Decay property of solutions to the wave equation with space-dependent damping, absorbing nonlinearity, and polynomially decaying data

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Abstract

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1. Introduction

We study the initial-boundary value problem of the wave equation with space-dependent damping and absorbing nonlinearity

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u + |u|^{p-1}u = 0, & t>0, x\in\Omega, \\ u(t,x) = 0, & t>0, x\in\partial\Omega, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega = \mathbb{R}^n$ with $n \ge 1$, or $\Omega \subset \mathbb{R}^n$ with $n \ge 2$ is an exterior domain, that is, $\mathbb{R}^n \setminus \Omega$ is compact. We also assume that the boundary $\partial\Omega$ of Ω is of class C^2. When $\Omega = \mathbb{R}^n$, the boundary condition is omitted and we consider the initial value problem. The unknown function $u = u(t,x)$ is assumed to be real-valued. The function $a(x)$ denotes the coefficient of the damping term. Throughout this paper, we assume that $a \in C(\mathbb{R}^n)$ is nonnegative and bounded. The semilinear term $|u|^{p-1}u$, where $p > 1$, is the so-called absorbing nonlinearity, which assists the decay of the solution.

The aim of this paper is to obtain the decay estimates of the energy

$$E[u](t) := \frac{1}{2}\int_\Omega (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)\,dx + \frac{1}{p+1}\int_\Omega |u(t,x)|^{p+1}\,dx \tag{1.2}$$

and the weighted L^2-norm

$$\int_\Omega a(x)|u(t,x)|^2\,dx$$

of the solution.

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First, for the energy $E[u](t)$, we observe from the equation (1.1) that

$$\frac{d}{dt}E[u](t) = -\int_{\Omega} a(x)|\partial_t u(t,x)|^2\,dx,$$

which gives the energy identity

$$E[u](t) + \int_0^t \int_{\Omega} a(x)|\partial_t u(s,x)|^2\,dxds = E[u](0).$$

Since $a(x)$ is nonnegative, the energy is monotone decreasing in time. Therefore, a natural question arises as to whether the energy tends to zero as time goes to infinity and, if that is true, what the actual decay rate is. Moreover, we can expect that the amplitude of the damping coefficient $a(x)$, the power p of the nonlinearity, and the spatial decay of the initial data (u_0, u_1) will play crucial roles for this problem. Our goal is to clarify how these three factors determine the decay property of the solution.

Before going to the main result, we shall review previous studies on the asymptotic behavior of solutions to linear and nonlinear damped wave equations.

The study of the asymptotic behavior of solutions to the damped wave equation goes back to the pioneering work by Matsumura [52]. He studied the initial value problem of the linear wave equation with the classical damping

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.3}$$

In this case the energy of the solution u is defined by

$$E_L(t) := \frac{1}{2}\int_{\mathbb{R}^n} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)\,dx. \tag{1.4}$$

By using the Fourier transform, he proved the so-called Matsumura estimates

$$\|\partial_t^k \partial_x^\gamma u(t)\|_{L^\infty} \le C(1+t)^{-\frac{n}{2m}-k-\frac{|\gamma|}{2}} \left(\|u_0\|_{L^m} + \|u_1\|_{L^m} + \|u_0\|_{H^{[\frac{n}{2}]+k+|\gamma|+1}} + \|u_1\|_{H^{[\frac{n}{2}]+k+|\gamma|}} \right),$$

$$\|\partial_t^k \partial_x^\gamma u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-k-\frac{|\gamma|}{2}} \left(\|u_0\|_{L^m} + \|u_1\|_{L^m} + \|u_0\|_{H^{k+|\gamma|}} + \|u_1\|_{H^{k+|\gamma|-1}} \right) \tag{1.5}$$

for $1\le m\le 2$, $k\in\mathbb{Z}_{\ge 0}$, and $\gamma\in\mathbb{Z}_{\ge 0}^n$, and applied them to semilinear problems. In particular, the above estimate implies

$$\begin{aligned} &(1+t)E_L(t) + \|u(t)\|_{L^2}^2 \\ &\le C(1+t)^{-n\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|u_0\|_{L^m} + \|u_1\|_{L^m} + \|u_0\|_{H^1} + \|u_1\|_{L^2} \right)^2. \end{aligned} \tag{1.6}$$

This indicates that the spatial decay of the initial data improves the time decay of the solution.

Moreover, the decay rate in the estimates (1.5) suggests that the solution of (1.3) is approximated by a solution of the corresponding heat equation

$$\partial_t v - \Delta v = 0, \quad t>0, x\in\mathbb{R}^n.$$

This is the so-called diffusion phenomenon and firstly proved by Hsiao and Liu [18] for the hyperbolic conservation law with damping.

There are many improvements and generalizations of the Matsumura estimates and the diffusion phenomenon for (1.3). We refer the reader to [7, 17, 20, 21, 28, 33, 41, 44, 51, 55, 59, 61, 76, 78, 86, 99] and the references therein.

Next, we consider the initial boundary value problem of the linear wave equation with space-dependent damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = 0, & t>0, x\in\Omega,\\ u(t,x)=0, & t>0, x\in\partial\Omega,\\ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x), & x\in\Omega. \end{cases} \tag{1.7}$$

Mochizuki [56] firstly studied the case $\Omega=\mathbb{R}^n$ $(n\neq 2)$ and showed that if $a(x)\le C\langle x\rangle^{-\alpha}$ with $\alpha>1$, then the wave operator exists and is not identically vanishing. Namely, the energy $E_L(t)$ defined by (1.4) of the solution does not decay to zero in general, and the solution behaves like a solution of the wave equation without damping. This means that if the damping is sufficiently small at the spatial infinity, then the energy of the solution does not decay to zero in general. His result actually includes the time and space dependent damping, and generalizations in the damping coefficients and domains can be found in Mochizuki and Nakazawa [57], Matsuyama [54], and Ueda [90].

On the other hand, for (1.7) with $\Omega=\mathbb{R}^n$, from the result by Matsumura [53], we see that if $u_0,u_1\in C_0^\infty(\mathbb{R}^n)$ and $a(x)\ge C\langle x\rangle^{-1}$, then $E_L(t)$ decays to zero as $t\to\infty$ (see also Uesaka [91]). These results indicate that for the damping coefficient $a(x)=\langle x\rangle^{-\alpha}$, the value $\alpha=1$ is critical for the energy decay or non-decay.

Regarding the precise decay rate of the solution to (1.7), Todorova and Yordanov [89] proved that if $\Omega=\mathbb{R}^n$, $a(x)$ is positive, radial and satisfies $a(x)=a_0|x|^{-\alpha}+o(|x|^{-\alpha})$ $(|x|\to\infty)$ with some $\alpha\in[0,1)$, and the initial data has compact support, then the solution satisfies

$$(1+t)E_L(t)+\int_{\mathbb{R}^n} a(x)|u(t,x)|^2\,dx \le C(1+t)^{-\frac{n-\alpha}{2-\alpha}+\delta}\left(\|u_0\|_{H^1}+\|u_1\|_{L^2}\right)^2,$$

where $\delta>0$ is arbitrary constant and C depends on δ and the support of the data. We note that if we formally take $\alpha=0$ and $\delta=0$, then the decay rate coincides with that of (1.6). The proof of [89] is based on the weighted energy method with the weight function

$$t^{-\frac{n-\alpha}{2-\alpha}+2\delta}\exp\left(-\left(\frac{n-\alpha}{2-\alpha}-\delta\right)\frac{A(x)}{t}\right),$$

where $A(x)$ is a solution of the Poisson equation $\Delta A(x)=a(x)$. Such weight functions were firstly introduced by Ikehata and Tanizawa [36] and Ikehata [32] for damped wave equations. Some generalizations of the principal part to variable coefficients were made by Radu, Todorova, and Yordanov [71, 72]. The assumption of the radial symmetry of $a(x)$ was relaxed by Sobajima and the author [81]. Moreover, in [83, 84], the compactness assumption on the support of the initial data was removed and polynomially decaying data were treated. The point is the use of a suitable supersolution of the corresponding heat equation

$$a(x)\partial_t v-\Delta v=0$$

having polynomial order in the far field. This approach is also a main tool in this paper. For the diffusion phenomenon, we refer the reader to [40, 68, 73, 74, 80, 82, 92].

When the damping coefficient is critical for the energy decay, the situation becomes more delicate. Ikehata, Todorova, and Yordanov [38] studied (1.7) in the case where $\Omega=\mathbb{R}^n$ $(n\ge 3)$, $a(x)$ satisfies $a_0\langle x\rangle^{-1}\le a(x)\le a_1\langle x\rangle^{-1}$ with some

$a_0, a_1 > 0$, and the initial data has compact support. They obtained the decay estimates

$$E_L(t) = \begin{cases} O(t^{-a_0}) & (1 < a_0 < n), \\ O(t^{-n+\delta}) & (a_0 \ge n) \end{cases}$$

as $t \to \infty$ with arbitrary small $\delta > 0$. This indicates that the decay rate depends on the constant a_0. Similar results in the lower dimensional cases and the optimality of the above estimates under additional assumptions were also obtained in [38].

We also mention that $a(x)$ is not necessarily positive everywhere. It is known that the so-called geometric control condition (GCC) introduced by Rauch and Taylor [75] and Bardos, Lebeau, and Rauch [2] is sufficient for the energy decay of solutions with initial data in the energy space. For the problem (1.7) with $\Omega = \mathbb{R}^n$, (GCC) is read as follows: There exist constants $T > 0$ and $c > 0$ such that for any $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$, we have

$$\frac{1}{T}\int_0^T a(x_0 + s\xi_0)\, ds \ge c.$$

For this and related topics, we refer the reader to [1, 5, 9, 29, 45, 58, 67, 68, 101]. We note that for $a(x) = \langle x\rangle^{-\alpha}$ with $\alpha > 0$, (GCC) is not fulfilled.

We note that for the linear wave equation with time-dependent damping

$$\partial_t^2 u - \Delta u + b(t)\partial_t u = 0,$$

the asymptotic behavior of the solution can be classified depending on the behavior of $b(t)$. See [93, 94, 95, 96, 97, 98].

Thirdly, we consider the semilinear problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = f(u), & t > 0, x \in \Omega, \\ u(t,x) = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{1.8}$$

When $f(u) = |u|^{p-1}u$ or $\pm|u|^p$ with $p > 1$, the nonlinearity works as a sourcing term and it may cause the singularity of the solution in a finite time. In this case, it is known that there exists the critical exponent $p_F(n) = 1 + \frac{2}{n}$, that is, if $p > p_F(n)$, then (1.8) admits the global solution for small initial data; if $p < p_F(n)$, then the solution may blow up in finite time even for the small initial data. The number $p_F(n)$ is the so-called Fujita critical exponent named after the pioneering work by Fujita [10] for the semilinear heat equation.

When $\Omega = \mathbb{R}^n$ and $f(u) = \pm|u|^p$, Todorova and Yordanov [87] determined the critical exponent for compactly supported initial data. Later on, Zhang [100] and Kirane and Qafsaoui [46] proved that the critical case $p = p_F(n)$ belongs to the blow-up case.

There are many improvements and related studies to the results above. The compactness assumption of the support of the initial data were removed by [13, 20, 21, 36, 60]. The diffusion phenomenon for the global solution was proved by [11, 13, 42, 43]. The case where Ω is the half space or the exterior domain was studied by [24, 26, 30, 31, 69, 70, 77] Also, estimates of lifespan for blowing-up solutions were obtained by [48, 49, 62, 27, 22, 24, 23].

When $f(u) = |u|^{p-1}u$, the global existence part can be proved completely the same way as in the case $f(u) = \pm|u|^p$. However, regarding the blow-up of solutions, the same proof as before works only for $n \le 3$, since the fundamental solution of the linear damped wave equation is not positive for $n \geq 4$, which follows from the explicit formula of the linear wave equation (see e.g., [76, p.1011]). Ikehata and Ohta [35] obtained the blow-up of solutions for the subcritical case $p < p_F(n)$. The critical case $p = p_F(n)$ with $n \geq 4$ seems to remain open.

When $f(u) = -|u|^{p-1}u$ with $p > 1$, the nonlinearity works as an absorbing term. In this case with $\Omega = \mathbb{R}^n$, Kawashima, Nakao, and Ono [44] proved the large data global existence. Moreover, decay estimates of solutions were obtained for $p > 1 + \frac{4}{n}$. Later on, Nishihara and Zhao [65] and Ikahata, Nishihara, and Zhao [34] studied the case $1 < p \leq 1 + \frac{4}{n}$. From their results, we have the energy estimate

$$(1+t)E[u](t) + \|u(t)\|_{L^2}^2 \leq C(I_0)(1+t)^{-2\left(\frac{1}{p-1}-\frac{n}{4}\right)}, \tag{1.9}$$

where

$$I_0 := \int_{\mathbb{R}^n} \left(|u_1(x)|^2 + |\nabla u_0(x)|^2 + |u_0(x)|^{p+1} + |u_0(x)|^2\right) \langle x \rangle^{2m} \, dx, \quad m > 2\left(\frac{1}{p-1} - \frac{n}{4}\right)$$

and we recall that $E[u](t)$ is defined by (1.2). Also, the asymptotic behavior was discussed by [41, 12, 15, 16, 34, 63]. There seems no result for exterior domain cases.

Finally, we consider the semilinear problem with space-dependent damping which is slightly more general than (1.1):

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = f(u), & t > 0, x \in \Omega, \\ u(t,x) = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

When the nonlinearity works as a sourcing term, we expect that there is the critical exponent as in the case $a(x) \equiv 1$. Indeed, in the case where $\Omega = \mathbb{R}^n$, $f(u) = \pm|u|^p$, the initial data has compact support, and $a(x)$ is positive, radial, and satisfies $a(x) = a_0|x|^{-\alpha} + o(|x|^{-\alpha})$ $(|x| \to \infty)$ with $\alpha \in [0,1)$, Ikehata, Todorova, and Yordanov [37] determined the critical exponent as $p_F(n-\alpha) = 1 + \frac{2}{n-\alpha}$. The estimate of lifespan for blowing-up solutions was obtained in [24, 27]. The blow-up of solutions for the case $f(u) = |u|^{p-1}u$ seems to be an open problem.

Recently, Sobajima [79] studied the critical damping case $a(x) = a_0|x|^{-1}$ in an exterior domain Ω with $n \geq 3$, and proved the small data global existence of solutions under the conditions $a_0 > n-2$ and $p > 1 + \frac{4}{n-2+\min\{n,a_0\}}$. The blow-up part was investigated by [25, 50, 79]. In particular, when Ω is the outside a ball with $n \geq 3$, $a_0 \geq n$, and $f(u) = \pm|u|^p$, the critical exponent is determined as $p = p_F(n-1)$. Moreover, in Ikeda and Sobajima [25], the blow-up of solutions was obtained for $\Omega = \mathbb{R}^n$ $(n \geq 3)$, $0 \leq a_0 < \frac{(n-1)^2}{n+1}$, $f(u) = \pm|u|^p$ with $\frac{n}{n-1} < p \leq p_S(n+a_0)$, where $p_S(n)$ is the positive root of the quadratic equation

$$2 + (n+1)p - (n-1)p^2 = 0$$

and is the so-called Strauss exponent. We remark that $p_S(n+a_0) > p_F(n-1)$ holds if $a_0 < \frac{(n-1)^2}{n+1}$. From this, we can expect that the critical exponent changes depending on the value a_0.

For the absorbing nonlinear term $f(u) = -|u|^{p-1}u$ in the whole space case $\Omega = \mathbb{R}^n$ was studied by Todorova and Yordanov [88] and Nishihara [64]. In [64], for compactly supported initial data, the following two results were proved:

(i) If $a(x) = a_0 \langle x \rangle^{-\alpha}$ with some $a_0 > 0$ and $\alpha \in [0,1)$, then we have

$$(1+t)E[u](t) + \int_{\mathbb{R}^n} a(x)|u(t,x)|^2\,dx \le C(1+t)^{-\frac{n-\alpha}{2-\alpha}+\delta}$$

with arbitrary small $\delta > 0$;

(ii) If $a_0\langle x \rangle^{-\alpha} \le a(x) \le a_1 \langle x \rangle^{-\alpha}$ with some $a_0, a_1 > 0$ and $\alpha \in [0,1)$, then we have

$$(1+t)E[u](t) + \int_{\mathbb{R}^n} a(x)|u(t,x)|^2\,dx \le C \begin{cases} (1+t)^{-\frac{4}{2-\alpha}\left(\frac{1}{p-1}-\frac{n-\alpha}{4}\right)} & (p > p_{subc}(n,\alpha)), \\ (1+t)^{-\frac{2}{p-1}}\log(2+t) & (p = p_{subc}(n,\alpha)), \\ (1+t)^{-\frac{2}{p-1}} & (p < p_{subc}(n,\alpha)), \end{cases}$$

where

$$p_{subc}(n,\alpha) := 1 + \frac{2\alpha}{n-\alpha}. \tag{1.10}$$

We note that the decay rate in (i) is the same as that of the linear problem (1.7) and it is better than that of (ii) if $p > p_F(n-\alpha)$. This means $p_F(n-\alpha)$ is critical in the sense of the effect of the nonlinearity to the decay rate of the energy. Moreover, (ii) shows that the second critical exponent $p_{subc}(n,\alpha)$ appears and it divides the decay rate of the energy. We also note that the estimate for the case $p > p_{subc}(n,\alpha)$ corresponds to the estimate (1.9). Thus, we may interpret the situation in the following way: When the damping is weak in the sense of $a(x) \sim \langle x \rangle^{-\alpha}$ with $\alpha \in (0,1)$, we cannot obtain the same type energy estimate as in (1.9) for all $p > 1$, and the decay rate becomes worse under or on the second critical exponent $p_{subc}(n,\alpha)$. Our main goal in this paper is to give a generalization of the results (i) and (ii) above.

In recent years, semilinear wave equations with time-dependent damping have been intensively studied. For the progress of this problem, we refer the reader to Sections 1 and 2 in Lai, Schiavone, and Takamura [47]. We also refer to [66] and the references therein for a recent study of semilinear wave equations with time and space dependent damping.

To state our results, we define the solution.

Definition 1.1 (Mild and strong solutions)**.** *Let $\mathcal{A}$ be the operator*

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -a(x) \end{pmatrix}$$

defined on $\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega)$ with the domain $D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Let $U(t)$ denote the C_0-semigroup generated by $\mathcal{A}$. Let $(u_0, u_1) \in \mathcal{H}$ and $T \in (0,\infty]$. A function

$$u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$$

is called a mild solution of (1.1) *on* $[0,T)$ *if $\mathcal{U} = {}^t(u, \partial_t u)$ satisfies the integral equation*

$$\mathcal{U}(t) = U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u|^{p-1}u \end{pmatrix} ds$$

in $C([0,T);\mathcal{H})$*. Moreover, when* $(u_0,u_1)\in D(\mathcal{A})$*, a function*

$$u\in C([0,T);H^2(\Omega))\cap C^1([0,T);H^1_0(\Omega))\cap C^2([0,T);L^2(\Omega))$$

is said to be a strong solution of (1.1) *on* $[0,T)$ *if* u *satisfies the equation of* (1.1) *in* $C([0,T);L^2(\Omega))$*. If* $T=\infty$*, we call* u *a global (mild or strong) solution.*

First, we prepare the existence and regularity of the global solution.

