# A class of differential hemivariational inequalities constrained on nonconvex star-shaped sets

Liang Lu[1], Lijie Li[2], and Jiangfeng Han[1]

[1]Guangxi University of Finance and Economics
[2]Yulin Normal University

February 22, 2024

**Abstract**

The purpose of this paper is to investigate a class of nonconvex constrainted differential hemivariational inequalities consisting of nonlinear evolution equations and evolutionary hemivariational inequalities. The admissible set of constraints is closed and star-shaped with respect to a certain ball in a reflexive Banach space. We construct an auxiliary inclusion problem and obtain the existence results by applying a surjectivity theorem for multivalued pseudomonotone operators and the properties of Clarke subgradient operator. Moreover, the existence of solution of original problem is established by hemivariational inequality approach and a penalization method in which a small parameter does not have to tend to zero. Finally, an application of the main results is provided.

# A class of differential hemivariational inequalities constrained on nonconvex star-shaped sets

*Liang Lu*[1] *Lijie Li*[2] *Jiangfeng Han*[1,*]

[1]Guangxi Key Laboratory of Cross-border E-commerce Intelligent Information Processing, Guangxi University of Finance and Economics, Nanning 530003, P.R. China

[2]Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China

**Abstract**

The purpose of this paper is to investigate a class of nonconvex constrainted differential hemivariational inequalities consisting of nonlinear evolution equations and evolutionary hemivariational inequalities. The admissible set of constraints is closed and star-shaped with respect to a certain ball in a reflexive Banach space. We construct an auxiliary inclusion problem and obtain the existence results by applying a surjectivity theorem for multivalued pseudomonotone operators and the properties of Clarke subgradient operator. Moreover, the existence of solution of original problem is established by hemivariational inequality approach and a penalization method in which a small parameter does not have to tend to zero. Finally, an application of the main results is provided.

**AMS Subject Classification:** 34G20, 47J20, 49J40, 49J52.

**Key words:** Differential hemivariational inequality; Clarke subgradient; star-shaped set; surjectivity result; penalization method.

*Corresponding author; Email: gxluliang@163.com (L. Lu), lilijiemaths@126.com (L.J. Li), hanjiangfeng2014@hotmail.com (J.F. Han)

# 1 Introduction

Let $E$, $X$, $Y$ and $V$ be reflexive, separable Banach spaces, $H$ be a reflexive Hilbert space. The dual of $V$ is denoted by $V^*$ and denote the dual pair between $V^*$ and $V$ by $\langle \cdot, \cdot \rangle$. Let $C \subset V$ be a nonempty, closed and star-shaped set. $I$ denotes a bounded interval $[0, b]$ with $b > 0$. Let $A : D(A) \subseteq E \to E$ be the infinitesimal generator of a $C_0-$semigroup $\{T(t)\}_{t \geq 0}$ on $E$ and $f : I \times E \times Y \to E$, $\omega : V \to Y$, $B : I \times E \to V^*$, $g : I \times V \to V^*$, $J : I \times X \to \mathbb{R}$ and $\vartheta : V \to X$ be given functions. With these data, we consider a class of differential hemivariational inequalities consisting of nonlinear abstract evolution equations and evolutionary hemivariational inequalities as follows.

**Problem $\mathcal{P}$.** Find a pair of functions $(x, u)$ with $x : I \to E$ and $u : I \to V$ such that

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), \omega u(t)) \text{ for a.e. } t \in I, \\ u(\cdot) \in SOL(C, B, g, J, x, u_0), \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Here, the notation $SOL(C, B, g, J, x, u_0)$ stands for the set of solutions for evolutionary hemivariational inequality constrained on nonconvex star-shaped set, consisting of functions $u : I \to V$ such that

$$\begin{cases} u(t) \in C \text{ for all } t \in I, \\ \langle u'(t) + B(t, x(t)) + g(t, u(t)), v\rangle + J^0(t, \vartheta u(t), \vartheta v) \geq 0 \text{ for a.e. } t \in I, \forall v \in T_C(u(t)), \\ u(0) = u_0. \end{cases} \tag{1.2}$$

We note that Problem $\mathcal{P}$ represents a system which couples the evolution equation (1.1) with the hemivariational inequality (1.2) of parabolic type, associated to the initial conditions. So we refer to Problem $\mathcal{P}$ as a differential hemivariational inequality which follows the terminology such as [14, 15, 21]. Also, the solution of problem $\mathcal{P}$ is understood in the following sense.

**Definition 1.1** *A pair of functions $(x, u)$, with $x \in C(I; E)$, $u \in L^2(I; V)$ and $u' \in L^2(I, V^*)$, is said to be a mild solution of Problem $\mathcal{P}$ if*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s), \omega u(s))ds \text{ for all } t \in I,$$

*where $u(t) \in C$ solves inequality (1.2) for a.e. $t \in I$.*

It is well known that the theory of variational inequalities began in the early 1960s, based on arguments of convexity and monotonicity. If the corresponding energy functionals are nonconvex, it arises another type of inequalities as variational formulation of a problem. Which is called hemivariational inequalities and based on properties of the Clarke subgradient defined for locally Lipschitz functions. The study of hemivariational inequality started with the pioneering works of Panagiotopoulos in the context of applications in engineering problems [26, 27]. The analysis of hemivariational inequality mainly uses the properties of the subdifferential in the sense of Clarke, which defined for locally Lipschitz functions which may be nonconvex. Some significant results on variational/hemivariational inequalities and their application can be found in [2, 6, 7, 8, 19, 20, 21, 22, 23, 30, 32, 33, 34, 35] and its references.

The notion of differential variational inequalities was introduced in [1] by Aubin and Cellina. Differential variational inequalities are systems which couple differential or partial differential equations with a time-dependent variational inequality. Various mathematical models arising in the study of contact and impact problems lead to differential variational inequalities. Since a systematic study was carry out by Pang and Stewart [29], there is a number of papers have been dedicated to the development of theory of differential variational inequalities and their applications [3, 11, 13, 14, 17, 18, 24]. Furthermore, differential hemivariational inequalities was firstly introduced by Liu et al. [15]. Interest in differential hemivariational inequalities and, more general, differential variational-hemivariational inequalities, represent an important extension, originated, similarly as in differential variational inequalities. Over the past decade, the theory of this topic grew rapidly. We refer the reader to some recent references [9, 10, 15, 16, 21, 37] and their references.

It is remarkable that some basic results concerning the properties of solution set were obtained under the assumption of compactness and convexity on constraint set [14]. Later, by relaxing the compactness of constraint set $K$, the existence and properties of solution set are obtained [12]. Also, in [15], the authors required that the constraint set $K$ is bounded. Furthermore, Liu et al. [11] established a general existence theorem for a differential variational inequality with constrainted on closed convex set. So far, in many of the existing articles, the authors obtained the existence and unique solution of differential variational inequality, hemivariational inequalities problems involving constraints on nonempty convex subset of Banach spaces. However, there are very few results concerning nonconvex constrained differential variational/hemivariational inequalities.

So far as we known, there are some references which study only a single variational or hemivariational inequalities constraints on nonconvex sets. we refer to [7, 22] for stationary problems, and to [8, 31] for evolution problems. The proof of the existence theorem in [8, 31] is based on the hemivariational inequality approach, surjectivity theorem and penalization method. This approach was initiated in [22, 23] to concern with existence of solutions to a problem, which can be seen as a nonconvex counterpart of a stationary problem. It was continued in [7] to nonconvex constrained problems in the theory of von Kqárman plates. Inspired from the above work, we apply this approach to study differential hemivariational inequality (1.1) involving constraint is not necessarily convex, but it is star-shaped with respect to a ball.

The main novelties of the recent paper are follows: First, for the first time, we study a system of a nonlinear abstract evolution equation driven by a hemivariational inequality of parabolic type constrained on nonconvex star-shaped set. Until now, (1.1) has not been studied in the literature. Second, there are a few papers devoted to this kind problems consisting of abstract evolution equations and evolutionary hemivariational inequalities (see e.g. [21]). Moreover, the main results can be applied to a special case of Problem $\mathcal{P}$, for instance, if the nonlinear function $B$ is assumed to be independent of the variable $x$, Problem $\mathcal{P}$ reduces to the hemivariational inequality constrained on nonconvex star-shaped set. Third, the admissible set of constraints is closed and star-shaped with respect to a certain ball, this allows one to use a discontinuity property of the generalized Clarke subdifferential of the distance function. The existence of the solutions will be proved by applying a surjectivity theorem for multivalued pseudomonotone operators, a differential hemivariational inequality approach and a penalization method in which a small parameter does not have to tend to zero. The penalty method in this paper is unlike the recent literature [5, 10, 13, 16] and its references.

The rest of the paper is structured as follows. In Section 2 we state notations, basic definitions and preliminaries. In Section 3 the necessary assumptions and the results on existence of mild solution are given and proven. Finally, an example is provided in Sections 4 and the proofs are based on our abstract results.

## 2 Notations and preliminaries

In this section, we review some notation, basic definitions and preliminaries that to be used in the next sections. More details can be found in [4, 20, 23, 34, 36].

Let $E$, $X$, $Y$ and $V$ be reflexive, separable Banach spaces, $H$ be a reflexive Hilbert space with the norms $\|\cdot\|_E$, $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_V$ and $\|\cdot\|_H$, respectively. The dual of $V$ is denoted by $V^*$ and denote the dual pair between $V^*$ and $V$ by $\langle\cdot,\cdot\rangle$. $0_V$ represents the zero element of the space $V$, a similar definition of the zero element of other spaces. $I = [0, b]$ denotes a bounded interval with $0 < b < +\infty$. In the sequel, we use the standard Bochner-Lebesgue function spaces $\mathcal{V} = L^2(I; V)$, $\mathcal{H} = L^2(I; H)$, $\mathcal{X} = L^2(I; X)$, $\mathcal{V}^* = L^2(I; V^*)$. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is given by

$$\langle u, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^b \langle u(t), v(t)\rangle dt,\ u \in \mathcal{V}^*,\ v \in \mathcal{V}.$$

Besides, $C(I; E)$ and $C(I; V)$ represent the space of continuous functions on $I$ with values in $E$ and $V$, respectively. We say that $V$ is embedded in $Y$ if, $V$ is a vector subspace of $Y$, and the embedding operator $\omega : V \to Y$ defined by $\omega(x) = x$ for all $x \in V$ is continuous. Moreover, we say that $V$ is compactly embedded in $Y$ if the embedding operator $\omega$ is compact. Since the embedding operator is linear, the continuity condition is equivalent to the existence of a constant $c > 0$ such that

$$\|x\|_Y \leq c\|x\|_V \text{ for all } x \in V.$$

Consider an evolution triple of spaces $(V, H, V^*)$, which means that $V$ is a reflexive separable Banach space, $H$ is a separable Hilbert space, The embedding $V \subset H$ is continuous and $V$ is dense in $H$. Now, introduce the space $\mathcal{W}$ defined by $\mathcal{W} := \{\nu \in \mathcal{V} | \nu' \in \mathcal{V}^*\}$, where the time derivative $\nu' = \partial\nu/\partial t$ is understood in the sense of vector-valued distributions. It is well known that the space $\mathcal{W}$ endowed with the graph norm $\|\nu\|_{\mathcal{W}} = \|\nu\|_{\mathcal{V}} + \|\nu'\|_{\mathcal{V}^*}$ is a Banach space, which is separable and reflexive due to the separability and reflexivity of $\mathcal{V}$ and $\mathcal{V}^*$. Furthermore, we have the following continuous embeddings $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ and the embedding $\mathcal{W} \subset C(I; H)$ is continuous.

