Some attributes of the matrix operators about the weighted generalized difference sequence space

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Abstract

We can describe the norm for an operator given as $T:X\rightarrow Y$ as follows: It is the most appropriate value of U that satisfies the following inequality $$\Vert Tx\Vert_{Y}\leq U\Vert x\Vert_{X}$$ and also for the lower bound of T we can say that the value of L agrees with the following inequality $$\Vert Tx\Vert_{Y}\geq L\Vert x\Vert_{X},$$ where $\Vert .\Vert_{X}$ and $\Vert .\Vert_{Y}$ stand for the norms corresponding to the spaces X and Y. The main feature of this article is that it converts the norms and lower bounds of those matrix operators used as weighted sequence space $\ell_p(w)$ into a new space. This new sequence space is the generalized weighted sequence space. For this purpose, the double sequential band matrix $\tilde{B}(\tilde{r},\tilde{s})$ and also the space consisting of those sequences whose $\tilde{B}(\tilde{r},\tilde{s})$ transforms lie inside $\ell_p(\tilde{w})$, where $\tilde{r}=(r_{n})$, $\tilde{s}=(s_{n})$ are convergent sequences of positive real numbers. When comparing with the corresponding results in the literature, it can be seen that the results of the present study are more general and comprehensive.

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$$\|Tx\|_Y \leq U\|x\|_X$$

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where $\|.\|_X$ and $\|.\|_Y$ stand for the norms corresponding to the spaces X and Y. The main feature of this article is that it converts the norms and lower bounds of those matrix operators used as weighted sequence space $\ell_p(w)$ into a new space. This new sequence space is the generalized weighted sequence space. For this purpose, the double sequential band matrix $\tilde{B}(\tilde{r}, \tilde{s})$ and also the space consisting of those sequences whose $\tilde{B}(\tilde{r}, \tilde{s})$ transforms lie inside $\ell_p(\tilde{w})$, where $\tilde{r} = (r_n)$, $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. When comparing with the corresponding results in the literature, it can be seen that the results of the present study are more general and comprehensive.

1. Introduction

Let us outline some fundamental definitions and results, which we will largely be used in the following sections. Primarily, we will offer the concept of the sequence, the details of which are well known in elementary analysis. Although there are many different ways to describe the sequence, all of which mean the same thing, we we have chosen to give the following definition here. The sentence " x is a sequence " means $x := \{x_n\} := \{x_0, x_1, \ldots, x_n, \ldots\}$, where each x_n is a complex number. In other words, a sequence is easily introduced as an ordered list of complex numbers. Thus if x is a sequence, then it can be viewed as a mapping of $x : \mathbb{N} := \{1, 2, \ldots\} \to \mathbb{C}$. More generally terms, every x in sequence X is a transformation $x : \mathbb{N} \to X$, where X is a non-empty set. The collection of all real or complex number sequences forms a vector space which we denote by w, under the operations of coordinate-wise addition and the familiar scalar multiplication. The subspaces of ω are significant in such applications because each of them is called a sequence space.

Given an infinite matrix $A = (a_{nk})$ having complex numbers a_{nk} as entries in which $n, k \in \mathbb{N}$, it can be written for a sequence x, as follows

$$(Ax)_n := \sum a_{nk}x_k; \ (n \in \mathbb{N}, x \in D_{00}(A)),$$

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in which $D_{00}(A)$ describes the defined subspace of ω consisting of $x \in \omega$ for which the summation exists as a finite sum. For a simple notation, the summation ranges without limits from 0 to ∞.

The X_A is known to be the matrix domain of an infinite matrix A for any subspace X of the all real-valued sequence space w is described as

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}$$

which is a sequence space. There are several techniques to create new sequence spaces from old ones like X. One of them is to use an arbitrary matrix domain generated by an infinite matrix A such as X_A. To briefly explain the topic, these sequence spaces, namely X and X_A, may overlap but in any case either of them may contain the other one. The reader can find detailed information in the book "Summability Theory and Its Applications" by Başar [1] and therein.

Recently we have seen a significant increase in the construction of new sequence spaces using matrix domain in summability areas such as sequence spaces.

Many of the works [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] we have studied so far have something in common, they use the matrix domain.

Attempts have been made to find the best upper bound for some well-known matrix operators denoted by T from $\ell_p(w)$ to $F_{w,p}$. In the context of this statement, note that an upper bound for a matrix operator denoted by T defined from one sequence space X into another denoted by Y can be given by the following value of U

$$\|Tx\|_Y \leq U\|x\|_X,$$

in which $\|.\|_X$ and $\|.\|_Y$ denote the commonly known norms prescribed for spaces X and Y, respectively. Here, U does not dependent on x. Among them, the best value of U can be called the operator norm for T.

In addition, several researchers have tried to figure out the lower bounds for these matrix operators. This concept was first discussed in Ref [12] on the Cesàro matrix. But after that, others such as in Refs [13, 14] and [15, 16] have studied the lower bounds for some matrix operators defined on the sequence space denoted by ℓ_p and simultaneously on the weighted sequence space denoted by $\ell_p(w)$ with the Lorentz sequence space. Similarly, a lower bound of a matrix operator defined as $T : X \to Y$ is defined as the value of L satisfying the following inequality

$$\|Tx\|_Y \geq L\|x\|_X.$$

This inequality can also be used for some applications of functional analysis. For example, for finding the necessary and sufficient conditions under which an operator has its inverse, and for simultaneously finding the operator kernel containing only the zero vector for this case. For these reasons, knowing the lower bound for an operator is significant. In recent years, Dehghan and Talebi [17] have worked on the largest possible lower bound for some matrices on the Fibonacci sequence spaces. Furthermore, Foroutannia and Roopaei [18] have considered the problem of computing both the norm and lower and upper bounds for some operators defined on weighted difference sequence spaces. One can refer to these works [19, 20, 21, 22, 23, 24, 25] and those contained therein for related problems over some classical sequence spaces.

