v_2 \ dx \\
&\quad + \int_{\Omega_1} \gamma_1 \mid u_{1\lambda} \mid^{r-2} u_{1\lambda} v_1 \ dx + \int_{\Omega_2} \gamma_2 \mid u_{2\lambda} \mid^{s-2} u_{2\lambda} v_2 \ dx + \int_{\Sigma} \beta \mid u_{2\lambda} \mid^{\zeta-2} u_{2\lambda} v_2 \ d\sigma \\
&\quad = \lambda\Big(\int_{\Omega_1} \mid u_{1\lambda} \mid^{p-2} u_{1\lambda} v_1 \ dx + \int_{\Omega_2} \mid u_{2\lambda} \mid^{q-2} u_{2\lambda} v_2 \ dx \Big).
\end{aligned} \tag{6}$$

The proof of this result is easy. It can be achieved by using arguments similar to those from the proof of Proposition 1.1 in Barbu-Moroşanu-Pintea[2], so we omit it.

For $\rho > 0$, consider the subset $\mathcal{M}_\rho$ of $\widetilde{W}$ defined by

$$\mathcal{M}_\rho := \Big\{ \widetilde{u} = (u_1, u_2) \in \widetilde{W};\ \frac{1}{p} \int_{\Omega_1} \mid u_1 \mid^p dx + \frac{1}{q} \int_{\Omega_2} \mid u_2 \mid^q dx = \rho \Big\}. \tag{7}$$

It is easy to verify that $\mathcal{M}_\rho$ has an infinite number of nonzero elements.

Our goal is to use the Lusternik-Schnirelmann theory on C^1-manifolds to investigate the eigenvalues of problem (1). Specifically, we shall prove the following result.

Theorem 1. Assume that (h_1) and (h_2) are fulfilled. Then, for any $\rho > 0$, there is a sequence of eigenpairs $\big(\lambda_n, \pm(u_{1n}, u_{2n})\big)_n$ of problem (1), with $\big((u_{1n}, u_{2n})\big)_n \subset \mathcal{M}_\rho$ and $\lambda_n \to \infty$ as $n \to \infty$.

Transmission problems arise in various applications in fluid mechanics, physics, chemistry, biology, etc. See, e.g., Fife[6], Nicaise[14], Pflüger[15]. So, it is important to investigate such kind of problems. Let us recall, for instance, that Figueiredo and Montenegro[7] proved that the following elliptic transmission problem in $\mathbb{R}^2$

$$\begin{cases}
-\Delta u_1 = f(x, u_1) \text{ in } \Omega_1, \\
-\Delta u_2 = g(x, u_2) \ \text{ in } \Omega_2, \\
u_1 = u_2,\ \ \frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} \ \text{ on } \Gamma, \\
u_2 = 0 \ \text{ on } \Sigma,
\end{cases}$$

with exponential nonlinearities of critical type, has a nontrivial solution. Also, the transmission problem,

$$\begin{cases} -\Delta u_1 = \lambda f(x, u_1) \text{ in } \Omega_1, \\ -\Delta u_2 =\mid u_2 \mid^{2^*-2} u_2 \ \text{ in } \Omega_2, \\ u_1 = u_2, \ \frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} \ \text{ on } \Gamma, \\ u_2 = 0 \ \text{ on } \Sigma, \end{cases}$$

with critical growth, was studied by the same authors in[8]. Other existence results for nonlinear transmission problems, approached by variational arguments, are treated for instance in[9,11,12,15], and the references therein.

The nonlinear transmission eigenvalue problem (1) we investigate here is closely related to the problems mentioned above.

2 | PRELIMINARIES

We start this section by recalling some basic notions on the Krasnosel'skiĭ's genus which will be used in the proof of our main result (Theorem 1).

Let X be a real Banach space. We denote by X^* the dual of X and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Consider $\mathcal{E} \subset X$ the set of all nonempty closed and symmetric subsets of $X \setminus \{0\}$. We say that the set $A \in \mathcal{E}$ has genus m and we denote $\gamma(A) = m$ if m is the smallest integer with the property that there exists an odd continuous map from A to $\mathbb{R}^m \setminus \{0\}$. If $A = \emptyset$ we have $\gamma(A) = 0$ and if there is no such a finite m we set $\gamma(A) = \infty$.

In the following lemma we will recall only two properties of the genus that will be used in this paper. More information on this subject may be found in the references[10],[17],[18],[20].

Lemma 1. [17, Lemma 1.1, Theorm 1.2]
Let $A, B \in \mathcal{E}$.
(1) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;
(2) Let D be a symmetric and bounded neighbourhood of the origin in $\mathbb{R}^N$ and let $A \in \mathcal{E}$ be homeomorphic to ∂D by an odd homeomorphism. Then $\gamma(A) = N$. In particular, the unit sphere $S \subset \mathbb{R}^N$ is a set of genus N.

In order to use variational methods, let us also recall some results related to the Palais-Smale compactness condition. First, we have the following definition (see, for example,[19, pg. 123],[22, Definition 44.13]).

Definition 3. Let $\mathbf{M}$ be a given subset of a real Banach space X and let $F : D(F) \subset X \to \mathbb{R}$ be a functional that has a tangential mapping $F'_{\mathbf{M}}$ with respect to $\mathbf{M}$ at each point $u \in \mathbf{M}$. Functional F satisfies the *local Palais-Smale condition* $(PS)_c$ *with respect to* $\mathbf{M}$ if and only if the condition

$$\begin{cases} \textit{each sequence } (u_n)_n \textit{ in } \mathbf{M} \textit{ such that} \\ \| F'_{\mathbf{M}}(u_n) \| \to 0 \text{ and } F(u_n) \to c \ \textit{ as } n \to \infty \\ \textit{has a convergent subsequence} \end{cases}$$

holds for a fixed $c \in \mathbb{R}$.