In what follows, we will assume that $V$ is densely and compactly embedded in $Y$. Let $\mathcal{Y} = L^2(I; Y)$ and $\mathcal{Y}^* = L^2(I; Y^*)$. Since the embedding $V \subset Y$ is compact, then by the Lions-Aubin lemma, we know that the embedding $\mathcal{W} \subset \mathcal{Y}$ is also compact. All the limits and upper limits are considered as $n \to \infty$, even if we do not mention it explicitly. We now proceed with the notions of pseudomonotonicity for multivalued operators.

**Definition 2.1** *An operator $A : V \to V^*$ is said to be:*

(a) *bounded, if $A$ maps bounded sets of $V$ into bounded sets of $V^*$;*

(b) *monotone, if $\langle Au - Av, u - v\rangle \geq 0$ for all $u,\ v \in V$;*

(c) *pseudomonotone, if $A$ is a bounded operator and for every sequence $\{x_n\} \subseteq V$ with $x_n \to x$ weakly in $V$, such that $\limsup\langle Ax_n, x_n - x\rangle \leq 0$, we have $\langle Ax, x - y\rangle \leq \liminf\langle Ax_n, x_n - y\rangle$ for all $y \in V$.*

(d) *demicontinuous, if $u_n \to u$ in $V$ implies $Au_n \to Au$ weakly in $V^*$.*

**Definition 2.2** *Let $L : V \supset D(L) \to V^*$ be a linear, maximal monotone operator. An operator $A : V \to V^*$ is said to be L-pseudomonotone, if for any sequence $\{v_n\} \subseteq V$ with $v_n \to v$ weakly in $V$, such that $\limsup\langle Av_n, v_n - v\rangle \leq 0$, it follows that $Av_n \to Av$ weakly in $V^*$ and $\lim\langle Av_n, v_n\rangle = \langle Av, v\rangle$.*

**Definition 2.3** *Let $L : V \supset D(L) \to V$ be a linear, maximal monotone operator. A multivalued operator $A : V \to 2^{V^*}$ is said to be L-pseudomonotone if*

(a) *for all $v \in V$ the set $Av$ is a nonempty, bounded, closed, and convex subset of $V^*$.*

(b) *$A$ is upper semicontinuous from each finite dimensional subspace of $V$ into $w$-$V^*$*

(c) *for any sequences $\{v_n\} \subset D(L)$ with $v_n \to v$ weakly in $V$, $Lv_n \to Lv$ weakly in $V^*$, $v_n^* \in Av_n$ is such that $v_n^* \to v^*$ weakly in $V^*$ and $\limsup\langle v_n^*, v_n - v\rangle \leq 0$, then $v^* \in Av$ and $\lim\langle v_n^*, v_n\rangle = \langle v^*, v\rangle$.*

The following surjectivity result will be crucial in our proof of the existence theorem.

**Theorem 2.4 ([28, Theorem 2.1])** *Assume that $V$ is a reflexive and strictly convex Banach space. Let $L : D(L) \subset V \to V^*$ be a linear, maximal monotone operator, and $A : V \to 2^{V^*}$ be a multivalued operator, which is bounded, L-pseudomonotone and coercive, i.e.,*

$$\lim_{\|u\|_V \to +\infty} \frac{\inf\{\langle u^*, u\rangle \mid u^* \in Au\}}{\|u\|} = +\infty.$$

*Then $L + A$ is a surjective operator.*

Now, we review the definition of generalized gradient in the Clarke sense and their properties.

Assume that $j : V \to \mathbb{R}$ be a locally Lipschitz functional. From [4], we denote by $j^0(x; v)$ the Clarke generalized directional derivative of $j$ at $x$ in the direction $v$, that is

$$j^0(x; v) = \limsup_{y \to x,\ \lambda \to 0^+} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

And the generalized Clarke subdifferential of $j$ at $x$ is a subset of $V^*$ given by

$$\partial j(x) = \{x^* \in V^* \mid j^0(x; v) \geq \langle x^*, v\rangle, \text{ for all } v \in V\}.$$

**Lemma 2.5 ([4, Proposition 2.1.2])** *If the functional $j : V \to \mathbb{R}$ is a locally Lipschitz continuous, then we have the following statements.*

(i) *There holds $j^0(x; v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial j(x)\}$ for all $x, v \in V$.*

(ii) *For each $x \in V$, $\partial j(x)$ is a nonempty, convex, weak*-compact subset of $V^*$.*

(iii) *For each $x \in V$, the function $U \ni v \mapsto j^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $j^0(x; \lambda v) = \lambda j^0(x; v)$ for all $\lambda \geq 0,\ v \in U$ and $j^0(x; v_1 + v_2) \leq j^0(x; v_1) + j^0(x; v_2)$ for all $v_1,\ v_2 \in V$, respectively.*

**Lemma 2.6 ([8, Lemma 2.1])** *If $V$ is a Banach space and $j : V \to \mathbb{R}$ be a locally Lipschitz continuous function with a Lipschitz constant $k > 0$, then*

$$\|\partial j(v)\|_{V^*} \leq k \text{ for all } v \in V.$$

The following result provides an example of a multivalued pseudomonotone operator which is a superposition of the Clarke subgradient with a compact operator.

**Lemma 2.7 ([2, Proposition 5.6])** *Let $V$ and $X$ be two reflexive Banach spaces, $\vartheta : V \to X$ be a linear, continuous, and compact operator. We denote by $\vartheta^* : X^* \to V^*$ the adjoint operator to $\vartheta$. Let $j : X \to R$ be a locally Lipschitz functional such that*

$$\|\partial j(v)\|_{X^*} \leq c_j(1 + \|v\|_X) \textit{ for all } v \in V$$

*with $c_j > 0$. Then the multivalued operator $G : V \to 2^{V^*}$ defined by*

$$G(v) = \vartheta^* \partial j(\vartheta(v)) \text{ for all } v \in V$$

*is pseudomonotone.*

Let $\overline{B_V}(v_0, r)$ denote the closed ball in $V$ with centre $v_0 \in V$ and radius $r > 0$, i.e.,

$$\overline{B_V}(v_0, r) := \{v \in V \mid \|v - v_0\|_V \leq r\}.$$

**Definition 2.8** *Let $C$ be a subset of $V$. We say that $C$ is star-shaped with respect to a ball $\overline{B_V}(v_0, r)$, iff*

$$\lambda v + (1 - \lambda)z \in C \text{ for all } v \in C,\ z \in \overline{B_V}(v_0, r), \lambda \in [0, 1].$$

For a nonempty set $C \subset V$, by $\bar{d} : V \to \mathbb{R}$, we denote the distance function of $C$, i.e.,

$$\overline{d}(z) := \inf_{v \in C} \|v - z\|_V \text{ for all } z \in V.$$

The Clarke tangent cone to $C$ at a point $u \in V$, denoted by $\overline{T}_C(u)$, is defined by

$$\overline{T}_C(u) := \{v \in V : \overline{d}^0(u; v) = 0\}.$$

Besides, we recall the discontinuity property of the generalized Clarke derivative of the distance function for a star-shaped set.

Assume that $V \subset Y$ and the embedding operator $\omega : V \to Y$ is dense and continuous. Let $C \subset V$ be a nonempty set. In what follows, we will write $\omega C$ and $\omega u$ instead of $\omega(C)$ (i.e. the image of the set $C$) and $\omega(u)$ (i.e. the value of $u$), respectively. We denote by $\tilde{d} : Y \to \mathbb{R}$ the distance function of $\omega C$ in $Y$, i.e.,

$$\tilde{d}(z) := \inf_{v \in \omega C} \|v - z\|_Y \text{ for all } z \in Y.$$

Let $d : V \to \mathbb{R}$ be defined by

$$d := \tilde{d}|_V = d \circ \omega.$$

It is easy to see that both functions $d$ and $\tilde{d}$ are Lipschitz continuous, with Lipschitz constants equal to one and $\|\omega\|$, respectively(see, e.g., [28, Proposition 2.4.1]). For any $v \in V$, we also have

$$d(v) = \tilde{d}(\omega v) = \inf_{y \in C} \|\omega v - \omega y\|_Z \leq \|\omega\| \inf_{y \in C} \|v - y\|_V = \|\omega\| \tilde{d}(v) \tag{2.1}$$

and for any $u, v \in V$, we have

$$d^0(u; v) \leq \tilde{d}^0(\omega u; \omega v), \tag{2.2}$$

see, e.g., [20, Proposition 3.37, p. 61].

The following property of the Clarke directional derivative of the distance function (see, e.g., [22, Lemma 2.1], [23, Lemma 7.2 and (7.2.47)]), will be crucial in our main theorem.

**Lemma 2.9** *Let $V$ and $Y$ be reflexive Banach spaces with the continuous and dense embedding $\omega : V \to Y$. Let $C \subset V$ be a closed set, which $C \subset V$ is star-shaped with respect to a ball $\overline{B_V}(v_0, r)$ for some $v_0 \in C$ and $r > 0$. Let $d$ be the function defined by* (2.1)*, then*

$$
\begin{aligned}
&d^0(u; v_0 - u) \le -d(u) - r\|\omega\| \text{ for all } u \notin C,\\
&d^0(u; v_0 - u) \le 0 \text{ for all } u \in C.
\end{aligned}
$$

Finally, for every $u \in C$, we introduce the following cone in $V$

$$
\begin{aligned}
T_C(u) :&= \{v \in V : \tilde{d}^0(\omega u; \omega v) = 0\} \\
&= \{v \in V : \omega v \in \overline{T}^Y_{\omega C}(\omega u) = 0\} = \omega^{-1}\overline{T}^Y_{\omega C}(\omega u).
\end{aligned} \tag{2.3}
$$

For this cone, we have the following property.