In this article, it is assumed that $w = (w_n)$ and also $\tilde{w} = (\tilde{w}_n)$ are sequences consisting of positive real terms. In this paper, a new space the generalized weighted difference sequence space, is introduced via the generalized difference matrix. Moreover, some properties of this sequence space are investigated. Among other things, it was found that although this space is semi-normed, it is not necessarily a normed space. Recall that a semi-normed satisfies every axiom of a norm, but the semi-norm of a vector must be zero without including the zero vector. Again, this is a semi-inner product space for the value of $p = 2$. Moreover, one obtains an isomorphism when using this space. Next, the norm for some matrix operators on the generalized weighted difference sequence space is defined. In the next step, we address the lower bound problem for the described operators of $\ell_p(w)$ in the generalized weighted difference sequence space.

2. The Sequence Space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

We examined in the former chapter that many topic lead to building new sequence spaces. Moreover, the concepts we offered were inherently large. Let us start by presenting the following matrix $\tilde{B} = (\tilde{b}_{nk}(\tilde{r}, \tilde{s}))$;

$$\tilde{b}_{nk}(\tilde{r}, \tilde{s}) = \begin{cases} s_n, & k = n+1 \\ r_n, & k = n \\ 0, & 0 \leq k < n \text{ or } k > n+1 \end{cases}$$

where $\tilde{r} = (r_n)$, $\tilde{s} = (s_n)$ are convergent sequences of positive real numbers. It should be noted at this point that many authors have described various sequence spaces and studied many different aspects of these spaces, using a different matrix similar to this matrix but actually different. Some of them are available in References [2, 3, 4, 5].

We will see later that this matrix allows us to construct an efficient structure for solving algebraic and topological properties. Applying the definition of matrix domain to this matrix, we define the new sequence space whose result lies in the $\ell_p(\tilde{w})$ space, as follows:

$$\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p < \infty \right\},$$

in which $1 \leq p < \infty$. We note here that, the space is a semi-normed space with the semi-norm defined by

$$\|x\|_{p,\tilde{w},\tilde{B}} = \left(\sum_{n=1}^{\infty} \tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p \right)^{1/p}.$$

To calculate the truth of this assertion, we now give an example. If we consider the sequence $x_n = \frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right)$, so due to $r_n x_n + s_n x_{n+1} = 0$ we obtain $\|x\|_{p,\tilde{w},\tilde{B}} = 0$, then it follows, from the definition of the norm, that $\|.\|_{p,\tilde{w},\tilde{B}}$ defined on $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is not a norm.

Before we begin with the general theory, we will first state the following basic theorem, which indicate that the set just described plays a significant role in its algebraic structure.

Theorem 2.1. *The set $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is linear space, that is, sequence space.*

Proof. We omit the proof which can be found in standard procedure. □

Let us proceed with the following theorem about an algebraic property of this newly defined sequence space.

Theorem 2.2. *It is true that the inclusion relation $\ell_p(\tilde{w}) \subset \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is strictly valid.*

Proof. If we take any $x \in \ell_p(\tilde{w})$, then the following calculation shows that the inclusion is valid

$$\begin{aligned}\tilde{w}_n \left| r_n x_n + s_n x_{n+1} \right|^p &\leq \tilde{w}_n 2^{p-1} \left(|r_n x_n|^p + |s_n x_{n+1}|^p \right) \\ &\leq 2^{p-1} max \left[|sup_{n\in\mathbb{N}} r_n|^p, |sup_{n\in\mathbb{N}} s_n|^p \right] \tilde{w}_n \left(|x_n|^p + |x_{n+1}|^p \right)\end{aligned}$$

by summing of n from 1 to ∞, in which $1 \leq p < \infty$.

To show that the inclusion relation is strictly valid. If the sequence $\tilde{w}$ with $(1, 1, 1, ...)$, we consider again the sequence $(x_n) = \left(\frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right) \right) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. From this it is easy to deduce that $(x_n) \notin \ell_p(\tilde{w})$. □

Theorem 2.3. *If $H = \left\{ x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s})) : r_n x_n + s_n x_{n+1} = 0 \text{ for all } n \in \mathbb{N} \right\}$, the quotient space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$.*

Proof. The basic approach to proving this theorem is to define a new T transformation from the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ to $\ell_p(\tilde{w})$ that exploits the definition of the fundamental matrix transformation, for all $x \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ uniquely $Tx = ((Tx)_n) = (r_n x_n + s_n x_{n+1})$. Since it is fairly obvious that T is linear, the first issue here is to show that T is surjective. One of the ways to accomlish this for any $y = (y_k) \in \ell_p(\tilde{w})$ is to say $x_n = \frac{1}{r_n} \sum_{k=n}^{\infty} \prod_{i=n}^{k-1} \left(\frac{-s_i}{r_{i+1}} \right) y_k$ for all $n \in \mathbb{N}$ in the norm of $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. In this case, by simple calculations, we obtain the following equations