Proposition 1.2. *Let* $\Omega=\mathbb{R}^n$ *with* $n\ge 1$*, or* $\Omega\subset\mathbb{R}^n$ *with* $n\ge 2$ *be an exterior domain with* C^2*-boundary. Let* $a(x)\in C(\mathbb{R}^n)$ *be nonnegative and bounded. Let*

$$1<p<\infty\ (n=1,2),\quad 1<p\le\frac{n}{n-2}\ (n\ge 3),\tag{1.11}$$

and let $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$*. Then, there exists a unique global mild solution* u *to* (1.1)*. If we further assume* $(u_0,u_1)\in(H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$*, then* u *becomes a strong solution to* (1.1)*.*

Remark 1.3. *The assumption* $\partial\Omega\in C^2$ *is used to ensure* $D(\mathcal{A})=(H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$ *(see Cazenave and Haraux* [6, Remark 2.6.3] *and Brezis* [4, Theorem 9.25]*). The restriction of the range of* p *in* (1.11) *is due to the use of Gagliardo–Nirenberg inequality (see Section A.2).*

The proof of Proposition 1.2 is standard. However, for reader's convenience, we will give an outline of the proof in the appendix.

To state our result, we recall that $E[u](t)$ and $p_{subc}(n,\alpha)$ are defined by (1.2) and (1.10), respectively. The main result of this paper reads as follows.

Theorem 1.4. *Let* $\Omega=\mathbb{R}^n$ *with* $n\ge 1$ *or* $\Omega\subset\mathbb{R}^n$ *with* $n\ge 2$ *be an exterior domain with* C^2*-boundary. Let* p *satisfy* (1.11) *and* $(u_0,u_1)\in H^1_0(\Omega)\times L^2(\Omega)$*, and let* u *be the corresponding global mild solution of* (1.1)*. Then, the followings hold.*

(i) *Assume that* $a\in C(\mathbb{R}^n)$ *is positive and satisfies*

$$\lim_{|x|\to\infty}|x|^\alpha a(x)=a_0\tag{1.12}$$

with some constants $\alpha\in[0,1)$ *and* $a_0>0$*. Moreover, we assume that the initial data satisfy*

$$\begin{aligned}&I_0[u_0,u_1]\\&:=\int_\Omega\left[(|u_1(x)|^2+|\nabla u_0(x)|^2+|u_0(x)|^{p+1})\langle x\rangle^\alpha+|u_0(x)|^2\langle x\rangle^{-\alpha}\right]\langle x\rangle^{\lambda(2-\alpha)}\,dx\\&<\infty\end{aligned}\tag{1.13}$$

with some $\lambda\in[0,\frac{n-\alpha}{2-\alpha})$*. Then, we have*

$$(1+t)E[u](t)+\int_\Omega a(x)|u(t,x)|^2\,dx\le CI_0[u_0,u_1](1+t)^{-\lambda}$$

for $t\ge 0$ *with some constant* $C=C(n,a,p,\lambda)>0$*.*

(ii) *Assume that* $a\in C(\mathbb{R}^n)$ *is positive and satisfies*

$$a_0\langle x\rangle^{-\alpha}\le a(x)\le a_1\langle x\rangle^{-\alpha}$$

with some constants $\alpha \in [0,1)$, $a_0, a_1 > 0$. Moreover, we assume that the initial data satisfy the condition $I_0[u_0,u_1] < \infty$ with some $\lambda \in [0,\infty)$, where $I_0[u_0,u_1]$ is defined by (1.13). Then, we have

$$
\begin{aligned}
&(1+t)E[u](t) + \int_\Omega a(x)|u(t,x)|^2\,dx \\
&\le C(I_0[u_0,u_1]+1) \\
&\quad\times \begin{cases}
(1+t)^{-\lambda} & (\lambda < \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\}),\\
(1+t)^{-\lambda}\log(2+t) & (\lambda = \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\},\ p \ne p_{subc}(n,\alpha)),\\
(1+t)^{-\lambda}(\log(2+t))^2 & (\lambda = \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}) = \frac{2}{p-1},\ \text{i.e., } p = p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4})} & (\lambda > \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}),\ p > p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{2}{p-1}}\log(2+t) & (\lambda > \frac{2}{p-1},\ p = p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{2}{p-1}} & (\lambda > \frac{2}{p-1},\ p < p_{subc}(n,\alpha))
\end{cases}
\end{aligned}
$$

for $t \ge 0$ with some constant $C = C(n,a,p,\lambda) > 0$.

Remark 1.5. *Under the assumptions of (i), the both conclusions of (i) and (ii) are true. In Figure 1, the decay rates of $\int_\Omega a(x)|u(t,x)|^2\,dx$ is classified in the case $(n,\alpha) = (3, 0.5)$ (for ease of viewing, the figure is multiplied by 7 and 0.75 in the horizontal and vertical axis, respectively).*

Remark 1.6. *From the proof of the above theorem, we also have the following estimates for the L^2-norm of u without the weight $a(x)$: Under the assumptions on (i) with $\lambda \in [\frac{\alpha}{2-\alpha}, \frac{n-\alpha}{2-\alpha})$, we have*

$$\int_\Omega |u(t,x)|^2\,dx \le C(1+t)^{-\lambda+\frac{\alpha}{2-\alpha}}$$

for $t > 0$; Under the assumptions on (ii) with $\lambda \in [\frac{\alpha}{2-\alpha}, \infty)$, we have

$$
\begin{aligned}
&\int_\Omega |u(t,x)|^2\,dx \\
&\le C \begin{cases}
(1+t)^{-\lambda+\frac{\alpha}{2-\alpha}} & (\lambda < \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\}),\\
(1+t)^{-\lambda+\frac{\alpha}{2-\alpha}}\log(2+t) & (\lambda = \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\},\ p \ne p_{subc}(n,\alpha)),\\
(1+t)^{-\lambda+\frac{\alpha}{2-\alpha}}(\log(2+t))^2 & (\lambda = \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}) = \frac{2}{p-1},\ \text{i.e., } p = p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4})+\frac{\alpha}{2-\alpha}} & (\lambda > \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}),\ p > p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{2}{p-1}+\frac{\alpha}{2-\alpha}}\log(2+t) & (\lambda > \frac{2}{p-1},\ p = p_{subc}(n,\alpha)),\\
(1+t)^{-\frac{2}{p-1}+\frac{\alpha}{2-\alpha}} & (\lambda > \frac{2}{p-1},\ p < p_{subc}(n,\alpha))
\end{cases}
\end{aligned}
$$

for $t > 0$.

Remark 1.7. *(i) Theorem 1.4 generalizes the result of Nishihara* [64] *to the exterior domain, general damping coefficient $a(x)$ satisfying* (1.12)*, and polynomially decaying initial data satisfying* (1.13).

(ii) For the simplest case $\Omega = \mathbb{R}^n$ and $a(x) \equiv 1$, the result of Theorem 1.4 (ii) extends that of Ikehata, Nishihara, and Zhao [34]*, in the sense that our estimate in the region $\lambda > 2\left(\frac{1}{p-1} - \frac{n}{4}\right)$ coincides with their estimate* (1.9)*. Moreover, the*

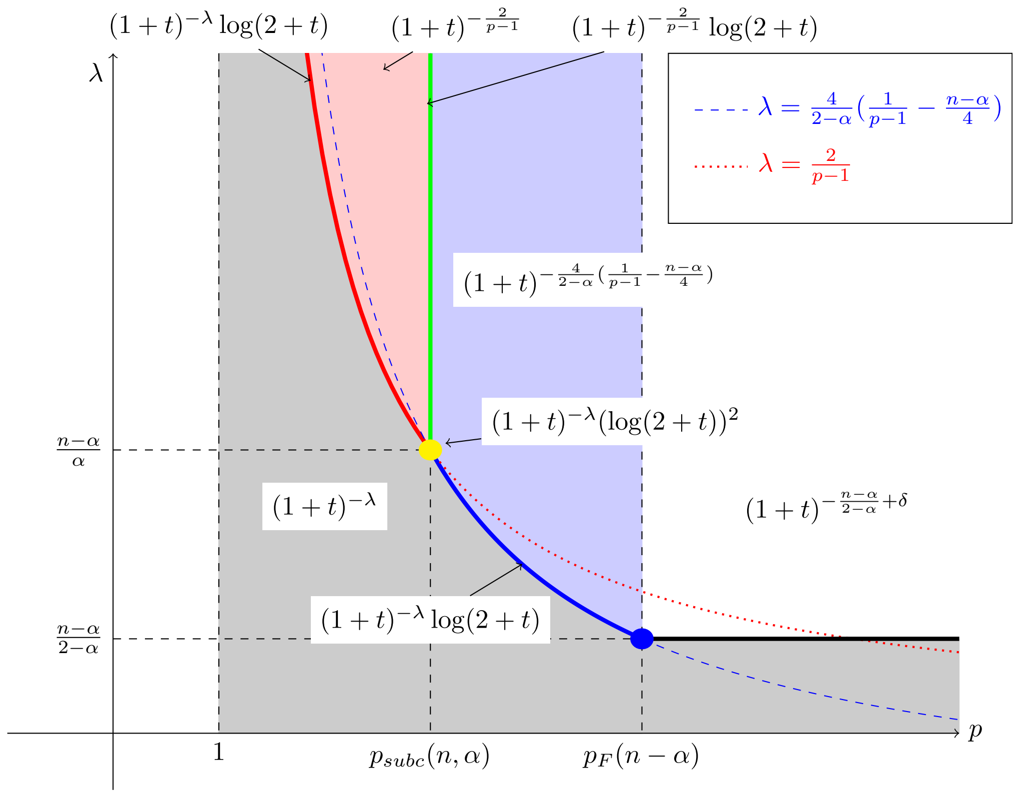

FIGURE 1. Classification of decay rates in p-λ plane when $(n, \alpha) = (3, \frac{1}{2})$

result of Theorem 1.4 (i) in the case $p > p_F(n)$ is better than the estimate obtained in [34]*. Hence, our result still has a novelty.*

Remark 1.8. *The optimality of the decay rates in Theorem 1.4 is an open problem. We expect that the estimate in the case (i) is optimal if $p > p_F(n-\alpha) = 1 + \frac{2}{n-\alpha}$, since the decay rate is the same as that of the linear problem* (1.7) *obtained by* [84]*. On the other hand, in the critical case $p = p_F(n-\alpha)$, the estimates in Theorem 1.4 will be improved in view of the known results* [15, 16] *for the classical damping* (1.8) *in the whole space. Moreover, the optimality in the subcritical case $p < p_F(n-\alpha)$ is a difficult problem even when $a(x) \equiv 1$ and $\Omega = \mathbb{R}^n$, and we have no idea so far.*

The strategy of the proof of Theorem 1.4 is as follows. For the both parts (i) and (ii), we apply the weighted energy method. The difficulty is how to estimate the weighted L^2-norm of the solution. To overcome it, we take different approaches for (i) and (ii). First, for the part (i), we apply the weighted energy method developed by [83, 84]. We shall use a suitable supersolution of the corresponding heat equation $a(x)\partial_t v - \Delta v = 0$ as the weight function. Next, for the part (ii), we shall use the same type weight function as in Ikehata, Nishihara, and Zhao [34] with a modification to fit the space-dependent damping case. In this case the absorbing semilinear term helps to estimate the weighted L^2-norm of the solution.

The rest of the paper is organized in the following way. In the next section, we prepare the definitions and properties of the weight functions used in the proof. Sections 3 and 4 are devoted to the proof of Theorem 1.4 (i) and (ii), respectively. In Appendix A, we give a proof of Proposition 1.2. Finally, in Appendix B, we prove the properties of weight functions stated in Section 2.

We end up this section with introducing notations used throughout this paper. The letter C indicates a generic positive constant, which may change from line to line. In particular, $C(*,\cdots,*)$ denotes a constant depending only on the quantities in the parentheses. For $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, we define $\langle x\rangle=\sqrt{1+|x|^2}$. We sometimes use $B_R(x_0)=\{x\in\mathbb{R}^n; |x-x_0|<R\}$ for $R>0$ and $x_0\in\mathbb{R}^n$.

Let $L^p(\Omega)$ be the usual Lebesgue space equipped with the norm

$$\|f\|_{L^p}=\begin{cases}\left(\displaystyle\int_\Omega |f(x)|^p\,dx\right)^{1/p} & (1<p<\infty),\\ \operatorname*{ess\,sup}_{x\in\Omega}|f(x)| & (p=\infty).\end{cases}$$

In particular, $L^2(\Omega)$ is a Hilbert space with the innerproduct

$$(f,g)_{L^2}:=\int_\Omega f(x)g(x)\,dx.$$

Let $H^k(\Omega)$ with a nonnegative integer k be the Sobolev space equipped with the innerproduct and the norm

$$(f,g)_{H^k}=\sum_{|\alpha|\le k}(\partial^\alpha f,\partial^\alpha g)_{L^2},\quad \|f\|_{H^k}=\sqrt{(f,f)_{H^k}},$$

respectively. $C_0^\infty(\Omega)$ denotes the space of smooth functions on Ω with compact support. $H_0^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^k}$. For an interval $I\subset\mathbb{R}$, a Banach space X, and a nonnegative integer k, $C^k(I;X)$ stands for the space of k-times continuously differentiable functions from I to X.

2. Preliminaries

In this section, we prepare weight functions for the weighted energy method used in the proof of Theorem 1.4.

These lemmas were shown in [77, 81, 83, 84], however, for the convenience, we give a proof of them in the appendix.

Following [81], we first take a suitable approximate solution of the Poisson equation $\Delta A(x)=a(x)$, which will be used for the construction of the weight function.

Lemma 2.1 ([81, 84])**.** *Assume that $a(x)\in C(\mathbb{R}^n)$ is positive and satisfies the condition $\lim_{|x|\to\infty}|x|^\alpha a(x)=a_0$ with some constants $\alpha\in(-\infty,\min\{2,n\})$ and $a_0>0$. Let $\varepsilon\in(0,1)$. Then, there exist a function $A_\varepsilon\in C^2(\mathbb{R}^n)$ and positive constants $c=c(n,a,\varepsilon)$ and $C=C(n,a,\varepsilon)$ such that for $x\in\mathbb{R}^n$, we have*

$$(1-\varepsilon)a(x)\le\Delta A_\varepsilon(x)\le(1+\varepsilon)a(x), \tag{2.1}$$

$$c\langle x\rangle^{2-\alpha}\le A_\varepsilon(x)\le C\langle x\rangle^{2-\alpha}, \tag{2.2}$$

$$\frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)}\le\frac{2-\alpha}{n-\alpha}+\varepsilon. \tag{2.3}$$

For the construction of our weight function, we also need the following Kummer's confluent hypergeometric function.

Definition 2.2 (Kummer's confluent hypergeometric functions)**.** *For $b, c \in \mathbb{R}$ with $-c \notin \mathbb{N} \cup \{0\}$, Kummer's confluent hypergeometric function of first kind is defined by*

$$M(b,c;s) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{s^n}{n!}, \quad s \in [0,\infty),$$

where $(d)_n$ is the Pochhammer symbol defined by $(d)_0 = 1$ and $(d)_n = \prod_{k=1}^{n}(d+k-1)$ for $n \in \mathbb{N}$; note that when $b = c$, $M(b,b;s)$ coincides with e^s.

For $\varepsilon \in (0,1/2)$, we define

$$\widetilde{\gamma}_\varepsilon = \left(\frac{2-\alpha}{n-\alpha} + 2\varepsilon \right)^{-1}, \quad \gamma_\varepsilon = (1-2\varepsilon)\widetilde{\gamma}_\varepsilon. \tag{2.4}$$

Definition 2.3. *For $\beta \in \mathbb{R}$, define*

$$\varphi_{\beta,\varepsilon}(s) = e^{-s} M\left(\gamma_\varepsilon - \beta, \gamma_\varepsilon; s\right), \quad s \geq 0.$$

Since $M(\gamma_\varepsilon, \gamma_\varepsilon, s) = e^s$, we remark that $\varphi_{0,\varepsilon}(s) \equiv 1$. Roughly speaking, if we formally take $\varepsilon = 0$, then $\{\varphi_{\beta,0}\}_{\beta\in\mathbb{R}}$ gives a family of self-similar profiles of the equation $|x|^{-\alpha}\partial_t v = \Delta v$ with the parameter β. See [83] for more detailed explanation. The next lemma states basic properties of $\varphi_{\beta,\varepsilon}$.

Lemma 2.4. *The function $\varphi_{\beta,\varepsilon}$ defined in Definition 2.3 satisfies the following properties.*

(i) *$\varphi_{\beta,\varepsilon}(s)$ satisfies the equation*

$$s\varphi''(s) + (\gamma_\varepsilon + s)\varphi'(s) + \beta\varphi(s) = 0. \tag{2.5}$$

(ii) *If $0 \leq \beta < \gamma_\varepsilon$, then $\varphi_{\beta,\varepsilon}(s)$ satisfies the estimates*

$$k_{\beta,\varepsilon}(1+s)^{-\beta} \leq \varphi_{\beta,\varepsilon}(s) \leq K_{\beta,\varepsilon}(1+s)^{-\beta}$$

with some constants $k_{\beta,\varepsilon}, K_{\beta,\varepsilon} > 0$.

(iii) *For every $\beta \geq 0$, $\varphi_{\beta,\varepsilon}(s)$ satisfies*

$$|\varphi_{\beta,\varepsilon}(s)| \leq K_{\beta,\varepsilon}(1+s)^{-\beta}$$

with some constant $K_{\beta,\varepsilon} > 0$.

(iv) *For every $\beta \in \mathbb{R}$, $\varphi_{\beta,\varepsilon}(s)$ and $\varphi_{\beta+1,\varepsilon}(s)$ satisfy the recurrence relation*

$$\beta\varphi_{\beta,\varepsilon}(s) + s\varphi'_{\beta,\varepsilon}(s) = \beta\varphi_{\beta+1,\varepsilon}(s).$$

(v) *For every $\beta \in \mathbb{R}$, we have*

$$\varphi'_{\beta,\varepsilon}(s) = -\frac{\beta}{\gamma_\varepsilon} e^{-s} M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s),$$
$$\varphi''_{\beta,\varepsilon}(s) = \frac{\beta(\beta+1)}{\gamma_\varepsilon(\gamma_\varepsilon+1)} e^{-s} M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 2; s).$$

In particular, if $0 < \beta < \gamma_\varepsilon$, then $\varphi'_{\beta,\varepsilon}(s)$ and $\varphi''_{\beta,\varepsilon}(s)$ satisfy

$$-K_{\beta,\varepsilon}(1+s)^{-\beta-1} \leq \varphi'_{\beta,\varepsilon}(s) \leq -k_{\beta,\varepsilon}(1+s)^{-\beta-1},$$
$$k_{\beta,\varepsilon}(1+s)^{-\beta-2} \leq \varphi''_{\beta,\varepsilon}(s) \leq K_{\beta,\varepsilon}(1+s)^{-\beta-2}$$

with some constants $k_{\beta,\varepsilon}, K_{\beta,\varepsilon} > 0$.