The above condition is a local version of the following *Palais-Smale compactness condition*:

$$\begin{cases} \textit{each sequence } (u_n)_n \textit{ in } \mathbf{M} \textit{ such that} \\ \| F'_{\mathbf{M}}(u_n) \| \to 0 \textit{ and } \big(F(u_n)\big)_n \textit{ is bounded}, \\ \textit{has a convergent subsequence}. \end{cases} \tag{PS}$$

For the definition of the tangential mapping $F'_{\mathbf{M}}$ (or the differential of F with respect to $\mathbf{M}$) see, for example,[22, Definition 43.18].

In order to solve the eigenvalue problem (1), the constrained variational method can be applied. We will use the following Lusternik–Schnirelmann principle on C^1-manifolds (Szulkin[19, Corollary 4.1]).

Theorem 2. Suppose that $\mathbf{M}$ is a closed symmetric C^1-submanifold of a real Banach space X and $0 \notin \mathbf{M}$. Suppose also that $F \in C^1(\mathbf{M}, \mathbb{R})$ is even and bounded below. Define

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} F(x),$$

where $\Gamma_j = \{A \subset \mathbf{M} : A \in \mathcal{E}, \gamma(A) \geq j,$ and A is compact $\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies $(PS)_c$ for all $c = c_j, j = 1, \cdots, k$, then F has at least k distinct pairs of critical points.

Next, we are going to exploit some properties of the set $\mathcal{M}_\rho$ (defined by (7)), which is evidently symmetric with respect to the origin. Let us first introduce some notations.

$$\begin{aligned} K_{pq}(u_1,u_2) &:= \frac{1}{p}\int_{\Omega_1} |\nabla u_1|^p\,dx + \frac{1}{q}\int_{\Omega_2} |\nabla u_2|^q\,dx, \\ k_{rs\zeta}(u_1,u_2) &:= \frac{1}{r}\int_{\Omega_1} \gamma_1 |u_1|^r\,dx + \frac{1}{s}\int_{\Omega_2}\gamma_2 |u_2|^s\,dx + \frac{1}{\zeta}\int_{\Sigma} |u_2|^\zeta\,d\sigma, \\ j_{pq}(u_1,u_2) &:= \frac{1}{p}\int_{\Omega_1} |u_1|^p\,dx + \frac{1}{q}\int_{\Omega_2} |u_2|^q\,dx\ \forall\,(u,u_2) \in \widetilde{W}. \end{aligned} \tag{8}$$

Since for all $\widetilde{u} = (u_1, u_2) \in \mathcal{M}_\rho$ we have $\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle \neq 0$, ρ is a regular value of the C^1 functional j_{pq}. Therefore, $\mathcal{M}_\rho = j^{-1}_{pq}(\rho)$ is a C^1-manifold of codimension 1 in $\widetilde{W}$ (see, for example,[13, Theorem 2.2.7]) with tangent space, in a point $\widetilde{u} = (u_1, u_2) \in \mathcal{M}_\rho$, given by $T_{\widetilde{u}}\mathcal{M}_\rho = \ker j'_{pq}(\widetilde{u})$.

Define the C^1 functional,

$$\mathcal{J} : \widetilde{W} \to \mathbb{R},\ \mathcal{J}(\widetilde{u}) = K_{pq}(u_1,u_2) + k_{rs\zeta}(u_1,u_2)\ \ \forall\,\widetilde{u} = (u_1,u_2) \in \widetilde{W}. \tag{9}$$

Obviously, $\mathcal{J} \in C^1(\mathcal{M}_\rho, \mathbb{R})$. We denote by $\mathcal{J}_{\mathcal{M}_\rho}$ the restriction of the functional $\mathcal{J}$ on $\mathcal{M}_\rho$ and by $\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u})$ the differential of $\mathcal{J}$ at $\widetilde{u} \in \mathcal{M}_\rho$ with respect to $\mathcal{M}_\rho$, i.e. the restriction of $\mathcal{J}'(\widetilde{u})$ on $T_{\widetilde{u}}\mathcal{M}_\rho$.

Remark 3. We are going to compute $\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u})$, $\widetilde{u} \in \mathcal{M}_\rho$. Obviously, $\widetilde{u} \notin T_{\widetilde{u}}\mathcal{M}_\rho$, thus $W = T_{\widetilde{u}}\mathcal{M}_\rho \oplus \{\alpha\widetilde{u}; \alpha \in \mathbb{R}\}$. Let $P : \widetilde{W} \to T_{\widetilde{u}}\mathcal{M}_\rho$ be the projection operator. Then, for every $\widetilde{v} \in \widetilde{W}$, there exists a unique $\alpha \in \mathbb{R}$ (which depends on $\widetilde{v}$) such that $\widetilde{v} = P\widetilde{v} + \alpha\widetilde{u}$. In particular, as $\langle j'_{pq}(\widetilde{u}), P\widetilde{v}\rangle = 0$, we obtain that $\alpha = \langle j'_{pq}(\widetilde{u}), \widetilde{v}\rangle / \langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle$. Therefore, if $\widetilde{v} \in T_{\widetilde{u}}\mathcal{M}$