$$\begin{aligned}\|x\|^p_{p,\tilde{w},\tilde{B}} &= \sum_{n=1}^{\infty} \tilde{w}_n \left| \frac{r_n}{r_n} \sum_{k=n}^{\infty} \prod_{i=n}^{k-1} \left(\frac{-s_i}{r_{i+1}} \right) y_k + \frac{s_n}{r_{n+1}} \sum_{k=n+1}^{\infty} \prod_{i=n+1}^{k-1} \left(\frac{-s_i}{r_{i+1}} \right) y_k \right|^p \\ &= \sum_{n=1}^{\infty} \tilde{w}_n \left| y_n + \left[\sum_{k=n+1}^{\infty} \prod_{i=n}^{k-1} \left(\frac{-s_i}{r_{i+1}} \right) y_k - \sum_{k=n+1}^{\infty} \prod_{i=n}^{k-1} \left(\frac{-s_i}{r_{i+1}} \right) y_k \right] \right|^p \\ &= \sum_{n=1}^{\infty} \tilde{w}_n \left| y_n \right|^p \\ &= \|y\|^p_{p,\tilde{w}} \\ &< \infty\end{aligned}$$

which implies that $x = (x_n) \in \ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. Returning back to the T transformation described above, it is very simple to say that $Tx = y$. Due to the fact that the image of the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ under the transformation T is $\ell_p(\tilde{w})$ and also $\ker T = H$, we have that $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$ under the first isomorphism theorem. □

We will use an example to show that the transformation T defined above is not injective. Namely, for $x = (x_n) = \left(\frac{1}{r_n} \prod_{i=1}^{n-1} \left(\frac{-r_{i+1}}{s_i} \right) \right)$ we get $Tx = 0$; in other words, $\ker T \neq \{0\}$.

Theorem 2.4. *If p is not equal to 2 and at the same time the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is not given as a semi-inner product space, then it is concluded that the space $\ell_2(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ is defined as a semi-inner product space.*

Proof. First, we will answer the question whether the semi-norm $\|.\|_{2,\tilde{w},\tilde{B}}$ can be induced with a semi-inner product. It is convenient at this point to use the notation $z_k = \tilde{w}_k^{1/2}(r_k x_k + s_k x_{k+1})$ for all $k \in \mathbb{N}$ and $\langle z, z\rangle_2 = \sum_{k=1}^{\infty} |z_k|^2$. Indeed taken arbitrary, $x \in \ell_2(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, we get

$$\|x\|_{2,\tilde{w},\tilde{B}} = \sqrt{\langle z, z\rangle_2}.$$

Moreover, it is easy to verify from the following equations that the semi-norm $\|.\|_{p,\tilde{w},\tilde{B}}$ cannot be obtained when considering a semi-inner product just defined as

$$\begin{aligned}
\|x+y\|^2_{p,\tilde{w},\tilde{B}} + \|x-y\|^2_{p,\tilde{w},\tilde{B}} &= 4(\tilde{w}_1^{2/p} + \tilde{w}_2^{2/p})\left(\frac{r_2}{r_1}\right)^2 \\
&\neq 4\left(\tilde{w}_1 + \frac{\tilde{w}_2}{2^p}\left|\frac{r_2}{r_1}\right|^p\right)^{2/p} \\
&= 2(\|x\|^2_{p,\tilde{w},\tilde{B}} + \|y\|^2_{p,\tilde{w},\tilde{B}}),
\end{aligned}$$

in which $x = \left(\frac{2r_1+s_1}{2r_1^2}, -\frac{1}{2r_1}, 0, 0, ...\right)$, $y = \left(\frac{2r_1-s_1}{2r_1^2}, \frac{1}{2r_1}, 0, 0, ...\right)$ and $p \neq 2$. □

3. The Norm of Matrix Operators from $\ell_1(w)$ to $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

Having defined a function from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, we will compute in this chapter that it is a norm. Before proceeding with the development of the general theory, let us start by presenting a very simple definition.

The matrix $A = (a_{nk})$ is said to be quasi-summable if A is an upper triangular matrix, namely, $a_{nk} = 0$ for $n > k$. As it can be clearly seen, the matrix satisfies $\sum_{n=1}^{k} a_{nk} = 1$ for all $k \in \mathbb{N}$.

Theorem 3.1. *The matrix $T = (t_{nk})$ is a bounded matrix operator from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ if $M = \sup_{k\in\mathbb{N}} \frac{\lambda_k}{w_k} < \infty$, in which $\lambda_k = \sum_{n=1}^{\infty} \tilde{w}_n |r_n t_{nk} + s_n t_{n+1,k}|$. In that case, the norm of operator is obtained as $\|T\|_{1,w,\tilde{w},\tilde{B}} = M$.*

For all $n \in \mathbb{N}$, taking both $w_n = 1$ and $\tilde{w}_n = 1$ specially, the transformation T is a bounded operator from the space ℓ_1 to the space $\ell_1(\tilde{B}(\tilde{r}, \tilde{s}))$ and also $\|T\|_{1,\tilde{B}} = \sup_{k\in\mathbb{N}} \lambda_k$.

Proof. We take into consideration a sequence $x = (x_n)$ in $\ell_1(w)$, thus

$$\begin{aligned}
\|Tx\|_{1,\tilde{w},\tilde{B}} &= \sum_{n=1}^{\infty} \tilde{w}_n \left| \sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k})\, x_k \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \tilde{w}_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| |x_k| \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| |x_k| \\
&= \sum_{k=1}^{\infty} \lambda_k |x_k| \\
&\leq M \sum_{k=1}^{\infty} w_k |x_k| \\
&= M \|x\|_{1,w}.
\end{aligned}$$

From these equations it follows that $\|T\|_{1,w,\tilde{w},\tilde{B}} \leq M$ since $\frac{\|Tx\|_{1,\tilde{w},\tilde{B}}}{\|x\|_{1,w}} \leq M$. We introduce the sequence $e^i = (0, 0, ..., 0, \overset{i.}{1}, 0, ...)$ for each $i \in \mathbb{N}$ to compute the inverse inequality, and then obtain $\|e^i\|_{1,w} = w_i$ and also $\|Te^i\|_{1,\tilde{w},\tilde{B}} = \lambda_i$. Therefore, it is easy to see that $\|T\|_{1,w,\tilde{w},\tilde{B}} \geq M$, and then $\|T\|_{1,w,\tilde{w},\tilde{B}} = M$.