Finally, we define the weight function which will be used for our energy method.

Definition 2.5. *For $\beta \in \mathbb{R}$ and $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we define*

$$\Phi_{\beta,\varepsilon}(x,t;t_0) = (t_0+t)^{-\beta}\varphi_{\beta,\varepsilon}(z), \quad z = \frac{\widetilde{\gamma}_\varepsilon A_\varepsilon(x)}{t_0+t},$$

where $\varepsilon \in (0,1/2)$, $\widetilde{\gamma}_\varepsilon$ is the constant given in (2.4), $t_0 \ge 1$, $\varphi_{\beta,\varepsilon}$ is the function defined by Definition 2.3, and $A_\varepsilon(x)$ is the function constructed in Lemma 2.1.

Since $\varphi_{0,\varepsilon}(s) \equiv 1$, we again remark that $\Phi_{0,\varepsilon}(x,t;t_0) \equiv 1$.
For $t_0 \ge 1$, $t > 0$, and $x \in \mathbb{R}^n$, we also define

$$\Psi(x,t;t_0) := t_0 + t + A_\varepsilon(x). \tag{2.6}$$

Proposition 2.6. *The function $\Phi_{\beta,\varepsilon}(x,t;t_0)$ satisfies the following properties:*

(i) *For every $\beta \ge 0$, we have*

$$\partial_t \Phi_{\beta,\varepsilon}(x,t;t_0) = -\beta\Phi_{\beta+1,\varepsilon}(x,t;t_0).$$

(ii) *If $\beta \ge 0$, then there exists a constant $C = C(n,\alpha,\beta,\varepsilon) > 0$ such that*

$$|\Phi_{\beta,\varepsilon}(x,t;t_0)| \le C\Psi(x,t;t_0)^{-\beta}$$

for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

(iii) *If $0 \le \beta < \gamma_\varepsilon$, then there exists a constant $c = c(n,\alpha,\beta,\varepsilon) > 0$ such that*

$$\Phi_{\beta,\varepsilon}(x,t;t_0) \ge c\Psi(x,t;t_0)^{-\beta}$$

for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

(iv) *For $\beta > 0$, there exists a constant $c = c(n,\alpha,\beta,\varepsilon) > 0$ such that*

$$a(x)\partial_t\Phi_{\beta,\varepsilon}(x,t;t_0) - \Delta\Phi_{\beta,\varepsilon}(x,t;t_0) \ge ca(x)\Psi(x,t;t_0)^{-\beta-1}$$

for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

Finally, we prepare a useful lemma for our weighted energy method. The proof can be found in [83, Lemma 3.6] or [77, Lemma 2.5]. However, for the convenience, we give its proof in the appendix.

Lemma 2.7. *Let $\Omega = \mathbb{R}^n$ with $n \ge 1$ or $\Omega \subset \mathbb{R}^n$ with $n \ge 2$ be an exterior domain with C^2-boundary. Let $\Phi \in C^2(\overline{\Omega})$ be a positive function and let $\delta \in (0,1/2)$. Then, for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying* $\operatorname{supp} u \Subset B_R(0) = \{x \in \mathbb{R}^n; |x| < R\}$ *with some $R > 0$, we have*

$$\int_\Omega (u\Delta u)\,\Phi^{-1+2\delta}\,dx \le -\frac{\delta}{1-\delta}\int_\Omega |\nabla u|^2\Phi^{-1+2\delta}\,dx + \frac{1-2\delta}{2}\int_\Omega u^2(\Delta\Phi)\Phi^{-2+2\delta}\,dx.$$

3. Proof of Theorem 1.4: first part

In this section, we prove Theorem 1.4 (i). First, we note that Proposition 1.2 implies the existence of the global mild solution u.

Following the argument in Sobajima [79], we first prove Theorem 1.4 (i) in the case of compactly supported initial data, and after that, we will treat the general case by an approximation argument.

3.1. **Proof for the compactly supported initial data.** We first consider the case where the initial data are compactly supported, that is, we assume that $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B_{R_0}(0) = \{x \in \mathbb{R}^n; |x| < R_0\}$. Then, by the finite propagation property (see Section A.2.7), the corresponding mild solution u satisfies $\operatorname{supp} u(t,\cdot) \subset B_{R_0+t}(0)$.

Let $T_0 > 0$ be arbitrary fixed and let $T \in (0, T_0)$. Then, we have $\operatorname{supp} u(t,\cdot) \subset B_{R_0+T_0}(0)$ for all $t \in [0,T]$. Let $D = \Omega \cap B_{R_0+T_0}(0)$. Then, for $t \in [0,T]$, we can convert the problem (1.1) to the problem in the bounded domain

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u + |u|^{p-1}u = 0, & t \in (0,T], x \in D, \\ u(t,x) = 0, & t \in (0,T], x \in \partial D, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in D \end{cases}$$

with $(u_0, u_1) \in \mathcal{H}_D := H_0^1(D) \times L^2(D)$.

Let $\mathcal{A}_D$ be the operator

$$\mathcal{A}_D = \begin{pmatrix} 0 & 1 \\ \Delta & -a(x) \end{pmatrix}$$

defined on $\mathcal{H}_D$ with the domain $D(\mathcal{A}_D) = (H^2(D) \cap H_0^1(D)) \times H_0^1(D)$. Then, from the argument in Section A.1, there exists $\lambda_* > 0$ such that for any $\lambda > \lambda_*$, the resolvent $J_\lambda = (I - \lambda^{-1}\mathcal{A}_D)^{-1}$ is defined as a bounded operator on $\mathcal{H}_D$. Take a sequence $\{\lambda_j\}_{j=1}^\infty$ such that $\lambda_j > \lambda_*$ for $j \ge 1$ and $\lim_{j\to\infty} \lambda_j = \infty$, and define

$$\begin{pmatrix} u_0^{(j)} \\ u_1^{(j)} \end{pmatrix} := J_{\lambda_j} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Then, we have

$$(u_0^{(j)}, u_1^{(j)}) \in D(\mathcal{A}_D), \quad \lim_{j\to\infty} (u_0^{(j)}, u_1^{(j)}) = (u_0, u_1) \text{ in } \mathcal{H}_D \tag{3.1}$$

(see e.g. the proof of [19, Theorem 2.18]). Therefore, Proposition 1.2 shows that the mild solution $u^{(j)}$ corresponding to the initial data $(u_0^{(j)}, u_1^{(j)})$ becomes a strong solution. Moreover, the continuous dependence on the initial data (see Section A.2.4) implies

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \|(u^{(j)}(t), \partial_t u^{(j)}(t)) - (u(t), \partial_t u(t))\|_{\mathcal{H}_D} = 0.$$

This means that, if we prove the conclusion of Theorem 1.4 (i) for $u^{(j)}$, that is,

$$(1+t)E[u^{(j)}](t) + \int_\Omega a(x)|u^{(j)}(t,x)|^2\,dx \le C I_0[u_0^{(j)}, u_1^{(j)}](1+t)^{-\lambda}$$

for $t \in [0,T]$, where the constant C is independent of j, T, T_0, R_0, then letting $j \to \infty$ and also using the Sobolev embedding $\|u\|_{L^{p+1}(D)} \le C\|u\|_{H^1(D)}$, we have the same estimate for the original mild solution u. Note that (3.1) implies $\lim_{j\to\infty} I_0[u_0^{(j)}, u_1^{(j)}] = I_0[u_0, u_1]$, since the integral is taken over the bounded region D. Finally, since T and T_0 are arbitrary and C is independent of them, we obtain the desired energy estimate for any $t \ge 0$.

Therefore, in the following argument, we may further assume $(u_0, u_1) \in D(\mathcal{A}_D)$ and u is the strong solution. This enables us to justify all the computations in this section.

In what follows, we shall use the weight functions $\Phi_{\beta,\varepsilon}(x,t;t_0)$ and $\Psi(x,t;t_0)$ defined by Definition 2.5 and (2.6), respectively. We also recall that the constant γ_ε is given by (2.4). Then, we define the following energies.

Definition 3.1. *For a function $u=u(t,x)$, $\alpha\in[0,1)$, $\delta\in(0,1/2)$, $\varepsilon\in(0,1/2)$, $\lambda\in[0,(1-2\delta)\gamma_\varepsilon)$, $\beta=\lambda/(1-2\delta)$, $\nu>0$, and $t_0\ge 1$, we define*

$$E_1(t;t_0,\lambda)=\int_\Omega\left[\frac{1}{2}\left(|\partial_t u(t,x)|^2+|\nabla u(t,x)|^2\right)+\frac{1}{p+1}|u(t,x)|^{p+1}\right]\Psi(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}}\,dx,$$

$$\begin{aligned}
E_0(t;t_0,\lambda)&=\int_\Omega\left(2u(t,x)\partial_t u(t,x)+a(x)|u(t,x)|^2\right)\Phi_{\beta,\varepsilon}(t,x;t_0)^{-1+2\delta}\,dx,\\
E_*(t;t_0,\lambda,\nu)&=E_1(t;t_0,\lambda)+\nu E_0(t;t_0,\lambda),\\
\tilde{E}(t;t_0,\lambda)&=(t_0+t)\int_\Omega\left[\frac{1}{2}\left(|\partial_t u(t,x)|^2+|\nabla u(t,x)|^2\right)+\frac{1}{p+1}|u(t,x)|^{p+1}\right]\Psi(t,x;t_0)^{\lambda}\,dx
\end{aligned}$$

for $t\ge 0$.

Since

$$2u\partial_t u\le\frac{a(x)}{2}|u|^2+\frac{2}{a(x)}|\partial_t u|^2\le\frac{a(x)}{2}|u|^2+C\Psi^{\frac{\alpha}{2-\alpha}}|\partial_t u|^2 \tag{3.2}$$

and $\Phi_{\beta,\varepsilon}^{-1+2\delta}\le C\Psi^\lambda$ (see (2.2) and Proposition 2.6 (iii)), we see that there exists a small constant $\nu_0=\nu_0(n,a,\delta,\varepsilon,\lambda)>0$ such that for any $\nu\in(0,\nu_0)$,

$$E_*(t;t_0,\lambda,\nu)\ge\frac{1}{2}E_1(t;t_0,\lambda)+\frac{\nu}{2}\int_\Omega a(x)|u(t,x)|^2\Psi(t,x;t_0)^\lambda\,dx \tag{3.3}$$

holds.

We first prepare the following energy estimates for $E_1(t;t_0,\lambda)$ and $E_0(t;t_0,\lambda)$.

Lemma 3.2. *Under the assumptions on Theorem 1.4 (i), there exists $t_1=t_1(n,a,\lambda,\varepsilon)\ge 1$ such that for $t_0\ge t_1$ and $t>0$, we have*

$$\begin{aligned}
\frac{d}{dt}E_1(t;t_0,\lambda)\le&-\frac{1}{2}\int_\Omega a(x)|\partial_t u(t,x)|^2\Psi(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\
&+C\int_\Omega\left(|\nabla u(t,x)|^2+|u(t,x)|^{p+1}\right)\Psi(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx
\end{aligned}$$

with some constant $C=C(n,\alpha,p,\lambda)>0$.

Proof. Differentiating $E_1(t;t_0,\lambda)$, one has

$$\begin{aligned}
\frac{d}{dt}E_1(t;t_0,\lambda)=&\int_\Omega\left[\partial_t u\partial_t^2 u+\nabla u\cdot\nabla\partial_t u+|u|^{p-1}u\partial_t u\right]\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\
&+\left(\lambda+\frac{\alpha}{2-\alpha}\right)\int_\Omega\left[\frac{1}{2}\left(|\nabla u|^2+|\partial_t u|^2\right)+\frac{1}{p+1}|u|^{p+1}\right]\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx.
\end{aligned}$$

The integration by parts and the equation (1.1) imply

$$\begin{aligned}\frac{d}{dt}E_1(t;t_0,\lambda) &= -\int_\Omega a(x)|\partial_t u|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\ &\quad -\left(\lambda+\frac{\alpha}{2-\alpha}\right)\int_\Omega \partial_t u(\nabla u\cdot\nabla\Psi)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx\\ &\quad +\left(\lambda+\frac{\alpha}{2-\alpha}\right)\int_\Omega \left[\frac{1}{2}\left(|\nabla u|^2+|\partial_t u|^2\right)+\frac{1}{p+1}|u|^{p+1}\right]\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx.\end{aligned} \tag{3.4}$$

Let us estimate the right-hand side. First, the Schwarz inequality gives

$$\left|-\left(\lambda+\frac{\alpha}{2-\alpha}\right)\partial_t u(\nabla u\cdot\nabla\Psi)\right| \le \frac{a(x)}{4}|\partial_t u|^2\Psi + C|\nabla u|^2\frac{|\nabla\Psi|^2}{a(x)\Psi}.$$

Moreover, by (2.3), we have

$$\frac{|\nabla\Psi|^2}{a(x)\Psi} \le \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} \le \frac{2-\alpha}{n-\alpha}+\varepsilon. \tag{3.5}$$

Also, from the definition of Ψ, (2.2), and $a(x)\sim\langle x\rangle^{-\alpha}$, one obtains

$$\Psi(t,x;t_0)^{-1} \le t_0^{-1+\frac{\alpha}{2-\alpha}}A_\varepsilon(x)^{-\frac{\alpha}{2-\alpha}} \le Ct_0^{-\frac{2(1-\alpha)}{2-\alpha}}a(x). \tag{3.6}$$

Therefore, taking $t_1\ge 1$ sufficiently large, we have, for $t_0\ge t_1$,

$$\left(\lambda+\frac{\alpha}{2-\alpha}\right)\int_\Omega |\partial_t u|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx \le \frac{1}{4}\int_\Omega a(x)|\partial_t u|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx.$$

Using the above estimates to (3.4), we deduce

$$\begin{aligned}\frac{d}{dt}E_1(t;t_0,\lambda) &\le -\frac{1}{2}\int_\Omega a(x)|\partial_t u|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\ &\quad + C\int_\Omega \left(|\nabla u|^2+|u|^{p+1}\right)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx,\end{aligned}$$

which completes the proof. □

Lemma 3.3. *Under the assumptions on Theorem 1.4 (i), for $t_0\ge 1$ and $t>0$, we have*

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) &\le -\eta\int_\Omega \left(|\nabla u(t,x)|^2+|u(t,x)|^{p+1}\right)\Psi(t,x;t_0)^\lambda\,dx\\ &\quad + C\int_\Omega |\partial_t u(t,x)|^2\Psi(t,x;t_0)^\lambda\,dx\end{aligned}$$

with some positive constants $\eta=\eta(n,\alpha,\delta,\varepsilon,\lambda)$ and $C=C(n,\alpha,\delta,\varepsilon,\lambda)$.

Proof. Differentiating $E_0(t;t_0,\lambda)$ and using the equation (1.1) yield

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) &= \int_\Omega \left(2|\partial_t u|^2+2u\partial_t^2 u+2a(x)u\partial_t u\right)\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\\ &\quad -(1-2\delta)\int_\Omega \left(2u\partial_t u+a(x)|u|^2\right)\Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}\,dx.\end{aligned}$$

Using the equation (1.1), we have

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) &= 2\int_\Omega |\partial_t u|^2 \Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx + 2\int_\Omega u\Delta u \Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\\ &\quad - 2\int_\Omega |u|^{p+1}\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\\ &\quad - (1-2\delta)\int_\Omega \left(2u\partial_t u + a(x)|u|^2\right)\Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}\,dx.\end{aligned}$$

Applying Lemma 2.7 with $\Phi = \Phi_{\beta,\varepsilon}$ to the second term of the right-hand side, one obtains

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) &\le 2\int_\Omega |\partial_t u|^2 \Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx - \frac{2\delta}{1-\delta}\int_\Omega |\nabla u|^2 \Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\\ &\quad - 2\int_\Omega |u|^{p+1}\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\\ &\quad - 2(1-2\delta)\int_\Omega u\partial_t u \Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}\,dx\\ &\quad - (1-2\delta)\int_\Omega |u|^2\Phi_{\beta,\varepsilon}^{-2+2\delta}\left(a(x)\partial_t\Phi_{\beta,\varepsilon} - \Delta\Phi_{\beta,\varepsilon}\right)\,dx. \qquad (3.7)\end{aligned}$$

Next, we estimate the terms in the right-hand side. First, we remark that if $\lambda = 0$ (i.e., $\beta = 0$), then the last two terms in (3.7) vanish, since $\Phi_{\beta,\varepsilon} \equiv 1$. For the case $\beta > 0$, by Proposition 2.6 (ii) and (iv), we have

$$\int_\Omega |u|^2\Phi_{\beta,\varepsilon}^{-2+2\delta}\left(a(x)\partial_t\Phi_{\beta,\varepsilon} - \Delta\Phi_{\beta,\varepsilon}\right)\,dx \ge \eta_1\int_\Omega a(x)|u|^2\Psi^{\lambda-1}\,dx$$

with some constant $\eta_1 = \eta_1(n,\alpha,\delta,\varepsilon,\lambda) > 0$. Moreover, Proposition 2.6 (i), (ii), and (iii) imply

$$|u\partial_t u \Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}| \le C|u||\partial_t u||\Phi_{\beta,\varepsilon}^{-2+2\delta}||\Phi_{\beta+1,\varepsilon}| \le C|u||\partial_t u|\Psi^{\lambda-1}.$$

This and the Schwarz inequality lead to

$$\begin{aligned}&\left|2(1-2\delta)\int_\Omega u\partial_t u\Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}\,dx\right|\\ &\le C\int_\Omega |u||\partial_t u|\Psi^{\lambda-1}\,dx\\ &\le C\left(\int_\Omega a(x)|u|^2\Psi^{\lambda-1}\,dx\right)^{1/2}\left(\int_\Omega a(x)^{-1}|\partial_t u|^2\Psi^{\lambda-1}\,dx\right)^{1/2}\\ &\le \frac{\eta_1}{2}\int_\Omega a(x)|u|^2\Psi^{\lambda-1}\,dx + C\int_\Omega |\partial_t u|^2\Psi^{\lambda}\,dx\end{aligned}$$

with some $C = C(n,a,\delta,\varepsilon,\lambda) > 0$. Summarizing the above computations, we see that for both cases $\lambda = 0$ and $\lambda > 0$, the last two terms of (3.7) can be estimated as

$$\begin{aligned}
&-2(1-2\delta)\int_\Omega u\partial_t u\Phi_{\beta,\varepsilon}^{-2+2\delta}\partial_t\Phi_{\beta,\varepsilon}\,dx\\
&\quad-(1-2\delta)\int_\Omega |u|^2\Phi_{\beta,\varepsilon}^{-2+2\delta}\left(a(x)\partial_t\Phi_{\beta,\varepsilon}-\Delta\Phi_{\beta,\varepsilon}\right)dx\\
&\le C\int_\Omega |\partial_t w|^2\Psi^\lambda\,dx.
\end{aligned}$$

Finally, from Proposition 2.6 (ii) and (iii), one obtains

$$2\int_\Omega |\partial_t u|^2\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\le C\int_\Omega |\partial_t u|^2\Psi^\lambda\,dx$$

and

$$\frac{2\delta}{1-\delta}\int_\Omega |\nabla u|^2\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx+2\int_\Omega |u|^{p+1}\Phi_{\beta,\varepsilon}^{-1+2\delta}\,dx\ge\eta\int_\Omega\left(|\nabla u|^2+|u|^{p+1}\right)\Psi^\lambda\,dx$$

with some positive constants $C=C(n,\alpha,\delta,\varepsilon,\lambda)$ and $\eta=\eta(n,\alpha,\delta,\varepsilon,\lambda)$. Putting this all together, we deduce from (3.7) that

$$\begin{aligned}
\frac{d}{dt}E_0(t;t_0,\lambda)\le&-\eta\int_\Omega\left(|\nabla u|^2+|u|^{p+1}\right)\Psi^\lambda\,dx\\
&+C\int_\Omega |\partial_t u|^2\Psi^\lambda\,dx,
\end{aligned}$$

and the proof is complete. □

Combining Lemmas 3.2 and 3.3, we have the following estimate for $E_*(t;t_0,\lambda,\nu)$.