$$\begin{aligned} \langle \mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}), \widetilde{v}\rangle = \langle \mathcal{J}'(\widetilde{u}), P\widetilde{v}\rangle &= \langle \mathcal{J}'(\widetilde{u}), \widetilde{v}\rangle - \frac{\langle j'_{pq}(\widetilde{u}), \widetilde{v}\rangle}{\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle}\langle \mathcal{J}'(\widetilde{u}), \widetilde{u}\rangle \\ &= \left\langle \mathcal{J}'(\widetilde{u}) - \frac{\langle \mathcal{J}'(\widetilde{u}), \widetilde{u}\rangle}{\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle} j'_{pq}(\widetilde{u}), \widetilde{v}\right\rangle \end{aligned}$$

which implies that

$$\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}) = \mathcal{J}'(\widetilde{u}) - \lambda(\widetilde{u}) j'_{pq}(\widetilde{u}),\ \lambda(\widetilde{u}) = \frac{\langle \mathcal{J}'(\widetilde{u}), \widetilde{u}\rangle}{\langle j'_{pq}(\widetilde{u}), \widetilde{u}\rangle}.$$

Moreover, $\widetilde{u} \in \mathcal{M}_\rho$ is a critical point of $\mathcal{J}_{\mathcal{M}_\rho}$ if and only if $\mathcal{J}'(\widetilde{u}) = \lambda j'_{pq}(\widetilde{u})$ for some $\lambda \in \mathbb{R}$. Thus, there is a one-to-one correspondence between critical points of $\mathcal{J}_{\mathcal{M}_\rho}$ and the weak solutions of problem (1) (see, for example[22, Proposition 43.21]).

The following lemma shows, essentially, that $\gamma(\mathcal{M}_\rho) = \infty$.

Lemma 2. For any positive integer k there exists a compact symmetric subset $K \subset \mathcal{M}_\rho$ such that $\gamma(K) = k$.

Proof. Let $\phi_1, \phi_2, \cdots, \phi_k \in C_0^\infty(\Omega)$ be nonnegative functions with disjoint compact supports, supp $\phi_j \subset \Omega_1$, $\forall\, j = 1, 2, \cdots, k$, such that $p^{-1}\int_{\Omega 1} \phi_j^p dx = \rho\ \forall\ j = 1, 2, \cdots, k$. Obviously, $\{\phi_1, \phi_2, \cdots, \phi_k\} \subset \mathcal{M}_\rho$ is a linearly independent set, thus $V_k :=$ Span $\{\phi_1, \phi_2, \cdots, \phi_k\}$ is a k dimensional space. It is clear that $\mathcal{M}_\rho \cap V_k$ is the sphere of radius $(p\rho)^{1/p}$ in V_k with respect to the L^p-norm. In particular, $\gamma(\mathcal{M}_\rho \cap V_k) = k$ and the proof is complete (see Lemma 1 (2)). □

Remark 4. From Lemma 2 we see that the manifold $\mathcal{M}_\rho$ contains compact subsets of arbitrarily large genus, i. e., $\Gamma_k \neq \emptyset$ for any $k \geq 1$ (the set Γ_k was defined in Theorem 2).

For the proof of the main result (Theorem 1), the following lemma will play an important role in computations (see[16, Lemma 3.1]).

Lemma 3. Let $D \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that

$$\theta \in (1, N),\ \eta \in (\theta, \theta^*),\ \xi \in \left(0, N\left(1 - \frac{\eta}{\theta^*}\right)\right). \tag{10}$$

Then there exists a positive constant C such that, for every $u \in W^{1,\theta}(D)$

$$\| u \|^{\eta}_{L^{\eta}(D)} \leq C\left(\| \nabla u \|^{\theta}_{L^{\theta}(D)} + \| u \|^{\theta}_{L^{\theta}(D)} \right)^{(\eta-\xi)/\theta} \| u \|^{\xi}_{L^{\theta}(D)} . \tag{11}$$

Remark 5. From $\xi < N\left(1 - \frac{\eta}{\theta^*}\right)$ we have $\xi < \eta$.

Inequality (11) is still valid in the case $\theta \geq N,\ \eta > \theta$, with $1 < \xi < \eta$.

3 | PROOF OF THEOREM 1

Throughout this section we assume that (h_1) and (h_2) are fulfilled and will be used without mentioning them in the statements below.

The proof of Theorem 1 will follow as a consequence of several intermediate results.

Lemma 4. The functional $\mathcal{J}_{\mathcal{M}_\rho}$ is coercive, i.e.,

$$\lim_{\|(u_1,u_2)\|\to\infty,(u_1,u_2)\in\mathcal{M}_\rho} \mathcal{J}(u_1, u_2) = \infty.$$

Proof. Arguing by contradiction, we assume that there exist a positive constant C and a sequence $\left(\widetilde{u}_n\right)_n = \left(u_{1n}, u_{2n}\right)_n \subset \mathcal{M}_\rho$ such that $\|\widetilde{u}_n\| \to \infty$ in $\widetilde{W}$ as $n \to \infty$ and

$$\mathcal{J}_\lambda(\widetilde{u}_n) \leq C \ \forall\, n \geq 1. \tag{12}$$

It is obvious that

$$\begin{aligned} \mathcal{J}(u_{1n}, u_{2n}) \geq \frac{1}{p} \| \nabla u_{1n} \|^p_{1p} + \frac{1}{q} \| \nabla u_{2n} \|^q_{2q} \\ - \frac{1}{r} \| \gamma_1 \|_{1\infty} \| u_{1n} \|^r_{1r} - \frac{1}{s} \| \gamma_2 \|_{2\infty} \| u_{2n} \|^s_{2s} \ \ \forall\, n \geq 1. \end{aligned} \tag{13}$$