Since special choices are made in the proof of the remaining part, no proof will be given here. $\square$

Theorem 3.2. *Let us assume that $T = (t_{nk})$ is the upper triangular matrix having the non-negative entries and also assume that (w_n) is an increasing given sequence. When the inequality $t_{nk} \geq t_{n+1,k}$ is valid for each values of $n \in \mathbb{N}$, constant $k \in \mathbb{N}$ and $M' = \sup_{k\in\mathbb{N}} \sum_{k=1}^{n} t_{nk} < \infty$, then T is defined as a bounded operator described from $\ell_1(w)$ to $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$. At the same time, the norm of this given operator satisfies the inequality given in the form $\|T\|_{1,w,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|)M'$. When the specific condition of T is being quasi summable matrix, also $r_k \geq -s_k > 0$ and $s_{k-1} + r_k = 1$ is taken into consideration, thus the condition $\|T\|_{1,w,\tilde{B}} = 1$ is satisfied.*

Proof. Given the hypothesis, we must say that the matrix $T = (t_{nk})$ satisfying the condition $t_{nk} \geq t_{n+1,k}$ (for all $n, k = 1, 2, ...$) is an upper triangular and also the sequence (w_n) is increasing. With simple calculations, we can derive the following

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| \\
&= \sum_{n=1}^{k-1} w_n \left| r_n t_{nk} + s_n t_{n+1,k} \right| + w_k |r_k| t_{kk} \\
&\leq w_k \left[\sum_{n=1}^{k-1} \left(|r_n| t_{nk} + |s_n| t_{n+1,k} \right) + |r_k| t_{kk} \right] \\
&= w_k \left[\left(|r_1| t_{1k} + |s_1| t_{2k} \right) + ... + \left(|r_{k-1}| t_{k-1,k} + |s_{k-1}| t_{kk} \right) + |r_k| t_{kk} \right] \\
&= w_k \left[|r_1| t_{1k} + \left(|s_1| + |r_2| \right) t_{2k} + ... + \left(|s_{k-1}| + |r_k| \right) t_{kk} \right] \\
&\leq \left(\sup_{k \in \mathbb{N}} |r_k| + \sup_{k \in \mathbb{N}} |s_k| \right) w_k \sum_{n=1}^{k} t_{nk}.
\end{aligned}$$

Obviously, $\|T\|_{1,w,\tilde{B}} = \sup_{k \in \mathbb{N}} \frac{s_k}{w_k} \leq (\sup_{k \in \mathbb{N}} |r_k| + \sup_{k \in \mathbb{N}} |s_k|) \sup_{k \in \mathbb{N}} \sum_{n=1}^{k} t_{nk} = (\sup_{k \in \mathbb{N}} |r_k| + \sup_{k \in \mathbb{N}} |s_k|) M'$ from Theorem 3.1.

Let us suppose that T is a quasi summable matrix, so $M' = 1$. If $r_k \geq -s_k > 0$ holds, then of course $r_n t_{nk} + s_n t_{n+1,k} > 0$ holds for every $k, n \in \mathbb{N}$ and also if the equality $s_{k-1} + r_k = 1$ is satisfied, then we can easily write $\lambda_k \leq w_k \sum_{n=1}^{k} t_{nk}$ thus $\|T\|_{1,w,\tilde{B}} \leq 1$. To obtain the inverse inequality, let us consider the sequence $e^1 = (1, 0, 0, ...)$. It follows that $\|e^1\|_{1,w} = w_1$ and $\|Te^1\|_{1,w,\tilde{B}} = w_1$, namely $\|T\|_{1,w,\tilde{B}} \geq 1$. As a result, we obtain $\|T\|_{1,w,\tilde{B}} = 1$. $\square$

In the light of the above theorems, we are concerned here with the computation of the norm of some specific quasi summable matrices. First, we consider the transpose of the well-known Riesz matrix $\tilde{R} = (\tilde{r}_{nk})$ which is described as follows:

$$\tilde{r}_{nk} = \begin{cases} \frac{q_n}{Q_k}, & n \leq k \\ 0, & n > k, \end{cases} \tag{1}$$

where (q_n) is a non-negative sequence with $q_1 > 0$ and $Q_k = q_1 + ... + q_k$ for all $k \in \mathbb{N}$.

Taking $q_n = 1$ for all $n \in \mathbb{N}$, we derive the transpose of the Cesáro matrix of order one, also known as the Copson matrix (see [16]). We denote this particular matrix by $\tilde{C} = (\tilde{c}_{nk})$, where

$$\tilde{c}_{nk} = \begin{cases} \frac{1}{k}, & n \leq k \\ 0, & n > k. \end{cases}$$

Corollary 3.3. *When (q_n) is a decreasing sequence and (w_n) is an increasing sequence, in that case $\tilde{R}$ is a bounded operator from the space $\ell_1(w)$ into the space $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$ and, also $\|\tilde{R}\|_{1,w,\tilde{B}} = 1$ for $r_n \geq -s_n > 0$ and $s_{n-1} + r_n = 1$ for every $n \in \mathbb{N}$.*