**Lemma 2.10 ([8, Lemma 2.3])** *If $C$ is a nonempty and convex subset of a Banach space $V$, then $\overline{T}_C(u) \subset T_C(u)$ for all $u \in C$.*

# 3 Hypothesis and main results

In this section, we establish the existence results for the Problem $\mathcal{P}$. For this goal, we first state the following hypothesis.

$$
\left\{
\begin{array}{l}
A : D(A) \subset E \to E \text{ is the generator of a } C_0\text{-semigroup of} \\
\qquad \text{linear and continuous operators } \{T(t)\}_{t\ge 0} \text{ on the space } E.
\end{array}
\right. \tag{3.1}
$$

$$
\left\{
\begin{array}{l}
f : I \times E \times Y \to E \text{ is such that :} \\
(a)\ f(\cdot, x, y) \text{ is measurable on } I \text{ for all } (x, y) \in E \times Y; \\
(b)\ f(\cdot, 0_E, 0_Y) \in L^2(I; E); \\
(c)\ \|f(t, x_1, y) - f(t, x_2, y)\|_E \le l_1\|x_1 - x_2\|_E \\
\qquad \text{for a.e. } t \in I \text{ and all } x_1, x_2 \in E, y \in Y \text{ with } l_1 \ge 0; \\
(d)\ f(t, x, \cdot) \text{ is } \alpha\text{-Hölder continous, i.e.,} \\
\qquad \|f(t, x, y_1) - f(t, x, y_2)\|_E \le l_2\|y_1 - y_2\|^\alpha_Y \\
\qquad \text{for a.e. } t \in I \text{ and all } x \in E,\ y_1, y_2 \in Y \text{ with } 0 < \alpha < 1,\ l_2 \ge 0.
\end{array}
\right. \tag{3.2}
$$

$$
\left\{
\begin{array}{l}
\text{The embedding } V \subset Y \text{ is compact and } \omega : V \to Y \text{ is} \\
\qquad \text{a embedded compact operatot from } V \text{ to } Y.
\end{array}
\right. \tag{3.3}
$$

$$\begin{cases} B : I \times E \to V^* \text{ is such that:} \\ (a)\ B(\cdot, x) \text{ is measurable on } I \text{ for all } x \in E; \\ (b)\ B(t, \cdot) \text{ is continous for a.e. } t \in I; \\ (c)\ \|B(t,x)\|_{V^*} \leq b_0(t) + b_1\|x\|_E \text{ for a.e. } t \in I, \\ \qquad \text{and all } x \in E \text{ with } b_0 \in L^2(I, \mathbb{R}^+) \text{ and } b_1 \geq 0. \end{cases} \tag{3.4}$$

$$\begin{cases} g : I \times V \to V^* \text{ is such that:} \\ (a)\ g(\cdot, u) \text{ is measurable on } I \text{ for all } u \in V; \\ (b)\ g(t, \cdot) \text{ is pesudomonotone on } V \text{ for a.e. } t \in I; \\ (c)\ \|g(t,u)\|_{V^*} \leq a_0(t) + a_1\|u\|_V \text{ for a.e. } t \in I, \text{ and all } u \in V \\ \qquad \text{with } a_0 \in L^2(I, R^+) \text{ and } a_1 > 0; \\ (d)\ \langle g(t,u), u\rangle \geq m_g\|u\|_V^2 - m_0\|u\|_V - m_1 \text{ for a.e. } t \in I \\ \qquad \text{and all } u \in V \text{ with } m_g > 0 \text{ and } m_0,\ m_1 \geq 0. \end{cases} \tag{3.5}$$

$$\begin{cases} J : I \times X \to \mathbb{R} \text{ is such that:} \\ (a)\ J(\cdot, v) \text{ is measurable on } I \text{ for all } v \in X; \\ (b)\ J(t, \cdot) \text{ is locally Lipschitz continous on } X \text{ for a.e. } t \in I; \\ (c)\ \|\xi\|_{X^*} \leq c_J(1 + \|v\|_X) \text{ for a.e. } t \in I \text{ and all} \\ \qquad v \in X,\ \xi \in \partial J(t,v) \text{ with } c_J > 0. \end{cases} \tag{3.6}$$

$$\begin{cases} \vartheta : V \to X \text{ is a linear, continuous and compact operator and} \\ \qquad \text{and } \vartheta^* : X^* \to V^* \text{ is its adjoint operator.} \end{cases} \tag{3.7}$$

$$m_g > 2c_J\|\vartheta\|^2. \tag{3.8}$$

$$\begin{cases} C \text{ is star-shaped set with respect to a ball } \overline{B_V}(v_0, r) \\ \qquad \text{with } v_0 \in C \text{ and } r > 0. \end{cases} \tag{3.9}$$

Firstly, we provide the follows preliminary lemma.

**Lemma 3.1** *Assume that* (3.1)-(3.3) *hold. Then, for each $u \in C(I; V)$, there exists a unique function $x \in C(I; E)$ such that*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s), \omega u(s))ds \text{ for all } t \in I. \tag{3.10}$$

*Define a solution operator $\mathcal{R} : C(I; V) \to C(I; E)$ of* (3.10) *as $x = \mathcal{R}u$. Then $\mathcal{R}$ is a history-dependent operator, i.e. there exists a constant $M_f > 0$ such that*

$$\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_E \leq M_f \int_0^t \|u_1(s) - u_2(s)\|_V^\alpha ds \quad \text{for all} \quad t \in I.$$

*for any $u_1, u_2 \in C(I; V)$.*

**Proof.** Firstly, for given $u \in C(I;V)$, from [25, Theorem 6.1.2], it is well-known that there exists a unique solution $x \in C(I;E)$ to (3.10).

Next, assume that $x_1, x_2 \in C(I;E)$ represent the solution of problem (3.10) corresponding to the functions $u_1, u_2 \in C(I;V)$. Then for $t \in I$, we have

$$\mathcal{R}u_i(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x_i(s), \omega u_i(s))ds \;\text{ for }\; i = 1, 2.$$

From assumption (3.2), we derive

$$\begin{aligned}
&\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_E = \|x_1(t) - x_2(t)\|_E \\
\le &\int_0^t \|T(t-s)\| \|f(s, x_1(s), \omega u_1(s)) - f(s, x_2(s), \omega u_2(s))\|_E ds \\
\le &M_A L_f \int_0^t \|x_1(s) - x_2(s)\|_E ds + M_A L_f \|\omega\|^\alpha \int_0^t \|u_1(s) - u_2(s)\|_V^\alpha ds.
\end{aligned}$$

Above inequality and the Gronwall argument (see [34, Lemma 2]) yields that

$$\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_E \le M_f \int_0^t \|u_1(s) - u_2(s)\|_V^\alpha ds,$$

where $M_f = M_A L_f \|\omega\|^\alpha e^{M_A L_f b}$ with $M_A = \sup_{t \in [0,b]} \|T(t)\|$. It completes the proof. $\square$

Moreover, from Lemma 3.1, we can see that there exist two constants $c_1$ and $c_2$ such that

$$\|\mathcal{R}u(t)\|_E \le c_1 + c_2 \int_0^t \|u(s)\|_V^\alpha ds \text{ for all } t \in I \text{ and any } u \in C(I, V). \tag{3.11}$$

To obtain the existence results for Problem $\mathcal{P}$, we will consider the auxiliary diferentail hemivariational inequality as follows.

**Problem** $\mathcal{P}_\rho$ : For $\rho > 0$, find a pair of function $(x_\rho, u_\rho)$ with $x_\rho : I \to E$ and $u_\rho : I \to V$ such that:

$$\begin{cases} x_\rho'(t) = Ax_\rho(t) + f(t, x_\rho(t), \omega u_\rho(t)), \; a.e.\ t \in I, \\ u_\rho(\cdot) \in SOL(B, g, x, u_0, \rho, d), \\ x_\rho(0) = x_0. \end{cases} \tag{3.12}$$

Here, $SOL(B, g, x, u_0, \rho, d)$ means the set of solution of the evolutionary hemivariational inequality: find $u_\rho(t) \in V$ such that

$$\begin{cases} \langle u_\rho'(t) + B(t, x_\rho(t)) + g(t, u_\rho(t)), v\rangle + J^0(t, \vartheta u_\rho(t), \vartheta v) + \dfrac{1}{\rho} d^0(u_\rho(t); v) \ge 0, \\ u_\rho(0) = u_0 \end{cases}$$

for all $v \in V$ and a.e. $t \in I$. From Lemma 3.1, for each $\rho > 0$, the Problem $\mathcal{P}_\rho$ can be converted into the following hemivariational inequality: find a function $u_\rho \in \mathcal{W}$ such that

$$\begin{cases} \langle u_\rho'(t) + B(t, \mathcal{R}u_\rho(t)) + g(t, u_\rho(t)), v\rangle + J^0(t, \vartheta u_\rho(t), \vartheta v) + \dfrac{1}{\rho} d^0(u_\rho(t); v) \geq 0, \\ u_\rho(0) = u_0 \end{cases} \tag{3.13}$$

for all $v \in V$ and a.e. $t \in I$, where $\mathcal{R}$ is the solution operator of evolution equation of (3.12). Moreover, to obtain the solution of (3.13), we shall study the following auxiliary inclusion problem:

$$\begin{cases} \text{Find } u_\rho \in \mathcal{W} \text{ such that} \\ u_\rho'(t) + B(t, \mathcal{R}u_\rho(t)) + g(t, u_\rho(t)) + \vartheta^* \partial J(t, \vartheta u_\rho(t)) + \dfrac{1}{\rho} \partial d(u_\rho(t)) \ni 0, \; t \in I, \\ u_\rho(0) = u_0. \end{cases} \tag{3.14}$$

We claim that every solution to (3.14) is also a solution to (3.13). In fact, according to the definition of solutions for (3.14), if $u_\rho$ is a solution of (3.14), it means that there exist $\xi \in \mathcal{X}^*$, $\zeta \in \mathcal{V}^*$ such that $\xi(t) \in \partial J(t, \vartheta u_\rho(t))$, $\zeta(t) \in \partial d(u_\rho(t))$ for a.e. $t \in I$ and

$$\begin{cases} u_\rho'(t) + B(t, \mathcal{R}u_\rho(t)) + g(t, u_\rho(t)) + \vartheta^* \xi(t) + \dfrac{1}{\rho} \zeta(t) = 0_{V^*} \text{ a.e. } \; t \in I, \\ u_\rho(0) = u_0, \end{cases}$$

which turns out

$$\begin{cases} \langle u_\rho'(t) + B(t, \mathcal{R}u_\rho(t)) + g(t, u_\rho(t)) + \vartheta^* \xi(t) + \dfrac{1}{\rho} \zeta(t), v\rangle = 0, \\ u(0) = u_0 \end{cases}$$

for all $v \in V$ and a.e. $t \in I$. Since $\xi(t) \in \partial J(\vartheta u_\rho(t))$ implies that $\langle \vartheta^* \xi(t), v\rangle_V = \langle \xi(t), \vartheta v\rangle_X \leq J^0(t, \vartheta u_\rho(t); \vartheta v)$ a.e. $t \in I$, and also $\zeta(t) \in \partial d(u_\rho(t))$ implies $\langle \zeta(t), v\rangle \leq d^0(u_\rho(t); v)$ a.e. $t \in I$, we obtain

$$\begin{cases} \langle u_\rho'(t) + B(t, \mathcal{R}u_\rho(t)) + g(t, u_\rho(t)), v\rangle + J^0(t, \vartheta u_\rho(t), \vartheta v) + \dfrac{1}{\rho} d^0(u_\rho(t); v) \geq 0, \\ u(0) = u_0 \end{cases}$$

for all $v \in V$ and a.e. $t \in I$. Therefore, in order to study the existence of the Problem $\mathcal{P}_\rho$, we only need to deal with the inclusions (3.14).