For $n \geq 1$, denote

$$\begin{aligned} T_{1n} &= \frac{1}{p} \| \nabla u_{1n} \|^p_{1p} - \frac{1}{r} \| \gamma_1 \|_{1\infty} \| u_{1n} \|^r_{1r}, \\ T_{2n} &= \frac{1}{q} \| \nabla u_{2n} \|^q_{2q} - \frac{1}{s} \| \gamma_2 \|_{2\infty} \| u_{2n} \|^s_{2s} . \end{aligned} \tag{14}$$

As $\| \widetilde{u}_n \| \to \infty$, taking into account the fact that $\widetilde{u}_n \in \mathcal{M}_\rho$, we derive that $\| \nabla u_{1n} \|_{1p} + \| \nabla u_{2n} \|_{2q} \to \infty$. Therefore, without loss of generality, we can assume that, up to a subsequence, $\| \nabla u_{1n} \|_{1p} \to \infty$.

Now, if $r \leq p$ we have that $L^r(\Omega_1)$ is continuously embedded into $L^p(\Omega_1)$. Thus, there exists a positive constant C independent of n such that

$$T_{1n} \geq \frac{1}{p} \| \nabla u_{1n} \|^p_{1p} - C \| \gamma_1 \|_{1\infty} \ \ \forall\, n \geq 1. \tag{15}$$

On the other hand, if $r > p$ and $p < N$, we make use of an argument in Figueiredo-Siciliano[9, lemma 2.2]. Thus, from the inequality $r < p\left(1 + \frac{p}{N}\right)$ (see assumptions (h_2)) we obtain that $r < p_*$ and $0 < r - p < N(1 - r/p_*)$, therefore there exists ξ_1 such that

$$r - p < \xi_1 < N\left(1 - \frac{r}{p_*}\right). \tag{16}$$

Now, for such a ξ_1, using Lemma 3 with $D = \Omega_1,\ \theta = p,\ \eta = r$ and $u = u_{1n}$, we obtain that there exists a positive constant C_1 (independent of n) such that

$$\begin{aligned} \| u_{1n} \|^r_{1r} \leq C_1\left(\| \nabla u_{1n} \|^p_{1p} + \| u_{1n} \|^p_{1p} \right)^{(r-\xi_1)/p} \| u_{1n} \|^{\xi_1}_{1p} \\ \leq C_1\left(\| \nabla u_{1n} \|^p_{1p} + p\rho\right)^{(r-\xi_1)/p} (p\rho)^{\xi_1/p}. \end{aligned} \tag{17}$$

Taking into account $(14)_1$ and (17) we have, for all $n \geq 1$,

$$T_{1n} \geq \frac{1}{p} \| \nabla u_{1n} \|^p_{1p} - \frac{C_1}{r} \| \gamma_1 \|_{1\infty} \left(\| \nabla u_{1n} \|^p_{1p} + p\rho\right)^{(r-\xi_1)/p} (p\rho)^{\xi_1/p}. \tag{18}$$

Finally, if $r > p$ and $p \geq N$, making use of Remark 5 we can choose ξ_1 such that $r - p < \xi_1 < r$. A similar argument to the one in the former case implies that (18) is still satisfied. Summing up, as $\| \nabla u_{1n} \|_{1p} \to \infty$ and $p > r - \xi_1$ if $r \geq p$, we obtain that $T_{1n} \to \infty$ (see (15) and (18)).

Obviously, if $q < N$, then T_{2n} satisfies an inequality similar to (15); in the contrary case, T_{2n} will satisfy an inequality similar to (18). It follows that $T_{1n} + T_{2n} \to \infty$.

Summing up, (13) implies that $\mathcal{J}(u_{1n}, u_{2n}) \to \infty$ which contradicts (12). This contradiction shows that $\mathcal{J}$ is coercive on $\mathcal{M}_\rho$ and the proof is complete.

□

Obviously, the functional $\mathcal{J}$ is even and since it is coercive on $\mathcal{M}_\rho$, it is also bounded below on $\mathcal{M}_\rho$. Thus, we can exploit the symmetry property in order to get multiplicity results for the critical points of $\mathcal{J}_{\mathcal{M}_\rho}$.

Remark 6. From Lemma 4 and Remark 1, it is easy to see that for every sequence $\left(\widetilde{u}_n\right)_n \subset \mathcal{M}_\rho, \widetilde{u}_n = (u_{1n}, u_{2n})$ such that $\left(\mathcal{J}(\widetilde{u}_n)\right)_n$ is bounded (thus, from Lemma 4, $\left(\widetilde{u}_n\right)_n$ is bounded) the sequences

$$\begin{gathered}\Big(\int_{\Omega_1} |\nabla u_{1n}|^p\,dx\Big)_n, \Big(\int_{\Omega_2} |\nabla u_{2n}|^q\,dx\Big)_n,\\ \Big(\int_{\Omega_1} \gamma_1 |u_{1n}|^r\,dx\Big)_n, \Big(\int_{\Omega_2} \gamma_2 |u_{2n}|^s\,dx\Big)_n, \Big(\int_{\Sigma} \beta |u_{2n}|^\zeta\,d\sigma\Big)_n\end{gathered} \tag{19}$$

are bounded.

For the proof of the main result, the following lemma will play an important role in computations.