Proof. First of all, since (q_n) is a decreasing sequence from the hypothesis the following inequality $\tilde{r}_{nk} = \frac{q_n}{Q_k} \geq \frac{q_{n+1}}{Q_k} = \tilde{r}_{n+1,k}$ holds for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$. For $\tilde{R}$ is a non-negative upper triangular matrix and (w_n) is an increasing sequence, it follows from Theorem 3.2 that $\tilde{R}$ is a bounded operator from $\ell_1(w)$ into $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$. Also due to the fact that $\sum_{n=1}^{k} \tilde{r}_{nk} = 1$ for every $k \in \mathbb{N}$, $\tilde{R}$ is a quasi summable matrix. If $r_n \geq -s_n > 0$ and $s_{n-1} + r_n = 1$ for every $n \in \mathbb{N}$, then it is clear that $\|\tilde{R}\|_{1,w,\tilde{B}} = 1$ from Theorem 3.2. $\square$

Corollary 3.4. *If* $\sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k} < \infty$*, then the matrix* $\tilde{C}$ *defined just above is a bounded operator from the space* $\ell_1(w)$ *into* $\ell_1(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ *and* $\|\tilde{C}\|_{1,w,\tilde{w},\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$.

Proof. We get the following inequality

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} \tilde{w}_n \left| r_n \tilde{c}_{nk} + s_n \tilde{c}_{n+1,k} \right| \\
&\leq \frac{1}{k} \left[\sum_{n=1}^{k-1} \tilde{w}_n \left(|r_n| + |s_n| \right) + \tilde{w}_k |r_n| \right] \\
&= \frac{\sup_{k\in\mathbb{N}} |r_k|}{k} \sum_{n=1}^{k} \tilde{w}_n + \frac{\sup_{k\in\mathbb{N}} |s_k|}{k} \sum_{n=1}^{k-1} \tilde{w}_n \\
&\leq \frac{\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|}{k} \sum_{n=1}^{k} \tilde{w}_n.
\end{aligned}$$

Therefore, we obtain that $\|\tilde{C}\|_{1,w,\tilde{w},B} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$ from Theorem 3.1. $\square$

Theorem 3.5. *Let us suppose that* $T = (t_{nk})$ *is a matrix having the non-negative entries and the inequalities* $t_{nk} \geq t_{n+1,k}$ *hold for all* $n \in \mathbb{N}$ *and each fixed* $k \in \mathbb{N}$*. If* $\sum_{n=1}^{\infty} t_{nk} < \infty$ *for each* $k \in \mathbb{N}$ *and also* $M'' = \sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} t_{nk} < \infty$*, then the matrix* T *is a bounded operator from the space* ℓ_1 *to* $\ell_1(\tilde{B}(\tilde{r}, \tilde{s}))$ *and the norm of operator is* $\|T\|_{1,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) M''$*. When the fact that the specific condition of* T *is being quasi summable matrix is taken into consideration for* $r_k \geq -s_k > 0$ *and* $s_{k-1} + r_k = 1$ *(for all* $k \in \mathbb{N}$*), then the condition* $\|T\|_{1,\tilde{B}} = 1$ *is derived.*

Proof. For any $k \in \mathbb{N}$, we get

$$\lambda_k = \sum_{n=1}^{\infty} \left| r_n t_{nk} + s_n t_{n+1,k} \right| = (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) \sum_{n=1}^{\infty} t_{nk}.$$

Using Theorem 3.1 here, we find that the norm $\|T\|_{1,\tilde{B}} \leq (\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|) M''$. The rest of the proof can be done similarly to the proof of Theorem 3.2. $\square$

The matrix $H = (h_{nk})$ defined as $h_{nk} = \frac{1}{n+k}$ for all $n, k \in \mathbb{N}$ is known to be the Hilbert matrix operator. Here, we will discover the norm of the operator just mentioned.

Now, let us give the following integral to be used in the proofs:

$$\int_0^{\infty} \frac{1}{t^{\alpha}(t+c)} dt = \frac{\pi}{c^{\alpha} \sin \alpha\pi},$$

in which $0 < \alpha < 1$.

Theorem 3.6. *Let $w_n = \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$, in which $0 < \alpha < 1$. In this case, the Hilbert matrix operator H just described is bounded from the space $\ell_1(w)$ to the space $\ell_1(w, \tilde{B}(\tilde{r}, \tilde{s}))$ and also the norm $\|H\|_{1,w,\tilde{B}} \leq \frac{\pi}{\sin \alpha\pi}(\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i|)$.*

Proof. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\lambda_n &= \sum_{i=1}^{\infty} w_i \left| r_i h_{in} + s_i h_{i+1,n} \right| \\
&\leq \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(\frac{|r_i|}{i+n} + \frac{|s_i|}{i+n+1} \right) \\
&\leq \int_0^\infty \frac{1}{t^\alpha} \left(\frac{\sup_{i\in\mathbb{N}} |r_i|}{t+n} + \frac{\sup_{i\in\mathbb{N}} |s_i|}{t+n+1} \right) dt \\
&= \frac{\pi}{\sin \alpha\pi} \left(\frac{\sup_{i\in\mathbb{N}} |r_i|}{n^\alpha} + \frac{\sup_{i\in\mathbb{N}} |s_i|}{(n+1)^\alpha} \right).
\end{aligned}$$

It follows that

$$n^\alpha \lambda_n \leq \frac{\pi}{\sin \alpha\pi} \left[\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i| \left(\frac{n}{n+1} \right)^\alpha \right] \leq \frac{\pi}{\sin \alpha\pi} \left(\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i| \right).$$

Considering Theorem 3.1, this means that $\|H\|_{1,w,\tilde{B}} \leq \frac{\pi}{\sin \alpha\pi} (\sup_{i\in\mathbb{N}} |r_i| + \sup_{i\in\mathbb{N}} |s_i|)$. □

4. The Norm of Matrix Operators from $\ell_p(w)$ to $\ell_p(w, \tilde{B}(\tilde{r}, \tilde{s}))$

In this section, we are going to discuss calculating the norm of some matrix operators from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. We now present an essential lemma which is obtained by Jameson and Lashkaripour, since this important result is used in the proofs.