Lemma 3.4. *Under the assumptions on Theorem 1.4 (i), there exist constants $\nu_*=\nu_*(n,a,\delta,\varepsilon,\lambda)\in(0,\nu_0)$ and $t_2=t_2(n,a,p,\delta,\varepsilon,\lambda,\nu_*)\ge 1$ such that for $t_0\ge t_2$ and $t>0$, we have*

$$\begin{aligned}
&E_*(t;t_0,\lambda,\nu_*)+\int_0^t\int_\Omega a(x)|\partial_t u(s,x)|^2\Psi(s,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}}\,dxds\\
&\quad+\int_0^t\int_\Omega(|\nabla u(s,x)|^2+|u(s,x)|^{p+1})\Psi(s,x;t_0)^\lambda\,dxds\\
&\le CE_*(0;t_0,\lambda,\nu_*)
\end{aligned}$$

with some constant $C=C(n,a,\delta,\varepsilon,\lambda,\nu_*)>0$.

Proof. Let $\nu\in(0,\nu_0)$, where ν_0 is taken so that (3.2) holds. From the definition of $E_*(t;t_0,\lambda,\nu)$ and Lemmas 3.2 and 3.3, one has

$$\begin{aligned}
\frac{d}{dt}E_*(t;t_0,\lambda,\nu)&=\frac{d}{dt}E_1(t;t_0,\lambda)+\nu\frac{d}{dt}E_0(t;t_0,\lambda)\\
&\le-\frac{1}{2}\int_\Omega a(x)|\partial_t u|^2\Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\
&\quad+C\int_\Omega\left(|\nabla u|^2+|u|^{p+1}\right)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx\\
&\quad-\nu\eta\int_\Omega\left(|\nabla u|^2+|u|^{p+1}\right)\Psi^\lambda\,dx\\
&\quad+C\nu\int_\Omega |\partial_t u|^2\Psi^\lambda\,dx
\end{aligned}\tag{3.8}$$

for $t_0 \geq t_1$ and $t > 0$, where $t_1 \geq 1$ is determined in Lemma 3.2. Noting that (1.12) and (2.2) imply

$$|\partial_t u|^2 \Psi^\lambda \leq C\langle x\rangle^{-\alpha} A_\varepsilon(x)^{\frac{\alpha}{2-\alpha}} |\partial_t u|^2 \Psi^\lambda \leq Ca(x)|\partial_t u|^2 \Psi^{\lambda+\frac{\alpha}{2-\alpha}}$$

with some constant $C = C(n, a, \alpha, \varepsilon) > 0$, and taking $\nu = \nu_*$ with sufficiently small $\nu_* \in (0, \nu_0)$, we deduce

$$-\frac{1}{2}\int_\Omega a(x)|\partial_t u|^2 \Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx + C\nu_* \int_\Omega |\partial_t u|^2 \Psi^\lambda\,dx \leq -\frac{1}{4}\int_\Omega a(x)|\partial_t u|^2 \Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx.$$

Next, by $\Psi^{\frac{\alpha}{2-\alpha}-1} \leq (t_0+t)^{\frac{\alpha}{2-\alpha}-1}$ and taking $t_2 \geq t_1$ sufficiently large depending on ν_*, one obtains

$$\begin{aligned} &C\int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Psi^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx - \nu_*\eta\int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Psi^\lambda\,dx \\ &\leq -\frac{\nu_*\eta}{2}\int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Psi^\lambda\,dx \end{aligned}$$

for $t_0 \geq t_2$. Finally, plugging the above estimates into (3.8) with $\nu = \nu_*$, we conclude

$$\begin{aligned} \frac{d}{dt}E_*(t; t_0, \lambda, \nu_*) \leq &-\frac{1}{4}\int_\Omega a(x)|\partial_t u|^2 \Psi^{\lambda+\frac{\alpha}{2-\alpha}}\,dx \\ &-\frac{\nu_*\eta}{2}\int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Psi^\lambda\,dx \end{aligned}$$

for $t_0 \geq t_2$ and $t > 0$. Integrating it over $[0, t]$, we have the desired estimate. □

Lemma 3.5. *Under the assumptions on Theorem 1.4 (i), there exists a constant $t_2 = t_2(n, a, p, \delta, \varepsilon, \lambda) \geq 1$ such that for $t_0 \geq t_2$ and $t > 0$, we have*

$$\tilde{E}(t; t_0, \lambda) + \int_\Omega a(x)|u(t,x)|^2 \Psi(t, x; t_0)^\lambda\,dx \leq CI_0[u_0, u_1]$$

with some constant $C = C(n, a, p, \delta, \varepsilon, \lambda, \nu_*, t_0) > 0$.

Proof. Take the same constants ν_* and t_2 as in Lemma 3.4. The integration by parts and the equation (1.1) imply

$$\begin{aligned} \frac{d}{dt}\tilde{E}(t; t_0, \lambda) &= \int_\Omega \left[\frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) + \frac{1}{p+1}|u|^{p+1}\right]\left(\Psi + \lambda(t_0+t)\right)\Psi^{\lambda-1}\,dx \\ &\quad + (t_0+t)\int_\Omega \left(\partial_t u \partial_t^2 u + \nabla u\cdot\nabla\partial_t u + |u|^{p-1}u\partial_t u\right)\Psi^\lambda\,dx \\ &= \int_\Omega \left[\frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) + \frac{1}{p+1}|u|^{p+1}\right]\left(\Psi + \lambda(t_0+t)\right)\Psi^{\lambda-1}\,dx \\ &\quad - (t_0+t)\int_\Omega a(x)|\partial_t u|^2\Psi^\lambda\,dx - \lambda(t_0+t)\int_\Omega \partial_t u(\nabla u\cdot\nabla\Psi)\Psi^{\lambda-1}\,dx. \end{aligned}$$

The last term of the right-hand side is estimated as

$$\begin{aligned} -\lambda(t_0+t)\int_\Omega \partial_t u(\nabla u\cdot\nabla\Psi)\Psi^{\lambda-1}\,dx &\leq \eta(t_0+t)\int_\Omega a(x)|\partial_t u|^2\frac{|\nabla\Psi|^2}{a(x)}\Psi^{\lambda-1}\,dx \\ &\quad + C(t_0+t)\int_\Omega |\nabla u|^2\Psi^{\lambda-1}\,dx \end{aligned}$$

for any $\eta > 0$. Using (3.5) and taking $\eta = \eta(n, \alpha, \varepsilon)$ sufficiently small, we have

$$\begin{aligned}\frac{d}{dt}\tilde{E}(t; t_0, \lambda) \le C \int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^{p+1} \right) (\Psi + (t_0 + t))\Psi^{\lambda-1}\, dx \\ - \frac{1}{2}(t_0 + t) \int_\Omega a(x) |\partial_t u|^2 \Psi^\lambda \, dx.\end{aligned}$$

Noting $t_0 + t \le \Psi$ and $a(x)^{-1} \le C\Psi^{\frac{\alpha}{2-\alpha}}$, we estimate

$$\int_\Omega |\partial_t u|^2 (\Psi + \lambda(t_0 + t))\Psi^{\lambda-1}\, dx \le C \int_\Omega a(x) |\partial_t u|^2 \Psi^{\lambda + \frac{\alpha}{2-\alpha}} \, dx.$$

Therefore, integrating over $[0, t]$ yield

$$\begin{aligned}&\tilde{E}(t; t_0, \lambda) + \frac{1}{2}\int_0^t (t_0 + s) \int_\Omega a(x)|\partial_t u|^2 \Psi^\lambda \, dxds \\ &\le \tilde{E}(0; t_0, \lambda) + C\int_0^t \int_\Omega a(x)|\partial_t u|^2 \Psi^{\lambda + \frac{\alpha}{2-\alpha}}\, dxds + C\int_0^t \int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) \Psi^\lambda \, dxds.\end{aligned}$$

Now, we multiply the both sides of above inequality by a sufficiently small constant $\mu > 0$, and add it and the conclusion of Lemma 3.4. Then, we obtain

$$\begin{aligned}&\mu\tilde{E}(t; t_0, \lambda) + E_*(t; t_0, \lambda, \nu_*) \\ &\quad + \int_0^t \int_\Omega a(x)|\partial_t u|^2 \left[\frac{\mu}{2}(t_0 + s) + (1 - C\mu)\Psi^{\frac{\alpha}{2-\alpha}} \right] \Psi^\lambda \, dxds \\ &\quad + (1 - C\mu) \int_0^t \int_\Omega \left(|\nabla u|^2 + |u|^{p+1} \right) \Psi^\lambda \, dxds \\ &\le \mu\tilde{E}(0; t_0, \lambda) + CE_*(0; t_0, \lambda, \nu_*)\end{aligned} \tag{3.9}$$

for $t_0 \ge t_2$ and $t > 0$. Let us take μ sufficiently small so that $1 - C\mu > 0$. Then, the last three terms in the left-hand side can be dropped. Finally, from the definitions of $E_*(t; t_0, \lambda)$ and $\tilde{E}(t; t_0, \lambda)$, we can easily verify

$$\mu\tilde{E}(0; t_0, \lambda) + E_*(0; t_0, \lambda, \nu_*) \le CI_0[u_0, u_1]$$

with some constant $C = C(a, p, \lambda, t_0) > 0$. Thus, we conclude

$$\tilde{E}(t; t_0, \lambda) + E_*(t; t_0, \lambda, \nu_*) \le CI_0[u_0, u_1]$$

for $t_0 \ge t_2$ and $t > 0$. This and the lower bound (3.3) of $E_*(t; t_0, \lambda, \nu_*)$ give the desired estimate. □

Proof of Theorem 1.4 (i) for compactly supported initial data. Take $\lambda \in [0, \frac{n-\alpha}{2-\alpha})$ as in the assumption (1.13), and then choose $\delta, \varepsilon \in (0, 1/2)$ so that $\lambda \in [0, (1-2\delta)\gamma_\varepsilon)$ holds. Moreover, take the same constants ν_* and t_2 as in Lemmas 3.4 and 3.5. By (3.3), Lemmas 3.4 and 3.5, Definition 3.1, and $(t_0 + t)^\lambda \le \Psi^\lambda$, we have

$$(t_0 + t)^{\lambda+1} E[u](t) + (t_0 + t)^\lambda \int_\Omega a(x)|u(t, x)|^2 \, dx \le CI_0[u_0, u_1] \tag{3.10}$$

for $t_0 \ge t_2$ and $t > 0$ with some constant $C = C(n, a, p, \delta, \varepsilon, \lambda, \nu_*, t_0) > 0$. This completes the proof. □

Remark 3.6. *From* (3.9), *we have a slightly more general estimate*

$$\begin{aligned}
&\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^{p+1}\right)\left[(t_0+t) + \Psi^{\frac{\alpha}{2-\alpha}}\right]\Psi^\lambda + \int_\Omega a(x)|u|^2\Psi^\lambda\,dx \\
&\quad + \int_0^t\int_\Omega a(x)|\partial_t u|^2\left[(t_0+s) + \Psi^{\frac{\alpha}{2-\alpha}}\right]\Psi^\lambda\,dxds \\
&\quad + \int_0^t\int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Psi^\lambda\,dxds \\
&\le CI_0[u_0,u_1]
\end{aligned}$$

for $t_0 \ge t_2$ *and* $t > 0$. *Moreover, from the proof of Lemma 3.3, we can add the term* $\int_0^t\int_\Omega a(x)|u|^2\Psi^{\lambda-1}\,dxds$ *to the left-hand side when* $\lambda > 0$.

3.2. **Proof for the general case.** Here, we give a proof of Theorem 1.4 (i) for non-compactly supported initial data.

Let $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ satisfy $I_0[u_0,u_1] < \infty$ and let u be the corresponding mild solution to (1.1). We take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \le \chi(x) \le 1\ (x\in\mathbb{R}^n), \quad \chi(x) = \begin{cases} 1 & (|x|\le 1), \\ 0 & (|x|\ge 2). \end{cases}$$

For each $j\in\mathbb{N}$, we define $\chi_j(x) = \chi(x/j)$. Then, we have

$$\begin{aligned}
&0 \le \chi_j(x) \le 1\ (x\in\mathbb{R}^n), \quad \chi_j(x) = \begin{cases} 1 & (|x|\le j), \\ 0 & (|x|\ge 2j), \end{cases} \\
&|\nabla\chi_j(x)| \le \frac{C}{j}\ (x\in\mathbb{R}^n), \quad \operatorname{supp}\nabla\chi_j \subset \overline{B_{2j}(0)\setminus B_j(0)},
\end{aligned}$$

where the constant C is independent of j.

Let $(u_0^{(j)}, u_1^{(j)}) = (\chi_j u_0, \chi_j u_1)$ and let $u^{(j)}$ be the corresponding mild solution to (1.1). First, by definition, it is easily seen that

$$\lim_{j\to\infty}(u_0^{(j)}, u_1^{(j)}) = (u_0,u_1) \quad \text{in} \quad H_0^1(\Omega)\times L^2(\Omega).$$

Therefore, the continuous dependence on the initial data (see Section A.2.4) yields

$$\lim_{j\to\infty}(u^{(j)}(t), \partial_t u^{(j)}(t)) = (u(t),\partial_t u(t)) \quad \text{in} \quad C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$$

for any fixed $T>0$. From this and the Sobolev embedding, we deduce

$$\lim_{j\to\infty} E[u^{(j)}](t) = E[u](t) \tag{3.11}$$

for any $t\ge 0$.

We next show

$$\lim_{j\to\infty} I_0[u_0^{(j)}, u_1^{(j)}] = I_0[u_0,u_1]. \tag{3.12}$$

To prove this, we use the notation

$$\begin{aligned}
&I_0[u_0,u_1;D] \\
&:= \int_D \left[(|u_1(x)|^2 + |\nabla u_0(x)|^2 + |u_0(x)|^{p+1})\langle x\rangle^\alpha + |u_0(x)|^2\langle x\rangle^{-\alpha}\right]\langle x\rangle^{\lambda(2-\alpha)}\,dx
\end{aligned}$$

for a region $D \subset \Omega$. Using the properties of χ_j described above and

$$|\nabla(\chi_j u_0)|^2 = \chi_j^2 |\nabla u_0|^2 + 2(\nabla \chi_j \cdot \nabla u_0)\chi_j u_0 + |\nabla \chi_j|^2 |u_0|^2,$$

we calculate

$$\begin{aligned} |I_0[u_0, u_1] - I_0[u_0^{(j)}, u_1^{(j)}]| &\le I_0[u_0, u_1; \Omega \setminus B_j(0)] \\ &\quad + \left| \int_{B_{2j}(0)\setminus B_j(0)} 2(\nabla \chi_j \cdot \nabla u_0)\chi_j u_0 \langle x \rangle^{\alpha+\lambda(2-\alpha)} \, dx \right| \\ &\quad + \int_{B_{2j}(0)\setminus B_j(0)} |\nabla \chi_j|^2 |u_0|^2 \langle x \rangle^{\alpha+\lambda(2-\alpha)} \, dx. \end{aligned} \tag{3.13}$$

The Schwarz inequality gives

$$\begin{aligned} &\left| \int_{B_{2j}(0)\setminus B_j(0)} 2(\nabla \chi_j \cdot \nabla u_0)\chi_j u_0 \langle x \rangle^{\alpha+\lambda(2-\alpha)} \, dx \right| \\ &\le I_0[u_0, u_1; \Omega \setminus B_j(0)] + \int_{B_{2j}(0)\setminus B_j(0)} |\nabla \chi_j|^2 |u_0|^2 \langle x \rangle^{\alpha+\lambda(2-\alpha)} \, dx. \end{aligned}$$

Furthermore, using the estimate of $\nabla \chi_j$, one sees that

$$\begin{aligned} &\int_{B_{2j}(0)\setminus B_j(0)} |\nabla \chi_j|^2 |u_0|^2 \langle x \rangle^{\alpha+\lambda(2-\alpha)} \, dx \\ &\le C j^{-2} (1 + |2j|^2)^{\alpha} \int_{B_{2j}(0)\setminus B_j(0)} |u_0|^2 \langle x \rangle^{-\alpha+\lambda(2-\alpha)} \, dx \\ &\le C I_0[u_0, u_1; \Omega \setminus B_j(0)], \end{aligned}$$

where the constant C is independent of j. Putting this all together into (3.13), we have

$$|I_0[u_0, u_1] - I_0[u_0^{(j)}, u_1^{(j)}]| \le C I_0[u_0, u_1; \Omega \setminus B_j(0)].$$

Since $I_0[u_0, u_1] < \infty$, the right-hand side tends to zero as $j \to \infty$. This proves (3.12).

Now we are at the position to proof Theorem 1.4 (i).