Next, we introduce the operators $\widehat{\mathcal{J}} : \mathcal{V} \to 2^{\mathcal{V}^*}$ and $\widehat{\mathcal{N}} : \mathcal{V} \to 2^{\mathcal{V}^*}$ by

$$\widehat{\mathcal{J}}(u) = \{w \in \mathcal{V}^* : w(t) = \vartheta^* \xi(t) \text{ with } \xi(t) \in \partial J(t, \vartheta u(t)) \text{ a.e. } t \in I\} \text{ for } u \in \mathcal{V},$$
$$\widehat{\mathcal{N}}(u) = \{w \in \mathcal{V}^* : w(t) \in \partial d(u(t)) \text{ a.e. } t \in I\} \text{ for } u \in \mathcal{V}.$$

By virtue of Lemma 2.5, Lemma 2.6 and [19, Lemma 11], we have the following lemmas by a similar proof.

**Lemma 3.2** *For each $u \in \mathcal{V}$, the sets $\widehat{\mathcal{J}}(u)$ and $\widehat{\mathcal{N}}(u)$ have nonempty, convex, bounded and weakly compact values.*

**Lemma 3.3** *The operator $\widehat{\mathcal{N}}$ satisfies: if $u_n \to u$ in $\mathcal{V}$, $w_n \to w$ weakly in $\mathcal{V}^*$ and $w_n \in \widehat{\mathcal{N}}(u_n)$, then we have $w \in \widehat{\mathcal{N}}(u)$. The operator $\widehat{\mathcal{J}}$ also has this property.*

From the hypothesis (3.4)–(3.6), we introduce the operators $\mathcal{G} : \mathcal{V} \to \mathcal{V}^*$, $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$, $\mathcal{J} : \mathcal{V} \to 2^{\mathcal{V}^*}$ and $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}^*}$ by

$$\mathcal{B}u(t) = B(t, \mathcal{R}(u(t) + u_0)) \text{ for all } u \in \mathcal{V},$$

$$\mathcal{G}u(t) = g(t, u(t) + u_0) \text{ for all } u \in \mathcal{V},$$
$$\mathcal{J}u(t) = \vartheta^* \partial J(\vartheta(u(t) + u_0)) \text{ for all } u \in \mathcal{V},$$
$$\mathcal{N}u(t) = \frac{1}{\rho} \partial d(u(t) + u_0) \text{ for all } u \in \mathcal{V},$$

and define the following operator $\mathcal{F} : \mathcal{V} \to 2^{\mathcal{V}^*}$ that assigns

$$\mathcal{F}u := \mathcal{B}u + \mathcal{G}u + \mathcal{J}u + \mathcal{N}u \text{ for all } u \in \mathcal{V}.$$

Carrying out the same arguments as in the proof of [20, Lemma 5.5], we have the following results.

**Lemma 3.4** *If* (3.5) *hold, then the operator $\mathcal{G} : \mathcal{V} \to \mathcal{V}^*$ has the following properties:*

(i) *$\mathcal{G}$ is $L$-pesudomonotone;*

(ii) *$\|\mathcal{G}u\|_{\mathcal{V}^*} \le \widehat{a_0} + \widehat{a_1}\|u\|_{\mathcal{V}}$ for all $u \in \mathcal{V}$ with $\widehat{a_0} \ge 0, \widehat{a_1} > 0$;*

(iii) *$\langle \mathcal{G}u, u\rangle_{\mathcal{V}^* \times \mathcal{V}} \ge \dfrac{1}{2} m_g \|u\|_{\mathcal{V}}^2 - \widehat{m_1}\|u\|_{\mathcal{V}} - \widehat{m_2}$ for all $u \in V$ with $\widehat{m_1},\ \widehat{m_2} \ge 0$;*

(iv) *$\mathcal{G}$ is demicontinous.*

Now, we present the first existence results.

**Theorem 3.5** *If* (3.1)-(3.8) *hold, then the inclusion problem* (3.14) *has at least a solution $u_\rho \in \mathcal{W}$ for any $\rho > 0$.*

**Proof.** Firstly, we define an operator $L : D(L) \subset \mathcal{V} \to \mathcal{V}^*$ by

$$Lu := u' \text{ for all } u \in D(L) = \{u \in \mathcal{W} \mid u(0) = 0\}.$$

Then, it is well known, see [36, Proposition 32.10, p. 855], that the operator $L$ is linear and maximal monotone and therefore $D(L)$ is dense in $\mathcal{V}$ and $L$ is graph closed (see [36, Theorem 32.L, p. 897]). Using above notation, it follows, that auxiliary problem (3.14) is equivalent to the following problem

$$\text{Find } u \in D(L) \text{ such that } Lu + \mathcal{F}u \ni 0_{\mathcal{V}^*}. \tag{3.15}$$

Then, $u_\rho = u + u_0 \in \mathcal{W}$ solves problem (3.14) if and only if $u \in D(L)$ solves problem (3.15). In order to show that (3.15) has a solution, we apply a surjectivity result of Theorem 2.4. For this purpose, we need to prove that $\mathcal{F}$ is bounded, coercive and $L$-pseudomonotone.

Next, we divide the rest of the proof into three steps.

**Step (i)**. $\mathcal{F}$ is bounded, i.e., $\mathcal{F}$ maps bounded sets of $\mathcal{V}$ into bounded sets of $\mathcal{V}^*$.

Firstly, taking advantage of (3.4)(c) and (3.11), the Minkowski inequality and Hölder inequality guarantee that

$$\begin{aligned}
\|\mathcal{B}u\|_{\mathcal{V}^*} &= \left(\int_0^b \|B(t,\mathcal{R}(u(t)+u_0))\|_{V^*}^2 ds\right)^{\frac{1}{2}} \\
&\le \left(\int_0^b [b_0(t) + b_1\|\mathcal{R}(u(t)+u_0)\|_E]^2 ds\right)^{\frac{1}{2}} \\
&\le \|b_0\|_{L^2(I,\mathbb{R}^+)} + b_1\left(\int_0^b [c_1 + c_2\|(u(t)+u_0)\|_V^\alpha]^2 ds\right)^{\frac{1}{2}} \\
&\le \|b_0\|_{L^2(I,\mathbb{R}^+)} + b_1c_1\sqrt{b} + b_1c_1\left(\int_0^b \|u(t)+u_0\|_V^{2\alpha} ds\right)^{\frac{1}{2}} \\
&= \|b_0\|_{L^2(I,\mathbb{R}^+)} + b_1c_1\sqrt{b} + b_1c_1 b^{\frac{1-\alpha}{2}}\|u(\cdot)+u_0\|_{\mathcal{V}}^\alpha.
\end{aligned}$$

Next, from the fact that the function $d$ is Lipschitz continuous with Lipschitz constant $\|\omega\|$ (see Sect. 2) and Lemma 2.6, we derive

$$\|\partial d(w)\|_{V^*} \le \|\omega\|, \text{ for all } v \in V,$$

and subsequently

$$\begin{aligned}
\|\mathcal{N}u\|_{\mathcal{V}^*} &= \sup\left\{\frac{1}{\rho}\left(\int_0^b \|\zeta(t)\|^2 dt\right)^{\frac{1}{2}} | \zeta \in \mathcal{V}^*, \zeta(t) \in \partial d(u(t)+u_0) \text{ a.e. } t \in I\right\} \\
&\le \frac{1}{\rho}\|\omega\|\sqrt{b}.
\end{aligned} \tag{3.16}$$

Moreover, by virtue of (3.6)(b) and a similar way as above, we conclude

$$\begin{aligned}
\|\mathcal{J}u\|_{\mathcal{V}^*} &= \sup\left\{\left(\int_0^b \|\vartheta^*\xi(t)\|^2 dt\right)^{\frac{1}{2}} | \xi \in \mathcal{X}^*,\ \xi(t) \in \partial J(t,\vartheta(u(t)+u_0)) \text{ a.e. } t \in I\right\} \\
&\le \|\vartheta\|c_J\sqrt{b} + c_J\|\vartheta\|^2\|u(\cdot)+u_0\|_{\mathcal{V}}.
\end{aligned} \tag{3.17}$$

Here we used the equality $\|\vartheta^*\| = \|\vartheta\|$, which is a consequence of [20, Proposition 1.51]. Therefore, applying above estimatation and Lemma 3.4(ii), we reach the estimate

$$\begin{aligned}\|\mathcal{F}u\|_{\mathcal{V}^*} &\le \|\mathcal{G}u\|_{\mathcal{V}^*} + \|\mathcal{B}u\|_{\mathcal{V}^*} + \|\mathcal{J}u\|_{\mathcal{V}^*} + \|\mathcal{N}u\|_{\mathcal{V}^*} \\ &\le \widehat{a_1}\|u\|_{\mathcal{V}} + b_1c_1\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha} + c_J\|\vartheta\|^2\|u(\cdot)+u_0\|_{\mathcal{V}} \\ &\quad + \widehat{a_0} + \|b_0\|_{L^2(I,R^+)} + b_1c_1\sqrt{b} + c_J\|\vartheta\|\sqrt{b} + \frac{1}{\rho}\|\omega\|\sqrt{b}\end{aligned}$$

for all $u \in \mathcal{V}$, which proves that $\mathcal{F}$ is bounded.

**Step (ii)**. $\mathcal{F}$ is coercive.

First, we can exploit hypotheses (3.4)(c), (3.11) and apply again the Hölder inequality to obtain the estimate

$$\begin{aligned}|\langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}}| &\le \int_0^b |\langle B(t, \mathcal{R}(u(t)+u_0)), u(t)\rangle| dt \\ &\le \int_0^b (b_0(t) + b_1\|\mathcal{R}(u(t)+u_0)\|_E)\|u(t)\|_V dt \\ &\le b_1\int_0^b (c_1 + c_2\int_0^t \|u(\tau)+u_0\|_V^{\alpha} d\tau \|u(t)\|_V dt + \int_0^b b_0(t)\|u(t)\|_V dt \\ &\le b_1c_2\int_0^b \|u(t)+u_0\|_V^{\alpha} dt \int_0^b \|u(t)+u_0-u_0\|_V dt \\ &\quad + b_1c_1\int_0^b \|u(t)\|_V dt + \|b_0\|_{L^2(I,R^+)}\|u\|_{\mathcal{V}} \\ &\le b_1c_2\Big[b^{\frac{2-\alpha}{2}}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha}\big(\sqrt{b}\|u(\cdot)+u_0\|_{\mathcal{V}} + b\|u_0\|_V\big)\Big] \\ &\quad + (b_1c_1\sqrt{b} + \|b_0\|_{L^2(I,\mathbb{R}^+)})\|u\|_{\mathcal{V}} \\ &= b_1c_2\big[b^{\frac{3-\alpha}{2}}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha+1} + b^{\frac{5-\alpha}{2}}\|u_0\|_V\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha}\big] \\ &\quad + (b_1c_1\sqrt{b} + \|b_0\|_{L^2(I,\mathbb{R}^+)})\|u\|_{\mathcal{V}}.\end{aligned}$$