Lemma 4.1. [16] *Let us suppose that $A = (a_{nk})$ is a matrix operator having the nonnegative entries $a_{nk} \geq 0$, also suppose that (u_n) and (v_k) are positive sequences given such that*

$$u_n^{1/p} \sum_{k=1}^{\infty} \frac{a_{nk}}{v_k^{1/p}} \leq K_1 \qquad (\textit{for } n \in \mathbb{N},\ K_1 \in \mathbb{R})$$

and

$$\frac{1}{v_k^{(1-p)/p}} \sum_{n=1}^{\infty} u_n^{(1-p)/p} a_{nk} \leq K_2 \qquad (\textit{for } k \in \mathbb{N},\ K_2 \in \mathbb{R})$$

in that case, that inequality $\|A\|_p \leq \frac{K_2^{1/p}}{K_1^{(1-p)/p}}$ is valid, in which $p > 1$.

Now, let us state and prove another necessary lemma.

Lemma 4.2. *Let us assume that the equality $a_{nk} = \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} (r_n t_{nk} + s_n t_{n+1,k})$ is valid for the matrix operators $T = (t_{nk})$ and $A = (a_{nk})$. At the same time, we have $\|T\|_{p,w,\tilde{w},\tilde{B}} = \|A\|_p$, for $p \geq 1$. Under the conditions of this hypothesis, T is bounded operator from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ iff A is bounded operator onto the space ℓ_p.*

Proof. If the x lying in the space $\ell_p(w)$ is taken as arbitrarily, and the sequence $y = (y_k)$ is defined as $y_k = w_k^{1/p} x_k$ for all $k \in \mathbb{N}$ by making use of it, then we derive that equality $\|x\|_{p,w} = \|y\|_p$. Therefore, the proof should be clear with the following basic calculations

$$\begin{aligned}
\|T\|^p_{p,w,\tilde{w},\tilde{B}} &= \sup_{x\in\ell_p(w), x\neq 0} \frac{\|Tx\|^p_{p,\tilde{w},\tilde{B}}}{\|x\|^p_{p,w}} \\
&= \sup_{x\in\ell_p(w), x\neq 0} \frac{\sum_{n=1}^{\infty} \tilde{w}_n \left|\sum_{k=1}^{\infty} \left(r_n t_{nk} + s_n t_{n+1,k}\right) x_k\right|^p}{\sum_{k=1}^{\infty} w_k |x_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} \left(r_n t_{nk} + s_n t_{n+1,k}\right) y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} \\
&= \sup_{y\in\ell_p} \frac{\sum_{n=1}^{\infty} \left|\sum_{k=1}^{\infty} a_{nk} y_k\right|^p}{\sum_{k=1}^{\infty} |y_k|^p} = \sup_{y\in\ell_p} \frac{\|Ay\|^p_p}{\|y\|^p_p} = \|A\|^p_p.
\end{aligned}$$

□

Theorem 4.3. *Let us assume that the matrix operator $\tilde{R}$ is as defined in* (1)*, and also assume that (q_n) is a decreasing sequence having $q_1 = q_2 = 2$ and $\lim_{n\to\infty} Q_n = \infty$. For all $n \in \mathbb{N}$, if the sequence (w_n) is taken as $\left(\frac{2Q_{n-1}}{q_n}\right)^p$ with $Q_0 = 1$, in that case, $\tilde{R}$ is bounded operator from the space $\ell_p(w)$ to the space $\ell_p(\tilde{B}(\tilde{r},\tilde{s}))$ and $\|\tilde{R}\|_{p,w,\tilde{B}} \leq \frac{\sup_{n\in\mathbb{N}}|r_n| + \sup_{n\in\mathbb{N}}|s_n|}{2}$ for $p > 1$.*

Proof. In Lemma 4.2, utilize the matrix $\tilde{R}$ in place of T. So, the matrix $A = (a_{nk})$ is described by

$$a_{nk} = \begin{cases} \frac{q_k}{2Q_{k-1}Q_k}\left(r_n q_n + s_n q_{n+1}\right), & n < k \\ \frac{1}{2} r_k \frac{q_k^2}{Q_{k-1}Q_k}, & n = k \\ 0, & n > k \end{cases}$$

and besides that, $\|\tilde{R}\|_{p,w,\tilde{B}} = \|A\|_p$ is obtained.

We derive

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk} &= \frac{r_n}{2} q_n \frac{q_n}{Q_{n-1}Q_n} + \frac{1}{2}\left(r_n q_n + s_n q_{n+1}\right) \sum_{k=n+1}^{\infty} \frac{q_k}{Q_{k-1}Q_k} \\
&= \frac{r_n}{2} q_n \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right) + \frac{1}{2}\left(r_n q_n + s_n q_{n+1}\right) \frac{1}{Q_n} \\
&= \frac{r_n}{2} \frac{q_n}{Q_{n-1}} + \frac{s_n}{2} \frac{q_{n+1}}{Q_n} \\
&\leq \frac{\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|}{2}
\end{aligned}$$