Lemma 5. Let $\mathcal{K} := K'_{pq} : \widetilde{W}^* \to \mathbb{R}$ be the derivative of K_{pq} defined in $(8)_1$. Then, for any $\widetilde{u} = (u_1, u_2), \widetilde{v} = (v_1, v_2) \in \widetilde{W}$ one has

$$\begin{aligned}&\langle \mathcal{K}(\widetilde{u}) - \mathcal{K}(\widetilde{v}), u - v\rangle\\ &\quad\geq \left(\|\nabla u_1\|_{1p}^{p-1} - \|\nabla v_1\|_{1p}^{p-1} \right)\left(\|\nabla u_1\|_{1p} - \|\nabla v_1\|_{1p} \right)\\ &\quad+ \left(\|\nabla u_2\|_{2q}^{q-1} - \|\nabla v_2\|_{2q}^{q-1} \right)\left(\|\nabla u_2\|_{2q} - \|\nabla v_2\|_{2q} \right) \geq 0.\end{aligned} \tag{20}$$

Proof. It is obvious that

$$\begin{aligned}&\langle \mathcal{K}(\widetilde{u}) - \mathcal{K}(\widetilde{v}), u - v\rangle\\ &\quad= \|\nabla u_1\|_{1p}^p + \|\nabla v_1\|_{1p}^p + \|\nabla u_2\|_{2q}^q + \|\nabla v_2\|_{2q}^q\\ &\quad- (T_1 + T_2) - (T_3 + T_4),\end{aligned} \tag{21}$$

where we have denoted

$$T_1 := \int_{\Omega_1} |\nabla u_1|^{p-2}\,\nabla u_1 \cdot \nabla v_1\,dx,\; T_2 := \int_{\Omega_1} |\nabla v_1|^{p-2}\,\nabla v_1 \cdot \nabla u_1\,dx,$$

T_3, T_4 are similarly defined, by replacing p, Ω_1 with q, Ω_2, and u_1, v_1 with u_2, v_2.

We have, by the Hölder inequality

$$T_1 \leq \Big(\int_{\Omega_1} |\nabla u_1|^p\,dx\Big)^{\frac{p-1}{p}} \Big(\int_{\Omega_1} |\nabla v_1|^p\,dx\Big)^{\frac{1}{p}}. \tag{22}$$

Similar inequalities can be obtained for the other terms, T_2, T_3, T_4 and using (21) we derive (20). □

Lemma 6. The functional $\mathcal{J}$ satisfies the Palais–Smale condition with respect to $\mathcal{M}_\rho$.

Proof. We already know that $\mathcal{M}_\rho$ is a C^1- manifold and $\mathcal{J}_{\mathcal{M}_\rho} \in C^1(\mathcal{M}_\rho, \mathbb{R})$. Thus, the proof amounts to showing that the functional $\mathcal{J}$ satisfies condition (PS).

Let $\left(\widetilde{u}_n\right)_n \subset \mathcal{M}_\rho$, $\widetilde{u}_n = (u_{1n}, u_{2n})$, and $\left(\lambda_n\right)_n \subset \mathbb{R}$ be such that $\left(\mathcal{J}(\widetilde{u}_n)\right)_n$ is bounded and $\mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}_n) \to 0$, i.e.,

$$\mathcal{J}'(\widetilde{u}_n) - \lambda_n j'_{pq}(\widetilde{u}_n) = K'_{pq}(\widetilde{u}_n) + k'_{rs\zeta}(\widetilde{u}_n) - \lambda_n j'_{pq}(\widetilde{u}_n) \to 0 \tag{23}$$

in $\widetilde{W}^*$ (see Remark 3).

We have already observed that $\mathcal{J}$ is coercive on $\mathcal{M}_\rho$ (see Lemma 4); this implies that the sequence $\left(\widetilde{u}_n\right)_n$ is bounded in $\widetilde{W}$. Therefore, we can assume that there is a subsequence, still denoted $\left(\widetilde{u}_n\right)_n$, such that

$$\begin{gathered}\widetilde{u} \rightharpoonup u_* = (u_{1*}, u_{2*}) \text{ in } \widetilde{W},\ u_{1n} \to u_{1*} \text{ in } L^{\theta_1}(\Omega_1),\\ u_{2n} \to u_{2*} \text{ in } L^{\theta_2}(\Omega_2),\ u_{2n} \to u_{2*} \text{ in } L^{\theta_3}(\Sigma),\end{gathered} \tag{24}$$

for some $\widetilde{u}_* \in \widetilde{W}$, with $\theta_1 < p^*,\ \theta_2 < q^*, \theta_3 < q_*$ (see Remark 1).

In particular, for $\theta_1 = p,\ \theta_2 = q$ and $\theta_1 = r,\ \theta_2 = s,\ \theta_3 = \zeta$, respectively, we obtain

$$\begin{gathered}\frac{1}{p}\int\limits_{\Omega_1} |\, u_{1*} \,|^p\, dx + \frac{1}{q}\int\limits_{\Omega_2} |\, u_{2*} \,|^q\, dx = \rho \ \Rightarrow\ u_* \in \mathcal{M}_\rho,\\ \int\limits_{\Omega_1} \gamma_1 \,|\, u_{1n} \,|^r\, dx \to \int\limits_{\Omega_1} \gamma_1 \,|\, u_{1*} \,|^r\, dx,\\ \int\limits_{\Omega_2} \gamma_2 \,|\, u_{2n} \,|^s\, dx \to \int\limits_{\Omega_2} \gamma_2 \,|\, u_{2*} \,|^s\, dx,\\ \int\limits_{\Sigma} \beta \,|\, u_{2n} \,|^\zeta\, d\sigma \to \int\limits_{\Sigma} \beta \,|\, u_{2*} \,|^\zeta\, d\sigma.\end{gathered} \tag{25}$$