Proof of Theorem 1.4 (i) for the general case. Take the same constant t_2 as in Lemmas 3.4 and 3.5. Let $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ be the sequence defined above and let $u^{(j)}$ be the corresponding mild solution to (1.1) with the initial data $(u_0^{(j)}, u_1^{(j)})$. Since each $(u_0^{(j)}, u_1^{(j)})$ has the compact support, one can apply the result (3.10) in the previous subsection to obtain

$$(t_0 + t)^{\lambda+1} E[u^{(j)}](t) + (t_0 + t)^{\lambda} \int_{\Omega} a(x) |u^{(j)}(t, x)|^2 \, dx \le C I_0[u_0^{(j)}, u_1^{(j)}]$$

for $t_0 \ge t_2$ and $t > 0$. Finally, using (3.11) and (3.12), we have

$$(t_0 + t)^{\lambda+1} E[u](t) + (t_0 + t)^{\lambda} \int_{\Omega} a(x) |u(t, x)|^2 \, dx \le C I_0[u_0, u_1]$$

for $t_0 \ge t_2$ and $t > 0$, which completes the proof. $\square$

4. Proof of Theorem 1.4: second part

In this section, we prove Theorem 1.4 (ii). By the same approximation argument described in Section 3, we may assume $(u_0, u_1) \in D(\mathcal{A}_D)$ and consider the strong solution u.

First, we note that, since the larger λ is, the stronger the assumption on the initial data is. Thus, without loss of generality, we may assume that λ always satisfies

$$\lambda < \min\left\{ \frac{2}{p-1}, \frac{4}{2-\alpha}\left(\frac{1}{p-1} - \frac{n-\alpha}{4} \right) \right\} + \varepsilon, \tag{4.1}$$

where $\varepsilon > 0$ is a sufficiently small constant specified later. This will be used for the estimate of the remainder term.

In contrast to the previous section, in the following, we shall use only

$$\Theta(x,t;t_0) := t_0 + t + \langle x \rangle^{2-\alpha} \tag{4.2}$$

as a weight function, and we define the following energies.

Definition 4.1. *For a function $u = u(t,x)$, $\alpha \in [0,1)$, $\lambda \in [0,\infty)$, $\nu > 0$, and $t_0 \ge 1$, we define*

$$E_1(t;t_0,\lambda) = \int_\Omega \left[\frac{1}{2}\left(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \right) + \frac{1}{p+1}|u(t,x)|^{p+1} \right] \Theta(t,x;t_0)^{\lambda + \frac{\alpha}{2-\alpha}}\, dx,$$

$$E_0(t;t_0,\lambda) = \int_\Omega \left(2u(t,x)\partial_t u(t,x) + a(x)|u(t,x)|^2 \right) \Theta(t,x;t_0)^{\lambda}\, dx,$$

$$E_*(t;t_0,\lambda,\nu) = E_1(t;t_0,\lambda) + \nu E_0(t;t_0,\lambda),$$

$$\tilde{E}(t;t_0,\lambda) = (t_0+t)\int_\Omega \left[\frac{1}{2}\left(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \right) + \frac{1}{p+1}|u(t,x)|^{p+1} \right] \Theta(t,x;t_0)^{\lambda}\, dx$$

for $t \ge 0$.

Similarly to (3.2) and (3.3), we can prove the lower bound

$$E_*(t;t_0,\lambda,\nu) \ge \frac{1}{2}E_1(t;t_0,\lambda) + \frac{\nu}{2}\int_\Omega a(x)|u(t,x)|^2 \Theta(t,x;t_0)^{\lambda}\, dx, \tag{4.3}$$

provided that $\nu \in (0,\nu_0)$ with some constant $\nu_0 > 0$.

We start with the following simple estimates for $E_1(t;t_0,\lambda)$ and $E_0(t;t_0,\lambda)$.

Lemma 4.2. *Under the assumptions on Theorem 1.4 (ii), there exists $t_1 = t_1(n,\alpha,a_0,\lambda,\varepsilon) \ge 1$ such that for $t_0 \ge t_1$ and $t > 0$, we have*

$$\begin{aligned} \frac{d}{dt}E_1(t;t_0,\lambda) &\le -\frac{1}{2}\int_\Omega a(x)|\partial_t u(t,x)|^2 \Theta(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}}\, dx \\ &\quad + C\int_\Omega \left(|\nabla u(t,x)|^2 + |u(t,x)|^{p+1} \right) \Theta(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}-1}\, dx \end{aligned}$$

with some constant $C = C(n,\alpha,a_0,p,\lambda) > 0$.

Proof. The proof is almost the same as that of Lemma 3.2. The only differences are the use of

$$\frac{|\nabla\Theta|^2}{a(x)\Theta} = (2-\alpha)^2 \frac{\langle x \rangle^{-2\alpha}|x|^2}{a(x)(t_0+t+\langle x \rangle^{2-\alpha})} \le \frac{(2-\alpha)^2}{a_0} \tag{4.4}$$

and

$$\Theta(t,x;t_0)^{-1} \le t_0^{-1+\frac{\alpha}{2-\alpha}} \langle x \rangle^{-\alpha} \le \frac{1}{a_0} t_0^{-1+\frac{\alpha}{2-\alpha}} a(x)$$

instead of (3.5) and (3.6), respectively. Thus, we omit the detail. □

Lemma 4.3. *Under the assumptions on Theorem 1.4 (ii), for $t_0 \ge 1$ and $t > 0$, we have*

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) \le &- \int_\Omega |\nabla u(t,x)|^2 \Theta(t,x;t_0)^\lambda \, dx - 2\int_\Omega |u(t,x)|^{p+1}\Theta(t,x;t_0)^\lambda \, dx \\ &+ C\int_\Omega a(x)|\partial_t u(t,x)|^2 \Theta(t,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}} \, dx + C\int_\Omega a(x)|u(t,x)|^2\Theta(t,x;t_0)^{\lambda-1} \, dx\end{aligned}$$

with some constant $C = C(n,\alpha,a_0,\lambda) > 0$.

Proof. The equation (1.1) and the integration by parts imply

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) &= 2\int_\Omega |\partial_t u|^2\Theta^\lambda \, dx + 2\int_\Omega \left(\partial_t^2 u + a(x)\partial_t u\right)\Theta^\lambda \, dx \\ &\quad + \lambda\int_\Omega \left(2u\partial_t u + a(x)|u|^2\right)\Theta^{\lambda-1} \, dx \\ &= 2\int_\Omega |\partial_t u|^2\Theta^\lambda \, dx + 2\int_\Omega \left(\Delta u - |u|^{p-1}u\right) u\Theta^\lambda \, dx \\ &\quad + \lambda\int_\Omega \left(2u\partial_t u + a(x)|u|^2\right)\Theta^{\lambda-1} \, dx \\ &= -2\int_\Omega |\nabla u|^2\Theta^\lambda \, dx - 2\int_\Omega |u|^{p+1}\Theta^\lambda \, dx \\ &\quad + 2\int_\Omega |\partial_t u|^2\Theta^\lambda \, dx - 2\lambda\int_\Omega (\nabla u\cdot\nabla\Theta)u\Theta^{\lambda-1} \, dx \\ &\quad + \lambda\int_\Omega \left(2u\partial_t u + a(x)|u|^2\right)\Theta^{\lambda-1} \, dx. \end{aligned} \tag{4.5}$$

Let us estimates the right-hand side. Applying the Schwarz inequality and (4.4), we obtain

$$\begin{aligned}-2\lambda\int_\Omega (\nabla u\cdot\nabla\Psi)u\Theta^{\lambda-1} \, dx &\le \frac{1}{2}\int_\Omega |\nabla u|^2\Theta^\lambda \, dx + C\int_\Omega |u|^2|\nabla\Theta|^2\Theta^{\lambda-2} \, dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2\Theta^\lambda \, dx + C\int_\Omega a(x)|u|^2\Theta^{\lambda-1} \, dx.\end{aligned}$$

Moreover, the Schwarz inequality and $\Theta^{-1} \le \frac{1}{a_0}a(x)$ imply

$$\begin{aligned}\lambda\int_\Omega 2u(t,x)\partial_t u(t,x)\Theta^{\lambda-1} \, dx &\le \frac{1}{2}\int_\Omega |\nabla u|^2\Theta^\lambda \, dx + C\int_\Omega |u|^2\Theta^{\lambda-2} \, dx \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^2\Theta^\lambda \, dx + C\int_\Omega a(x)|u|^2\Theta^{\lambda-1} \, dx.\end{aligned}$$

From $1 \le \frac{1}{a_0}a(x)\Theta^{\frac{\alpha}{2-\alpha}}$, we also obtain

$$2\int_\Omega |\partial_t u|^2\Theta^\lambda \, dx \le C\int_\Omega a(x)|\partial_t u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}} \, dx.$$

Putting them all together into (4.5), we conclude

$$\begin{aligned}\frac{d}{dt}E_0(t;t_0,\lambda) \le &-\int_\Omega |\nabla u|^2\Theta^\lambda\,dx - 2\int_\Omega |u|^{p+1}\Theta^\lambda\,dx\\ &+ C\int_\Omega a(x)|\partial_t u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}}\,dx + C\int_\Omega a(x)|u|^2\Theta^{\lambda-1}\,dx.\end{aligned}$$

This completes the proof. □

Combining Lemmas 4.2 and 4.3, we have the following.

Lemma 4.4. *Under the assumptions on Theorem 1.4 (ii), there exist constants $\nu_* = \nu_*(n,\alpha,a_0,\lambda) \in (0,\nu_0)$ and $t_2 = t_2(n,\alpha,a_0,p,\lambda,\nu_*) \ge 1$ such that for $t_0 \ge t_2$, and $t > 0$, we have*

$$\begin{aligned}&E_*(t;t_0,\lambda,\nu_*) + \int_0^t\int_\Omega a(x)|\partial_t u(s,x)|^2\Theta(s,x;t_0)^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\,ds\\ &\quad+\int_0^t\int_\Omega \left(|\nabla u(s,x)|^2 + |u(s,x)|^{p+1}\right)\Theta(s,x;t_0)^\lambda\,dxds\\ &\le CE_*(0;t_0,\lambda,\nu) + C\int_0^t\int_\Omega a(x)|u(s,x)|^2\Theta(s,x;t_0)^{\lambda-1}\,dxds\end{aligned}$$

with some constant $C = C(n,\alpha,a_0,p,\lambda,\nu_*) > 0$.

Proof. Let $\nu \in (0,\nu_0)$, where ν_0 is taken so that (4.3) holds. Let t_1 be the constant determined by Lemma 4.2. Then, by Lemmas 4.2 and 4.3, we obtain for $t_0 \ge t_1$ and $t > 0$,

$$\begin{aligned}\frac{d}{dt}E_*(t;t_0,\lambda,\nu) &= \frac{d}{dt}E_1(t;t_0,\lambda) + \nu\frac{d}{dt}E_0(t;t_0,\lambda)\\ &\le -\frac{1}{2}\int_\Omega a(x)|\partial_t u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}}\,dx\\ &\quad+ C\int_\Omega |\nabla u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx + C\int_\Omega |u|^{p+1}\Theta^{\lambda+\frac{\alpha}{2-\alpha}-1}\,dx\\ &\quad-\nu\int_\Omega |\nabla u|^2\Theta^\lambda\,dx - 2\nu\int_\Omega |u|^{p+1}\Theta^\lambda\,dx\\ &\quad+ C\nu\int_\Omega a(x)|\partial_t u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}}\,dx + C\nu\int_\Omega a(x)|u|^2\Theta^{\lambda-1}\,dx.\end{aligned}$$

We take $\nu = \nu_*$ with sufficiently small $\nu_* \in (0,\nu_0)$ such that the constants in front of the last two terms satisfy $C\nu_* < \frac{1}{2}$. Moreover, taking $t_2 > 0$ sufficiently large depending on ν_* so that $C\Theta^{\frac{\alpha}{2-\alpha}-1} < \nu_*$ for $t_0 \ge t_2$, we conclude

$$\begin{aligned}\frac{d}{dt}E_*(t;t_0,\lambda,\nu) \le &-\eta\int_\Omega a(x)|\partial_t u|^2\Theta^{\lambda+\frac{\alpha}{2-\alpha}}\,dx - \eta\int_\Omega |\nabla u|^2\Theta^\lambda\,dx\\ &-\eta\int_\Omega |u|^{p+1}\Theta^\lambda\,dx + C\nu\int_\Omega a(x)|u|^2\Theta^{\lambda-1}\,dx\end{aligned}$$

with some constant $\eta = \eta(n,\alpha,a_0,p,\lambda,\nu_*) > 0$. Finally, integrating the above inequality over $[0,t]$ gives the desired estimate. □

Besed on Lemma 4.4, we show the following estimate for $\tilde{E}(t;t_0,\lambda)$.

Lemma 4.5. *Under the assumptions on Theorem 1.4 (ii), there exists a constant* $t_2 = t_2(n, \alpha, a_0, p, \lambda) \geq 1$ *such that for* $t_0 \geq t_2$ *and* $t > 0$*, we have*

$$\begin{aligned}
&\tilde{E}(t; t_0, \lambda) + \int_\Omega a(x)|u(t,x)|^2 \Theta(t,x;t_0)^\lambda \, dx \\
&\quad + \int_0^t \int_\Omega a(x)|\partial_t u(s,x)|^2 \left[(t_0+s) + \Theta(s,x;t_0)^{\frac{\alpha}{2-\alpha}}\right] \Theta(s,x;t_0)^\lambda \, dx\, ds \\
&\quad + \int_0^t \int_\Omega \left(|\nabla u(s,x)|^2 + |u(s,x)|^{p+1}\right) \Theta(s,x;t_0)^\lambda \, dxds \\
&\leq CI_0[u_0,u_1] + C\int_0^t \int_\Omega a(x)^{\frac{p+1}{p-1}} \Theta(s,x;t_0)^{\lambda - \frac{p+1}{p-1}} \, dxds
\end{aligned}$$

with some constant $C = C(n, \alpha, a_0, a_1, p, \lambda, t_0) > 0$.

Proof. Take the same constants ν_* and t_2 as in Lemma 4.4. By the same computation as in Lemma 3.5, we can obtain

$$\begin{aligned}
&\tilde{E}(t;t_0,\lambda) + \frac{1}{2}\int_0^t (t_0+s) \int_\Omega a(x)|\partial_t u|^2 \Theta^\lambda \, dxds \\
&\leq \tilde{E}(0;t_0,\lambda) + C\int_0^t \int_\Omega a(x)|\partial_t u|^2 \Theta^{\lambda + \frac{\alpha}{2-\alpha}} \, dxds + C\int_0^t \int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Theta^\lambda \, dxds.
\end{aligned}$$

We multiply the both sides by a sufficiently small constant $\mu > 0$, and add it and the conclusion of Lemma 4.4. Then, we obtain

$$\begin{aligned}
&\mu\tilde{E}(t;t_0,\lambda) + E_*(t;t_0,\lambda,\nu_*) \\
&\quad + \int_0^t \int_\Omega a(x)|\partial_t u|^2 \left[\frac{\mu}{2}(t_0+s) + (1-C\mu)\Theta^{\frac{\alpha}{2-\alpha}}\right]\Theta^\lambda \, dxds \\
&\quad + (1-C\mu)\int_0^t \int_\Omega \left(|\nabla u|^2 + |u|^{p+1}\right)\Theta^\lambda \, dxds \\
&\leq \mu\tilde{E}(0;t_0,\lambda) + CE_*(0;t_0,\lambda,\nu_*)
\end{aligned}$$

for $t_0 \geq t_2$ and $t > 0$. By taking μ sufficiently small so that $1 - C\mu > 0$ holds, the terms including $|\partial_t u|^2$ and $|\nabla u|^2$ in the left-hand side can be dropped. Since both $\tilde{E}(0;t_0,\lambda)$ and $E_*(0;t_0,\lambda,\nu_*)$ are bounded by $CI_0[u_0,u_1]$ with some constant $C = C(a_1, p, \lambda, t_0) > 0$, one obtains

$$\begin{aligned}
&\tilde{E}(t;t_0,\lambda) + \int_\Omega a(x)|u(t,x)|^2\Theta(t,x;t_0)^\lambda \, dx + \int_0^t \int_\Omega |u|^{p+1}\Theta^\lambda \, dxds \\
&\leq CI_0[u_0,u_1] + C\int_0^t \int_\Omega a(x)|u|^2\Theta^{\lambda-1} \, dxds
\end{aligned} \tag{4.6}$$

with some $C = C(n, \alpha, a_0, a_1, p, \lambda, t_0) > 0$. Finally, applying the Young inequality to the last term of the right-hand side, we deduce

$$\begin{aligned}
C\int_0^t \int_\Omega a(x)|u|^2\Theta^{\lambda-1} \, dxds &= C\int_0^t \int_\Omega |u|^2 \Theta^{\frac{2}{p+1}\lambda} \cdot a(x)\Theta^{\lambda(1-\frac{2}{p+1})-1} \, dxds \\
&\leq \frac{1}{2}\int_0^t \int_\Omega |u|^{p+1}\Theta^\lambda \, dxds + C\int_0^t \int_\Omega a(x)^{\frac{p+1}{p-1}}\Theta^{\lambda - \frac{p+1}{p-1}} \, dxds.
\end{aligned}$$

This and (4.6) give the conclusion. □

By virtue of Lemma 4.5, it suffices to estimate the term

$$C\int_0^t\int_\Omega a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dxds.$$

For this, we have the following lemma.