Then, through the fact that $\|u+v\|_{\mathcal{V}}^{\alpha+1} \le (\|u\|_{\mathcal{V}} + \|v\|_{\mathcal{V}})^{\alpha+1} \le 2^{\alpha}(\|u\|_{\mathcal{V}}^{\alpha+1} + \|v\|_{\mathcal{V}}^{\alpha+1})$, we conclude

$$\begin{aligned}|\langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}}| &\le b_1c_2\Big[b^{\frac{3-\alpha}{2}}2^{\alpha}(\|u\|_{\mathcal{V}}^{\alpha+1} + (\sqrt{b}\|u_0\|_V)^{\alpha+1}) + \|u_0\|_V b^{2-\frac{\alpha}{2}}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha}\Big] \\ &\quad + (b_1c_1\sqrt{b} + \|b_0\|_{L^2(I,\mathbb{R}^+)})\|u\|_{\mathcal{V}} \\ &= \widehat{b_1}\|u\|_{\mathcal{V}}^{\alpha+1} + \widehat{b_2}\|u\|_{\mathcal{V}} + \widehat{b_3}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha} + \widehat{b_4} \qquad (3.18)\end{aligned}$$

with $\widehat{b_1} = 2^{\alpha}b_1c_2b^{\frac{3-\alpha}{2}}$, $\widehat{b_2} = b_1c_1\sqrt{b} + \|b_0\|_{L^2(I,\mathbb{R}^+)}$, $\widehat{b_3} = b_1c_2\|u_0\|_V b^{2-\frac{\alpha}{2}}$, $\widehat{b_4} = 2^{\alpha}b_1c_2b^2\|u_0\|^{\alpha+1}$.

Finally, combining the inequalities (3.16), (3.17), (3.18) and Lemma 3.4(iii), for

$u^* \in \mathcal{F}u$, there exist $z^* \in \mathcal{J}u$ and $\mu^* \in \mathcal{N}u$ such that $u^* = \mathcal{G}u + \mathcal{B}u + z^* + \mu^*$ and

$$\begin{aligned}
\langle u^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \mathcal{G}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle z^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mu^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} \\
&\geq \frac{1}{2} m_g \|u\|_{\mathcal{V}}^2 - \widehat{m_1}\|u\|_{\mathcal{V}} - \widehat{m_2} \\
&\quad - (\widehat{b_1}\|u\|_{\mathcal{V}}^{\alpha+1} + \widehat{b_2}\|u\|_{\mathcal{V}} + \widehat{b_3}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha} + \widehat{b_4}) - \|z^*\|_{\mathcal{V}^*}\|u\|_{\mathcal{V}} - \|\mu^*\|_{\mathcal{V}^*}\|u\|_{\mathcal{V}} \\
&\geq \|u\|_{\mathcal{V}}^{\alpha+1}\big((\frac{1}{2}m_g - c_J\|\vartheta\|^2)\|u\|_{\mathcal{V}}^{1-\alpha} - \widehat{b_1}\big) - \widehat{b_3}\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha} \\
&\quad - \big(\widehat{m_1} + \widehat{b_2} + c_J\|\vartheta\|\sqrt{b} + \frac{1}{\rho}\|\omega\|\sqrt{b}\big)\|u\|_{\mathcal{V}} - c_J\|\vartheta\|^2\sqrt{b}\|u_0\| - \big(\widehat{m_2} + \widehat{b_4}\big).
\end{aligned}$$

Thus above inequality and (3.8) imply that

$$\begin{aligned}
&\lim_{\|u\|_{\mathcal{V}}\to+\infty} \frac{\inf\{\langle u^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} | u^* \in \mathcal{F}u\}}{\|u\|_{\mathcal{V}}} \\
&\geq \lim_{\|u\|_{\mathcal{V}}\to+\infty} \|u\|_{\mathcal{V}}^{\alpha}\big((\frac{1}{2}m_g - c_J\|\vartheta\|^2)\|u\|_{\mathcal{V}}^{1-\alpha} - \widehat{b_1}\big) - \big(\widehat{m_1} + \widehat{b_2} + c_J\|\vartheta\|\sqrt{b} + \frac{1}{\rho}\|\omega\|\sqrt{b}\big) \\
&\quad - \widehat{b_3}\lim_{\|u\|_{\mathcal{V}}\to+\infty} \frac{\|u(\cdot)+u_0\|_{\mathcal{V}}^{\alpha}}{\|u\|_{\mathcal{V}}} - \lim_{\|u\|_{\mathcal{V}}\to+\infty} \frac{c_J\|\vartheta\|^2\sqrt{b}\|u_0\| + \widehat{m_2} + \widehat{b_4}}{\|u\|_{\mathcal{V}}} \\
&= +\infty.
\end{aligned}$$

Therefore, we prove that $\mathcal{F}$ is coercive.

**Step (iii)**. $\mathcal{F}$ is $L$-pseudomonotone.

As known from Lemma 3.2 and Step (i), $\mathcal{J}u$ and $\mathcal{N}u$ are nonempty, bounded, closed and convex subset of $\mathcal{V}^*$ and so is the operator $\mathcal{F}u$ for each $u \in \mathcal{V}$.

Next, we prove that the operator $\mathcal{F}$ satisfies the condition (c) of the definition of $L$-pseudomonotonicity (see Definition 2.3). Assume that the sequence $\{u_n\} \subset D(L)$ with $u_n \to u$ weakly in $\mathcal{V}$, $u_n' \to u'$ weakly in $\mathcal{V}^*$, $u_n^* \in \mathcal{F}u_n$ such that $u_n^* \to u^*$ weakly in $\mathcal{V}^*$ and $\limsup\langle u_n^*, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq 0$.

Then there exists two sequences $\{z_n^*\} \subset \mathcal{J}u_n$ and $\{\mu_n^*\} \subset \mathcal{N}u_n$ such that $u_n^* = \mathcal{G}u_n + \mathcal{B}u_n + z_n^* + \mu_n^*$. Notice that $u_n \to u$ weakly in $\mathcal{V}$, $u_n' \to u'$ weakly in $\mathcal{V}^*$ and form [20, Lemma 2.55(i)], we can see that $u_n \to u$ weakly in $\mathcal{W}$ and $u_n(t) \to u(t)$ weakly in $V$ for all $t \in I$. From condition (3.3), it is obvious that $\omega u_n(t) \to \omega u(t)$ in $Y$ for all $t \in I$. Moreover, Lemma 3.1 and assumption 3.4(b) guarantees that $\mathcal{R}u_n(t) \to \mathcal{R}u(t)$ in $E$ for all $t \in I$ and

$$B(t, \mathcal{R}(u_n(t) + u_0)) \to B(t, \mathcal{R}(u_n(t) + u_0)) \text{ in } V^* \text{ for all } t \in I.$$

By virtue of the Lebesgue-dominated convergence theorem, we are aware of

$$\mathcal{B}u_n \to \mathcal{B}u \text{ in } \mathcal{V}^* \text{ and } \langle \mathcal{B}u_n, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{3.19}$$

From the compactness of embedding $\mathcal{W} \subset \mathcal{Y}$, we infer that $u_n \to u$ in $\mathcal{Y}$. Since the function $\tilde{d}$ is Lipschitz continuous with Lipschitz constant one, from Lemma 2.6, we note that

$$\|\partial\tilde{d}(y)\|_{Y^*} \leq \|\omega\| \text{ for all } y \in Y.$$

Carrying out the same arguments as (3.16), it is true that

$$\|\mu_n^*\|_{\mathcal{Y}^*} \leq \|\mathcal{N}u_n\|_{\mathcal{Y}^*} \leq \frac{1}{\rho}\|\omega\|\sqrt{b}.$$

Which implies that the sequence $\{\mu_n^*\} \subset \mathcal{Y}^*$ is bounded, so passing to a subsequence if necessary, we may assume that $\mu_n^* \to \mu^*$ weakly in $\mathcal{Y}^*$. Because $\mathcal{N}$ has a closed graph with respect to the strong topology in $\mathcal{Y}$ and weak topology in $\mathcal{Y}^*$ (see Lemma 3.3), it is immediate that $\mathcal{N}$ is upper semicontinuous in these topologies and $\mu^* \in \mathcal{N}u$. Moreover, we have

$$\langle \mu_n^*, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mu_n^*, u_n\rangle_{\mathcal{Y}^*\times\mathcal{Y}} \to \langle \mu^*, u\rangle_{\mathcal{Y}^*\times\mathcal{Y}} = \langle \mu^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{3.20}$$

By a similar way, based on (3.6), (3.7) and Lemma 2.7, it follows that $\mathcal{J}$ is upper semicontinuous from $\mathcal{V}$ to $w$-$\mathcal{V}^*$ and there is an element $z^* \in \mathcal{J}u$ such that

$$\langle z_n^*, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle z^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{3.21}$$

Besides, through (3.19), (3.20) and (3.21), we obtain

$$\begin{aligned}
0 &\geq \limsup\langle u_n^*, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}}\\
&= \limsup\langle \mathcal{G}u_n, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \lim\langle \mathcal{B}u_n, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \lim\langle z_n^* + \mu_n^*, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}}\\
&= \limsup\langle \mathcal{G}u_n, u_n - u\rangle_{\mathcal{V}^*\times\mathcal{V}}.
\end{aligned}$$

Next, the $L$-pseudomonotonicity of the operator $\mathcal{G}$ (see Lemma 3.4(i)) implies that $\mathcal{G}u_n \to \mathcal{G}u$ weakly in $\mathcal{V}^*$ and $\lim\langle \mathcal{G}u_n, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle \mathcal{G}u, v\rangle_{\mathcal{V}^*\times\mathcal{V}}$. Therefore, $u_n^* = \mathcal{G}u_n + \mathcal{B}u_n + z_n^* + \mu_n^* \to \mathcal{G}u + \mathcal{B}u + z^* + \mu^* = u^* \in \mathcal{F}u$ and

$$\begin{aligned}
\langle u_n^*, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \mathcal{G}u_n, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}u_n, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle z_n^*, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mu_n^*, u_n\rangle_{\mathcal{V}^*\times\mathcal{V}}\\
&\to \langle \mathcal{G}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}u, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle z^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mu^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle u^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}}.
\end{aligned}$$

Finally, keeping in mind that $\mathcal{B}$ is continuous from $w$-$\mathcal{V}$ to $\mathcal{V}^*$, $\mathcal{G}$ is demicontinuous, $\mathcal{J}$ and $\mathcal{N}$ is upper semicontinuous from $\mathcal{V}$ to $w$-$\mathcal{V}^*$, we can immediately conclude that $\mathcal{F}$ is upper semicontinuous from each finite dimensional subspace of $V$ into $w$-$V^*$. Therefore, we prove that the operator $\mathcal{F}$ is $L$-pseudomonotone.

According to the Step (i)-(iii), we can apply the surjectivity result of Theorem 2.4 to get that (3.15) has a solution $u_\rho \in \mathcal{W}$. Thus it immediately to know that (3.14) has a solution $u_\rho \in \mathcal{W}$ for $\rho > 0$. $\square$

The main result in the section is the following.

