Exsistence, Blow up and Numerical approximations of Solutions for a Biharmonic Coupled System with Variable exponents

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Abstract

In this paper, we consider a coupled system of two biharmonic equations with damping and source terms of variable-exponents nonlinearities, supplemented with initial and mixed boundary conditions. We establish an existence and uniqueness result of a weak solution, under suitable assumptions on the variable exponents. Then, we show that solutions with negative-initial energy blow up in finite time. To illustrate our theoritical findings, we present two numerical examples.

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1. Introduction

In this work, we study the following biharmonic (or Petrovsky) coupled system with initial and boundary conditions:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(x)-2} u_t = f_1(x,u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} + \Delta^2 v + |v_t|^{r(x)-2} v_t = f_2(x,u,v) & \text{in } \Omega \times (0,T), \\ u = v = 0 & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ (u(0), v(0)) = (u_0, v_0) \text{ and } (u_t(0), v_t(0)) = (u_1, v_1) & \text{in } \Omega \times \Omega, \end{cases} \tag{1.1}$$

where $T > 0, \Omega$ is a smooth and bounded domain of $\mathbb{R}^n, \left(n = \overline{1,6}\right)$, the exponents m and r are continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later, $\frac{\partial u}{\partial \eta}$ and $\frac{\partial v}{\partial \eta}$ denote the external normal derivatives of u

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and v, respectively, on the boundary $\partial\Omega$ and the coupling terms f_1 and f_2 are given as follows: for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$,

$$f_1(x,u,v) = \frac{\partial}{\partial u} F(x,u,v) \text{ and } f_2(x,u,v) = \frac{\partial}{\partial v} F(x,u,v), \tag{1.2}$$

with

$$F(x,u,v) = a\,|u+v|^{p(x)+1} + 2b\,|uv|^{\frac{p(x)+1}{2}}, \tag{1.3}$$

where $a, b > 0$ are two positive constants and p is a given continuous function on $\overline{\Omega}$ satisfying the condition $(H.3)$ bellow.

The fourth single-order nonlinear equations arise in various physical phenomena such as the study of travelling waves in suspnssion bridges [21], micro electro mechanical systems [33], bending behaviour of a thin elastic rectangular plate [35], geometric and functional design [9], radar imaging [3],..., etc.

Other physical phenomena like flows of electro-rheological fluids, fluids with temperature dependent viscocity, filtration processes through a porous media, image processing and thermorheological fluids give rise to mathematical models of hyperbolic, parabolic and biharmonic equations with variable exponents of nonlinearity. See [4, 5, 34] for more details.

Recently, the hyperbolic equations with nonlinearties of variable exponents type had received a considerable amount of attention. Treating this class of problems, the researchers in [16, 27, 28, 31, 30, 32] investigated the local existence and blow up of solutions, whereas in [13, 22, 26, 36, 17], they estabilshed several uniform estimates of decay rates of the solution energy.

Concerning coupled systems of wave equations in the variable-exponents case, we have only few works. In [10], Bouhoufani and Hamchi obtained the global existence of a weak solution and established decay rates of the solution in a bounded domain. Messaoudi et al. [27] studied the same system and proved a theorem of existence and uniqueness of a weak solution, established a blow-up result for certain solutions with positive-initial energy and gave some numerical applications for their theoritical resuls. Also, Messaoudi and Talahmeh [29] treated a system of hyperbolic equations with variable exponents in the damping and source terms, and established a blow-up result for solutions with negative initial energy. In [30], Messaoudi et al. considered the following system

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m(x)-2} u_t + f_1(u,v) = 0 & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r(x)-2} v_t + f_2(u,v) = 0 & \text{in } \Omega \times (0,T), \end{cases} \tag{1.4}$$

with initial and Dirichlet-boundary conditions (here f_1 and f_2 are the coupling terms introduced in (1.2)). The authors proved the existence of global solutions, obtained explicit decay rate estimates, under suitable assumptions on the variable exponents m, r and p and presented some numrical tests. Recently, Bouhoufani et al. [11] treated a similar system to (1.4), where

$$f_1(u,v) = |u|^{p(x)-2} u |v|^{p(x)} \text{ and } f_2(u,v) = |v|^{p(x)-2} v |u|^{p(x)}$$

and the damping term, in each equation, is modilated by a time-dependent coefficients $\alpha(t)$ and $\beta(t)$. They established decay rate results, under appropriate assumptions on the coefficient functions and the variable exponents and illustrated their results by some examples and numerical tests.

For equations and systems with biharmonic operator and constant exponents of nonlinearity, we mention the work by Komornik [18], in which he proved the well-posedness for a Petrovsky equation, by using the nonlinear semigroup theory, and established the energy decay estimates for a weak solutions. Guesmia [14] used the same approach to obtain a global existence, uniqueness and regularity results for Petrovsky equation, in a more general setting. He, established decay estimates of weak, as well as strong solutions, under suitable conditions on the damping term. In [15], the same author proved the well-posedness and uniform stabilisation for a damped nonlinear coupled system of two Petrovsky equations, under appropriate assumptions. After that, Assila and Guesmia [7] considered the following problem

$$\begin{cases} u_{tt} + k_1\Delta^2 u + k_2\Delta^2 u_t + \Delta g(\Delta u) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{on } \Omega, \end{cases}$$

where k_1 and k_2 are two positive constants, and g is C^2-class real valued function. By invoking an important Lemma of Komornik [19], they showed that the solution energy decays exponentially. The well-posedness of this type of problems has been studied in many papers; the reader can see, for example, the work by Banks et al. [8]. For the Petrovsky equation with nonlinear source term, we have the work of Messaoudi [25], in which he studied the problem:

$$\begin{cases} u_{tt} + \Delta^2 u + a u_t \left|u_t\right|^{m-2} = b u \left|u\right|^{\rho-2} & \text{in } \Omega \times \mathbb{R}^+, \\ u = \partial_\eta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{on } \Omega, \end{cases}$$

where a is a positive constant and $m > 2$. He obtained an existence result and showed that the solution blows up, in finite time, if $m < P$ and exists globally otherwise.

Very recently, Antontsev and al. [6] studied the following Petrovsky equation

$$u_{tt} + \Delta^2 u - \Delta u_t + \left|u_t\right|^{m(x)-2} u_t = \left|u\right|^{p(x)-2} u. \tag{1.5}$$

They proved the existence of local weak solutions by using the Banach fixed-point theorem, and gave a blow-up result for negative-initial-energy solutions, under suitable assumptions. In [23], Liao and Tan treated a similar system with $M(\|\nabla u\|_2^2)\Delta u$ in the left-hand side of the equation (1.5), where $M(s) = a + bs^\gamma$ is a positive C^1-function, $a > 0, b > 0, \gamma \geq 1$, and m, p are given measurable functions. The upper and lower bounds of the blow-up time, as well as some uniform decay rates have been established.

To the best of our knowledge, the Petrovsky (or biharmonic) coupled system with variable exponents of nonlinearty given by (1.2) and (1.3), has never been studied. Our aim in this work is to prove the existence and uniqueness of a weak solution to the Petrovsky system (1.1), by using the Faedo-Galerkin method, together with a fixed-point principle. We also establish a blow-up result for negative-initial-energy solutions, under appropriate conditions on the variable exponents. We note here that the well-posedness is established only for $n \leq 6$. For dimensions higher than 6, the problem remains open, see Remark 3.4 below.

The paper is devided into three sections, in addition to the introduction. In Section 2, we present some notations, definitions and important properties and tools of the variable-exponent Lebesgue and Sobolev spaces. We also introduce our assumptions. Section 3 is devoted to the statement and proof of the well-posedness. Our blow-up result will be given in Section 4. Finally, some numerical tests to verfy the finite time blow-up result, are presented in Section 5.

2. Preliminaries

In this section, we define the variable-exponent Lebesgue and Sobolev spaces and, then, present some of their propereties and facts. For more details, see [5, 12, 20].

Let $q : \Omega \longrightarrow [1, \infty)$ be a measurable function. We define the Lebesgue space with a variable exponent by

$$L^{q(.)}(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable in } \Omega : \ \varrho_{q(.)}(\lambda f) < +\infty, \text{ for some } \lambda > 0 \right\},$$

where

$$\varrho_{q(.)}(f) = \int_\Omega |f(x)|^{q(x)} dx.$$

$L^{q(.)}(\Omega)$ is a Banach space with respect to the following Luxembourg-type norm

$$\|f\|_{q(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

Let $k \in \mathbb{N}$. We define the variable exponent Sobolev space $W^{k,p(.)}(\Omega)$ as follows:

$$W^{k,q(.)}(\Omega) = \left\{ u \in L^{q(.)}(\Omega) : \partial^{|\alpha|} u \in L^{q(.)}(\Omega), \ \textit{with } |\alpha| \leq k \right\}.$$

$W^{k,q(.)}(\Omega)$ is a Banach space equipped with the following norm

$$\|u\|_{W^{k,q(.)}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \|\partial_\alpha u\|_{q(.)},$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$.

Lemma 2.1. (Young's Inequality [5, 20]**)**
Let $r, q, s \geq 1$ be measurable functions defined on Ω, such that

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)}, \ \textit{for a.e } y \in \Omega.$$

Then, for all $a, b \geq 0$, *we have*

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{r(.)}}{r(.)} + \frac{(b)^{q(.)}}{q(.)}.$$

Lemma 2.2. (Hölder's Inequality [5, 20]**)** *Let* $r, q, s : \Omega \longrightarrow [1, \infty)$ *be measurable functions, such that*

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)}, \ for \ \ a.e. \ y \in \Omega.$$

If $f \in L^{r(.)}(\Omega)$ *and* $g \in L^{q(.)}(\Omega)$, *then* $fg \in L^{s(.)}(\Omega)$, *with*

$$\|fg\|_{s(.)} \leq 2\|f\|_{r(.)}\|g\|_{q(.)}.$$

Lemma 2.3. [5, 20] *If* $1 < q^- \leq q(x) \leq q^+ < +\infty$ *holds then, for any* $f \in L^{q(.)}(\Omega)$,

$$min\left\{\|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+}\right\} \leq \varrho_{q(.)}(f) \leq max\left\{\|f\|_{q(.)}^{q^-}, \|f\|_{q(.)}^{q^+}\right\},$$

where

$$q^- = ess \inf_{x\in\Omega} \ q(x) \ and \ q^+ = ess \sup_{x\in\Omega} \ q(x).$$

Lemma 2.4. [20] *If* $q^+ < +\infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^{q(.)}(\Omega)$.