Lemma 4.6. *Under the assumptions on Theorem 1.4 (ii) and* (4.1)*, we have for any* $t_0>0$ *and* $t\ge 0$*,*

$$\begin{aligned}&\int_0^t\int_\Omega a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dxds\\ &\le C\begin{cases}1 & (\lambda<\min\{\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4}),\frac{2}{p-1}\}),\\ \log(t_0+t) & (\lambda=\min\{\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4}),\frac{2}{p-1}\},\ p\ne p_{subc}(n,\alpha)),\\ (\log(t_0+t))^2 & (\lambda=\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4})=\frac{2}{p-1},\ \text{i.e., } p=p_{subc}(n,\alpha)),\\ (1+t)^{\lambda-\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4})} & (\lambda>\frac{4}{2-\alpha}(\frac{1}{p-1}-\frac{n-\alpha}{4}),\ p>p_{subc}(n,\alpha)),\\ (1+t)^{\lambda-\frac{2}{p-1}}\log(t_0+t) & (\lambda>\frac{2}{p-1},\ p=p_{subc}(n,\alpha)),\\ (1+t)^{\lambda-\frac{2}{p-1}} & (\lambda>\frac{2}{p-1},\ p<p_{subc}(n,\alpha))\end{cases}\end{aligned}$$

with some constant $C=C(n,\alpha,a_1,p,\lambda)>0$*.*

Proof. Let $s\in(0,t)$. First, we divide Ω into $\Omega=\Omega_1(s)\cup\Omega_2(s)$, where

$$\begin{aligned}\Omega_1(s)&=\left\{x\in\Omega;\ \langle x\rangle^{2-\alpha}\le t_0+s\right\},\\ \Omega_2(s)&=\Omega\setminus\Omega_1(s)=\left\{x\in\Omega;\ \langle x\rangle^{2-\alpha}>t_0+s\right\}.\end{aligned}$$

The corresponding integral is also decomposed into

$$\begin{aligned}\int_\Omega a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dx&=\int_{\Omega_1(s)}a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dx\\ &\quad+\int_{\Omega_2(s)}a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dx\\ &=:I(s)+II(s).\end{aligned}$$

Note that, in $\Omega_1(s)$, the function $\Theta(s,x;t_0)=t_0+s+\langle x\rangle^{2-\alpha}$ is bounded from both above and below by t_0+s. Therefore, we estimate

$$\begin{aligned}I(s)&\le C(t_0+s)^{\lambda-\frac{p+1}{p-1}}\int_{\Omega_1(s)}a(x)^{\frac{p+1}{p-1}}\,dx\\ &\le C(t_0+s)^{\lambda-\frac{p+1}{p-1}}\int_{\Omega_1(s)}\langle x\rangle^{-\alpha\frac{p+1}{p-1}}\,dx\\ &\le C(t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s),\end{aligned}\tag{4.7}$$

where

$$h(s)=\begin{cases}1 & (p<p_{subc}(n,\alpha)),\\ \log(t_0+s) & (p=p_{subc}(n,\alpha)),\\ (t_0+s)^{\frac{1}{2-\alpha}(n-\alpha\frac{p+1}{p-1})} & (p>p_{subc}(n,\alpha)).\end{cases}\tag{4.8}$$

On the other hand, in $\Omega_2(s)$, the function Θ is bounded from both above and below by $\langle x\rangle^{2-\alpha}$. Thus, we have

$$II(s) \le C \int_{\Omega_2(s)} \langle x\rangle^{-\alpha\frac{p+1}{p-1}+(2-\alpha)\left(\lambda-\frac{p+1}{p-1}\right)}\,dx.$$

Here, we remark that the condition (4.1) ensures the finiteness of the above integral, provided that ε is taken sufficiently small depending on n and α. A straightforward computation shows

$$II(s) \le C(t_0+s)^{\lambda-\frac{p+1}{p-1}+\frac{1}{2-\alpha}\left(n-\alpha\frac{p+1}{p-1}\right)}.$$

Since the above estimate is better than (4.7) if $p \le p_{subc}(n,\alpha)$ and is the same if $p > p_{subc}(n,\alpha)$, we conclude

$$\int_\Omega a(x)^{\frac{p+1}{p-1}}\Theta(s,x;t_0)^{\lambda-\frac{p+1}{p-1}}\,dx \le C(t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s).$$

Next, we compute the integral of the function $(t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s)$ over $[0,t]$. From the definition (4.8) of $h(s)$, one has the following: If $p < p_{subc}(n,\alpha)$, then

$$\int_0^t (t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s)\,ds \le C\begin{cases} 1 & \left(\lambda < \dfrac{2}{p-1}\right), \\ \log(t_0+t) & \left(\lambda = \dfrac{2}{p-1}\right), \\ (t_0+t)^{\lambda-\frac{2}{p-1}} & \left(\lambda > \dfrac{2}{p-1}\right); \end{cases}$$

If $p = p_{subc}(n,\alpha)$, then

$$\int_0^t (t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s)\,ds \le C\begin{cases} 1 & \left(\lambda < \dfrac{2}{p-1}\right), \\ (\log(t_0+t))^2 & \left(\lambda = \dfrac{2}{p-1}\right), \\ (t_0+t)^{\lambda-\frac{2}{p-1}}\log(t_0+t) & \left(\lambda > \dfrac{2}{p-1}\right); \end{cases}$$

If $p > p_{subc}(n,\alpha)$, then

$$\int_0^t (t_0+s)^{\lambda-\frac{p+1}{p-1}}h(s)\,ds \le C\begin{cases} 1 & \left(\lambda < \dfrac{4}{2-\alpha}\left(\dfrac{1}{p-1}-\dfrac{n-\alpha}{4}\right)\right), \\ \log(t_0+t) & \left(\lambda = \dfrac{4}{2-\alpha}\left(\dfrac{1}{p-1}-\dfrac{n-\alpha}{4}\right)\right), \\ (t_0+t)^{\lambda-\frac{4}{2-\alpha}\left(\frac{1}{p-1}-\frac{n-\alpha}{4}\right)} & \left(\lambda > \dfrac{4}{2-\alpha}\left(\dfrac{1}{p-1}-\dfrac{n-\alpha}{4}\right)\right). \end{cases}$$

This completes the proof. $\square$

We are now at the position to prove Theorem 1.4 (ii):

Proof of Theorem 1.4 (ii). By Lemmas 4.5 and 4.6 with the constant $t_2 \ge 1$ determined in Lemma 4.5, we have

$$\begin{aligned}
&\tilde{E}(t;t_0,\lambda) + \int_\Omega a(x)|u(t,x)|^2 \Theta(t,x;t_0)^\lambda \, dx \\
&\le C I_0[u_0,u_1] + C \begin{cases}
1 & (\lambda < \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\}), \\
\log(t_0+t) & (\lambda = \min\{\frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}), \frac{2}{p-1}\},\ p \ne p_{subc}(n,\alpha)), \\
(\log(t_0+t))^2 & (\lambda = \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}) = \frac{2}{p-1},\ \text{i.e., } p = p_{subc}(n,\alpha)), \\
(1+t)^{\lambda - \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4})} & (\lambda > \frac{4}{2-\alpha}(\frac{1}{p-1} - \frac{n-\alpha}{4}),\ p > p_{subc}(n,\alpha)), \\
(1+t)^{\lambda - \frac{2}{p-1}} \log(t_0+t) & (\lambda > \frac{2}{p-1},\ p = p_{subc}(n,\alpha)), \\
(1+t)^{\lambda - \frac{2}{p-1}} & (\lambda > \frac{2}{p-1},\ p < p_{subc}(n,\alpha))
\end{cases}
\end{aligned}$$

for $t_0 \ge t_2$ and $t \ge 0$. On the other hand, the definition (4.2) of Θ immediately gives the lower bound

$$\begin{aligned}
&\tilde{E}(t;t_0,\lambda) + \int_\Omega a(x)|u(t,x)|^2 \Theta(t,x;t_0)^\lambda \, dx \\
&\ge (t_0+t)^{\lambda+1} E[u](t) + (t_0+t)^\lambda \int_\Omega a(x)|u(t,x)|^2 \, dx,
\end{aligned}$$

where $E(t)$ is defined by (1.2). Combining them, we have the desired estimate. □

Appendix A. Outline of the proof of Proposition 1.2

In this section, we give a proof of Proposition 1.2. The solvability and basic properties of the solution of the linear problem (A.1) below can be found in, for example, [8, 19, 25, 68]. Here, we give an outline of the argument along with [19]. The existence of the unique mild solution of the semilinear problem (1.1) is proved by the contraction mapping principle. This argument can be found in, e.g., [6, 25, 36, 85]. Here, we will give a proof based on [6].

A.1. **Linear problem.** Let $n \in \mathbb{N}$, and let Ω be an open set in $\mathbb{R}^n$ with a compact C^2-boundary $\partial\Omega$ or $\Omega = \mathbb{R}^n$. We discuss the linear problem

$$\begin{cases}
\partial_t^2 u - \Delta u + a(x)\partial_t u = 0, & t > 0, x \in \Omega, \\
u(x,t) = 0, & t > 0, x \in \partial\Omega, \\
u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega.
\end{cases} \tag{A.1}$$

The function $a(x)$ is nonnegative, bounded, and continuous in $\mathbb{R}^n$. Let $\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega)$ be the real Hilbert space equipped with the inner product

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right)_{\mathcal{H}} = (u,w)_{H^1} + (v,z)_{L^2}.$$

Let $\mathcal{A}$ be the operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -a(x) \end{pmatrix}$$

defined on $\mathcal{H}$ with the domain $D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, which is dense in $\mathcal{H}$.

We first show the estimate

$$\left(\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}u\\v\end{pmatrix}\right)_{\mathcal{H}}\le\|(u,v)\|_{\mathcal{H}}^2$$

for $(u,v)\in D(\mathcal{A})$. Indeed, we calculate

$$\begin{aligned}\left(\mathcal{A}\begin{pmatrix}u\\v\end{pmatrix},\begin{pmatrix}u\\v\end{pmatrix}\right)_{\mathcal{H}}&=\left(\begin{pmatrix}v\\\Delta u-a(x)v\end{pmatrix},\begin{pmatrix}u\\v\end{pmatrix}\right)_{\mathcal{H}}\\&=(v,u)_{H^1}+(\Delta u-a(x)v,v)_{L^2}\\&=(\nabla v,\nabla u)_{L^2}+(v,u)_{L^2}-(\nabla v,\nabla u)_{L^2}-(a(x)v,v)_{L^2}\\&\le(v,u)_{L^2}\le\|(u,v)\|_{\mathcal{H}}^2.\end{aligned}$$

Next, we prove that there exists $\lambda_0\in\mathbb{R}$ such that for any $\lambda\ge\lambda_0$, the operator $\lambda-\mathcal{A}$ is invertible, that is, for any $(f,g)\in\mathcal{H}$, we can find a unique $(u,v)\in D(\mathcal{A})$ satisfying

$$(\lambda-\mathcal{A})\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}f\\g\end{pmatrix}.\tag{A.2}$$

Indeed, the above equation is equivalent with

$$\begin{cases}\lambda u-v=f,\\\lambda v-\Delta u+a(x)v=g.\end{cases}$$

We remark that the first equation implies $v=\lambda u-f$. Substituting this into the second equation, one has

$$(\lambda^2+\lambda a(x))u-\Delta u=h,\tag{A.3}$$

where $h=g+(\lambda+a(x))f\in L^2(\Omega)$. Take an arbitrary constant $\lambda_0>0$ and let $\lambda\ge\lambda_0$ be fixed. Associated with the above equation, we define the bilinear functional

$$\mathfrak{a}(z,w)=((\lambda^2+\lambda a(x))z,w)_{L^2}+(\nabla z,\nabla w)_{L^2}$$

for $z,w\in H_0^1(\Omega)$. Since $\lambda>0$ and $a(x)$ is nonnegative and bounded, $\mathfrak{a}$ is bounded: $\mathfrak{a}(z,w)\le C\|z\|_{H^1}\|w\|_{H^1}$, and coercive: $\mathfrak{a}(z,z)\ge C\|z\|_{H^1}^2$. Therefore, by the Lax–Milgram theorem (see, e.g., [6, Theorem 1.1.4]), there exists a unique $u\in H_0^1(\Omega)$ satisfying $\mathfrak{a}(u,\varphi)=(h,\varphi)_{H^1}$ for any $\varphi\in H_0^1(\Omega)$. In particular, u satisfies the equation (A.3) in the distribution sense. This shows $\Delta u\in L^2(\Omega)$, and hence, a standard elliptic estimate implies $u\in H^2(\Omega)$ (see, for example, Brezis [4, Theorem 9.25]). Defining v by $v=\lambda u-f\in H_0^1(\Omega)$, we find the solution $(u,v)\in D(\mathcal{A})$ to the equation (A.2).

The above properties enable us to apply the Hille–Yosida theorem (see, e.g., [19, Theorem 2.18]), and there exists a C_0-semigroup $U(t)$ on $\mathcal{H}$ satisfying the estimate

$$\left\|U(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\right\|_{\mathcal{H}}\le e^{Ct}\|(u_0,u_1)\|_{\mathcal{H}}\tag{A.4}$$

with some constant $C>0$. Moreover, if $(u_0,u_1)\in D(\mathcal{A})$, then $\mathcal{U}(t):=U(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}$ satisfies

$$\frac{d}{dt}\mathcal{U}(t)=\mathcal{A}\mathcal{U}(t),\quad t>0.\tag{A.5}$$

Therefore, the first component $u(t)$ of $\mathcal{U}(t)$ satisfies

$$u \in C([0,\infty); H^2(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$$

and the equation (A.1) in $C([0,\infty); L^2(\Omega))$.

For $(u_0, u_1) \in \mathcal{H}$, let $\mathcal{U}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} := U(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$. We next show that u satisfies

$$u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)). \tag{A.6}$$

The property $u \in C([0,\infty); H_0^1(\Omega))$ is obvious from $\mathcal{U} \in C([0,\infty); \mathcal{H})$. In order to prove $u \in C^1([0,\infty); L^2(\Omega))$, we employ an approximation argument. Let $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ be a sequence in $D(\mathcal{A})$ such that $\lim_{j\to\infty}(u_0^{(j)}, u_1^{(j)}) = (u_0, u_1)$ in $\mathcal{H}$, and let $\mathcal{U}^{(j)}(t) = \begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} := U(t) \begin{pmatrix} u_0^{(j)} \\ u_1^{(j)} \end{pmatrix}$. From $(u_0^{(j)}, u_1^{(j)}) \in D(\mathcal{A})$, $\mathcal{U}^{(j)}$ satisfies the equation (A.5), and hence, one obtains $v^{(j)} = \partial_t u^{(j)}$. For any fixed $T > 0$, the estimate (A.4) implies

$$\sup_{t\in[0,T]} \|u^{(j)}(t) - u(t)\|_{L^2} \le e^{CT} \|(u_0^{(j)} - u_0, u_1^{(j)} - u_1)\|_{\mathcal{H}} \to 0,$$
$$\sup_{t\in[0,T]} \|\partial_t u^{(j)}(t) - v(t)\|_{L^2} \le e^{CT} \|(u_0^{(j)} - u_0, u_1^{(j)} - u_1)\|_{\mathcal{H}} \to 0$$

as $j \to \infty$. This shows $u \in C^1([0,T]; L^2(\Omega))$ and $\partial_t u = v$. Since $T > 0$ is arbitrary, we obtain (A.6).

A.2. **Semilinear problem.** Let us turn to study the semilinear problem (1.1).

A.2.1. *Uniqueness of the mild solution.* We first show the uniqueness of the mild solution of the integral equation

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = U(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t U(t-s) \begin{pmatrix} 0 \\ -|u(s)|^{p-1}u(s) \end{pmatrix} ds \tag{A.7}$$

in $C([0,T_0); \mathcal{H})$ for arbitrary fixed $T_0 > 0$. Hereafter, as long as there is no risk of confusion, we call both $\mathcal{U}$ and the first component u of $\mathcal{U}$ mild solutions. Let $T_0 > 0$ and $C_0 = e^{CT_0}$, where C is the constant in (A.4). Let $\mathcal{U}(t) = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\mathcal{W}(t) = \begin{pmatrix} w \\ z \end{pmatrix}$ be two solutions to (A.7) in $C([0,T_0); \mathcal{H})$. Take $T \in (0, T_0)$ arbitrary and put $K := \sup_{t\in[0,T]}(\|\mathcal{U}(t)\|_{\mathcal{H}} + \|\mathcal{W}(t)\|_{\mathcal{H}}$. Then, the estimate (A.4) implies

$$\|\mathcal{U}(t) - \mathcal{W}(t)\|_{\mathcal{H}} \le C_0 \int_0^t \||w(s)|^{p-1}w(s) - |u(s)|^{p-1}u(s)\|_{L^2}\, ds.$$

Since the nonlinearity satisfies

$$||w|^{p-1}w - |u|^{p-1}u| \le C(|w| + |u|)^{p-1}|u - w|$$

and p fulfills the condition (1.11), we apply the Hölder and the Gagliardo–Nirenberg inequality $\|u\|_{L^{2p}} \le C\|u\|_{H^1}$ to obtain

$$\begin{aligned}\|\mathcal{U}(t) - \mathcal{W}(t)\|_{\mathcal{H}} &\le C_0 \int_0^t \||u(s)|^{p-1}u(s) - |w(s)|^{p-1}w(s)\|_{L^2}\, ds \\ &\le C_0 C \int_0^t (\|u(s)\|_{L^{2p}} + \|w(s)\|_{L^{2p}})^{p-1}\|u(s) - w(s)\|_{L^{2p}}\, ds \\ &\le C_0 C \int_0^t (\|u(s)\|_{H^1} + \|w(s)\|_{H^1})^{p-1}\|u(s) - w(s)\|_{H^1}\, ds \\ &\le C_0 C K^{p-1} \int_0^t \|\mathcal{U}(s) - \mathcal{W}(s)\|_{\mathcal{H}}\, ds \end{aligned} \tag{A.8}$$

for $t \in [0, T]$. Therefore, by the Gronwall inequality, we have $\|\mathcal{U}(t) - \mathcal{W}(t)\|_{\mathcal{H}} = 0$ for $t \in [0, T]$. Since $T \in (0, T_0)$ is arbitrary, we conclude $\mathcal{U}(t) = \mathcal{W}(t)$ for all $t \in [0, T_0)$.

A.2.2. *Existence of the mild solution.* Here, we show the existence of the mild solution.

Let $T_0 > 0$ be arbitrarily fixed. For $T \in (0, T_0)$ and $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} \in C([0, T]; \mathcal{H})$, we define the mapping

$$\mathbf{\Phi}(\mathcal{U})(t) = U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u(s)|^{p-1}u(s) \end{pmatrix} ds.$$

Let $C_0 = e^{CT_0}$, where C is the constant in (A.4). Then, we have

$$\left\| U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\mathcal{H}} \le C_0 \|(u_0, u_1)\|_{\mathcal{H}}$$

for $t \in (0, T_0)$. Let $K = 2C_0\|(u_0, u_1)\|_{\mathcal{H}}$ and define

$$M_{T,K} := \left\{ \mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} \in C([0, T]; \mathcal{H});\ \sup_{t \in [0,T]} \|(u(t), v(t))\|_{\mathcal{H}} \le K \right\}.$$

$M_{T,K}$ is a complete metric space with respect to the metric

$$d(\mathcal{U}, \mathcal{W}) = \sup_{t \in [0,T]} \|(u(t) - w(t), v(t) - z(t))\|_{\mathcal{H}}$$

for $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\mathcal{W} = \begin{pmatrix} w \\ z \end{pmatrix}$. We shall prove that Φ is the contraction mapping on $M_{T,R}$, provided that T is sufficiently small.