**Theorem 3.6** *If* (3.1)-(3.9) *hold, Problem* $\mathcal{P}$ *has at least a mild solution* $(x, u)$ *with* $x \in C(I; E)$ *and* $u \in \mathcal{W}$*. Moreover, there exists a constant* $M > 0$ *such that*

$$\|u\|_{\mathcal{V}} \leq M \text{ and } \|x(t)\|_E \leq M \text{ for all } t \in I. \tag{3.22}$$

**Proof.** The proof consists of three claims.

**Claim 1**. The solution $u_\rho$ of (3.14) satisfying $\|u_\rho\|_{\mathcal{V}} \leq M_0$ with a constant $M_0 > 0$.

From Theorem 3.5, we know that (3.14) has at least one solution and so is (3.13). Thus assume that for $\rho > 0$, $u_\rho \in \mathcal{W}$ satisfies (3.13). Then the auxiliary inclusion problem (3.14) implies that there is a function $\zeta \in \mathcal{V}^*$ such that $\zeta(t) \in \partial d(u_\rho(t))$ and $\langle \zeta(t), v\rangle \leq d^0(u_\rho(t); v)$ for a.e. $t \in I$, thus we have

$$\langle u_\rho'(t) + g(t, u_\rho(t)) + B(t, \mathcal{R}u_\rho(t)) + \vartheta^* \xi(t), v\rangle + \frac{1}{\rho} d^0(u_\rho(t); v) \geq 0 \tag{3.23}$$

for a.e. $t \in I$ and all $v \in V$, where $\xi \in \mathcal{X}$ such that $\xi(t) \in \partial J(t, \vartheta u_\rho(t))$ for a.e. $t \in I$. Inserting $v = v_0 - u_\rho(t)$ in (3.23), we reach that

$$\langle u_\rho'(t) + g(t, u_\rho(t)) + B(t, \mathcal{R}u_\rho(t)) + \vartheta^* \xi(t), v_0 - u_\rho(t)\rangle + \frac{1}{\rho} d^0(u_\rho(t); v_0 - u_\rho(t)) \geq 0$$

for a.e. $t \in I$. Then it ensures that

$$\begin{aligned}
&\int_0^b \langle g(t, u_\rho(t)) + B(t, \mathcal{R}u_\rho(t)) + \vartheta^* \xi(t), u_\rho(t) - v_0\rangle dt \\
&\leq \int_0^b (\langle u_\rho'(t), v_0\rangle - \langle u_\rho'(t), u_\rho(t)\rangle) dt + \frac{1}{\rho}\int_0^b d^0(u_\rho(t); v_0 - u_\rho(t)) dt. \qquad (3.24)
\end{aligned}$$

Moreover, taking into account the identity, we obtain

$$\begin{aligned}
&\int_0^b \langle u_\rho'(t), u_\rho(t)\rangle dt = \frac{1}{2}\|u_\rho(b)\|_H^2 - \frac{1}{2}\|u_\rho(0)\|_H^2 = \frac{1}{2}\|u_\rho(b)\|_H^2 - \frac{1}{2}\|u_0\|_H^2 \\
&\int_0^b \langle u_\rho'(t), v_0\rangle dt = \langle u_\rho(b), v_0\rangle_{H\times H} - \langle u_0, v_0\rangle_{H\times H} \\
&\qquad\qquad \leq \frac{1}{2}\|u_\rho(b)\|_H^2 + \frac{1}{2}\|v_0\|_H^2 + \|u_0\|_H \|v_0\|_H.
\end{aligned}$$

Combining (3.4)(c), (3.5)(d), (3.6)(c), (3.11) and using the Hölder inequality, we are led to

$$
\begin{aligned}
&\int_0^b \langle g(t,u_\rho(t)) + B(t,\mathcal{R}u_\rho(t)) + \vartheta^*\xi(t), u_\rho(t) - v_0\rangle dt \\
\geq & m_g \int_0^b \|u_\rho(t)\|_V^2 dt - \|v_0\|_V \int_0^b \|g(t,u_\rho(t))\|_{V^*} dt \\
&- \int_0^b (\|B(t,\mathcal{R}u_\rho(t))\|_{V^*} + \|\vartheta^*\xi(t)\|)(\|u_\rho(t)\|_V + \|v_0\|_V) dt \\
\geq & m_g \|u_\rho\|_{\mathcal{V}}^2 - \|v_0\|_V \int_0^b (a_0(t) + a_1\|u_\rho(t)\|_V]) dt \\
&- \int_0^b [b_0(t) + b_1(c_1 + c_2\|u_\rho(t)\|_V^\alpha + \|\vartheta^*\| c_J(1 + \|\vartheta u_\rho(t)\|)](\|u_\rho(t)\|_V + \|v_0\|_V) dt \\
\geq & (m_g - c_J\|\vartheta\|^2)\|u_\rho\|_{\mathcal{V}}^2 - \kappa_1\|u_\rho\|_{\mathcal{V}}^{\alpha+1} - \kappa_2\|u_\rho\|_{\mathcal{V}} - \kappa_3\|u_\rho\|_{\mathcal{V}}^\alpha - \kappa_4,
\end{aligned}
$$

where

$\kappa_1 = b_1c_2b^{\frac{1-\alpha}{2}}$, $\kappa_2 = \sqrt{b}(a_1\|v_0\|_V + b_1c_1 + c_J(\|\vartheta\| + \|\vartheta\|^2\|v_0\|) + \|b_0\|_{L^2(I,R^+)}$,

$\kappa_3 = b_1c_2\|v_0\|_V b^{\frac{2-\alpha}{2}}$, $\kappa_4 = \sqrt{b}(\|a_0\|_{L^2(I,R^+)} + \|b_0\|_{L^2(I,R^+)})\|v_0\|_V + (b_1c_1 + c_J\|\vartheta\|)b\|v_0\|_V)$.
Therefore, it reads

$$
\begin{aligned}
(m_g - c_J\|\vartheta\|^2)\|u_\rho\|_{\mathcal{V}}^2 \leq & \kappa_1\|u_\rho\|_{\mathcal{V}}^{\alpha+1} + \kappa_2\|u_\rho\|_{\mathcal{V}} + \kappa_3\|u_\rho\|_{\mathcal{V}}^\alpha \\
&+ \widehat{\kappa_4} + \frac{1}{\rho}\int_0^b d^0(u_\rho(t); v_0 - u_\rho(t)) dt
\end{aligned} \tag{3.25}
$$

with $\widehat{\kappa_4} = \kappa_4 + \frac{1}{2}(\|u_0\|_H + \|v_0\|_H)^2$. Moreover, Lemma 2.9 guarantees that $d^0(u_\rho(t); v_0 - u_\rho(t)) \leq 0$ for $t \in I$. Thus the inequality (3.25) and (3.8) reveal that there exists a constant $M_0 > 0$ such that

$$
\|u_\rho\|_{\mathcal{V}} \leq M_0 \text{ for all } \rho > 0. \tag{3.26}
$$

**Claim 2**. There exists a constant $\epsilon_0 > 0$ such that $u_\rho(t) \in C$ for all $t \in I$ and $\rho \in (0, \epsilon_0)$.

We proceed the by contradiction. Suppose that for any $\epsilon > 0$, there exists a point $t_0 \in I$ and a constant $\rho_0 \in (0,\epsilon)$ such that $u_{\rho_0}(t_0) \notin C$. Since $u_\rho \in \mathcal{W} \subset C(I;V)$ for any $\rho > 0$ and the set $C$ is closed, we can find a set $I_0 \subset I$ with $meas(I_0) > 0$ such that $t_0 \in I_0$ and

$$
u_{\rho_0}(t) \notin C \text{ for all } t \in I_0.
$$

Now, we can choose $\rho_0 = \frac{1}{2}\min\left\{\epsilon_0, r\|\omega\| meas(I_0)(\kappa_1 M_0^{\alpha+1} + \kappa_2 M_0 + \kappa_3 M_0^\alpha + \widehat{\kappa_4})^{-1}\right\}$. Obviously, $\rho_0 \in (0, \epsilon_0)$. Moreover, from Lemma 2.9, we know that

$$
d^0(u_{\rho_0}(t); v_0 - u_{\rho_0}(t)) \leq -r\|\omega\| \text{ for all } t \in I_0.
$$

Then, taking account of (3.25) and (3.26), we infer that

$$
\begin{aligned}
0 &\le \kappa_1 M_0^{\alpha+1} + \kappa_2 M_0 + \kappa_3 M_0^{\alpha} + \widehat{\kappa_4} \\
&+ \frac{1}{\rho}\bigg( \int_{I_0} d^0(u_{\rho_0}(t); v_0 - u_{\rho_0}(t))dt + \int_{I\setminus I_0} d^0(u_{\rho_0}(t); v_0 - u_{\rho_0}(t))dt \bigg) \\
&\le \kappa_1 M_0^{\alpha+1} + \kappa_2 M_0 + \kappa_3 M_0^{\alpha} + \widehat{\kappa_4} - \frac{r}{\rho_0}\|\omega\| meas(I_0),
\end{aligned}
$$

which equivalent to

$$
\rho_0 \ge \frac{r\|\omega\| meas(I_0)}{\kappa_1 M_0^{\alpha+1} + \kappa_2 M_0 + \kappa_3 M_0^{\alpha} + \widehat{\kappa_4}}.
$$

However, it is a contradiction to the choose of $\rho_0$. Therefore, there exists a $\epsilon_0 > 0$ such that $u_\rho(t) \in C$ for all $t \in I$ and $\rho \in (0, \epsilon_0)$.