Definition 2.5. We say that a function $q : \Omega \longrightarrow \mathbb{R}$ is log-Hölder continuous on Ω, if there exists constant $\theta > 0$ such that for all $0 < \delta < 1$, we have

$$|q(x) - q(y)| \leq -\frac{\theta}{log|x - y|}, \ for \ a.e. \ x, y \in \Omega, \ with \ |x - y| < \delta.$$

Remark 2.6. *The log-Hölder continuity condition on* q *can be replaced by* $q \in C(\overline{\Omega})$, *if* Ω *is bounded.*

Definition 2.7. The closure of the set of $W^{k,q(.)}(\Omega)$-functions with compact support in $W^{k,q(.)}(\Omega)$ is the Sobolev space $W_0^{k,q(.)}(\Omega)$ "with zero boundary trace",
i.e.,

$$W_0^{k,q(.)}(\Omega) = \overline{\{u \in W^{k,q(.)}(\Omega) : u = u\chi_K \ for \ a \ compact \ K \subset \Omega\}}.$$

Furtheremore, we denote by $H_0^{k,q(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,q(.)}(\Omega)$ and by $W^{-k,q'(.)}(\Omega)$ the dual space of $W_0^{k,q(.)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$.

Lemma 2.8. (Embedding Property [12]**)** *Let* $q : \Omega \longrightarrow [1, \infty)$ *be a measurable function and* $k \geq 1$ *be an integer. Suppose that* r *is a log-Hölder continuous function on* Ω, *such that, for all* $x \in \Omega$, *we have*

$$\begin{cases} k \leq q^- \leq q(x) \leq q^+ < \frac{nr(x)}{n-kr(x)}, & if \ r^+ < \frac{n}{k}, \\ k \leq q^- \leq q^+ < \infty, & if \ r^+ \geq \frac{n}{k}. \end{cases}$$

Then, the embedding $W_0^{k,r(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ *is continuous and compact.*

Throughout this paper, we denote by $\mathcal{V}$ the following space

$$\mathcal{V} = \{u \in H^2(\Omega) : \ u = \frac{\partial u}{\partial \eta} = 0 \ on \ \partial\Omega\} = H_0^2(\Omega).$$

So, $\mathcal{V}$ is a separable Hilbert space endowed with the inner product and norm, respectively,

$$(u, v)_{\mathcal{V}} = \int_\Omega \Delta u \Delta v dx \ and \ \|u\|_{\mathcal{V}} = \|\Delta u\|_2 \,,$$

where $\|\Delta u\|_k = \|\Delta u\|_{L^k(\Omega)}$.

Now, we present our assumptions on the variable exponents m, r and p, that will be used in the sequel. So, for all $x \in \overline{\Omega}$, we assume that

$$\left| \begin{array}{ll} 2 \leq m^-, & \text{if } \ n \leq 4, \\ 2 \leq m^- \leq m(x) \leq m^+ \leq 10, & \text{if } \ n = 5, \\ 2 \leq m^- \leq m(x) \leq m^+ \leq 6, & \text{if } \ n = 6, \end{array} \right. \tag{H.1}$$

$$\left| \begin{array}{ll} 2 \leq r^-, & \text{if } \ n \leq 4, \\ 2 \leq r^- \leq r(x) \leq r^+ \leq 10, & \text{if } \ n = 5, \\ 2 \leq r^- \leq r(x) \leq r^+ \leq 6, & \text{if } \ n = 6 \end{array} \right. \tag{H.2}$$

and

$$\left| \begin{array}{ll} 3 \leq p^-, & \text{if } \ n \leq 4, \\ 3 \leq p^- \leq p(x) \leq p^+ \leq 5, & \text{if } \ n = 5, \\ p(x) = 3, & \text{if } \ n = 6, \end{array} \right. \tag{H.3}$$

where

$$m^- = \inf_{x \in \overline{\Omega}} m(x), \ m^+ = \sup_{x \in \overline{\Omega}} m(x),$$

$$r^- = \inf_{x \in \overline{\Omega}} r(x), \ r^+ = \sup_{x \in \overline{\Omega}} r(x),$$

$$p^- = \inf_{x \in \overline{\Omega}} p(x) \ \text{and} \ p^+ = \sup_{x \in \overline{\Omega}} p(x).$$

3. Existence of weak solution

Before starting our study, we introduce the definition of a weak solution for system (1.1). We multiply the first equation in (1.1) by $\Phi \in C_0^\infty(\Omega)$ and the second equation by $\Psi \in C_0^\infty(\Omega)$, integrate each result over Ω, use of Green's formula and the boundary conditions to obtain the following.

Definition 3.1. (Weak Solution of (1.1)**)**
Let $(u_0, u_1), (v_0, v_1) \in \mathcal{V} \times L^2(\Omega)$. Any pair of functions (u, v), such that

$$\left| \begin{array}{l} u, v \in L^\infty([0, T]; \mathcal{V}), \\ u_t \in L^\infty\left([0, T]; L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0, T)), \\ v_t \in L^\infty\left([0, T]; L^2(\Omega)\right) \cap L^{r(.)}(\Omega \times (0, T)) \end{array} \right. \tag{3.1}$$

is called a weak solution of (1.1) on $[0,T)$, if

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t\Phi dx + \int_\Omega \Delta u\Delta\Phi dx + \int_\Omega |u_t|^{m(x)-2}u_t\Phi dx \\ = \int_\Omega f_1\Phi dx, \\ \frac{d}{dt}\int_\Omega v_t\Psi dx + \int_\Omega \Delta v\Delta\Psi dx + \int_\Omega |v_t|^{r(x)-2}u_t\Psi dx \\ = \int_\Omega f_2\Psi dx, \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1, \end{cases}$$

for a.e. $t \in (0,T)$ and all test functions $\Phi, \Psi \in \mathcal{V}$.
Note that $C_0^\infty(\Omega)$ is dense in $\mathcal{V}$ and $\mathcal{V} \subset L^{m(.)}(\Omega) \cap L^{r(.)}(\Omega)$.

In order to establish an existence result of a local weak solution for system (1.1), we, first, consider the following initial-boundary-value problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u_t\,|u_t|^{m(x)-2} = f(x,t) & \text{in } \Omega \times (0,T), \\ v_{tt} + \Delta^2 v + v_t\,|v_t|^{r(x)-2} = g(x,t) & \text{in } \Omega \times (0,T), \\ u = v = \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1 & \text{in } \Omega, \end{cases} \tag{S}$$

for given $f, g \in L^2(\Omega \times (0,T))$ and $T > 0$.

We have the following theorem of existence and uniqueness for problem (S).

Theorem 3.2. *Let $n = \overline{1,6}$ and $(u_0,u_1),(v_0,v_1) \in \mathcal{V} \times L^2(\Omega)$. Assume that assumptions $(H.1)$-$(H.2)$ hold. Then, the problem (S) admits a unique weak solution on $[0,T)$, in the sense of Definition* 3.1, *having the regularity* (3.1).

Proof. **UNIQUENESS**

Suppose that (S) has two weak solutions (u_1,v_1) and (u_2,v_2), in the sense of Definition 3.1. Taking, $(\Phi,\Psi) = (u_{1t}-u_{2t}, v_{1t}-v_{2t})$, in this definition, it follows that $(u,v) = (u_1-u_2, v_1-v_2)$ satisfies the following identities, for all $t \in (0,T)$,

$$\frac{d}{dt}\left[\int_\Omega \left(|u_t|^2 + (\Delta u)^2\right)dx\right] + 2\int_\Omega \left(|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\right)(u_{1t}-u_{2t})dx = 0 \tag{3.2}$$

and

$$\frac{d}{dt}\left[\int_\Omega \left(|v_t|^2 + (\Delta v)^2\right)dx\right] + 2\int_\Omega \left(|v_{1t}|^{r(x)-2}v_{1t} - |v_{2t}|^{r(x)-2}v_{2t}\right)(v_{1t}-v_{2t})dx = 0. \tag{3.3}$$

Integrating (3.2) and (3.3) over $(0,t)$, with $t \le T$, we obtain

$$\|u_t\|_2^2 + \|u\|_{\mathcal{V}}^2 + 2\int_0^t\int_\Omega \left(|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\right)(u_{1t}-u_{2t})dxd\tau = 0 \tag{3.4}$$

and

$$\|v_t\|_2^2 + \|v\|_{\mathcal{V}}^2 + 2\int_0^t \int_\Omega \left(|v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t} \right) (v_{1t} - v_{2t}) dx d\tau = 0. \tag{3.5}$$

But we have, for all $x \in \Omega, Y, Z \in \mathbb{R}$ and $q(x) \geq 2$,

$$\left(|Y|^{q(x)-2} Y - |Z| Z \right)^{q(x)} (Y - Z) \geq 0, \tag{3.6}$$

then, estimates (3.4) and (3.5) yield

$$\|u_t\|^2 + \|u\|_{\mathcal{V}}^2 = \|v_t\|^2 + \|v\|_{\mathcal{V}}^2 = 0.$$

Thus, $u_t(.,t) = v_t(.,t) = 0$ and $u(.,t) = v(.,t) = 0,$ for all $t \in (0,T)$. Thanks to the boundary conditions, we conclude $u = v = 0$ on $\Omega \times (0,T)$, which proves the uniqueness of the solution.

EXISTENCE:

To prove the existence of the solution for (S), we use the Faedo-Galerkin method. It will be carried out in the following steps.