We also have

$$\| \, u_{1n} \,\|_{1p} + \| \, u_{2n} \,\|_{2q} \to \| \, u_{1*} \,\|_{1p} + \| \, u_{2*} \,\|_{2q} \,. \tag{26}$$

We claim that the sequence $\left(\lambda_n\right)_n$ is bounded. Indeed, multiplying (23) by $\widetilde{u}_n \in \mathcal{M}_\rho$ and taking into account that the sequence $\left(\widetilde{u}_n\right)_n$ is bounded, we have

$$\begin{aligned}\int\limits_{\Omega_1} |\, \nabla u_{1n} \,|^p\, dx + \int\limits_{\Omega_2} |\, \nabla u_{2n} \,|^q\, dx + \int\limits_{\Sigma} \beta \,|\, u_{2n} \,|^\zeta\, d\sigma&\\ + \int\limits_{\Omega_1} \gamma_1 \,|\, u_{1n} \,|^p\, dx + \int\limits_{\Omega_2} \gamma_2 \,|\, u_{2n} \,|^q\, dx - \lambda_n \langle j'_{pq}(\widetilde{u}_n), \widetilde{u}_n \rangle \to 0.&\end{aligned}$$

Now, since $\langle j'_{pq}(\widetilde{u}_n), \widetilde{u}_n \rangle \in (\rho, (p+q)\rho)$, making use of Remark 6 we derive that $\left(\lambda_n\right)_n$ is bounden. Thus, up to a subsequence, we can assume $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$.

Next, we are going to prove that $\widetilde{u}_n \to u_*$ in $\widetilde{W}$. Since $\widetilde{W}$ is a reflexive Banach space and $\widetilde{u}_n \rightharpoonup u_*$, using the Lindenstrauss-Asplund-Troyanski theorem (see[21]), it is enough to prove that $\| \, \widetilde{u}_n \,\| \to \| \, \widetilde{u}_* \,\|$ in order to obtain the strong convergence $\widetilde{u}_n \to \widetilde{u}$. Moreover, using (26) we only need to show that

$$\| \, \nabla u_{1n} \,\|_{1p} + \| \, \nabla u_{2n} \,\|_{2q} \to \| \, \nabla u_{1*} \,\|_{1p} + \| \, \nabla u_{2*} \,\|_{2q} \,. \tag{27}$$

Note first that, since $\left(\widetilde{u}_n\right)_n$ is bounden in $\widetilde{W}$, (23) implies

$$|\, \langle \mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle \,| \leq \| \, \mathcal{J}'_{\mathcal{M}_\rho}(\widetilde{u}_n) \,\|_{T_{\widetilde{u}_n}\mathcal{M}^*_\rho} \left(\| \, \widetilde{u}_n \,\| + \| \, \widetilde{u}_* \,\| \right) \to 0. \tag{28}$$

Next, we claim that

$$\langle k'_{rs\zeta}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle \to 0. \tag{29}$$

Indeed, applying the Hölder inequality we have

$$
\begin{aligned}
| \langle k'_{rs\zeta}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle | \leq & \int_{\Omega_1} | \gamma_1(u_{1n} - u_{1*}) | \cdot | u_{1n} |^{r-1} \, dx \\
& + \int_{\Omega_2} | \gamma_2(u_{2n} - u_{2*}) | \cdot | u_{2n} |^{s-1} \, dx \\
& + \int_{\Sigma} | \beta(u_{1n} - u_{1*}) | \cdot | u_{1n} |^{\zeta-1} \, d\sigma \\
& \leq \| \gamma_1 \|_{1\infty} \| u_{1n} \|_{1r}^{r-1} \| u_{1n} - u_{1*} \|_{1r} \\
& + \| \gamma_2 \|_{2\infty} \| u_{2n} \|_{2s}^{s-1} \| u_{2n} - u_{2*} \|_{2s} \\
& + \| \beta \|_{\partial\infty} \| u_{2n} \|_{\partial\zeta}^{\zeta-1} \| u_{2n} - u_{2*} \|_{\partial\zeta} .
\end{aligned}
\tag{30}
$$

Since $\left(\| u_{1n} \|_{1r}^{r-1} \right)_n$, $\left(\| u_{2n} \|_{2s}^{s-1} \right)_n$ and $\left(\| u_{2n} \|_{\partial\zeta}^{\zeta-1} \right)_n$ are bounded (see Remark 6), using (24) we derive (29).

In a similar way, as $\left(\lambda_n\right)_n$ is bounded, we obtain

$$
\langle \lambda_n j'_{pq}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle \to 0. \tag{31}
$$

Now, (29) and (31) along with (28) and (23), imply

$$
\langle K'_{pq}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle \to 0. \tag{32}
$$

Then, using (32) and the convergence $\widetilde{u}_n \rightharpoonup \widetilde{u}_*$, we first notice that

$$
\begin{aligned}
& \lim_{n\to\infty} \langle K'_{pq}(\widetilde{u}_n) - K'_{pq}(\widetilde{u}_*), \widetilde{u}_n - \widetilde{u}_* \rangle \\
& \quad = \lim_{n\to\infty} \left(\langle K'_{pq}(\widetilde{u}_n), \widetilde{u}_n - \widetilde{u}_* \rangle - \langle K'_{pq}(\widetilde{u}_*), \widetilde{u}_n - \widetilde{u}_* \rangle \right) = 0.
\end{aligned}
\tag{33}
$$