**Claim 3**. Problem $\mathcal{P}$ has a mild solution $(x, u) \in C(I; E) \times \mathcal{W}$ satisfying (3.22).

Fix $\rho \in (0, \epsilon_0)$ with $\epsilon_0 > 0$ defined in Claim 2. From Theorem 3.5, let $u = u_\rho \in \mathcal{W}$ be a solution to Problem $\mathcal{P}_\rho$ satisfying $u(t) \in C$ for all $t \in I$. Moreover, Lemma 3.1 implies that there exists a solution $x \in C(I; E)$ such that $x(t) = \mathcal{R}u(t)$ corresponding to $u$ the intergal equation

$$
x(t) = T(t)x_0 + \int_0^t T(t-s) f(s, x(s), \omega u(s)) ds \text{ for all } t \in I.
$$

Moreover, since $\xi(t) \in \partial J(\vartheta u_\rho(t))$ implies that $\langle \vartheta^* \xi(t), v\rangle_V = \langle \xi(t), \vartheta v\rangle_X \le J^0(t, \vartheta u_\rho(t); \vartheta v)$ a.e. $t \in I$, we now use (3.23) and (2.2) to derive

$$
\begin{cases}
u(t) \in C, \\
\langle u'(t) + g(t, u(t)) + B(t, x(t)), v\rangle + J^0(t, \vartheta u_\rho(t); \vartheta v) + \dfrac{1}{\rho}\tilde{d}^0(\omega u(t); \omega v) \ge 0, \\
u'(0) = u_0.
\end{cases} \tag{3.27}
$$

for a.e. $t \in I$ and all $\in V$. Next, we are in the position to choose $v \in T_C(u(t))$ in (3.27). Then the definition of $T_C(\cdot)$ of (2.3) lead to $\tilde{d}^0(\omega u; \omega v) = 0$ for any $v \in T_C(u)$ and the follows

$$
\begin{cases}
u(t) \in C, \\
\langle u'(t) + g(t, u(t)) + B(t, x(t)), v\rangle + J^0(t, \vartheta u_\rho(t); \vartheta v) \ge 0, \ \ t \in I, \ \forall v \in T_C(u(t)) \\
u(0) = u_0.
\end{cases}
$$

Therefore, we obtain that $(x, u) \in C(I; E) \times \mathcal{W}$ is a mild solution of Problem $\mathcal{P}$. Finally, Claim 1 allow us to invoke inequality (3.11) obtaining

$$
\begin{aligned}
\|x(t)\|_E &\le c_1 + c_2 \int_0^b \|u(s)\|_V^\alpha ds \\
&\le c_1 + c_2 \sqrt{b} M_0^\alpha \text{ for all } t \in I.
\end{aligned}
$$

Now, setting $M = \max\{M_0, c_1 + c_2\sqrt{b}M_0^\alpha\}$, we infer that (3.22) holds. The proof of the theorem is complete.

□

# 4 An application

In this section, we provide an example of particular problem, for which our previous result can be applied. Let $I = [0, b]$ with $0 < b < \infty$ and $\Omega$ be an open bounded domain in $\mathbb{R}^n (n \geq 2)$ with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. The boundary is composed of two disjoint relatively open parts $\Gamma_D$ and $\Gamma_N$, such that $meas(\Gamma_D) > 0$. Let $\nu$ denote the unit outward normal vector at the boundary $\Gamma$. We now consider the following parabolic initial-boundary value problem.

**Problem $\mathcal{Q}$.** Find $x : [0, b] \times \Omega \to \mathbb{R}$ and $u : [0, b] \times \Omega \to \mathbb{R}$ such that

$$\begin{cases} x_t(z,t) - \Delta x(z,t) = \varepsilon\big(z, t, x(z,t), u(z,t)\big) & \text{in } \Omega \times I, \\ x(z,0) = \delta(z) & \text{in } \Omega, \\ x(z,t) = 0 & \text{in } \Gamma \times I \end{cases} \tag{4.1}$$

$$\begin{cases} u_t(z,t) + \mathrm{div}\varpi(z,t) + \beta(z,t,x(z,t)) = \phi_1(z,t) + h(z,t) & \text{in } \Omega \times I, \\ \varpi(z,t) = -\psi(z,t,u(z,t),\nabla u(z,t)) & \text{in } \Omega \times I, \\ \varpi(t) \cdot \nu = \phi_2(t) & \text{on } \Gamma_N \times I, \\ \dfrac{\partial u(z,t)}{\partial \nu} = \phi_3(z,t) & \text{on } \Gamma_D \times I, \\ -\dfrac{\partial u(z,t)}{\partial \nu} \in \partial j(u(z,t)) & \text{on } \Gamma_N \times I, \\ u(z,0) = \zeta(z) & \text{in } \Omega \end{cases} \tag{4.2}$$

In order to provide the differential hemivariational formulation of problem $\mathcal{Q}$, we need the Lebesgue spaces $E = Y = L^2(\Omega), X = L^2(\Gamma_D)$ with standard norm and the Sobolev space

$$V = \{\, v \in H^1(\Omega) \;:\; v(z) = 0 \quad \text{a.e. } z \in \Gamma_D \,\} \tag{4.3}$$

endowed with the inner product

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx \tag{4.4}$$

and the associated norm $\|\cdot\|_V$. It is well known that $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, as usual, we use $V^*$ for the dual of $V$ and $\langle \cdot, \cdot \rangle$ for the duality pairing mapping between $V^*$ and $V$.

Moreover, from Rellich-Kondrachov compactness theorem, $V$ is compactly embedded into $Y$. This means the inclusion operator $\omega : V \to Y$ is compact. Let $\gamma : V \to X$ denote the trace operator, so $\gamma u$ stands for the trace of $u \in V$ at the boundary $\partial\Omega$. Hence, the symbol $u$ in the boundary conditions of Problem $\mathcal{Q}$ should be understood in the sense of trace. Further, $C \subset V$ is a nonempty, closed set of constraints which can be convex or nonconvex. We now provide the following hypotheses.

$$\begin{cases} \text{The function } e : \Omega \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ (a)\ e(\cdot,\cdot,\eta,\xi) \text{ is measurable on } \Omega \times I \text{ for all } (\eta,\xi) \in \mathbb{R}^2; \\ (b)\ |e(z,t,0,0)| \le a_0(z,t) \text{ for a.e. } (z,t) \in \Omega \times I \\ \qquad \text{with } a_0 \in L^2_+(\Omega \times I); \\ (c)\ |e(z,t,\eta_1,\xi_1) - e(z,t,\eta_2,\xi_2)| \le l_1(z)|\eta_1 - \eta_2| + l_2(z)|\xi_1 - \xi_2|^\alpha \\ \qquad \text{for a.e. } (z,t) \in \Omega \times I \text{ and all } (\eta_1,\xi_1),(\eta_2,\xi_2) \in \mathbb{R}^2 \\ \qquad \text{with } l_1, l_2 \in L^2_+(\Omega) \text{ and } \alpha \in [0,1). \text{ This means } e(z,t,\eta,\xi) \text{ is} \\ \qquad \text{Lipschitz continuous in } \eta \text{ and is } \alpha\text{-Hölder continous in } \xi. \end{cases} \tag{4.5}$$

$$\begin{cases} H(\psi) : \text{The function } \psi : \Omega \times I \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ \psi = (\psi_1, \cdots, \psi_n) \\ \ \text{is such that, for } k = 1, \cdots, n, \\ (a)\ \psi_k(\cdot,\cdot,\eta,\xi) \text{ is measurable on } \Omega \times I \text{ for all } (\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n; \\ \qquad \psi_k(z,t,\cdot,\cdot) \text{ is continuous on } \mathbb{R} \times \mathbb{R}^n \text{ for a.e. } (z,t) \in \Omega \times I; \\ (b)\ |\psi_k(z,t,\eta,\xi)| \le m_0\big(a(z,t) + |\eta| + \sum_{k=1}^n |\xi_k|\big) \text{ for a.e. } (z,t) \in \Omega \times I \\ \qquad \text{and all } (\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n \text{ with } m_0 > 0,\ a \in L^2(\Omega \times I); \\ (c)\ \sum_{k=1}^n \big((\psi_k(z,t,\eta,\xi^1) - \psi_k(z,t,\eta,\xi^2))(\xi^1_k - \xi^2_k)\big) \ge 0 \\ \qquad \text{for all } \eta \in \mathbb{R}, \xi^1, \xi^2 \in \mathbb{R}^n \text{ and for a.e. } (z,t) \in \Omega \times I, \\ (d)\ \sum_{k=1}^n \psi_k(z,t,\eta,\xi)\xi_k \ge m_1(|\eta|^2 + \sum_{k=1}^n |\xi_k|^2) \text{ for all } \xi \in \mathbb{R}^n \\ \qquad \text{and for a.e. } (z,t) \in \Omega \times I \text{ with } m_1 > 0. \end{cases} \tag{4.6}$$

$$\begin{cases} H(\beta) : \text{The function } \beta : \Omega \times I \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ (a)\ \beta(\cdot,\cdot,\eta) \text{ is measurable on } \Omega \times I \text{ for all } \eta \in \mathbb{R}; \\ (b)\ \beta(z,t,\cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } (z,t) \in \Omega \times I; \\ (c)\ |\beta(z,t,\eta)| \le b_0(z,t) + b_1|\eta| \text{ for a.e. } (z,t) \in \Omega \times I \\ \qquad \text{and all } \eta \in \mathbb{R} \text{ with } b_0 \in L^2(\Omega \times I) \text{ and } b_1 \ge 0. \end{cases} \tag{4.7}$$

$$\begin{cases} H(j) : \text{The function } j : \Omega \times I \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ (a)\ j(\cdot,\cdot,\eta) \text{ is measurable on } \Omega \times I \text{ for all } \eta \in \mathbb{R}; \\ (b)\ j(z,t,\cdot) \text{ is locally Lipschitz continuous on } \mathbb{R}; \\ (c)\ \||\xi| \le c_j(1 + |\eta|),\ \xi \in \partial j(z,t,\eta), \ \text{ for a.e. } (z,t) \in \Omega \times I \\ \qquad \text{and all } \eta \in \mathbb{R} \text{ with } c_j > 0. \end{cases} \tag{4.8}$$

$$\phi_1 \in L^2(\Omega \times I), \phi_2 \in L^2(I; L^2(\Gamma_N)),\ \phi_3 \in L^2(I; L^2(\Gamma_D)). \tag{4.9}$$

$$\delta \in E,\ \zeta \in V. \tag{4.10}$$

Define the operator $A : D(A) \subset E \to E$ as follows

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega), \quad Ax = \Delta x \quad \forall\, x \in D(A). \tag{4.11}$$

Obviously, one can see that $A$ generates a $C_0$-semigroup semigroup $\{T(t)\}_{t\geq 0}$ on $E$ (see [25], for instance). Assume now that $u$ is sufficiently smooth solution to (4.2) and $v \in V$. We multiply the first equation of (4.2) by $v$ and use the Green formula to find

$$\int_\Omega u'(t)v\,dz + \int_\Omega \psi(t,u(t),\nabla u(t))\cdot\nabla v\,dz + \int_\Omega \beta(t,x(t))v\,dz + \int_{\Gamma_N} j^0(z,t,\gamma u(t));\gamma v)d\Gamma \\ \geq \int_\Omega \phi_1(t)v\,dz + \int_\Omega h(t)v\,dz + \int_{\Gamma_N}\phi_2(t)v\,d\Gamma + \int_{\Gamma_D} f_N(t)v\,d\Gamma \tag{4.12}$$

Under above assumptions, let $f: I\times E\times Y \to E$, $g: I\times V \to V^*$, $B: I\times E\to V^*$, $J: I\times X\to \mathbb{R}$ and $\varphi: I\to V^*$ be defined as follows:

$$f(t,x,y)(z) = e(z,t,x(z),y(z)) \qquad \forall\, t\in I,\ x\in E,\ y\in Y,\ \text{a.e. } z\in\Omega, \tag{4.13}$$

$$\langle g(t,u),v\rangle = \int_\Omega \psi(t,u(t),\nabla u)\cdot\nabla v\,dz - \int_\Omega \phi_1(t)v\,dz - \int_{\Gamma_N}\phi_2(t)v\,d\Gamma \\ - \int_{\Gamma_N}\phi_3(t)v\,d\Gamma \qquad \forall\, t\in I,\ \ u,\ v\ \in V, \tag{4.14}$$

$$\langle B(t,x),v\rangle = \int_\Omega \beta(t,x)v\,dz \qquad \forall\, t\in I,\ x\in E,\ v\ \in V, \tag{4.15}$$

$$J(t,u) = \int_{\Gamma_N} j(z,t,u(z))\,d\Gamma \text{ for all } u\in X, \tag{4.16}$$

$$\langle \varphi(t),v\rangle = \int_\Omega h(t)v\,dz \qquad \forall\, t\in I,\ v\ \in V. \tag{4.17}$$

We are in a position to formulate more general problem.