Approximate Problem. Consider $\{\omega_j\}_{j=1}^\infty$ an orthonormal basis of $\mathcal{V}$ and define, for all $k \geq 1$, (u^k, v^k) a sequence in $\mathcal{V}_k = span\{\omega_1, \omega_2, ..., \omega_k\} \subset \mathcal{V}$, given by

$$u^k(x,t) = \Sigma_{j=1}^k a_j(t)\omega_j(x) \text{ and } v^k(t) = \Sigma_{j=1}^k b_j(t)\omega_j(x)$$

for all $x \in \Omega$ and $t \in (0,T)$ and solves the following approximate problem:

$$\begin{cases} \int_\Omega u_{tt}^k(x,t)\omega_j dx + \int_\Omega \Delta u^k(x,t)\Delta\omega_j dx + \int_\Omega \left| u_t^k(x,t) \right|^{m(x)-2} u_t^k(x,t)\omega_j dx \\ = \int_\Omega f(x,t)\omega_j, \\ \int_\Omega v_{tt}^k(x,t)\omega_j dx + \int_\Omega \Delta v^k(x,t)\Delta\omega_j dx + \int_\Omega \left| v_t^k(x,t) \right|^{r(x)-2} v_t^k(x,t)\omega_j dx \\ = \int_\Omega g(x,t)\omega_j, \end{cases} \tag{S_k}$$

for all $j = 1, 2, ..., k$, with

$$\left| \begin{array}{l} u^k(0) = u_0^k = \Sigma_{i=1}^k \langle u_0, \omega_i \rangle \omega_i, \ u_t^k(0) = u_1^k = \Sigma_{i=1}^k \langle u_1, \omega_i \rangle \omega_i \\ v^k(0) = v_0^k = \Sigma_{i=1}^k \langle v_0, \omega_i \rangle \omega_i, \ v_t^k(0) = v_1^k = \Sigma_{i=1}^k \langle v_1, \omega_i \rangle \omega_i, \end{array} \right. \tag{3.7}$$

such that

$$\left| \begin{array}{l} u_0^k \longrightarrow u_0 \text{ and } v_0^k \longrightarrow v_0 \ \ in\ H_0^1(\Omega), \\ u_1^k \longrightarrow u_1 \text{ and } v_1^k \longrightarrow v_1 \ \ in\ L^2(\Omega) \end{array} \right. \tag{3.8}$$

For any $k \geq 1$, problem (S_k) generates a system of k nonlinear ordinary differential equations. The ODE's standard existence theory assures the existence of a unique local solution (u^k, v^k) for (S_k) on $[0, T_k)$, with $0 < T_k \leq T$.

Next, we have to show, by a priory estimates, that $T_k = T, \forall k \geq 1$.

A Priori Estimates. Multiplying $(S_k)_1$ and $(S_k)_2$ by $a_j'(t)$ and $b_j'(t)$, respectively, using Green's formula and the boundary conditions, and then summing each result over $j = \overline{1,k}$, we obtain, for all $0 < t \leq T_k$,

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|u_t^k(x,t)|^2 + (\Delta u^k)^2(x,t)\right) dx\right] + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)} dx$$
$$= \int_\Omega f(x,t)u_t^k(x,t)dx \tag{3.9}$$

and

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left(|v_t^k(x,t)|^2 + (\Delta v^k)^2(x,t)\right) dx\right] + \int_0^t \int_\Omega \left|v_t^k(x,t)\right|^{r(x)} dx$$
$$= \int_\Omega g(x,t)v_t^k(x,t)dx. \tag{3.10}$$

The addition of (3.9) and (3.10), and then the integration of the result, over $(0,t)$, lead to

$$\begin{aligned}&\frac{1}{2}\left[\|u_t^k(t)\|_2^2 + \|u^k(t)\|_{\mathcal{V}}^2 + \|v_t^k(t)\|_2^2 + \|v^k(t)\|_{\mathcal{V}}^2\right]\\ &+ \int_0^t \int_\Omega \left(\left|u_t^k(x,s)\right|^{m(x)} + \left|v_t^k(x,s)\right|^{r(x)}\right) dxds\\ &= \frac{1}{2}\left[\|u_1^k\|_2^2 + \|u_0^k\|_{\mathcal{V}}^2 + \|v_1^k\|_2^2 + \|v_0^k\|_{\mathcal{V}}^2\right]\\ &+ \int_0^t \int_\Omega \left[f(x,s)u_t^k(x,s) + g(x,s)v_t^k(x,s)\right] dxds.\end{aligned}$$

From the convergences (3.8) and exploiting Young's inequality, this gives, for some $C > 0$,

$$\begin{aligned}&\frac{1}{2}\left[\|u_t^k(t)\|_2^2 + \|v_t^k(t)\|_2^2 + \|u^k(t)\|_{\mathcal{V}}^2 + \|v^k(t)\|_{\mathcal{V}}^2\right]\\ &+ \int_0^{T_k} \int_\Omega \left(\left|u_t^k(x,s)\right|^{m(x)} + \left|v_t^k(x,s)\right|^{r(x)}\right) dxds\\ &\le C + \varepsilon \int_0^{T_k} \left(\left\|u_t^k(s)\right\|_2^2 + \left\|v_t^k(s)\right\|_2^2\right) ds\\ &+ C_\varepsilon \int_0^T \int_\Omega \left(|f(x,s)|^2 + |g(x,s)|^2\right) dxds.\end{aligned}$$

In fact that $f, g \in L^2\left(\Omega \times (0,T)\right)$, we infer

$$\frac{1}{2}\sup_{(0,T_k)}\left[\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|u^k\|_{\mathcal{V}}^2 + \|v^k\|_{\mathcal{V}}^2\right] + \int_0^{T_k}\int_\Omega \left(\left|u_t^k(x,s)\right|^{m(x)} + \left|v_t^k(x,s)\right|^{r(x)}\right)$$
$$\le C_\varepsilon + T\varepsilon \sup_{(0,T_k)}\left(\|u_t^k\|_2^2 + \|v_t^k\|_2^2\right). \tag{3.11}$$

Choosing $\varepsilon = \frac{1}{4T}$, estimate (3.11) yields, for all $T_k \le T$,

$$\frac{1}{2}\sup_{(0,T_k)}\left[\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|u^k\|_{\mathcal{V}}^2 + \|v^k\|_{\mathcal{V}}^2\right] + \int_0^{T_k}\int_\Omega \left(\left|u_t^k(x,s)\right|^{m(x)} + \left|v_t^k(x,s)\right|^{r(x)}\right)$$
$$\le C_T,$$

where $C_T > 0$ is a constant depending on T only. Consequently, the solution (u^k, v^k) can be extended to $(0,T)$, for any $k \geq 1$. In addition, we have

$$\left| \begin{array}{l} (u^k)_k, (v^k)_k \text{ are bounded in } L^\infty((0,T),\mathcal{V}), \\ (u_t^k)_k \text{ is bounded in } L^\infty((0,T),L^2(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ (v_t^k)_k \text{ is bounded in } L^\infty((0,T),L^2(\Omega)) \cap L^{r(\cdot)}(\Omega \times (0,T)). \end{array} \right.$$

Therefore, we can extract two subsequences, denoted by $(u_l)_l$ and $(v_l)_l$, respectively, such that, when $l \to \infty$, we have

$$\left| \begin{array}{l} u^l \to u \text{ and } v^l \to v \text{ weakly * in } L^\infty((0,T),\mathcal{V}), \\ u_t^l \to u_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{m(\cdot)}(\Omega \times (0,T)), \\ v_t^l \to v_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{r(\cdot)}(\Omega \times (0,T)). \end{array} \right.$$

Passage to the limit in the Nonlinear Terms. Under the assumptions $(H.1)$-$(H.2)$ and using symilar ideas and arguments as in [[27], Theorem 3.2, p. 6], one can see that

$$| u_t^l |^{m(.)-2} u_t^l \to | u_t |^{m(.)-2} u_t \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T)),$$

$$| v_t^l |^{r(.)-2} v_t^l \to | v_t |^{r(.)-2} v_t \text{ weakly in } L^{\frac{r(.)}{r(.)-1}}(\Omega \times (0,T))$$

and establish that (u,v) satisfies the two differential equations in (S), on $\Omega \times (0,T)$.

The Initial Conditions. By repeating the same steps of [27], we easily conclude that (u,v) satisfies the initial conditions.

Therefore, (u,v) is the unique local solution of (S), in the sense of Definition 3.1, having the regularity (3.1). □

Now, we state and prove our main result of existence related to system (1.1).

Theorem 3.3. *Let* $n = \overline{1,6}$. *Under the assumptions* $(H.1)$-$(H.3)$ *and for any* (u_0, u_1) *and* (v_0, v_1) *in* $\mathcal{V} \times L^2(\Omega)$, *the problem* (1.1) *admits a unique weak solution* (u,v), *in the sense of Definition* 3.1, *having the regularity* (3.1), *for* T *small enough.*

Proof. From (1.2) and (1.3), we have, for all $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$,

$$f_1(x,u,v) = (p(x)+1)\left[a\,|u+v|^{p(x)-1}(u+v) + bu\,|u|^{\frac{p(x)-3}{2}}\,|v|^{\frac{p(x)+1}{2}}\right] \tag{3.12}$$

and

$$f_2(x,u,v) = (p(x)+1)\left[a\,|u+v|^{p(x)-1}(u+v) + bv\,|v|^{\frac{p(x)-3}{2}}\,|u|^{\frac{p(x)+1}{2}}\right]. \tag{3.13}$$

Let $y, z \in L^\infty((0,T),\mathcal{V})$. In what follows, our task is to show that

$$f_1(y,z), f_2(y,z) \in L^2(\Omega \times (0,T)).$$

Applying Young's inequality (Lemma 2.1) and the Sobolev embeddings (Lemma 2.8), we obtain, for all $t \in (0,T)$ and some $C_1, C_2 > 0$, the following results:

- When $n = 5$ and $3 \leq p(x) \leq 5$ on Ω, we have

$$\int_\Omega |f_1(x,y,z)|^2\,dx \leq 2(p^+ + 1)\left[a^2 \int_\Omega |y+z|^{2p(x)}\,dx + b^2 \int_\Omega |y|^{p(x)-1}\,|z|^{p(x)+1}\,dx\right]$$
$$\leq C_1\left[\int_\Omega |y+z|^{2p^-}\,dx + \int_\Omega |y+z|^{2p^+}\,dx + \int_\Omega |y|^{\frac{5}{2}(p(x)-1)}\,dx + \int_\Omega |z|^{\frac{5}{3}(p(x)+1)}\,dx\right]$$
$$\leq C_1C_e\left[\|y+z\|_{\mathcal{V}}^{2p^-} + \|y+z\|_{\mathcal{V}}^{2p^+} + \|y\|_{\mathcal{V}}^{\frac{5}{2}(p^--1)}\right]$$
$$+ C_1C_e\left[\|y\|_{\mathcal{V}}^{\frac{5}{2}(p^+-1)} + \|z\|_{\mathcal{V}}^{\frac{5}{3}(p^-+1)} + \|z\|_{\mathcal{V}}^{\frac{5}{3}(p^++1)}\right] < \infty, \tag{3.14}$$

since

$$2 < \frac{5}{2}(p^- - 1) \leq \frac{5}{2}(p^+ - 1) \leq 2p^- \leq 2p^+ \leq \frac{5}{3}(p^- + 1) \leq \frac{5}{3}(p^+ + 1) \leq 10.$$