First, we show that $\mathbf{\Phi}(\mathcal{U}) \in M_{T,K}$ for $\mathcal{U} \in M_{T,K}$. By the estimate (A.4) and the Gagliardo–Nirenberg inequality, we obtain for $t \in [0, T]$,

$$\begin{aligned}\|\mathbf{\Phi}(\mathcal{U})(t)\|_{\mathcal{H}} &\le \frac{K}{2} + C_0 \int_0^t \||u(s)|^{p-1}u(s)\|_{L^2}\, ds \\ &\le \frac{K}{2} + C_0 \int_0^t \|u(s)\|_{L^{2p}}^p\, ds \\ &\le \frac{K}{2} + C_0 C \int_0^t \|u(s)\|_{H^1}^p\, ds \\ &\le \frac{K}{2} + C_0 C T K^p. \end{aligned} \tag{A.9}$$

Therefore, taking T sufficiently small so that

$$\frac{K}{2} + C_0 C T K^p \le K$$

holds, we see that $\mathbf{\Phi}(\mathcal{U}) \in M_{T,K}$. Moreover, for $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathcal{W} = \begin{pmatrix} w \\ z \end{pmatrix} \in M_{T,R}$, the same computation as in (A.8) yields for $t \in [0,T]$,

$$d(\mathbf{\Phi}(\mathcal{U}), \mathbf{\Phi}(\mathcal{W})) \le C_0 C T K^{p-1} d(\mathcal{U}, \mathcal{W}).$$

Thus, retaking T smaller if needed so that

$$C_0 C T K^{p-1} \le \frac{1}{2},$$

we have the contractivity of $\mathbf{\Phi}$. Thus, by the contraction mapping principle, we see that there exists a fixed point $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} \in M_{T,K}$, that is, $\mathcal{U}$ satisfies the integral equation (A.7). We postpone to verify $u \in C^1([0,T]; L^2(\Omega))$ and $\partial_t u = v$ after proving the approximation property below.

A.2.3. *Blow-up alternative.* Let $T_{\max} = T_{\max}(u_0, u_1)$ be the maximal existence time of the mild solution defined by

$$T_{\max} = \sup \left\{ T \in (0,\infty];\ {}^{\exists}\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix} \in C([0,T); \mathcal{H}) \text{ satisfies (A.7)} \right\}.$$

We show that if $T_{\max} < \infty$, the corresponding unique mild solution $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ must satisfy

$$\lim_{t \to T_{\max}-0} \|\mathcal{U}(t)\|_{\mathcal{H}} = \infty. \tag{A.10}$$

Indeed, if $m := \liminf_{t \to T_{\max}-0} \|\mathcal{U}(t)\|_{\mathcal{H}} < \infty$, then there exists a monotone increasing sequence $\{t_j\}_{j=1}^{\infty}$ in $(0, T_{\max})$ such that $\lim_{j\to\infty} t_j = T_{\max}$ and $\lim_{j\to\infty} \|\mathcal{U}(t_j)\|_{\mathcal{H}} = m$. Let $T_0 > T_{\max}$ be arbitrary fixed and let $C_0 = e^{CT_0}$ as in Section A.2.2. Applying the same argument as in Section A.2.2 with replacement (u_0, u_1) by $\mathcal{U}(t_j)$, one can find there exists T depending only on p, m, and C_0 such that there exists a mild solution on the interval $[t_j, t_j + T]$. However, this contradicts the definition of $T_{\max}$ when j is large. Thus, we have (A.10).

A.2.4. *Continuous dependence on the initial data.* Let $(u_0, u_1) \in \mathcal{H}$ and $T < T_0 < T_{\max}(u_0, u_1)$. We take $C_0 = e^{CT_0}$ as in Section A.2.2. Let $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ be a sequence in $\mathcal{H}$ such that $(u_0^{(j)}, u_1^{(j)}) \to (u_0, u_1)$ in $\mathcal{H}$ as $j \to \infty$. Then, we will prove that, for sufficiently large j, $T_{\max}(u_0^{(j)}, u_1^{(j)}) > T$ and the corresponding solution $\mathcal{U}^{(j)}$ with the initial data $(u_0^{(j)}, u_1^{(j)})$ satisfies

$$\lim_{j\to\infty} \sup_{t \in [0,T]} \|\mathcal{U}^{(j)}(t) - \mathcal{U}(t)\|_{\mathcal{H}} = 0. \tag{A.11}$$

Let $C_1 = 2 \sup_{t\in[0,T]} \|\mathcal{U}(t)\|_{\mathcal{H}}$ and let

$$\tau_j := \sup \left\{ t \in [0, T_{\max}(u_0^{(j)}, u_1^{(j)}));\ \sup_{t \in [0,T]} \|\mathcal{U}^{(j)}(t)\|_{\mathcal{H}} \le 2C_1 \right\}.$$

Since $(u_0^{(j)}, u_1^{(j)}) \to (u_0, u_1)$ in $\mathcal{H}$ as $j \to \infty$, we have $\|(u_0^{(j)}, u_1^{(j)})\|_{\mathcal{H}} \le C_1$ for large j, which ensures $\tau_j > 0$ for such j. Moreover, the same computation as in (A.8) and the Gronwall inequality imply, for $t \in [0, \min\{\tau_j, T\}]$,

$$\|\mathcal{U}^{(j)}(t) - \mathcal{U}(t)\|_{\mathcal{H}} \le C_0 \|\mathcal{U}^{(j)}(0) - \mathcal{U}(0)\|_{\mathcal{H}} \exp\left(CC_1^{p-1}T\right). \tag{A.12}$$

Note that the right-hand side tends to zero as $j \to \infty$. From this and the definition of C_1, we obtain

$$\|\mathcal{U}^{(j)}(t)\|_{\mathcal{H}} \le C_1 \quad (t \in [0, \min\{\tau_j, T\}])$$

for large j. By the definition of τ_j, the above estimate implies $\tau_j > T$, and hence, $T_{\max}(u_0^{(j)}, u_1^{(j)}) > T$. From this, the estimate (A.12) holds for $t \in [0, T]$. Letting $j \to \infty$ in (A.12) gives (A.11).

A.2.5. *Regularity of solution.* Next, we discuss the regularity of the solution. Let $(u_0, u_1) \in D(\mathcal{A})$ and $T_{\max} = T_{\max}(u_0, u_1)$. Then, we will show that the corresponding mild solution $\mathcal{U}$ satisfies

$$\mathcal{U} \in C([0, T_{\max}); D(\mathcal{A})) \cap C^1([0, T_{\max}); \mathcal{H}).$$

Take $T \in (0, T_{\max})$ arbitrary. First, from Section A.1, the linear part of the mild solution satisfies $\mathcal{U}_L(t) = U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in C([0,\infty); D(\mathcal{A})) \cap C^1([0,\infty); \mathcal{H})$. This implies, for $h > 0$ and $t \in [0, T-h]$,

$$\|\mathcal{U}_L(t+h) - \mathcal{U}_L(t)\|_{\mathcal{H}} \le Ch. \tag{A.13}$$

Thus, it suffices to show

$$\begin{aligned} \mathcal{U}_{NL}(t) &:= \int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u(s)|^{p-1}u(s) \end{pmatrix} ds \\ &\in C([0,T]; D(\mathcal{A})) \cap C^1([0,T]; \mathcal{H}). \end{aligned} \tag{A.14}$$

By the changing variable $t + h - s \mapsto s$, we calculate

$$\begin{aligned} \mathcal{U}_{NL}(t+h) - \mathcal{U}_{NL}(t) &= \int_0^{t+h} U(t-s)\begin{pmatrix} 0 \\ -|u(s)|^{p-1}u(s) \end{pmatrix} ds \\ &\quad - \int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u(s)|^{p-1}u(s) \end{pmatrix} ds \\ &= \int_0^t U(s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(t+h-s) + |u|^{p-1}u(t-s) \end{pmatrix} ds \\ &\quad + \int_t^{t+h} U(s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(t+h-s) \end{pmatrix} ds. \end{aligned}$$

Therefore, the same computation as in (A.8) and (A.9) implies

$$\|\mathcal{U}_{NL}(t+h) - \mathcal{U}_{NL}(t)\|_{\mathcal{H}} \le C\int_0^t \|u(s+h) - u(s)\|_{H^1}\, ds + Ch.$$

Combining this with (A.13), one obtains

$$\|\mathcal{U}(t+h) - \mathcal{U}(t)\|_{\mathcal{H}} \le Ch + \int_0^t \|\mathcal{U}(s+h) - \mathcal{U}(s)\|_{\mathcal{H}}\, ds.$$

The Gronwall inequality implies

$$\|\mathcal{U}(t+h) - \mathcal{U}(t)\|_{\mathcal{H}} \le Ch.$$

This further yields

$$\| - |u|^{p-1}u(t+h) + |u|^{p-1}u(t)\|_{H^1} \le Ch,$$

that is, the nonlinearity is Lipschitz continuous in $H_0^1(\Omega)$. From this, we can see $-|u|^{p-1}u \in W^{1,\infty}(0,T;H_0^1(\Omega))$ (see e.g. [6, Corollary 1.4.41]). Thus, we can differentiate the expression

$$\int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(s) \end{pmatrix} ds = \int_0^t U(s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(t-s) \end{pmatrix} ds$$

with respect to t in $\mathcal{H}$, and it implies $\mathcal{U}_{NL} \in C^1([0,T];\mathcal{H})$. Finally, for $h > 0$ and $t \in [0, T-h]$, we have

$$\begin{aligned} \frac{1}{h}\left(U(t) - I\right)\mathcal{U}_{NL}(t) &= \frac{1}{h}\int_0^t U(t+h-s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(s) \end{pmatrix} ds - \frac{1}{h}\int_0^t U(t-s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(s) \end{pmatrix} ds \\ &= \frac{1}{h}\left(\mathcal{U}_{NL}(t+h) - \mathcal{U}_{NL}(t)\right) - \frac{1}{h}\int_t^{t+h} U(t+h-s)\begin{pmatrix} 0 \\ -|u|^{p-1}u(s) \end{pmatrix} ds. \end{aligned}$$

This implies $\mathcal{U}(t) \in D(\mathcal{A})$ and

$$\frac{d}{dt}\mathcal{U}_{NL}(t) = \mathcal{A}\mathcal{U}_{NL}(t) + \begin{pmatrix} 0 \\ -|u|^{p-1}u(t) \end{pmatrix}.$$

Moreover, the above equation and $\mathcal{U} \in C^1([0,T];\mathcal{H})$ lead to $\mathcal{U} \in C([0,T];D(\mathcal{A}))$. This proves the property (A.14). We also remark that the first component u of $\mathcal{U}$ is a strong solution to (1.1).

A.2.6. *Approximation of the mild solution by strong solutions.* Let $(u_0, u_1) \in \mathcal{H}$ and $T_{\max} = T_{\max}(u_0,u_1)$. Let $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ be a sequence in $D(\mathcal{A})$ satisfying $\lim_{j\to\infty}(u_0^{(j)}, u_1^{(j)}) = (u_0, u_1)$ in $\mathcal{H}$. Take $T \in (0, T_{\max})$ arbitrary. Then, the results of Sections A.2.4 and A.2.5 imply that $T_{\max}(u_0^{(j)}, u_1^{(j)}) > T$ for large j, and the corresponding mild solution $\mathcal{U}^{(j)} = \begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix}$ with the initial data $(u_0^{(j)}, u_1^{(j)})$ satisfies $\mathcal{U}^{(j)} \in C([0,T];D(\mathcal{A})) \cap C^1([0,T];\mathcal{H})$. Moreover, $\partial_t u^{(j)} = v^{(j)}$ holds and $u^{(j)}$ is a strong solution to (1.1). By the result of Section A.2.4, we see that

$$\begin{aligned} &\lim_{j\to\infty}\sup_{t\in[0,T]}\|u^{(j)}(t) - u(t)\|_{H^1} = 0, \\ &\lim_{j\to\infty}\sup_{t\in[0,T]}\|\partial_t u^{(j)}(t) - v(t)\|_{L^2} = 0, \end{aligned}$$

which yields $u \in C^1([0,T];L^2(\Omega))$ and $\partial_t u = v$. Namely, we have the property stated at the end of Section A.2.2.

A.2.7. *Finite propagation property.* Here, we show the finite propagation property for the mild solution. In what follows, we use the notations $B_R(x_0) := \{x \in \mathbb{R}^n; |x-x_0| < R\}$ for $x_0 \in \mathbb{R}^n$ and $R > 0$. Let $T \in (0, T_{\max}(u_0,u_1))$ and $R > 0$. Assume that $(u_0,u_1) \in \mathcal{H}$ satisfies $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B_R(0) \cap \Omega$. Let $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ be the mild solution of (1.1). Then, we have

$$\operatorname{supp} u(t,\cdot) \subset B_{t+R}(0) \cap \Omega \quad (t \in [0,T]). \tag{A.15}$$

To prove this, we modify the argument of [39] in which the classical solution is treated. Let $(t_0,x_0) \in [0,T] \times \Omega$ be a point such that $|x_0| > t_0 + R$ and define

$$\begin{aligned}\Lambda(t_0,x_0) &= \{(t,x) \in (0,T) \times \Omega; \, 0 < t < t_0, |x-x_0| < t_0 - t\} \\ &= \bigcup_{t \in (0,t_0)} (\{t\} \times (B_{t_0-t}(x_0) \cap \Omega))).\end{aligned}$$

It suffices to show $u = 0$ in $\Lambda(t_0,x_0)$. We also put $S_{t_0-t} := \partial B_{t_0-t}(x_0) \cap \Omega$ and $S_{b,t_0-t} := B_{t_0-t}(x_0) \cap \partial\Omega$. Note that $\partial(B_{t_0-t}(x_0) \cap \Omega) = \overline{S_{t_0-t} \cup S_{b,t_0-t}}$ holds.

First, we further assume $(u_0,u_1) \in D(\mathcal{A})$. Then, by the result of Section A.2.5, u becomes the strong solution. This ensures that the following computations make sense.

Define

$$\mathcal{E}(t;t_0,x_0) := \frac{1}{2} \int_{B_{t_0-t}(x_0) \cap \Omega} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2) \, dx$$

for $t \in [0,t_0]$. By differentiating in t and applying the integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t;t_0,x_0) = &\int_{B_{t_0-t}(x_0) \cap \Omega} \left(\partial_t^2 u - \Delta u + u\right) \partial_t u \, dx \\ &- \frac{1}{2} \int_{S_{t_0-t} \cup S_{b,t_0-t}} (|\partial_t u|^2 + |\nabla u|^2 + |u|^2 - 2(\mathbf{n} \cdot \nabla u) \partial_t u) \, dS,\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector of $S_{t_0-t} \cup S_{b,t_0-t}$ and dS denotes the surface measure. The Schwarz inequality implies the second term of the right-hand side is nonpositive, and hence, we can omit it. Using the equation (1.1) to the first term and the Gagliardo–Nirenberg inequality $\|u(t)\|_{L^{2p}(B_{t_0-t}(x_0) \cap \Omega)} \le C\|u(t)\|_{H^1(B_{t_0-t}(x_0) \cap \Omega)}$, we can see that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t;t_0,x_0) &\le C \left(\|u(t)\|_{H^1(B_{t_0-t}(x_0) \cap \Omega)}^{2p} + \|\partial_t u(t)\|_{L^2(B_{t_0-t}(x_0) \cap \Omega)}^2 + \|u(t)\|_{L^2(B_{t_0-t}(x_0) \cap \Omega)}^2 \right) \\ &\le C\mathcal{E}(t;t_0,x_0),\end{aligned}$$

where we have also used $\|u(t)\|_{H^1(B_{t_0-t}(x_0) \cap \Omega)}$ is bounded for $t \in (0,t_0)$. Noting that the support condition of the initial data implies $\mathcal{E}(0;t_0,x_0) = 0$, we obtain from the above inequality that $\mathcal{E}(t;t_0,x_0) = 0$ for $t \in [0,t_0]$. This yields $u = 0$ in $\Lambda(t_0,x_0)$.

Finally, for the general case $(u_0,u_1) \in \mathcal{H}$, we take an arbitrary small $\varepsilon > 0$ and a sequence $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ in $D(\mathcal{A})$ such that $\operatorname{supp} u_0^{(j)} \cup \operatorname{supp} u_1^{(j)} \subset B_{R+\varepsilon}(0) \cap \Omega$ and $\lim_{j\to\infty} (u_0^{(j)}, u_1^{(j)}) = (u_0,u_1)$ in $\mathcal{H}$. Here, we remark that such a sequence can be constructed by the form $(u_0^{(j)}, u_1^{(j)}) = (\phi_\varepsilon \tilde{u}_0^{(j)}, \phi_\varepsilon \tilde{u}_1^{(j)})$, where $\{(\tilde{u}_0^{(j)}, \tilde{u}_1^{(j)})\}$ is a sequence in $D(\mathcal{A})$ which converges to (u_0,u_1) in $\mathcal{H}$ as $j \to \infty$, and $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function satisfy $0 \le \phi_\varepsilon \le 1$, $\phi_\varepsilon = 1$ on $B_R(0)$, and $\phi_\varepsilon = 0$ on $\mathbb{R}^n \setminus B_{R+\varepsilon}(0)$.

Then, the result of Section A.2.5 shows that the corresponding strong solution $u^{(j)}$ to $(u_0^{(j)}, u_1^{(j)})$ satisfies $\operatorname{supp} u^{(j)}(t,\cdot) \subset B_{R+\varepsilon+t}(0)$. Moreover, the result of Section A.2.6 leads to $\lim_{j\to\infty} u^{(j)} = u$ in $C([0,T]; H_0^1(\Omega))$. Hence, we conclude $\operatorname{supp} u(t,\cdot) \subset B_{R+\varepsilon+t}(0)$. Since ε is arbitrary, we have (A.15).

A.2.8. *Existence of the global solution.* Finally, we show the existence of the global solution to (1.1). Let $(u_0, u_1) \in \mathcal{H}$ and suppose that $T_{\max}(u_0, u_1)$ is finite. Then, by the blow-up alternative (Section A.2.3), the corresponding mild solution u must satisfy

$$\lim_{t\to T_{\max}-0} \|(u(t), \partial_t u(t))\|_{\mathcal{H}} = \infty. \tag{A.16}$$

Let $\{(u_0^{(j)}, u_1^{(j)})\}_{j=1}^{\infty}$ be a sequence in $D(\mathcal{A})$ such that $\lim_{j\to\infty}(u_0^{(j)}, u_1^{(j)}) = (u_0, u_1)$ in $\mathcal{H}$, and let $u^{(j)}$ be the corresponding strong solution with the initial data $(u_0^{(j)}, u_1^{(j)})$.