Using inequality (20) with $\widetilde{u} = \widetilde{u}_n = (u_{1n}, u_{2n}), \widetilde{v} = \widetilde{u}_* = (u_{1*}, u_{2*})$ and (33) we obtain

$$
\begin{aligned}
0 \leq & \left(\| \nabla u_{1n} \|_{1p}^{p-1} - \| \nabla u_{1*} \|_{1p}^{p-1} \right) \left(\| \nabla u_{1n} \|_{1p} - \| \nabla u_{1*} \|_{1p} \right) \\
& + \left(\| \nabla u_{2n} \|_{2q}^{q-1} - \| \nabla u_{2*} \|_{2q}^{q-1} \right) \left(\| \nabla u_{2n} \|_{2q} - \| \nabla u_{2*} \|_{2q} \right) \\
& \leq \langle K'_{pq}(\widetilde{u}_n) - K'_{pq}(\widetilde{u}_*), \widetilde{u}_n - \widetilde{u}_* \rangle \to 0,
\end{aligned}
\tag{34}
$$

and we conclude that

$$
\| \nabla u_{1n} \|_{1p} + \| \nabla u_{2n} \|_{2q} \to \| \nabla u_{1*} \|_{1p} + \| \nabla u_{2*} \|_{2q} . \tag{35}
$$

According to (26) and (35) we finally obtain the strong convergence of $\left(\widetilde{u}_n\right)_n$. □

Since the functional $\mathcal{J}$ satisfies the Palais–Smale condition with respect to $\mathcal{M}_\rho$ and is bounded from below, it has sublevels with finite genus.

Lemma 7. For any $c \in \mathbb{R}$, the set $\mathcal{J}_c = \{u \in \mathcal{M}_\rho;\ \mathcal{J}(u) \leq c\}$ has finite genus.

For the proof of this result we refer the reader to Benci-Frotunato[3, Lemma 9].

The existence of infinitely many critical points $\pm\widetilde{u}_n,\ n \geq 1$, for $\mathcal{J}$ on $\mathcal{M}_\rho$ is a consequence of Lemma 2, Lemma 4, Lemma 6 and Theorem 2. These critical points $\pm\widetilde{u}_n,\ n \geq 1$, give rise to Lagrange multipliers $\lambda_n,\ n \geq 1$, and then to infinitely many solutions $(\lambda_n, \pm\widetilde{u}_n),\ n \geq 1$, of problem (1).

In order to complete the proof of Theorem 1, we only need to prove that $\lambda_n \to \infty$. For this purpose, let $k \geq 1$ be an arbitrary but fixed integer. By Lemma 7 we deduce that $\gamma(\mathcal{J}_k) = n_k$ for some integer n_k. Now, from Lemma 2, there exists a compact $K_k \in \mathcal{M}_\rho \cap \mathcal{E}$ such that $\gamma(K_k) = n_k + 1$. In particular, this implies that Γ_{n_k+1} is nonempty (for the definition of this set see Theorem 2). Using property (1) from Lemma 1, we obtain that for any $A \in \Gamma_{n_k+1}$, we have $\sup_A \mathcal{J} > k$, and consequently $c_k \geq k$ (c_k was defined in Theorem 2). In addition, since $\mathcal{J}$ is bounded below we have that $c_1 > -\infty$, therefore $-\infty < c_1 \leq \cdots \leq c_k < \infty$. Since, from Lemma 6, $\mathcal{J}$ satisfies the Palais–Smale condition with respect to $\mathcal{M}_\rho$ it is known that c_k is a critical value of $\mathcal{J}_{\mathcal{M}_\rho}$ (see, for example,[17] and[20]).

Summing up, for any positive integer k there are $\lambda_k \in \mathbb{R}$ and $\widetilde{u}_k = (u_{1k}, u_{2k}) \in \mathcal{M}_\rho$ such that

$$
\mathcal{J}'(\widetilde{u}_k) = \lambda_k j'_{pq}(\widetilde{u}_k),\ \mathcal{J}(\widetilde{u}_k) = c_k \geq k. \tag{36}
$$

In particular, (36) implies that

$$\lambda_k \geq \frac{\langle \mathcal{J}'(\widetilde{u}_k), \widetilde{u}_k \rangle}{\rho(p+q)} \ \forall\, k \geq 1,$$

$$\mathcal{J}(\widetilde{u}_k) \to \infty \text{ as } k \to \infty. \tag{37}$$

Thus, in order to complete the proof it remains to show that (37) implies

$$\begin{aligned}\langle \mathcal{J}'(\widetilde{u}_k), \widetilde{u}_k \rangle =& \| \nabla u_{1k} \|_{1p}^{p} + \| \nabla u_{2k} \|_{2q}^{q} + \int_{\Sigma} \beta \mid u_{2k} \mid^{\zeta} d\sigma \\ &+ \int_{\Omega_1} \gamma_1 \mid u_{1k} \mid^{r} dx + \int_{\Omega_2} \gamma_2 \mid u_{2k} \mid^{s} dx \to \infty \text{ as } k \to \infty.\end{aligned} \tag{38}$$

On the one hand, we have

$$\begin{aligned}\mathcal{J}(\widetilde{u}_k) \leq& \| \nabla u_{1k} \|_{1p}^{p} + \| \nabla u_{2k} \|_{2q}^{q} + \int_{\Sigma} \beta \mid u_{2k} \mid^{\zeta} d\sigma \\ &+ \| \gamma_1 \|_{1\infty} \| u_{1k} \|_{1r}^{r} + \| \gamma_2 \|_{2\infty} \| u_{2k} \|_{2s}^{s} \to \infty.\end{aligned} \tag{39}$$