- If $n = 6$ and $p^- = p^+ = 3$. Then,

$$\int_\Omega |f_1(x,y,z)|^2\,dx \leq 2(p^+ + 1)\left[a^2 \int_\Omega |y+z|^6\,dx + b^2 \int_\Omega |y|^2\,|z|^4\,dx\right]$$
$$\leq C_2\left[\|y+z\|_{\mathcal{V}}^6 + \left(\int_\Omega |y|^6\,dx\right)^{\frac{1}{3}} x + \left(\int_\Omega |z|^6\,dx\right)^{\frac{2}{3}}\right]$$
$$\leq C_2C_e\left[\|y+z\|_{\mathcal{V}}^6 + \|y\|_{\mathcal{V}}^2 + \|z\|_{\mathcal{V}}^4\right] < \infty. \tag{3.15}$$

Remark 3.4. *The above embeddings remain valid even for $n \leq 4$, however, they will no longer be satisfied when $n \geq 7$, since $\mathcal{V}$ is not embed in $L^{\frac{n}{2}(p^+-1)}(\Omega)$ and in $L^{\frac{n}{n-2}(p^++1)}(\Omega)$ when $p^- \geq 3$.*

So, under the assumption $(H.3)$, we have

$$\int_\Omega |f_1(x,y,z)|^2\,dx < \infty$$

and similarly,

$$\int_\Omega |f_1(x,y,z)|^2\,dx < \infty,$$

for all $t \in (0,T)$. Thus, the claim is immediate.
Therefore, by invoking Theorem 3.2, there exists a unique (u,v) solution of the problem:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(x)-2}\,u_t = f_1(y,z), & \text{in } \Omega \times (0,T)\,, \\ v_{tt} + \Delta^2 v + |v_t|^{r(x)-2}\,v_t = f_2(y,z), & \text{in } \Omega \times (0,T)\,, \\ u = v = 0, & \text{on } \partial\Omega \times (0,T) \\ u\,(0) = u_0 \text{ and } u_t\,(0) = u_1 & \text{in } \Omega, \\ v\,(0) = v_0 \text{ and } v_t\,(0) = v_1, & \text{in } \Omega, \end{cases} \tag{R}$$

in the sense of Definition 3.1 and having the regularity 3.1. Now, consider the following Banach space

$$A_T = \left\{w \in L^\infty((0,T),\mathcal{V})/w_t \in L^\infty((0,T),L^2(\Omega))\right\},$$

equipped with the norm:

$$\|w\|_{A_T}^2 = \sup_{(0,T)} \|w\|_{\mathcal{V}}^2 + \sup_{(0,T)} \|w_t\|_2^2$$

and define a map $F : A_T \times A_T :\longrightarrow A_T \times A_T$ by $F(y,z) = (u,v)$.
For $d > 0$ sufficiently large and $T \leq T_0$ (T_0 to be fixed later), our goal is to prove that F is a contraction mapping from $D(0,d)$ into itself, where $D(0,d)$ is the set of $(w,\tilde{w}) \in A_T \times A_T$, such that

$$\|(w,\tilde{w})\|_{A_T \times A_T} \leq d.$$

F maps $\mathbf{D(0,d)}$ into itself:

Let (y,z) be in $D(0,d)$ and (u,v) be the corresponding solution of problem (Q) (i.e. $F(y,z) = (u,v)$). Taking $(\Phi,\Psi) = (u_t,v_t)$ in Definition 3.1 and integrating each identity over $(0,t)$, we obtain, for all $t \leq T$,

$$\begin{aligned} &\frac{1}{2}\left[\|u_t\|_2^2 - \|u_1\|_2^2 + \|\Delta u\|_2^2 - \|\Delta u_0\|_2^2\right] + \int_0^t \int_\Omega |u_t(x,t)|^{m(x)} \\ &= \int_0^t \int_\Omega u_t f_1(y,z) dx ds \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\|v_t\|_2^2 - \|v_1\|_2^2 + \|\Delta v\|_2^2 - \|\Delta v_0\|_2^2\right] + \int_0^t \int_\Omega |v_t(x,t)|^{r(x)} \\ &= \int_0^t \int_\Omega v_t f_2(y,z) dx ds. \end{aligned} \tag{3.17}$$

The addition of (3.16) and (3.17) lead to

$$\begin{aligned} &\frac{1}{2}\left[\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2\right] \\ &\leq \frac{1}{2}\left[\|u_1\|_2^2 + \|v_1\|_2^2 + \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2\right] \\ &+ \int_0^t \left(\left|\int_\Omega u_t f_1(y,z) dx\right| + \left|\int_\Omega v_t f_2(y,z) dx\right|\right) ds. \end{aligned}$$

for all $t \in (0,T)$. Therefore,

$$\begin{aligned} &\sup_{0\leq t\leq T} \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{\mathcal{V}}^2 + \|v\|_{\mathcal{V}}^2\right) \\ &\leq \gamma + 2 \sup_{0\leq t\leq T} \int_0^t \left(\left|\int_\Omega u_t f_1(y,z) dx\right| + \left|\int_\Omega v_t f_2(y,z) dx\right|\right) d\tau, \end{aligned} \tag{3.18}$$

where $\gamma = \|u_1\|_2^2 + \|v_1\|_2^2 + \|u_0\|_{\mathcal{V}}^2 + \|v_0\|_{\mathcal{V}}^2$. We have to handle the last term in (3.18). From the restrictions $(H.3)$, on p and n, and using the same arguments as those used to establish (3.14) and (3.15), we get, for all $t \in [0,T]$, the following,

- If $n = 5$, then

$$\left|\int_\Omega u_t f_1(y,z)dx\right| \le (p^+ + 1)\left[a\int_\Omega |u_t|\,|y+z|^{p(x)}\,dx + b\int_\Omega |u_t|\,.\,|y|^{\frac{p(x)-1}{2}}\,|z|^{\frac{p(x)+1}{2}}\,dx\right]$$
$$\le C_3\left[\varepsilon\,\|u_t\|_2^2 + C_\varepsilon\left(\|y\|_{\mathcal{V}}^{2p^-} + \|z\|_{\mathcal{V}}^{2p^-} + \|y\|_{\mathcal{V}}^{2p^+} + \|z\|_{\mathcal{V}}^{2p^+}\right)\right]$$
$$+ C_\varepsilon\left[\|y\|_{\mathcal{V}}^{\frac{5}{2}(p^- -1)} + \|y\|_{\mathcal{V}}^{\frac{5}{2}(p^+ -1)} + \|z\|_{\mathcal{V}}^{\frac{5}{3}(p^- +1)} + \|z\|_{\mathcal{V}}^{\frac{5}{3}(p^+ +1)}\right], C_3 > 0. \tag{3.19}$$

The fact that $(y,z) \in D(0,d)$ yields

$$\max\{\|y\|_{\mathcal{V}}^\alpha, \|z\|_{\mathcal{V}}^\alpha\} \le \|(y,z)\|_{\mathcal{V}\times\mathcal{V}}^\alpha \le d^\alpha,\ \forall \alpha \ge 0.$$

Thus, for d large enough, estimates (3.19) leads to

$$\left|\int_\Omega u_t f_1(y,z)dx\right| \le \varepsilon C_3\,\|u_t\|_2^2 + C_\varepsilon d^{\frac{5}{3}(p^+ +1)}.$$

- When $n = 6$, it comes, for some $C_4 > 0$,

$$\left|\int_\Omega u_t f_1(y,z)dx\right| \le 4\left[a\int_\Omega |u_t|\,|y+z|^3\,dx + b\int_\Omega |u_t|\,.\,|y|\,|z|^2\,dx\right]$$
$$\le C_4\left[\varepsilon\,\|u_t\|_2^2 + C_\varepsilon\left(\|y\|_{\mathcal{V}}^6 + \|z\|_{\mathcal{V}}^6 + \|y\|_{\mathcal{V}}^2 + \|z\|_{\mathcal{V}}^4\right)\right]$$
$$\le \varepsilon C_4\,\|u_t\|_2^2 + C_\varepsilon d^6$$
$$\le \varepsilon C_4\,\|u_t\|_2^2 + C_\varepsilon d^{\frac{5}{3}(p^+ +1)}.$$

Consequently, when $n \in \{5,6\}$ (and also for $n = \overline{1,4}$), we have

$$\left|\int_\Omega u_t f_1(y,z)dx\right| \le \varepsilon C_5\,\|u_t\|_2^2 + C_\varepsilon d^{\frac{5}{3}(p^+ +1)} \tag{3.20}$$

and similarly,

$$\left|\int_\Omega v_t f_2(y,z)dx\right| \le \varepsilon C_5\,\|v_t\|_2^2 + C_\varepsilon d^{\frac{5}{3}(p^+ +1)}, \tag{3.21}$$

for some $C_5 > 0$ and all $t \in [0,T]$. Thus, by combining (3.20) and (3.21), it results

$$\sup_{0\le t\le T}\int_0^t\left(\left|\int_\Omega u_t f_1(y,z)dx\right| + \left|\int_\Omega v_t f_2(y,z)dx\right|\right)ds$$
$$\le T\left(\varepsilon C_5 \sup_{0\le t\le T}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + C_\varepsilon d^{\frac{5}{3}(p^+ +1)}\right). \tag{3.22}$$

Now, inserting (3.22) into (3.18), we arrive at

$$\sup_{0\le t\le T}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{\mathcal{V}}^2 + \|v\|_{\mathcal{V}}^2\right)$$
$$\le \gamma + 2T\left(\varepsilon C_5 \sup_{0\le t\le T}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + C_\varepsilon d^{\frac{5}{3}(p^+ +1)}\right). \tag{3.23}$$

By taking $\varepsilon = \frac{1}{4TC_5}$, estimate (3.23) leads to, for some $C_6 > 0$,

$$\begin{aligned}\|(u,v)\|^2_{A_T\times A_T} &\le 2\gamma + 4TC_6 d^{\frac{5}{3}(p^+ +1)}\\ &\le 2\gamma + 4T_0C_6 d^{\frac{5}{3}(p^+ +1)}.\end{aligned}$$

So, if we take (d, T_0) such that $d^2 >> 2\gamma$ and $T_0 \le \frac{1}{4}\left(\frac{d^2-2\gamma}{C_6 d^{\frac{5}{3}(p^+ +1)}}\right)$, it yields

$$\|(u,v)\|^2_{A_T\times A_T} \le d^2,$$

which means that (u, v) belongs to $D(0, d)$. Consequently, F maps $D(0, d)$ into itself.