Using the integration by parts and the equation (1.1), we calculate

$$\frac{d}{dt}\left[\frac{1}{2}\left(\|\partial_t u^{(j)}(t)\|_{L^2}^2 + \|\nabla u^{(j)}(t)\|_{L^2}^2\right) + \frac{1}{p+1}\|u^{(j)}(t)\|_{L^{p+1}}^{p+1}\right] = -\|\partial_t u^{(j)}(t)\|_{L^2}^2.$$

This and the Gagliardo–Nirenberg inequality imply

$$\|\partial_t u^{(j)}(t)\|_{L^2}^2 + \|\nabla u^{(j)}(t)\|_{L^2}^2 \le C\left(\|u_1^{(j)}\|_{L^2}^2 + \|\nabla u_0^{(j)}\|_{L^2}^2 + \|u_0^{(j)}\|_{H^1}^{p+1}\right).$$

Moreover, by

$$u(t) = u_0 + \int_0^t \partial_t u(s)\,ds,$$

one obtains the bound

$$\|(u^{(j)}(t), \partial_t u^{(j)}(t))\|_{\mathcal{H}}^2 \le C(1+T)^2\left(\|u_1^{(j)}\|_{L^2}^2 + \|\nabla u_0^{(j)}\|_{L^2}^2 + \|u_0^{(j)}\|_{H^1}^{p+1}\right) \tag{A.17}$$

for $t \in [0,T]$. This and the blow-up alternative (Section A.2.3) show $T_{\max}(u_0^{(j)}, u_1^{(j)}) = \infty$ for all j. The bound (A.17) with $T = T_{\max}(u_0, u_1)$ also yields that

$$\sup_{j\in\mathbb{N}} \sup_{t\in[0,T_{\max}(u_0,u_1)]} \|(u^{(j)}(t), \partial_t u^{(j)}(t))\|_{\mathcal{H}}^2 < \infty. \tag{A.18}$$

On the other hand, from the result of Section A.2.6, we have

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \|(u^{(j)}(t) - u(t), \partial_t u^{(j)}(t) - \partial_t u(t))\|_{\mathcal{H}} = 0 \tag{A.19}$$

for any $T \in (0, T_{\max}(u_0, u_1))$. However, (A.18) and (A.19) contradict (A.16). Thus, we conclude $T_{\max}(u_0, u_1) = \infty$.

Appendix B. Proof of Preliminary lemmas

B.1. **Proof of Lemma 2.1.**

Proof of Lemma 2.1. We define

$$b_1(x) = \Delta\left(\frac{a_0}{(n-\alpha)(2-\alpha)}\langle x\rangle^{2-\alpha}\right) = a_0\langle x\rangle^{-\alpha} + \frac{a_0\alpha}{n-\alpha}\langle x\rangle^{-\alpha-2}$$

and $b_2(x) = a(x) - b_1(x)$. By

$$\frac{b_2(x)}{a(x)} = \frac{1}{\langle x\rangle^{\alpha} a(x)}\left(\langle x\rangle^{\alpha} a(x) - a_0 - \frac{a_0\alpha}{n-\alpha}\langle x\rangle^{-2}\right)$$

and the assumption (1.12), there exists a constant $R_\varepsilon > 0$ such that $|b_2(x)| \le \varepsilon a(x)$ holds for $|x| > R_\varepsilon$. Let $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \eta_\varepsilon(x) \le 1$ for $x \in \mathbb{R}^n$ and $\eta_\varepsilon(x) = 1$ for $|x| < R_\varepsilon$. Let $N(x)$ denote the Newton potential, that is,

$$N(x) = \begin{cases} \dfrac{|x|}{2} & (n=1), \\ \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & (n=2), \\ \dfrac{\Gamma(n/2+1)}{n(n-2)\pi^{n/2}}|x|^{2-n} & (n \ge 3). \end{cases}$$

We define

$$A_\varepsilon(x) = A_0 + \frac{a_0}{(n-\alpha)(2-\alpha)}\langle x\rangle^{2-\alpha} - N * (\eta_\varepsilon b_2),$$

where $A_0 > 0$ is a sufficiently large constant determined later. We show that the above $A_\varepsilon(x)$ has the desired properties. First, we compute

$$\Delta A_\varepsilon(x) = b_1(x) + \eta_\varepsilon(x) b_2(x) = a(x) - (1-\eta_\varepsilon) b_2(x),$$

which implies (2.1). Next, since $\eta_\varepsilon b_2$ has the compact support, $N * (\eta_\varepsilon b_2)$ satisfies

$$|N * (\eta_\varepsilon b_2)(x)| \le C \begin{cases} 1 + \log\langle x\rangle & (n=2) \\ \langle x\rangle^{2-n} & (n=1, n\ge 3) \end{cases}, \quad |\nabla N * (\eta_\varepsilon b_2)(x)| \le C\langle x\rangle^{1-n}$$

with some constant $C = C(n, R_\varepsilon, \|a\|_{L^\infty}, \alpha, a_0, \varepsilon) > 0$, and the former estimate leads to (2.2), provided that A_0 is sufficiently large. Moreover, the latter estimate shows

$$\begin{aligned} \lim_{|x|\to\infty} \frac{|\nabla A_\varepsilon(x)|^2}{a(x) A_\varepsilon(x)} &= \lim_{|x|\to\infty} \frac{1}{\langle x\rangle^{\alpha} a(x)} \cdot \frac{1}{\langle x\rangle^{\alpha-2} A_\varepsilon(x)} \left| \frac{a_0}{n-\alpha}\langle x\rangle^{-1} x - \langle x\rangle^{\alpha-1} \nabla N * (\eta_\varepsilon b_2) \right|^2 \\ &= \frac{2-\alpha}{n-\alpha}, \end{aligned}$$

which implies the inequality (2.3) for sufficiently large x. Finally, taking A_0 sufficiently large, we have (2.3) for any $x \in \mathbb{R}^n$. □

B.2. **Properties of Kummer's function.** To prove Lemma 2.4, we prepare some properties of Kummer's function.

Lemma B.1. *Kummer's confluent hypergeometric function $M(b,c;s)$ satisfies the properties listed as follows.*

(i) *$M(b,c;s)$ satisfies Kummer's equation*

$$su''(s) + (c-s)u'(s) - bu(s) = 0.$$

(ii) *If $c \ge b > 0$, then $M(b,c;s) > 0$ for $s \ge 0$ and*

$$\lim_{s\to\infty} \frac{M(b,c;s)}{s^{b-c}e^s} = \frac{\Gamma(c)}{\Gamma(b)}. \tag{B.1}$$

In particular, $M(b,c;s)$ satisfies

$$C(1+s)^{b-c}e^s \le M(b,c;s) \le C'(1+s)^{b-c}e^s \tag{B.2}$$

with some positive constants $C = C(b,c)$ and $C' = C(b,c)'$.

(iii) *More generally, if* $-c \notin \mathbb{N} \cup \{0\}$ *and* $c \ge b$*, then, while the sign of* $M(b,c;s)$ *is indefinite, it still has the asymptotic behavior*

$$\lim_{s\to\infty} \frac{M(b,c;s)}{s^{b-c}e^s} = \frac{\Gamma(c)}{\Gamma(b)}, \tag{B.3}$$

where we interpret that the right-hand side is zero if $-b \in \mathbb{N} \cup \{0\}$*. In particular,* $M(b,c;s)$ *has a bound*

$$|M(b,c;s)| \le C(1+s)^{b-c}e^s \tag{B.4}$$

with some positive constant $C = C(b,c)$*.*

(iv) $M(b,c;s)$ *satisfies the relations*

$$sM(b,c;s) = sM'(b,c;s) + (c-b)M(b,c;s) - (c-b)M(b-1,c;s),$$
$$cM'(b,c;s) = cM(b,c;s) - (c-b)M(b,c+1;s).$$

Proof. The property (i) is directly obtained from the definition of $M(b,c;s)$. When $c = b > 0$, (ii) is obvious from $M(b,b;s) = e^s$. When $c > b > 0$, we have the integral representation (see [3, (6.1.3)])

$$M(b,c;s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}e^{ts}\,dt,$$

which implies $M(b,c;s) > 0$. Moreover, [3, (6.1.8)] shows the asymptotic behavior (B.1). The estimate (B.2) is obvious, since the right-hand side of (B.1) is positive and $M(b,c;s) > 0$ for $s \ge 0$. Next, the property (iii) clearly holds if $c = b$ or $-b \in \mathbb{N} \cup \{0\}$, since $M(b,c;s)$ is a polynomial of order $-b$ if $-b \in \mathbb{N} \cup \{0\}$. For the cases $c > b$ and $-b \notin \mathbb{N} \cup \{0\}$, note that for any $m \in \mathbb{N} \cup \{0\}$ we have

$$\frac{d^m}{ds^m} M(b,c;s) = \frac{(b)_m}{(c)_m} M(b+m, c+m; s),$$

which implies $|\frac{d^m}{ds^m} M(b,c;s)| \to \infty$ as $s \to \infty$. By taking $m \in \mathbb{N} \cup \{0\}$ so that $b + m > 0$ and applying l'Hôpital theorem we deduce

$$\begin{aligned}\lim_{s\to\infty} \frac{M(b,c;s)}{s^{b-c}e^s} &= \lim_{s\to\infty} \frac{\frac{d^m}{ds^m} M(b,c;s)}{\frac{d^m}{ds^m}(s^{b-c}e^s)} = \frac{(b)_m}{(c)_m} \lim_{s\to\infty} \frac{M(b+m,c+m;s)}{s^{b-c}e^s + o(s^{b-c}e^s)} \\ &= \frac{(b)_m \Gamma(c+m)}{(c)_m \Gamma(b+m)} = \frac{\Gamma(c)}{\Gamma(b)}.\end{aligned}$$

The estimate (B.4) is easily follows from the asymptotic behavior (B.3) and we have (iii). Finally, the property (iv) can be found in [3, p.200]. □

B.3. **Proof of Lemma 2.4.**

Proof of Lemma 2.4. The property (i) is directly follows from Lemma B.1 (i). For (ii), noting that $0 \le \beta < \gamma_\varepsilon$ and applying Lemma B.1 (ii) with $b = \gamma_\varepsilon - \beta$ and $c = \gamma_\varepsilon$, we have $\varphi_\beta(s) > 0$ for $s \ge 0$ and

$$\lim_{s\to\infty} s^\beta \varphi_{\beta,\varepsilon}(s) = \frac{\Gamma(\gamma_\varepsilon)}{\Gamma(\gamma_\varepsilon - \beta)}.$$

This proves the property (ii). Next, by Lemma B.1 (iii) with $b = \gamma_\varepsilon - \beta$ and $c = \gamma_\varepsilon$, one still obtains $\lim_{s\to\infty} s^\beta \varphi_\varepsilon(s) = \Gamma(\gamma_\varepsilon)/\Gamma(\gamma_\varepsilon - \beta)$, where the right-hand side is interpreted as zero if $\beta - \gamma_\varepsilon \in \mathbb{N} \cup \{0\}$. In particular, this (or the estimate (B.4)) gives

$$|\varphi_{\beta,\varepsilon}(s)| \le K_{\beta,\varepsilon}(1+s)^{-\beta}$$

with some constant $K_{\beta,\varepsilon} > 0$. Thus, we have (iii). Noting that

$$\varphi'_{\beta,\varepsilon}(s) = e^{-s}\left[-M(\gamma_\varepsilon - \beta, \gamma_\varepsilon; s) + M'(\gamma_\varepsilon - \beta, \gamma_\varepsilon; s)\right] \tag{B.5}$$

and applying the first assertion of Lemma B.1 (iv), we have the property (iv). Finally, from (B.5) and the second assertion of Lemma B.1 (iv), we obtain

$$\gamma_\varepsilon \varphi'_{\beta,\varepsilon}(s) = -\beta e^{-s} M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s).$$

Differentiating again the above identity gives

$$\gamma_\varepsilon \varphi''_{\beta,\varepsilon}(s) = -\beta e^{-s}\left[-M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s) + M'(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s)\right].$$

Therefore, the second assertion of Lemma B.1 (iv) implies

$$\gamma_\varepsilon(\gamma_\varepsilon + 1)\varphi''_{\beta,\varepsilon}(s) = \beta(\beta+1)e^{-s} M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 2; s).$$

In particular, if $0 < \beta < \gamma_\varepsilon$, then Lemma B.1 (ii) shows that $M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s)$ (resp. $M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 2; s)$) is bounded from above and below by $(1+s)^{-\beta-1}e^s$ (resp. $(1+s)^{-\beta-2}e^s$), and hence, we have the assertions of (v). □

B.4. **Proof of Proposition 2.6.** We are now in a position to prove Proposition 2.6.

Proof of Proposition 2.6. Let $z = \widetilde{\gamma}_\varepsilon A_\varepsilon(x)/(t_0+t)$. From Definition 2.5 and Lemma 2.4 (iv), one obtains

$$\begin{aligned}
\partial_t \Phi_{\beta,\varepsilon}(t,x;t_0) &= -(t_0+t)^{-\beta-1}\left[\beta\varphi_{\beta,\varepsilon}(z) + z\varphi'_{\beta,\varepsilon}(z)\right] \\
&= -(t_0+t)^{-\beta-1}\beta\varphi_{\beta+1,\varepsilon}(z) \\
&= -\beta\Phi_{\beta+1,\varepsilon}(t,x;t_0),
\end{aligned}$$

which proves (i). Applying Lemma 2.4 (iii), we have

$$\begin{aligned}
|\Phi_{\beta,\varepsilon}(t,x;t_0)| &\le K_{\beta,\varepsilon}(t_0+t)^{-\beta}\left(1 + \frac{\widetilde{\gamma}_\varepsilon A_\varepsilon(x)}{t_0+t}\right)^{-\beta} \\
&\le C\left(t_0 + t + A_\varepsilon(x)\right)^{-\beta} \\
&= C\Psi(t,x;t_0)^{-\beta}
\end{aligned}$$

with some constant $C = C(n,\alpha,\beta,\varepsilon) > 0$. This implies (ii). Next, by Lemma 2.4 (ii), $\Phi_{\beta,\varepsilon}(t,x;t_0)$ satisfies

$$\begin{aligned}
\Phi_{\beta,\varepsilon}(t,x;t_0) &\ge k_{\beta,\varepsilon}(t_0+t)^{-\beta}\left(1 + \frac{\widetilde{\gamma}_\varepsilon A_\varepsilon(x)}{t_0+t}\right)^{-\beta} \\
&\ge c\left(t_0 + t + A_\varepsilon(x)\right)^{-\beta} \\
&= c\Psi(t,x;t_0)^{-\beta}
\end{aligned}$$

with some constant $c = c(n,\alpha,\beta,\varepsilon) > 0$, and (iii) is verified. For (iv), we again put $z = \tilde{\gamma}_\varepsilon A_\varepsilon(x)/(t_0+t)$ and compute

$$\begin{aligned}
&a(x)\partial_t \Phi_{\beta,\varepsilon}(x,t;t_0) - \Delta \Phi_{\beta,\varepsilon}(x,t;t_0) \\
&= -a(x)(t_0+t)^{-\beta-1} \\
&\times \left(\beta \varphi_{\beta,\varepsilon}(z) + z\varphi'_{\beta,\varepsilon}(z) + \tilde{\gamma}_\varepsilon \frac{\Delta A_\varepsilon(x)}{a(x)} \varphi'_{\beta,\varepsilon}(z) + \tilde{\gamma}_\varepsilon \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} z\varphi''_{\beta,\varepsilon}(z) \right).
\end{aligned}$$

Using the equation (2.5) and the definition (2.4), we rewrite the right-hand side as

$$\begin{aligned}
&\tilde{\gamma}_\varepsilon a(x)(t_0+t)^{-\beta-1} \left(1 - 2\varepsilon - \frac{\Delta A_\varepsilon(x)}{a(x)} \right) \varphi'_{\beta,\varepsilon}(z) \\
&\quad + a(x)(t_0+t)^{-\beta-1} \left(1 - \tilde{\gamma}_\varepsilon \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} \right) \varphi''_{\beta,\varepsilon}(z).
\end{aligned}$$

By (2.1) and (2.3) in Lemma 2.1, we have

$$\begin{aligned}
&1 - 2\varepsilon - \frac{\Delta A_\varepsilon(x)}{a(x)} \le -\varepsilon, \\
&1 - \tilde{\gamma}_\varepsilon \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} \ge \varepsilon \left(\frac{2-\alpha}{n-\alpha} + 2\varepsilon \right)^{-1} > 0.
\end{aligned}$$

From them and the property (v) of Lemma 2.4, we conclude

$$\begin{aligned}
a(x)\partial_t \Phi_{\beta,\varepsilon}(x,t;t_0) - \Delta \Phi_{\beta,\varepsilon}(x,t;t_0) &\ge -\varepsilon \tilde{\gamma}_\varepsilon a(x)(t_0+t)^{-\beta-1} \varphi'_{\beta,\varepsilon}\left(\frac{\tilde{\gamma}_\varepsilon A_\varepsilon(x)}{t_0+t} \right) \\
&\ge \varepsilon k_{\beta,\varepsilon} a(x)(t_0+t)^{-\beta-1} \left(1 + \frac{\tilde{\gamma}_\varepsilon A_\varepsilon(x)}{t_0+t} \right)^{-\beta-1} \\
&\ge c a(x) \left(t_0 + t + A_\varepsilon(x) \right)^{-\beta-1} \\
&= c a(x) \Psi(x,t;t_0)^{-\beta-1}
\end{aligned}$$

with some constant $c = c(n,\alpha,\beta,\varepsilon) > 0$, which completes the proof. □

B.5. **Proof of Lemma 2.7.**

Proof of Lemma 2.7. Putting $v = \Phi^{-1+\delta}u$, noting $\nabla u = (1-\delta)\Phi^{-\delta}(\nabla\Phi)v + \Phi^{1-\delta}\nabla v$, and applying integration by parts imply

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 \Phi^{-1+2\delta}\,dx \\
&= \int_\Omega |\nabla v|^2 \Phi\,dx + 2(1-\delta)\int_\Omega v(\nabla v\cdot\nabla\Phi)\,dx + (1-\delta)^2 \int_\Omega |v|^2 \frac{|\nabla\Phi|^2}{\Phi}\,dx \\
&= \int_\Omega |\nabla v|^2 \Phi\,dx - (1-\delta)\int_\Omega |v|^2 \Delta\Phi\,dx + (1-\delta)^2 \int_\Omega |v|^2 \frac{|\nabla\Phi|^2}{\Phi}\,dx \\
&\ge -(1-\delta)\int_\Omega |u|^2 (\Delta\Phi)\Phi^{-2+2\delta}\,dx + (1-\delta)^2 \int_\Omega |u|^2 |\nabla\Phi|^2 \Phi^{-3+2\delta}\,dx.
\end{aligned}$$

By $u\Delta u = -|\nabla u|^2 + \Delta(\frac{u^2}{2})$, integration by parts, and applying the above estimate, we have

$$\begin{aligned}
&\int_\Omega u\Delta u \Phi^{-1+2\delta}\,dx \\
&= -\int_\Omega |\nabla u|^2 \Phi^{-1+2\delta}\,dx + \frac{1}{2}\int_\Omega |u|^2 \Delta(\Phi^{-1+2\delta})\,dx \\
&= -\int_\Omega |\nabla u|^2 \Phi^{-1+2\delta}\,dx - \frac{1-2\delta}{2}\int_\Omega |u|^2 (\Delta\Phi)\Phi^{-2+2\delta}\,dx \\
&\quad + (1-\delta)(1-2\delta)\int_\Omega |u|^2 |\nabla\Phi|^2 \Phi^{-3+2\delta}\,dx \\
&\le -\frac{\delta}{1-\delta}\int_\Omega |\nabla u|^2 \Phi^{-1+2\delta}\,dx + \frac{1-2\delta}{2}\int_\Omega |u|^2 (\Delta\Phi)\Phi^{-2+2\delta}\,dx.
\end{aligned}$$

This completes the proof. □

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