for all $n \in \mathbb{N}$. Also, we derive

$$\begin{aligned}
\sum_{n=1}^{\infty} a_{nk} &= \frac{1}{2} \frac{q_k}{Q_{k-1}Q_k} \left[\sum_{n=1}^{k-1} (r_n q_n + s_n q_{n+1}) \right] + \frac{r_k}{2} \frac{q_k}{Q_{k-1}Q_k} q_k \\
&= \frac{1}{2} \frac{q_k}{Q_{k-1}Q_k} \left[r_1 q_1 + \sum_{n=1}^{k-1} (r_{n+1} + s_n)\, q_{n+1} \right] \\
&\leq \frac{\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|}{2} \frac{q_k}{Q_{k-1}Q_k} \sum_{n=1}^{k} q_n \\
&\leq \frac{\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|}{2}
\end{aligned}$$

for all $k \in \mathbb{N}$. Now, In Lemma 4.1, if we take $u_n = v_n = 1$ for all $n \in \mathbb{N}$, we get $K_1 \leq \frac{\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|}{2}$ and $K_2 \leq \frac{\sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|}{2}$ which require that $\|\tilde{R}\|_{p,w,\tilde{B}} \leq \frac{\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|}{2}$ for $p > 1$. □

Theorem 4.4. *Let $w_n = \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$, in which $1 - p < \alpha < 1$ and $p > 1$. In that case, the Hilbert matrix operator H is a bounded operator from the space $\ell_p(w)$ to the space $\ell_p(w, \tilde{B}(\tilde{r}, \tilde{s}))$ also following inequality*

$$\|H\|_{p,w,\tilde{B}} \leq \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n| \right) \max \left\{ \frac{\pi}{\sin \beta\pi}, \frac{\pi}{\sin \gamma\pi} \right\},$$

is valid, in which $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$.

Proof. Let us define the matrix $A = (a_{nk})$ as follows

$$a_{nk} = \left(\frac{k}{n} \right)^{\alpha/p} \left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1} \right)$$

for all $n, k \in \mathbb{N}$. In this case, $\|H\|_{p,w,\tilde{B}} = \|A\|_p$ which obtained by using Lemma 4.2. Specifically, when we choose $u_n = v_n = n$ in Lemma 4.1 for all $n \in \mathbb{N}$, we find that

$$\begin{aligned}
u_n^{\frac{1}{p}} \sum_{k=1}^{\infty} \frac{a_{nk}}{v_k^{\frac{1}{p}}} &= n^{1/p} \sum_{k=1}^{\infty} \frac{1}{k^{1/p}} \left(\frac{k}{n} \right)^{\alpha/p} \left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1} \right) \\
&\leq n^{\beta} \sum_{k=1}^{\infty} \frac{1}{k^{\beta}} \left(\frac{|r_n|}{n+k} + \frac{|s_n|}{n+k+1} \right) \\
&\leq n^{\beta} \int_{t=0}^{\infty} \frac{1}{t^{\beta}} \left(\frac{\sup_{n\in\mathbb{N}} |r_n|}{t+n} + \frac{\sup_{n\in\mathbb{N}} |s_n|}{t+(n+1)} \right) dt \\
&= n^{\beta} \left(\frac{\sup_{n\in\mathbb{N}} |r_n| \pi}{n^{\beta} \sin \beta\pi} + \frac{\sup_{n\in\mathbb{N}} |s_n| \pi}{(n+1)^{\beta} \sin \beta\pi} \right) \\
&\leq \frac{\pi}{\sin \beta\pi} \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n| \right)
\end{aligned}$$

for all $n \in \mathbb{N}$ also

$$\begin{aligned}
\frac{1}{v_k^{\frac{1-p}{p}}} \sum_{n=1}^{\infty} u_n^{\frac{1-p}{p}} a_{nk} &= \frac{1}{k^{(1-p)/p}} \sum_{n=1}^{\infty} n^{(1-p)/p} \left(\frac{k}{n}\right)^{\alpha/p} \left(\frac{r_n}{n+k} + \frac{s_n}{n+k+1}\right) \\
&\le k^{\gamma} \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} \left(\frac{|r_n|}{n+k} + \frac{|s_n|}{n+k+1}\right) \\
&\le k^{\gamma} \int_{t=0}^{\infty} \frac{1}{t^{\gamma}} \left(\frac{\sup_{n\in\mathbb{N}} |r_n|}{t+k} + \frac{\sup_{n\in\mathbb{N}} |s_n|}{t+(k+1)}\right) dt \\
&= k^{\gamma} \left(\frac{\sup_{n\in\mathbb{N}} |r_n| \pi}{k^{\gamma} \sin \gamma\pi} + \frac{\sup_{n\in\mathbb{N}} |s_n| \pi}{(k+1)^{\gamma} \sin \gamma\pi}\right) \\
&\le \frac{\pi}{\sin \gamma\pi} \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|\right)
\end{aligned}$$

for all $k \in \mathbb{N}$, where $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$. We therefore obtain that

$$\|H\|_{p,w,\tilde{B}} \le \left(\sup_{n\in\mathbb{N}} |r_n| + \sup_{n\in\mathbb{N}} |s_n|\right) \max\left\{\frac{\pi}{\sin \beta\pi}, \frac{\pi}{\sin \gamma\pi}\right\}$$

from Lemma 4.1. □

5. Lower Bounds of Matrix Operators from $\ell_p(w)$ to $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$

An important problem posed in this paper is to calculate the lower bound of an operator T from the space $\ell_p(w)$ to space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$. Thus, the goal is to obtain the lower bound of the operator T for the largest value L satisfying the following inequality

$$\|Tx\|_{p,\tilde{w},\tilde{B}} \ge L\|x\|_{p,w}$$

for every decreasing sequence $x = (x_k)$ with $x_k \ge 0$.

We need the following lemma to perform the calculations in the proofs in this section.