$\mathbf{F : D(0, d) \longrightarrow D(0, d)}$ is a contraction:

Let (y_1, z_1) and (y_2, z_2) be in $D(0, d)$ and set $(u_1, v_1) = F(y_1, z_1)$ and $(u_2, v_2) = F(y_2, z_2)$. Clearely, $(U, V) = (u_1 - u_2, v_1 - v_2)$ is a weak solution of the following system

$$\begin{cases} U_{tt} + \Delta^2 U + |u_{1t}|^{m(x)-2}\, u_{1t} - |u_{2t}|^{m(x)-2}\, u_{2t} & \\ = f_1(y_1,z_1) - f_1(y_2,z_2) & \text{in } \Omega\times(0,T), \\ V_{tt} + \Delta^2 V + |v_{1t}|^{r(x)-2}\, v_{1t} - |v_{2t}|^{r(x)-2}\, v_{2t} & \\ = f_2(y_1,z_1) - f_2(y_2,z_2) & \text{in } \Omega\times(0,T), \\ U = V = 0 & \text{on } \partial\Omega\times(0,T), \\ (U(0), V(0)) = (U_t(0), V_t(0)) = (0,0) & \text{in } \Omega. \end{cases} \tag{S}$$

in the sense of Definition 3.1. So, taking $(\Phi, \Psi) = (U_t, V_t)$, in this definition, using Green's formula together with the boundary conditions and then, integrating each result over $(0, t)$, we obtain, for a.e. $t \le T$,

$$\begin{aligned}&\frac{1}{2}\left(\|U_t\|_2^2 + \|\Delta U\|_2^2\right) + \int_0^t\int_\Omega \left(u_{1t}\,|u_{1t}|^{m(x)-2} - u_{2t}\,|u_{2t}|^{m(x)-2}\right) U_t dxds\\ &\le \int_0^t\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|\,|U_t| dxds\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\left(\|V_t\|_2^2 + \|\Delta V\|_2^2\right) + \int_0^t\int_\Omega \left(v_{1t}\,|v_{1t}|^{r(x)-2} - v_{2t}\,|v_{2t}|^{r(x)-2}\right) V_t dxds\\ &\le \int_0^t\int_\Omega |f_2(y_1,z_1) - f_2(y_2,z_2)|\,|V_t| dxds.\end{aligned}$$

Under the condition $(H.3)$, using Hölder's inequality (Lemma 2.2) and inequality (3.6), these two estimates give, for $n = \overline{1,6}$,

$$\|U_t\|_2^2 + \|U\|_{\mathcal{V}}^2 \le 4\int_0^t \|U_t\|_2 \|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 ds \tag{3.24}$$

and

$$\|V_t\|_2^2 + \|V\|_{\mathcal{V}}^2 \le 4\int_0^t \|V_t\|_2 \|f_2(y_1,z_1) - f_2(y_2,z_2)\|_2 ds. \tag{3.25}$$

Tha addition of (3.24) and (3.25) imply

$$\|U_t\|_2^2 + \|V_t\|_2^2 + \|U\|_{\mathcal{V}}^2 + \|V\|_{\mathcal{V}}^2 \leq 4\int_0^t \|U_t\|_2 \|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 ds$$
$$+ 4\int_0^t \|V_t\|_2 \|f_2(y_1,z_1) - f_2(y_2,z_2)\|_2 ds, \tag{3.26}$$

for all $t \in (0,T)$. Now, we estimate the terms:

$$\|f_1(y_1,z_1) - f_1(y_2,z_2)\|_2 \text{ and } \|f_2(y_1,z_1) - f_2(y_2,z_2)\|_2.$$

Using an appropriate algebric inequalities (see [1]), we obtain for two constants $C_1, C_2 > 0$ and for all $x \in \Omega$ and $t \in (0,T)$,

$$\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2 dx \leq I_1 + I_2 + I_3 + I_4, \tag{3.27}$$

where

$$\begin{aligned}
I_1 &= C_1 \int_\Omega |y_1 - y_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_1 \int_\Omega |y_1 - y_2|^2 (|y_2|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx, \\
I_2 &= C_1 \int_\Omega |z_1 - z_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_1 \int_\Omega |z_1 - z_2|^2 (|y_1|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx, \\
I_3 &= C_2 \int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} \left(|z_1|^{p(x)-1} + |z_2|^{p(x)-1}\right) dx, \\
I_4 &= C_2 \int_\Omega |y_1 - y_2|^2 |z_2|^{p(x)+1} \left(|y_1|^{p(x)-3} + |y_2|^{p(x)-3}\right) dx.
\end{aligned}$$

As in above, from assumption $(H.3)$ and Remark 3.4, we get the following estimate for a typical term in I_1 and I_2, when $n \in \{5,6\}$,

$$\begin{aligned}
&\int_\Omega |y_1 - y_2|^2 |y_1|^{2(p(x)-1)} dx \\
&\leq 2\left(\int_\Omega |y_1 - y_2|^{\frac{2n}{n-4}} dx\right)^{\frac{n-4}{n}} \left(\int_\Omega |y_1|^{\frac{n}{2}(p(x)-1)} dx\right)^{\frac{4}{n}} \\
&\leq C||y_1 - y_2||^2_{\frac{2n}{n-4}} \left[\left(\int_\Omega |y_1|^{\frac{n}{2}(p^+-1)} dx\right)^{\frac{4}{n}} + \left(\int_\Omega |y_1|^{\frac{n}{2}(p^--1)} dx\right)^{\frac{4}{n}}\right] \\
&\leq C||\Delta(y_1 - y_2)||_2^2 \left(||\Delta y_1||_2^{2(p^+-1)} + ||\Delta y_1||_2^{2(p^--1)}\right) \\
&\leq C||\Delta Y||_2^2 \left(||(y_1,z_1)||^{2(p^+-1)}_{A_T \times B_T} + ||(y_1,z_1)||^{2(p^--1)}_{A_T \times B_T}\right) \\
&\leq C||\Delta Y||_2^2,
\end{aligned}$$

where $C > 0$ is, from now on, used to denote a positive generic constant, $Y = y_1 - y_2$ and $Z = z_1 - z_2$. In a similar way, we find

$$\begin{aligned}\int_\Omega |z_1 - z_2|^2 \, |y_2|^{2(p(x)-1)} \, dx &\le C||\Delta Z||_2^2 \left(||\Delta y_2||_2^{2(p^+-1)} + ||\Delta y_2||_2^{2(p^--1)} \right) \\ &\le C||\Delta Z||_2^2 \left(||(y_2, z_2)||_{A_T \times B_T}^{2(p^+-1)} + ||(y_2, z_2)||_{A_T \times B_T}^{2(p^--1)} \right) \\ &\le C||\Delta Z||_2^2.\end{aligned}$$

We conclude that, for $n \in \{5, 6\}$ and all $t \in (0, T)$,

$$\begin{aligned}I_1 + I_2 &\le C||\Delta Y||_2^2 \left(||(y_1, z_1)||_{A_T \times B_T}^{2(p^+-1)} + ||(y_1, z_1)||_{A_T \times B_T}^{2(p^--1)} \right) \\ &\quad + C||\Delta Z||_2^2 \left(||(y_2, z_2)||_{A_T \times B_T}^{2(p^+-1)} + ||(y_2, z_2)||_{A_T \times B_T}^{2(p^--1)} \right) \\ &\le C \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right). \end{aligned} \tag{3.28}$$

Using the same arguments, also, when $n \in \{5, 6\}$, a typical term in I_3 can be handled as follows:

$$\begin{aligned}&\int_\Omega |z_1 - z_2|^2 \, |y_1|^{p(x)-1} \, |z_1|^{p(x)-1} \, dx \\ &\le 2 \left(\int_\Omega |z_1 - z_2|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}} \left(\int_\Omega |y_1|^{\frac{n}{2}(p(x)-1)} dx \right)^{\frac{2}{n}} \left(\int_\Omega |z_1|^{\frac{n}{2}(p(x)-1)} dx \right)^{\frac{2}{n}} \\ &\le C||z_1 - z_2||_{\frac{2n}{n-4}}^2 \left(||y_1||_{\frac{n}{2}(p^+-1)}^{p^+-1} + ||y_1||_{\frac{n}{2}(p^--1)}^{p^--1} \right) \left(||z_1||_{\frac{n}{2}(p^+-1)}^{p^+-1} + ||z_1||_{\frac{n}{2}(p^--1)}^{p^--1)} \right) \\ &\le C||\Delta(z_1 - z_2)||_2^2 \left(||\Delta y_1||_2^{p^+-1} + ||\Delta y_1||_2^{p^--1} \right) \left(||\Delta z_1||_2^{p^+-1} + ||\Delta z_1||_2^{p^--1)} \right) \\ &\le C||\Delta Z||_2^2.\end{aligned}$$

Thus,

$$I_3 \le C \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right). \tag{3.29}$$

Next, we estimate a typical term in I_4:
Case 1: If $n = 5$, we have $3 \le p^- \le p^+ \le 5$ (by $(H.3)$). Therefore,

$$\begin{aligned}&\int_\Omega |y_1 - y_2|^2 \, |z_2|^{p(x)+1} \, |y_1|^{p(x)-3} \, dx \\ &\le 2 \left(\int_\Omega |y_1 - y_2|^{10} dx \right)^{\frac{1}{5}} \left(\int_\Omega |z_2|^{\frac{5}{4}(p(x)+1)} |y_1|^{\frac{5}{4}(p(x)-3)} \right)^{\frac{4}{5}} \\ &\le C||y_1 - y_2||_{10}^2 \left(\int_\Omega |z_2|^{\frac{5}{3}(p(x)+1)} dx \right)^{\frac{3}{5}} \left(\int_\Omega |y_1|^{5(p(x)-3)} dx \right)^{\frac{1}{5}} \\ &\le C||Y||_{10}^2 \left(||z_2||_{\frac{5}{3}(p^++1)}^{p^++1} + ||z_2||_{\frac{5}{3}(p^-+1)}^{p^-+1} \right) \left(||y_1||_{5(p^+-3)}^{p^+-3} + ||y_1||_{5(p^--3)}^{p^--3} \right) \\ &\le C||\Delta Y||_2^2 \left(||\Delta z_2||_2^{p^++1} + ||\Delta z_2||_2^{p^-+1} \right) \left(||\Delta y_1||_2^{p^+-3} + ||\Delta y_1||_2^{p^--3} \right) \\ &\le C||\Delta Y||_2^2.\end{aligned}$$

Case 2: If $n = 6,\ p(x) = 3$ on Ω (by $(H.3)$). Then,

$$\begin{aligned}\int_\Omega |y_1 - y_2|^2 |z_2|^{p(x)+1} |y_1|^{p(x)-3} \, dx &= \int_\Omega |y_1 - y_2|^2 |z_2|^4 \, dx \\ &\leq C \left(\int_\Omega |y_1 - y_2|^6 \, dx \right)^{\frac{1}{3}} \left(\int_\Omega |z_2|^6 \, dx \right)^{\frac{2}{3}} \\ &\leq C ||y_1 - y_2||_6^2 ||z_2||_6^4 \\ &\leq C ||\Delta Y||_2^2 ||\Delta z_2||_2^4 \\ &\leq C ||\Delta Y||_2^2.\end{aligned}$$

Consequently, for $n \in \{5, 6\}$ and all $t \in (0, T)$, we have

$$I_4 \leq C ||\Delta Y||_2^2. \tag{3.30}$$

Remark 3.5. *By looking carefully at the above calculations, one can easily obtain the previous estimates of* I_i $(i = \overline{1, 4})$, *for* $n \leq 4$, *since* p *is bounded on* $\overline{\Omega}$ *and* $p^- \geq 3$.