**Problem $\mathcal{Q}_I$.** Find $x:[0,T]\to E$ and $u:[0,T]\to V$ such that

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t),\omega u(t)),\ \text{for } t\in I,\\ u'(t) + g(t,u(t)) + B(t,x(t)) + \gamma^*\partial J(t,\gamma u(t)) \ni \varphi(t) \text{ a.e. } t\in I,\\ x(0)=\delta,\ u(0)=\zeta. \end{cases} \tag{4.18}$$

In fact, let $(x,u)$ be a solution of $\mathcal{Q}_I$ and let $v\in V$. Then, there exists $\eta\in X^*$, such that

$$\begin{cases} \langle u'(t)+g(t,u(t))+B(t,x(t)),v\rangle + \langle \eta(t),\gamma v\rangle_{X^*\times X} = \langle\varphi(t),v\rangle \ \text{ a.e. } t\in I,\\ \eta(t)\in\partial J(\gamma u(t)) \text{ for a.e. } \ t\in I. \end{cases} \tag{4.19}$$

Hence, by Theorem 3.47 of [20] , it follows that for a.e.$t\in(0,b)$ , $\eta(t)$ can be treated as a function $\eta(t):\Gamma_N\to\mathbb{R}$ , such that $\eta(t)\in L^2(\Gamma_N)$ and it satisfies

$$\langle \eta(t),\gamma v\rangle_{X^*\times X} = \int_{\Gamma_N}\eta(z,t)\cdot v(z)d\Gamma \leq \int_{\Gamma_N} j^0(\gamma u(z,t));\gamma v(z))d\Gamma.$$

Next, we suppose that $\varphi(t) \in V$ is a function of $u(t)$, introduced in order to incorporate constraints to the model. We will assume that $u(t) \in C$ and $-\varphi(t) \in N_C(u(t))$, where $N_C(u(t))$ is Clarke's normal cone to $C$ at $u$. Therefore

$$\langle \varphi(t), v\rangle \geq 0 \text{ for all } t \in I \text{ and } v \in \overline{T}_C(u(t)), \tag{4.20}$$

where $\overline{T}_C(v)$ is Clarke's tangent cone of $C$ at $v$. Therefore, from (4.12)–(4.17), (4.20), the problem (4.1)–(4.2) can be transformed into the following abstract differential hemivariational inequality:

**Problem $\mathcal{Q}_V$.** Find $x : [0, T] \to E$ and $u : [0, T] \to V$ such that

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), \omega u(t)), \text{ for } t \in I, \\ x(0) = \delta, \\ u(t) \in C, \\ \langle u'(t) + g(t, u(t)) + B(t, x(t)), v\rangle + J^0(t, \gamma u(t), \gamma v) \geq 0, \; t \in I, \; \forall v \in \overline{T_C}(u(t)), \\ u(0) = \zeta. \end{cases} \tag{4.21}$$

Now, we can consider the concept of mild solution following Definition 1.1 for problem $\mathcal{Q}_V$. Moreover, we have the following existence result.

**Theorem 4.1** *Assume that* (4.5)–(4.10) *hold. If $m_0 > 2c_j$, then the Problem $\mathcal{Q}_V$ has at least a mild solution $(x, u) \in C(I, E) \times \mathcal{W}$.*

**Proof.** The proof can be obtained by using the abstract result of Theorem 3.6 with $E = Y = L^2(\Omega)$, $X = L^2(\Gamma_D)$, $V$ defined above. Let $x_0 = \delta$, $u_0 = \zeta$. $\vartheta = \gamma$, and $A$, $f$, $g$, $B$ and $J$ given by (4.11), (4.13)–(4.16). Indeed, it is easy to check that in this case assumptions (3.1)–(3.9) are satisfied. Therefore, we obtain the conclusion and omit the detail of proof. □

# Acknowledgement

This project is supported by the National Natural Science Foundation of China Grant No. 12001120, Guangxi Natural Science Foundation under Grant Nos. 2021GXNSFAA075022, 2022GXNSFAA035617, Middle-aged and Young Teachers' Basic Ability Promotion Project of Guangxi under Grant No. 2020KY16017, High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35), Hundred Talent Program for Introducing the Overseas High-Level Talents of Guangxi Colleges and Universities, Guangxi First-class Discipline Statistics Construction Project Fund.

# References

[1] J.P. Aubin, A. Cellina, Differential Inclusions, Springer-Verlag, New York, 1984.

[2] K. Bartosz, Numerical methods for evolution hemivariational inequalities, Advances in Variational and Hemivariational Inequalities, Springer, Cham, 2015, 111-144.

[3] X.J. Chen, Z.Y. Wang, Differential variational inequality approach to dynamic games with shared constraints, Mathematical Programming, 2014, 146: 379–408.

[4] F.H. Clarke, Optimization and nonsmooth analysis, Wiley, New York, 1983.

[5] Z. Faiz, O. Baiz, H. Benaissa, D. Moutawakil, Penalty method for a class of differential hemivariational inequalities with application, Numerical Functional Analysis and Optimization, 2021, 42(10): 1178-1200.

[6] J.F. Han, L. Lu, S.D. Zeng, Evolutionary variational-hemivariational inequalities with applications to dynamic viscoelastic contact mechanics, Zeitschrift für angewandte Mathematik und Physik, 2020, 71(1): 1-23.

[7] D. Goeleven, On the hemivariational inequality approach to nonconvex constrained problems in the theory ofvon Kárman plates. Zeitschrift für Angewandte Mathematik und Mechanik, 1995, 75: 861–866.

[8] L. Gasiński, Z.H. Liu Z, S. Migórski, A. Ochal, Z.J. Peng, Hemivariational inequality approach to evolutionary constrained problems on star-shaped sets, Journal of Optimization Theory and Applications, 2015, 164(2): 514-533.

[9] X.W. Li, Z.H. Liu, Sensitivity analysis of optimal control problems described by differential hemivariational inequalities, SIAM Journal on Control and Optimization, 2018, 56(5): 3569-3597.

[10] Z.H. Liu, D. Motreanu, S.D. Zeng, Generalized penalty and regularization method for differential variational-hemivariational inequalities, SIAM Journal on Optimization, 2021, 31(2): 1158-1183.

[11] Z.H. Liu, S. Migóski, S.D. Zeng, Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces, Journal of Differential Equations, 2017, 263: 3989–4006.

[12] Z.H. Liu, S.D. Zeng, Differential variational inequalities in infinite Banach spaces, Acta Mathematica Scientia, 2017, 37(1): 26-32.

[13] Z.H. Liu, S.D. Zeng, Penalty method for a class of differential variational inequalities, Applicable Analysis, 2021, 100(7): 1574-1589.

[14] Z.H. Liu, S.D. Zeng, D. Motreanu, Evolutionary problems driven by variational inequalities, Journal of Differential Equations, 2016, 260: 6787–6799.

[15] Z.H. Liu, S.D. Zeng, D. Motreanu, Partial differential hemivariational inequalities, Advances in Nonlinear Analysis, 2018, 7(4): 571-586.

[16] L. Lu, L.J. Li, M. Sofonea, A generalized penalty method for differential variational-hemivariational inequalities, Acta Mathematica Scientia, 2022, 42(1): 247-254.

[17] L. Lu, Z.H. Liu, D. Motreanu, Existence results of semilinear differential variational inequalities without compactness, Optimization, 2019, 68(5): 1017-1035.

[18] L. Lu, Z.H. Liu, V. Obukhovskii, Second order differential variational inequalities involving anti-periodic boundary value conditions, Journal of Mathematical Analysis and Applications, 2019: 473(2): 846-865.

[19] S. Migórski, A. Ochal, Quasi-static hemivariational inequality via vanishing acceleration approach, SIAM Journal on Mathematical Analysis, 2009, 41: 1415–1435.

[20] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.

[21] S. Migórski, S.D. Zeng, A class of differential hemivariational inequalities in Banach spaces, Journal of Global Optimization, 2018, 72(4): 761-779.

[22] Z. Naniewicz, Hemivariational inequality approach to constrained problems for star-shaped admissible sets, Journal of Optimization Theory and Applications, 1994, 83: 97-112.

[23] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications. (Monographs and textbooks in pure and applied mathematics; vol. 188). New York: Marcel Dekker; 1995.

[24] T.V.A. Nguyen, D.K. Tran, On the differential variational inequalities of parabolic elliptic type, Mathematical Methods in the Applied Sciences, 2017, 40: 4683–4695.

[25] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[26] P.D. Panagiotopoulos, Nonconvex superpotentials in sense of F.H. Clarke and applications, Mechanics Research Communications, 1981, 8: 335-340.

[27] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer, Berlin, 1993.

[28] N.S. Papageorgiou, F. Papalini, F. Renzacci, Existence of solutions and periodic solutions for nonlinear evolution inclusions, Rendiconti del Circolo Matematico di Palermo, 1999, 48: 341–364.

[29] J.S. Pang, D.E. Stewart, Differential variational inequalities, Mathematical Programming, 2008, 113: 345–424.

[30] Z.J. Peng, Optimal obstacle control problems involving nonsmooth cost functionals and quasilinear variational inequalities, SIAM Journal on Control and Optimization, 2020, 58(4): 2236-2255.

[31] Z.J. Peng, L. Gasiński, S. Migórski, A. Ochal, A class of evolution variational inequalities with nonconvex constraints, Optimization, 2019, 68(10): 1881-1895.

[32] Z.J. Peng, K. Kunisch, Optimal control of a class of elliptic variational- hemivariational inequalities, Journal of Optimization Theory and Applications, 2018, 178: 1-25.

[33] Z.J. Peng , Z.H. Liu, X.Y. Liu, Boundary hemivariational inequality problems with doubly nonlinear operators, Mathematische Annalen, 2013, 356(4): 1339-1358.

[34] M. Sofonea, S. Migórski, Variational-Hemivariational Inequalities with Applications, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London, 2018.

[35] M. Sofonea, S. Migórski S, W. Han, A penalty method for history-dependent variational-hemivariational inequalities, Computers and Mathematics with Applications, 2018, 75: 2561-2573.

[36] E. Zeidler, Nonlinear Functional Analysis and Applications II A/B, Springer, New York, 1990.

[37] S.D. Zeng, S. Migórski, Z.H. Liu, Well-Posedness, optimal control, and sensitivity analysis for a class of differential variational-Hemivariational inequalities, SIAM Journal on Optimization, 2021, 31(4): 2829-2862.