Lemma 5.1. [16] *Let us assume that both (q_n) and (x_n) are non-negative sequences, and that (x_n) is also a decreasing sequence satisfying condition $\lim_{n\to\infty} x_n = 0$. For $Q_n = \sum_{i=1}^{n} q_i$ with $Q_0 = 1$ also $R_n = \sum_{i=1}^{n} q_i x_i$, the following statements holds, in which $p \ge 1$ and $n \in \mathbb{N}$.*

1. $R_n^p - R_{n-1}^p \ge (Q_n^p - Q_{n-1}^p) x_n^p$.
2. *When the series $\sum_{i=1}^{\infty} q_i x_i$ converges, the following inequality is satisfied.*

$$\left(\sum_{i=1}^{\infty} q_i x_i\right)^p \ge \sum_{n=1}^{\infty} Q_n^p (x_n^p - x_{n+1}^p).$$

Theorem 5.2. *When $T = (t_{nk})$ is a matrix operator with $t_{nk} \ge 0$ from the space $\ell_p(w)$ into the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, in which $p \ge 1$, the following inequality $t_{nk} \ge t_{n+1,k}$ is valid for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$ also the series $\sum_{n=1}^{\infty} w_n$ diverges to infinity, in that case, for every decreasing sequence $x = (x_k)$ having $x_k \ge 0$, we have*

$$\|Tx\|_{p,\tilde{w},\tilde{B}} \ge L\|x\|_{p,w}$$

in which $L^p = \inf_{n\in\mathbb{N}} \frac{S_n}{W_n}$, $W_n = \sum_{k=1}^{n} w_k$ and $S_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(\sum_{k=1}^{n} (r_i t_{ik} + s_i t_{i+1,k})\right)^p$ where $r_n \ge -s_n > 0$ for all $n \in \mathbb{N}$.

Proof. Under the conditions of the hypothesis formulated in the theorem, we can give the proof as follows. Since $\sum_{n=1}^{\infty} w_n = \infty$, we obtain $\lim_{k\to\infty} x_k = 0$, and also, we can be establish that the series $\sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k}) x_k$ is convergent for all $n \in \mathbb{N}$. On the other hands, using Lemma 5.1 and Abel summation, we have

$$\begin{aligned}
\|Tx\|_{p,\tilde{w},B}^p &= \sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{\infty} (r_n t_{nk} + s_n t_{n+1,k}) x_k \right)^p \\
&\geq \sum_{n=1}^{\infty} \tilde{w}_n \sum_{i=1}^{\infty} \left(\sum_{k=1}^{i} (r_n t_{nk} + s_n t_{n+1,k}) \right)^p (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} \left[\sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{i} (r_n t_{nk} + s_n t_{n+1,k}) \right)^p \right] (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p) \geq L^p \sum_{i=1}^{\infty} W_i (x_i^p - x_{i+1}^p) = L^p \|x\|_{p,w}^p
\end{aligned}$$

which completes the proof. □

The following lemma can be verified using a technique similar to the proof of Proposition 1 in [16].

Lemma 5.3. *Let us assume that $T = (t_{nk})$ be a non-negative matrix operator defined from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$, in which $p \geq 1$. If the following inequality*

$$r_n t_{nk} + s_n t_{n+1,k} \geq r_n t_{n,k+1} + s_n t_{n+1,k+1}$$

is valid also $t_{nk} \geq t_{n+1,k}$ for all $k \in \mathbb{N}$, each fixed $n \in \mathbb{N}$ and $r_n \geq -s_n > 0$, if the series $\sum_{n=1}^{\infty} w_n$ is divergent the infinity, then we have

$$L^p \geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p] \frac{t_n}{w_n},$$

in which $t_n = \sum_{i=1}^{\infty} \tilde{w}_i (r_i t_{in} + s_i t_{i+1,n})^p$.

Theorem 5.4. *Let $H = (h_{nk})$ is the Hilbert matrix operator, $w_n = \frac{1}{n^{p+\alpha}}$ and $\tilde{w}_n = \frac{1}{n^{\alpha}}$ for every $n \in \mathbb{N}$, in which $p \geq 1$, $0 \leq p + \alpha \leq 1$ and $r_n \geq -s_n > 0$. For every decreasing sequences $x = (x_k)$ that are not negative terms, we have*

$$\|Hx\|_{p,\tilde{w},\tilde{B}} \geq L \|x\|_{p,w}$$

in which $L^p \geq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p(i+2)^p}$.

Proof. It is clear that both the Hilbert matrix $H = (h_{nk})$ and the sequence (w_n) satisfy the conditions listed in Lemma 5.3, therefore, we obtain

$$\begin{aligned}
L^p &\geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p] \frac{t_n}{w_n} \\
&\geq \inf_{n\in\mathbb{N}} n^{p-1} n^{p+\alpha} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r_i}{i+n} + \frac{s_i}{i+n+1} \right)^p \\
&\geq \inf_{n\in\mathbb{N}} n^{2p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} \left(\frac{r_i}{i+n} + \frac{s_i}{i+n+1} \right)^p.
\end{aligned}$$

The rest of the proof can be obtained in the same way as in the proof of Theorem 4.3 in [18]. □

Conclusion. In this manuscript, we have presented the norms for matrix operators which are defined between the weighted sequence space denoted by $\ell_p(w)$ and the weighted difference sequence space $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ which is valid for $1 \leq p < \infty$. To make the presentation more understandable, we have used some specific matrices like quasi summable ones (that is the transposes of Riesz and Cesàro matrices of the first order) and Hilbert matrix. Firstly, $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$ space has been presented and its properties have been scrutinized. Next, we have tried to compute the lower bound for the matrix given from $\ell_p(w)$ into $\ell_p(\tilde{w}, \tilde{B}(\tilde{r}, \tilde{s}))$.

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