By inserting (3.28), (3.29) and (3.30) into (3.27), we obtain

$$\int_\Omega |f_1(y_1, z_1) - f_1(y_2, z_2)|^2 \, dx \leq C \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right) \tag{3.31}$$

and likewise,

$$\int_\Omega |f_2(y_1, z_1) - f_2(y_2, z_2)|^2 \, dx \leq C \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right), \tag{3.32}$$

for all $t \in (0, T)$. The substitution of (3.31) and (3.32) into (3.26) yields

$$\begin{aligned}\|U_t\|_2^2 + \|V_t\|_2^2 + \|U\|_{\mathcal{V}}^2 + \|V\|_{\mathcal{V}}^2 &\leq C \int_0^t \|U_t\|_2 \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right)^{\frac{1}{2}} ds \\ &+ C \int_0^t \|V_t\|_2 \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right)^{\frac{1}{2}} ds.\end{aligned}$$

Exploiting Young's inequality, this latter estimate gives

$$\begin{aligned}&\|U_t\|_2^2 + \|V_t\|_2^2 + \|U\|_{\mathcal{V}}^2 + \|V\|_{\mathcal{V}}^2 \\ &\leq \varepsilon C \int_0^t \left(\|U_t\|_2^2 + \|V_t\|_2^2 \right) ds + C_\varepsilon \int_0^t \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right) ds,\end{aligned}$$

for all $t \in [0, T)$. Therefore,

$$\begin{aligned}&\sup_{0 \leq t \leq T} \left(\|U_t\|_2^2 + \|V_t\|_2^2 + \|U\|_{\mathcal{V}}^2 + \|V\|_{\mathcal{V}}^2 \right) \\ &\leq \varepsilon CT \sup_{0 \leq t \leq T} \left(\|U_t\|_2^2 + \|V_t\|_2^2 \right) + C_\varepsilon T \sup_{0 \leq t \leq T} \left(||\Delta Y||_2^2 + ||\Delta Z||_2^2 \right).\end{aligned}$$

Thus, by choosing ε such that $\varepsilon CT = \frac{1}{2}$, we arrive at

$$\begin{aligned}\|(U, V)\|_{A_T \times A_T}^2 &\leq CT \|(Y, Z)\|_{A_T \times A_T}^2 \\ &\leq CT_0 \|(Y, Z)\|_{A_T \times A_T}^2 \\ &\leq k \,\|(Y, Z)\|_{A_T \times B_T}^2,\end{aligned} \tag{3.33}$$

with $k = CT_0$. So, by taking T_0 so small that $0 < k < 1$, inquality (3.33) shows that F is a contraction mapping from $D(0,d)$ into itself. Therefore, the fixed-point theorem assures the existence of a unique $(u,v) \in D(0,d)$, such that $F(u,v) = (u,v)$. Hence, (u,v) is, obviously, a weak solution of system (1.1), in the sense of Definition 3.1, satisfying (3.1).
The uniqueness of this solution can be obtained by applying the energy method. $\square$

4. Blow up of Negative Initial Energy Solution

In this Section, we show that any solution (u,v) of problem (1.1) blows up in finite time, i.e, there exists $T^* \in (0,T)$, such that

$$lim_{t\to T^*}\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) = +\infty,$$

if

$$E(0) < 0 \text{ and } max\{m^+ - 1, r^+ - 1\} < p^-, \quad (H.4)$$

in addition to the assumptions (H.1)-(H.3), where E is the energy functional associated to system (P) defined, for all $t \in [0,T)$, by

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) - \int_\Omega F(x,u,v)\,dx. \tag{4.1}$$

A simple computation shows that E is a decreasing function, with

$$E'(t) = -\int_\Omega |u_t|^{m(x)}\,dx - \int_\Omega |v_t|^{r(x)}\,dx, \tag{4.2}$$

for all $t \in [0,T)$, thanks to Green's formula and the boundary conditions in (1.1).

Lemma 4.1. [2]
1- There exist $C_1, C_2 > 0$ such that, for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$ we have

$$C_1\left(|u|^{p(x)+1} + |v|^{p(x)+1}\right) \le F(x,u,v) \le C_2\left(|u|^{p(x)+1} + |v|^{p(x)+1}\right). \tag{4.3}$$

2- For all $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$, we have

$$u\, f_1(x,u,v) + v f_2(x,u,v) = (p(x)+1)\,F(x,u,v), \tag{4.4}$$

where f_1 and f_2 are defined by (3.1) *and F by* (3.2).

Let us define H by

$$H(t) = -E(t), \text{ for all } t \in [0,\ T). \tag{4.5}$$

Remark 4.2. *1. From* (4.1, (4.2, (4.3 *and* (4.4, *we have*

$$0 < H(0) \le H(t) \le C_3(\rho(u) + \rho(v)), \text{ for all } t \in [0,\ T), \tag{4.6}$$

where $C_3 > 0$ is a constant and

$$\rho(u) = \int_\Omega |u|^{p(x)+1}\,dx \text{ and } \rho(v) = \int_\Omega |v|^{p(x)+1}\,dx.$$

2.

$$H'(t) \geq \max\left\{\int_\Omega |u_t|^{m(x)}\,dx, \int_\Omega |v_t|^{r(x)}\,dx\right\}. \tag{4.7}$$

Hence, we can establish the following result.

Lemma 4.3. [27] *There exists $C_4 > 0$ such that*

$$\|u\|_{p^-+1}^{p^-+1} + \|v\|_{p^-+1}^{p^-+1} \leq C_4\left(\rho(u) + \rho(v)\right). \tag{4.8}$$

Consequently and in fact that $\max\{m^+ - 1, r^+ - 1\} < p^-$, it yields

Corollary 4.4. *There exist two constants $C_5, C_6 > 0$ such that*

$$\int_\Omega |u|^{m(x)}\,dx \leq C_5\left[\left(\rho(u) + \rho(v)\right)^{\frac{m^+}{p^-+1}} + \left(\rho(u) + \rho(v)\right)^{\frac{m^-}{p^-+1}}\right], \tag{4.9}$$

and

$$\int_\Omega |v|^{r(x)}\,dx \leq C_6\left[\left(\rho(u) + \rho(v)\right)^{\frac{r^+}{p^-+1}} + \left(\rho(u) + \rho(v)\right)^{\frac{r^-}{p^-+1}}\right]. \tag{4.10}$$

Now, we present and prove the blow-up result.

Theorem 4.5. *Suppose that assumptions (H.1)-(H.4) hold. Then, the solution of system* (1.1) *blows up in finite time.*

Proof. For small $\varepsilon > 0$ to be fixed later, we define the following functional

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t)\,dx, \text{ for all } t \in [0, T),$$

where

$$0 < \sigma \leq \min\left\{\frac{p^- - m^+ + 1}{(p^- + 1)(m^+ - 1)}, \frac{p^- - r^+ + 1}{(p^- + 1)(r^+ - 1)}, \frac{p^- - 1}{2(p^- + 1)}\right\}. \tag{4.11}$$

Our goal is to show that L satisfies a differential inequality which leads to a blow up in finite time. So, we will prove that, for some $C > 0$,

$$L'(t) \geq CL^{1/(1-\sigma)}(t), \text{ for all } t \in [0, T). \tag{4.12}$$

Step 1. We estimate $\mathbf{L'(t)}$:

Using (1.1) and Green's formula, we obtain for all $t \in (0, T)$,

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + \varepsilon\int_\Omega \left(uf_1(x,u,v) + vf_2(x,u,v)\right)dx - \varepsilon\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\ &\quad - \varepsilon\int_\Omega \left(|u_t|^{m(x)-2} u_t u + |v_t|^{r(x)-2} v_t v\right)dx. \end{aligned} \tag{4.13}$$

By the definitions of E and H, we have

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 = 2\int_\Omega F(x,u,v)\,dx - \left[\|u_t\|_2^2 + \|v_t\|_2^2 + 2H(t)\right]. \tag{4.14}$$

Thanks to (4.3), (4.4) and (4.14), identity (4.13) leads to

$$L'(t) \geq (1-\sigma) H^{-\sigma}(t) H'(t) + 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + \varepsilon c_1 (\rho(u) + \rho(v)) + 2\varepsilon H(t) - \varepsilon \int_\Omega \left(|u| |u_t|^{m(x)-1} + |v| |v_t|^{r(x)-1} \right) dx, \tag{4.15}$$

where $c_1 = C_2(p^- - 1) > 0$. Next, we estimate the last two terms in the right hand-side of (4.15); namely

$$I_1 := \int_\Omega |u| |u_t|^{m(x)-1} dx \text{ and } I_2 := \int_\Omega |v| |v_t|^{r(x)-1} dx.$$

Exploiting the following Young inequality

$$XY \leq \frac{\delta^\lambda}{\lambda} X^\lambda + \frac{\delta^{-\beta}}{\beta} Y^\beta, \; X, \; Y \geq 0, \;\; \delta > 0 \text{ and } \frac{1}{\lambda} + \frac{1}{\beta} = 1,$$

with

$$X = |u|, \; Y = |u_t|^{m(x)-1}, \; \lambda = m(x), \; \beta = \frac{m(x)}{m(x)-1} \text{ and } \delta > 0,$$

we find

$$I_1 \leq \int_\Omega \frac{\delta^{m(x)}}{m(x)} |u|^{m(x)} dx + \int_\Omega \frac{m(x)-1}{m(x)} \delta^{-m(x)/(m(x)-1)} |u_t|^{m(x)} dx. \tag{4.16}$$