On the other hand, using Lemma 3, there exist $\xi_1 < r,\ \xi_2 < s$ with $r - \xi_1 < p$, $s - \xi_2 < q$ such that for all $k \geq 1$ we have the following inequalities (see also the proof of Lemma 4)

$$\begin{aligned}&\| u_{1k} \|_{1r}^{r} \leq C_1 \big(\| \nabla u_{1k} \|_{1p}^{p} + p\rho \big)^{\frac{r-\xi_1}{p}} \text{ if } p \geq N, \\ &\| u_{2k} \|_{2s}^{s} \leq C_2 \big(\| \nabla u_{2k} \|_{2q}^{q} + q\rho \big)^{\frac{s-\xi_2}{q}} \text{ if } q \geq N, \\ &\| u_{1k} \|_{1r}^{r} \leq C_3 \text{ if } p < N,\ \ \| u_{2k} \|_{2s}^{r} \leq C_4 \text{ if } q < N \ \forall\, k \geq 1,\end{aligned} \tag{40}$$

where $C_1, \cdots, C_4$ are positive constants independent of k. Thus (39) and (40) imply

$$\| \nabla u_{1k} \|_{1p} + \| \nabla u_{2k} \|_{2q} + \int_{\Sigma} \beta \mid u_{2k} \mid^{\zeta} d\sigma \to \infty \text{ as } k \to \infty. \tag{41}$$

Finally, since

$$\begin{aligned}\langle \mathcal{J}'(\widetilde{u}_k), \widetilde{u}_k \rangle \geq& \| \nabla u_{1k} \|_{1p}^{p} + \| \nabla u_{2k} \|_{2q}^{q} + \int_{\Sigma} \beta \mid u_{2k} \mid^{\zeta} d\sigma \\ &- \| \gamma_1 \|_{1\infty} \| u_{1k} \|_{1r}^{r} - \| \gamma_2 \|_{2\infty} \| u_{2k} \|_{2s}^{s} \ \ \forall\, k \geq 1,\end{aligned} \tag{42}$$

using (40) and (41) we obtain (38) which completes the proof.

References

1. Adams R, Fournier J. *Sobolev Spaces, 2nd ed, Pure and Applied Mathematics*, Vol. 140. New York–London: Academic Press; 2003.
2. Barbu L, Moroşanu G, Pintea C. A nonlinear elliptic eigenvalue–transmission problem with Neumann boundary condition. *Ann Mat Pura Appl.* 2019;198:821–836.
3. Benci V, Fortunato D. An eigenvalue problem for the Schrödinger–Maxwell equations. *Topol Methods Nonlinear Anal.* 1998;11:283–293.
4. Brézis H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer; 2011.
5. Denkowski Z, Migórski S, Papageorgiou NS. *An Introduction to Nonlinear Analysis: Theory.* New York: Springer; 2003.
6. Fife PC. *Dynamics of internal layers and diffusive interfaces*. Society for industrial and applied mathematics; 1988.
7. Figueiredo GM, Montenegro M. A transmission problem on $\mathbb{R}^2$ with critical exponential growth. *Arch Math.* 2012;99(3):271–279.

8. Figueiredo GM, Montenegro M. On a nonlinear elliptic transmission problem with critical growth. *J Convex Anal.* 2013;20:947–954. 2013;

9. Figueiredo GM, Siciliano G. Normalized solutions for an horizontal transmission problem. *Appl Anal.* 2021;100(15):3174-3181.

10. Krasnosel'skiĭ MA. *Topological methods in the theory of nonlinear integral equations*. New York: MacMillan; 1964.

11. Li FY, Zhang Y, Zhu XL, Liang ZP. Ground-state solutions to Kirchhoff-type transmission problems with critical perturbation. *J Math Anal Appl.* 2020;482(2):123568.

12. Ma TF, Muñoz Rivera JE. Positive solutions for a nonlinear nonlocal elliptic transmission problem. *Appl Math Lett*. 2003;16:243–248.

13. Papageorgiou NS, Kyritsi ST. *Handbook of Applied Analysis, Advances in Mechanics and Mathematics*. New York: Springer; 2009.

14. Nicaise S. *Polygonal interface problems*. Lang; 1993.

15. Pflüger K. Nonlinear transmission problem in bounded domains of $\mathbb{R}^n$. *Appl Anal*. 1996;62:391–403.

16. Pisani L, Siciliano G. Neumann condition in the Schrödinger-Maxwell system. *Topol Methods Nonlinear Anal*. 2007;29:251–264.

17. Rabinowitz PH, *Variational methods for nonlinear eigenvalue problems*. Rome: Edizioni Cremonese; 139-195; 1974. (1974), 139–195.

18. Rabinowitz PH, et al. (ed.). *Minimax methods in critical point theory with applications to differential equations*. American Mathematical Society; 1986.

19. Szulkin A. Ljusternik–Schnirelmann theory on C^1-manifolds. *Ann Inst H. Poincaré Anal Non Linéaire*. 1998;5(2):119–139.

20. Struwe M. *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Berlin: Springer; 2000.

21. Troyanski SL. On locally uniformly convex and differentiable norms in certain non-separable Banach spaces. *Studia Math*. 1970/71;37:173–180.

22. Zeidler E. *Nonlinear functional analysis and its applications: III: variational methods and optimization*. New York: Springer; 2013.

23. Zeidler E. Ljusternik-Schnirelman theory on general level sets. *Math Nachr.* 1986;129:235-259.