Taking

$$\delta = \left[K H^{-\sigma}(t) \right]^{\frac{1-m(x)}{m(x)}},$$

where K is a large constant, estimates (4.16) becomes

$$I_1 \leq \frac{K^{1-m^-}}{m^-} \int_\Omega [H(t)]^{\sigma(m(x)-1)} |u|^{m(x)} dx + \frac{m^+ - 1}{m^-} K H^{-\sigma}(t) \int_\Omega |u_t|^{m(x)} dx.$$

By virtue of Remark 4.2 and since m is bounded on Ω, this gives, for some $c_2 > 0$,

$$I_1 \leq c_2 \frac{K^{1-m^-}}{m^-} [H(t)]^{\sigma(m^+ - 1)} \int_\Omega |u|^{m(x)} dx + \frac{m^+ - 1}{m^-} K H^{-\sigma}(t) H'(t). \tag{4.17}$$

Similarly and since r is bounded in Ω, we have, for some $c_3 > 0$,

$$I_2 \leq c_3 \frac{K^{1-r^-}}{r^-} [H(t)]^{\sigma(r^+ - 1)} \int_\Omega |v|^{r(x)} dx + \frac{r^+ - 1}{r^-} K H^{-\sigma}(t) H'(t). \tag{4.18}$$

On the other hand, estimate (4.9), implies, for some $c_4 > 0$,

$$[H(t)]^{\sigma(m^+-1)} \int_\Omega |u|^{m(x)}\,dx \le c_4\left(\rho(u)+\rho(v)\right)^{\sigma(m^+-1)+\frac{m^+}{p^-+1}} + c_4\left(\rho(u)+\rho(v)\right)^{\sigma(m^+-1)+\frac{m^-}{p^-+1}}. \tag{4.19}$$

From the conditions on σ and using the following algebraic inequality

$$z^\tau \le z+1 \le \left(1+\frac{1}{a}\right)(z+a), \text{ for all } z \ge 0,\ 0<\tau\le 1,\ a>0, \tag{4.20}$$

with

$$z=\rho(u)+\rho(v),\ a=H(0),\ \tau=\sigma\left(m^+-1\right)+\frac{m^+}{p^-+1}$$

and then with $\tau=\sigma(m^+-1)+\frac{m^-}{p^-+1}$, respectively, we get

$$\begin{aligned}\left(\rho(u)+\rho(v)\right)^{\sigma(m^+-1)+\frac{m^+}{p^-+1}} &\le \left[1+\frac{1}{H(0)}\right]\left(\rho(u)+\rho(v)+H(0)\right)\\ &\le \gamma\left(\rho(u)+\rho(v)+H(t)\right)\end{aligned} \tag{4.21}$$

and

$$\left(\rho(u)+\rho(v)\right)^{\sigma(m^+-1)+\frac{m^-}{p^-+1}} \le \gamma\left(\rho(u)+\rho(v)+H(t)\right), \tag{4.22}$$

where $\gamma = 1+\frac{1}{H(0)}$. By adding (4.21) and (4.22), estimate (4.19) takes the form

$$[H(t)]^{\sigma(m^+-1)} \int_\Omega |u|^{m(x)}\,dx \le c_5\left(\rho(u)+\rho(v)+H(t)\right), \tag{4.23}$$

where $c_5>0$ is a constant. Likewise, we obtain, for some $c_6>0$,

$$[H(t)]^{\sigma(r^+-1)} \int_\Omega |v|^{r(x)}\,dx \le c_6\left(\rho(u)+\rho(v)+H(t)\right). \tag{4.24}$$

By inserting (4.23) into (4.17), and (4.24) into (4.18), respectively, we find for some $c_7, c_8 > 0$,

$$I_1 \le c_7\frac{K^{1-m^-}}{m^-}\left(\rho(u)+\rho(v)+H(t)\right)+\frac{m^+-1}{m^-}KH^{-\sigma}(t)H'(t). \tag{4.25}$$

and

$$I_2 \le c_8\frac{K^{1-r^-}}{r^-}\left(\rho(u)+\rho(v)+H(t)\right)+\frac{r^+-1}{r^-}KH^{-\sigma}(t)H'(t). \tag{4.26}$$

So, the substitution of (4.25) and (4.26) into (4.15) yields

$$\begin{aligned}L'(t) \ge{}& (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t)+2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+2\varepsilon H(t)\\ &+\varepsilon c_1\left(\rho(u)+\rho(v)\right)-\varepsilon c_9\frac{K^{1-m^-}}{m^-}\left[\rho(u)+\rho(v)+H(t)\right]\\ &-\varepsilon c_{10}\frac{K^{1-r^-}}{r^-}\left[\rho(u)+\rho(v)+H(t)\right],\end{aligned} \tag{4.27}$$

for some $c_9, c_{10} > 0$ and $M = K\left(\frac{m^+-1}{m^-} + \frac{r^+-1}{r^-}\right)$. Therefore,

$$\begin{aligned} L'(t) &\geq (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) + 2\varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &+ \varepsilon\left(2 - \frac{K^{1-m^-}}{m^-}c_9 - \frac{K^{1-r^-}}{r^-}c_{10}\right) H(t) \\ &+ \varepsilon\left(c_1 - \frac{K^{1-m^-}}{m^-}c_9 - \frac{K^{1-r^-}}{r^-}c_{10}\right)(\rho(u) + \rho(v)). \end{aligned} \tag{4.28}$$

For large value of K, we can find $c_{11} > 0$, such that

$$\begin{aligned} L'(t) &\geq (1-\sigma-\varepsilon M) H^{-\sigma}(t) H'(t) \\ &+ \varepsilon c_{11}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + H(t) + \rho(u) + \rho(v)\right). \end{aligned} \tag{4.29}$$

Once K is fixed, we pick ε sufficiently small so that

$$1 - \sigma - \varepsilon M \geq 0 \text{ and } L(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega (u_0u_1 + v_0v_1)\,dx > 0.$$

By recalling that $H'(t) \geq 0$, then there exists $\Upsilon > 0$ such that

$$L'(t) \geq \varepsilon\Upsilon\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \rho(u) + \rho(v)\right). \tag{4.30}$$

Consequently,

$$L(t) \geq L(0) > 0, \text{ for all } t \in [0, T).$$

Step 2. We estimate $\mathbf{L^{1/(1-\sigma)}(t)}$:

By the definition of L, we have, for some $c_{12} > 0$,

$$\begin{aligned} L^{1/(1-\sigma)}(t) &\leq \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega |uu_t + vv_t|\,dx\right)^{1/(1-\sigma)} \\ &\leq 2^{\sigma/(1-\sigma)}\left(H(t) + \left(\varepsilon\int_\Omega (|uu_t| + |vv_t|)\,dx\right)^{1/(1-\sigma)}\right) \\ &\leq c_{12}\left(H(t) + \left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\,dx\right)^{1/(1-\sigma)}\right), \end{aligned} \tag{4.31}$$

since

$$(X+Y)^\delta \leq 2^{\delta-1}\left(X^\delta + Y^\delta\right), \text{ for all } X, Y \geq 0 \text{ and } \delta > 1. \tag{4.32}$$

Also, we have

$$\begin{aligned} \left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\,dx\right)^{1/(1-\sigma)} &\leq 2^{\sigma/(1-\sigma)}\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} \\ &+ 2^{\sigma/(1-\sigma)}\left(\int_\Omega |v|\,|v_t|\,dx\right)^{1/(1-\sigma)}. \end{aligned} \tag{4.33}$$

From the conditions on p, Hölder's and Young's inequalities imply, for some $c_{13}, c_{14} > 0$,

$$\begin{aligned}\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} &\leq \|u\|_2^{1/(1-\sigma)}\,\|u_t\|_2^{1/(1-\sigma)}\\ &\leq c_{13}\,\|u\|_{p^-+1}^{1/(1-\sigma)}\,\|u_t\|_2^{1/(1-\sigma)}\\ &\leq c_{14}\left(\|u\|_{p^-+1}^{\mu/(1-\sigma)} + \|u_t\|_2^{\beta/(1-\sigma)}\right), \end{aligned} \tag{4.34}$$

where $\frac{1}{\mu}+\frac{1}{\beta}=1$. By taking $\beta = 2(1-\sigma)$, then $\mu/(1-\sigma) = 2/(1-2\sigma)$ and hence, inequality (4.34) becomes

$$\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} \leq c_{14}\left(\|u\|_{p^-+1}^{2/(1-2\sigma)} + \|u_t\|_2^2\right). \tag{4.35}$$

Invoking Lemma 4.3, estimate (4.35) leads to

$$\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} \leq c_{15}\left((\rho(u)+\rho(v))^\tau + \|u_t\|_2^2\right),$$

where $c_{15} > 0$ and $\tau = 2/(p^-+1)(1-2\sigma)$. Again, by using (4.11) and (4.20), we get, for some $c_{16} > 0$,

$$\left(\int_\Omega |u|\,|u_t|\,dx\right)^{1/(1-\sigma)} \leq c_{16}\left(\rho(u)+\rho(v)+H(t)+\|u_t\|_2^2\right) \tag{4.36}$$

and

$$\left(\int_\Omega |v|\,|v_t|\,dx\right)^{1/(1-\sigma)} \leq c_{16}\left(\rho(v)+\rho(v)+H(t)+\|v_t\|_2^2\right). \tag{4.37}$$

By substituting (4.37) and (4.36) into (4.33), it results, for some $c_{17} > 0$,

$$\left(\int_\Omega (|u|\,|u_t| + |v|\,|v_t|)\,dx\right)^{1/(1-\sigma)} \leq c_{17}\left(\rho(u)+\rho(v)+\|u_t\|_2^2+\|v_t\|_2^2+H(t)\right).$$

Hence, inequality (4.31) becomes, for some $c_{18} > 0$,

$$L^{1/(1-\sigma)}(t) \leq c_{18}\left(\rho(u)+\rho(v)+H(t)+\|u_t\|_2^2+\|v_t\|_2^2\right). \tag{4.38}$$

Finally, by combining (4.38) and (4.30), we infer that, for all $t \in [0,T)$,

$$L'(t) \geq C L^{1/(1-\sigma)}(t), \; C > 0.$$

A simple integration over $(0,t)$ gives

$$L^{\sigma/(1-\sigma)}(t) \geq \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma C t}{1-\sigma}}.$$

Therefore,

$$\lim_{t\longrightarrow T^*} L(t) = +\infty, \; T^* \leq \frac{1-\sigma}{\sigma C\left[L^{\frac{\sigma}{(1-\sigma)}}(0)\right]}.$$

This completes the proof. □

5. Numerical Tests

In this section, some numerical experiments are performed to illustrate the theoretical results in Theorem 4.5. We solve the system (1.1) under the assumptions $(H.1)-(H.4)$, using a numerical scheme based on the finite element method in space and the Newmark method in time.

For the numerical tests, we consider the system (1.1) in two-dimension space and take the functions m, r and p as follows:

$$m(x,y)=2+\frac{1}{1+x^2},\quad r(x,y)=2+\frac{1}{1+y^2},\quad p(x,y)=3+\frac{1}{1+x^2+y^2},$$

and the source terms f_1 and f_2 are given by 1.2 and 1.3 with $a=b=1$.

Since we are dealing here with a higher order term, which is the bi-Laplacian $\Delta^2 u$, it is impossible to solve the problem by using linear finite elements. Using quadratic triangular elements [37], the discretized system is written as:

$$\begin{cases} M\ddot{U}_h + RU_h + M\left|\ddot{U}_h\right|^{m(x)-2}\ddot{U}_h = MF_1\left(U_h, V_h\right), \\ M\ddot{V}_h + RV_h + M\left|\ddot{V}_h\right|^{r(x)-2}\ddot{V}_h = MF_2\left(U_h, V_h\right), \end{cases} \tag{5.1}$$

where M, R are the mass and the stiffness matrices, respectively, (U_h, V_h) is the approximate solution of the system (1.1), and F_1, F_2 are the approximate source terms.

We perform two tests by running our code with a time step $\Delta t = 5\cdot 10^{-4}$, which is small enough to catch the blow-up behavior.

Test 1: For the first test, we consider a rectangular domain

$$\Omega_1 = \{(x,y)/-1<x<1 \text{ and } 0<y<1\}$$

with a triangulation discretization (see the mesh-grid in Figure 1), which consists of 3766 nodes and 1819 elements, and take the following initial conditions:

$$u_0(x,y)=y^2(1-y)^2(1-x^2)^2,\ v_0(x,y)=\frac{3}{2}y^2(1-y)^2(1-x^2)^2,\ u_1=v_1=0.$$

Figure 2 shows the approximate numerical results of the solution (u,v) at different time iterations $t=0$, $t=0.025$, $t=0.043$ and $t=0.0445$, where the left column shows the approximate values of u and the right column shows the approximate values of v.

Figure 3 presents the numerical values of the functional $H(t)$ defined by (4.5) during the time iterations. It shows the blow-up of the energy of the system (1.1). Notice that the blow-up is occurring at instant $t=0.0425$.

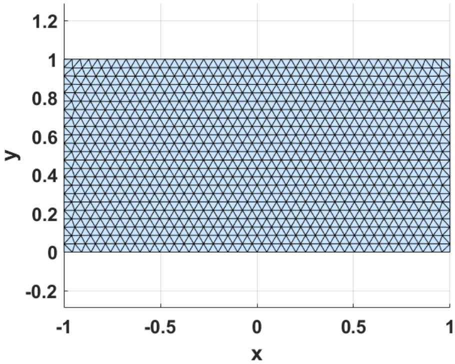
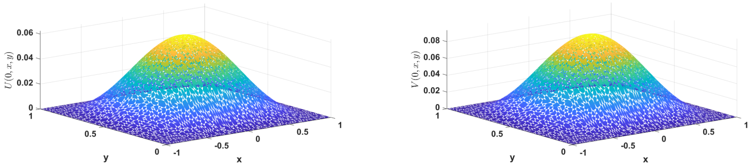
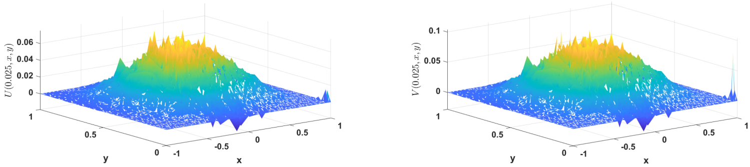


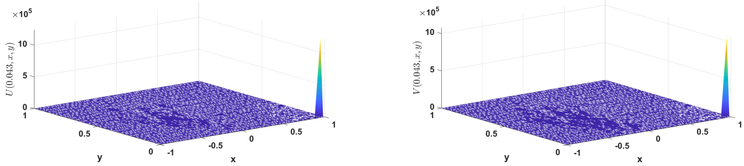
FIGURE 1. Uniform mesh grid of Ω_1.



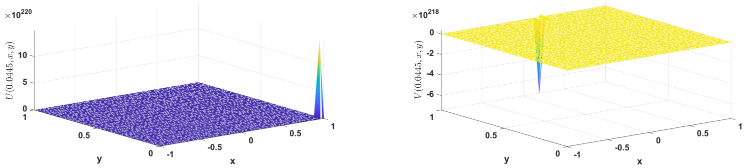
(A) $t = 0$



(B) $t = 0.025$



(C) $t = 0.043$



(D) $t = 0.0445$

FIGURE 2. The numerical results of Test 1 at different times.

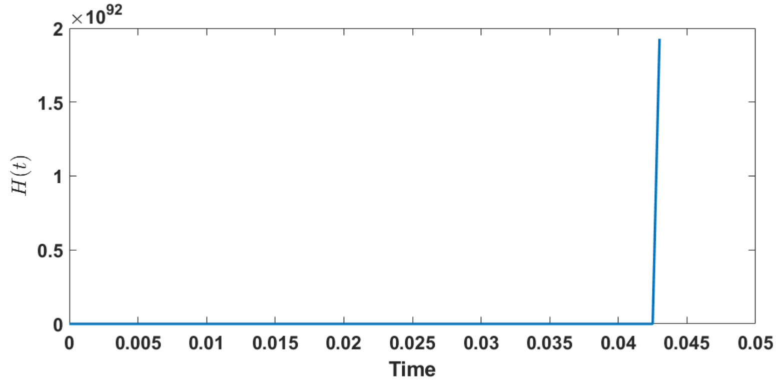


FIGURE 3. Test 1: The blow-up of H in finite time.

Test 2: For the second test, we consider an elliptical domain

$$\Omega_2 = \left\{ (x,y) / \frac{x^2}{4} + y^2 < 1 \right\}$$

with a triangulation discretization (see the mesh-grid in Figure 4), which consists of 2792 nodes and 1349 elements, and take the following initial conditions:

$$u_0(x,y) = 2(1 - \frac{x^2}{4} - y^2), \;\; v_0(x,y) = 3(1 - \frac{x^2}{4} - y^2), \;\; u_1 = v_1 = 0.$$

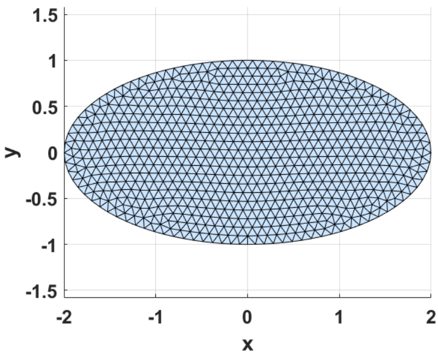
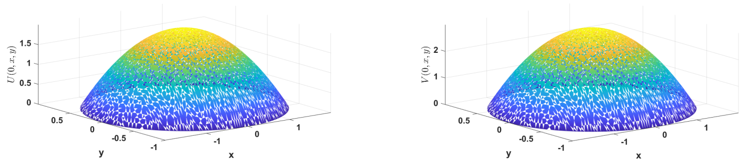
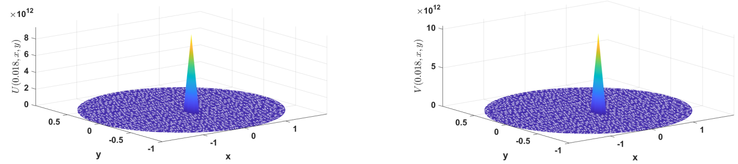


FIGURE 4. Uniform mesh grid of Ω_2.

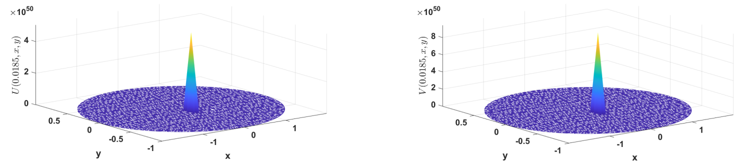
For Test 2, Figure 5 presents the approximate numerical results of the solution (u, v) at different time iterations $t = 0$, $t = 0.018$, $t = 0.0185$ and $t = 0.019$, where the left column shows the approximate values of u and the right column shows the approximate values of v. The numerical values of the functional $H(t)$ are presented in Figure 6. We observe the blow-up of the energy from $t = 0.0175$.



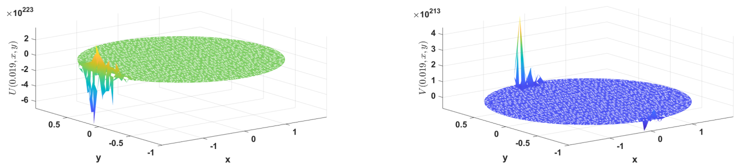
(A) $t = 0$



(B) $t = 0.018$



(C) $t = 0.0185$



(D) $t = 0.019$

FIGURE 5. The numerical results of Test 2 at different times.

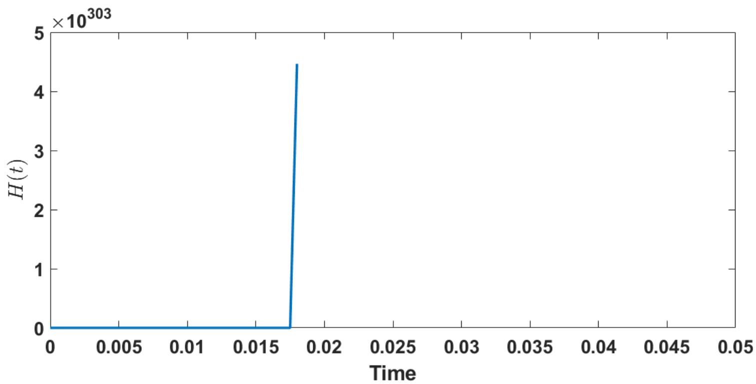


FIGURE 6. Test 2: The blow-up of H in finite time.

As a conclusion, the computational simulations show the blow-up of the solution of system (1.1) at finite time, which is compatible with the theoretical results.

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