Some Inequalities and Optimal Estimation on the Ruin Probability for Light Tail Distributions with Restrictions

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Abstract

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Abstract. Sometimes when there are some restrictions on the random variables of insurance risk model, it is impossible to calculate the exact value of ruin probabilities. For these cases, even finding a suitable approximation, is very important from a practical point of view. In the present paper, we consider the classical insurance surplus model with light tailed claim amount distributions and try to find some inequalities and optimal estimation on the infinite time ruin probability depending on the amount of initial reserve when the assumption of net profit does not hold but there exist some other restrictions on the mathematical functions of random variables of model. The obtained assertions depend on the amount of initial reserve, distribution of nonnegative claim amounts and claim inter-arrival times. Finally, to show the application and effectiveness of results some examples are presented.

Key words and phrases: Classical risk model, Interest rate, Monte Carlo simulation, Probability density function, Ruin probability.

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1. Introduction

In studying a company insurance portfolio, it is important to know how the portfolio may be expected over a certain period of time. A common criterion to assess risks an insurer is called ruin probability. Ruin is a technical term which does not necessarily mean that the company is bankrupt but rather that bankruptcy is at hand and that the company should therefore be prompted to take action to improve its solvency status. Thus, insurance companies customarily take precautions to avoid ruin. Risk theory is a branch of mathematics science that is an important part of actuarial education, as it uses statistical and mathematical approaches to explain about the financial reserve of insurance company to ruin. This theory for a portfolio of business is concerned with the excess of the income over the costs or claims paid and insurance companies have to use this theory to protect their financial reserve against possible loss.

Recently, within the actuarial world, modern risk management techniques play a central role. Finding adequate models for the claim arrival and claim size distributions of a risk process is essential in the theoretical and application of ruin probabilities. Risk assessment and management was established as a scientific field about 40–50 years ago. Approaches and methods were developed for how to conceptualise, assess and manage risk. Risk management of an insurance studies the impact of deductible and policy limit on the ruin probability. For more results on the risk management one can see for example Better, et al. (2008).

In the financial literature, there are many models and approaches that have been adopted to measure risks. Prominent among them are ruin theory models. In risk management, insurance companies start to set risk limits. Originally developed for the insurance industry, the ruin probability is used to investigate the stochastic processes that represent the time evolution of the surplus and serve as the main risk measure to quantify the solvency of the company. Thus the ruin probability is considered as an important type of risk measure. In classical risk theory by adding to the previous surplus the current premium flow and deducting the claims made during the period, the process gives the value of the capital that is available to the insurer at each point in time. The risk process is an important stochastic model for the fluctuations of the insurer surplus over the time.

Lundberg's (1903) pioneer work in risk theory received rigorous mathematical treatment first by Cramer in (1930) and (1955) and later, by many authors. Lundberg's contributions were presented in his monograph collective risk theory. Lundberg's model, expounded by Cramer, is termed the Cramer-Lundberg model or the classical risk model. The classical Cramer–Lundberg is a compound Poisson model, accounting for claims arriving independently at exponential times, random in size, but independent and identical distributed. More generalizations of this model exist and most of the research has been inclined to computing infinite time ruin probabilities in preference to finite time ruin probabilities. The evaluation of ruin probabilities strongly depends on the distribution of the claim amounts. Also, very advanced models of the classical continuous risk process have been developed. They exist to pool together risks faced by individuals or companies who in the event of a loss are compensated by the insurer to reduce the financial burden.

Several authors have presented different approaches for calculating finite and infinite time ruin probabilities. For the classical surplus process, recursive algorithms for the calculation of this probability have been developed by, for example, De Vylder and Goovaerts (1988) and Dickson and Waters (1999). An improvement to the algorithms above was done by Cardoso and Waters (2003). This topic has been investigated for a long time (see e.g. the book of Goovaerts et al., 1990). Embrechts and Klüppelberg (1993) showed how the theory of probability and mathematical statistics are applied for solving problems of the insurance field.

Wang, et al. (2004) studied the compound Poisson risk model with a constant interest force for an insurance portfolio and computed the distribution of the surplus immediately after ruin. Burnecki, et al. (2005) compared different approximations of infinite time ruin probability and showed that approximations based on the Pollaczek-Khinchin formula obtain the most accurate results. Kasozi and Paulsen (2005) obtained the numerical results for infinite time ruin probabilities under interest force. Shimizu (2009), introduced a new aspect of a risk process, which is a macro approximation of the flow of a risk reserve. Choi, et al. (2010) considered a continuous time risk process, where the premium rate is constant and claim process forms a compound Poisson process and introduced new approximations of the ruin probability. Santana, et al. (2017) computed the approximation of infinite time ruin probability in the Cramer-Lundberg risk model with any arbitrary continuous distribution of claim sizes. Dong, et al. (2018) calculated the infinite ruin probability in the classical risk mode using Laplace transform inversion and Fourier transform. Yuen, et al. (2020) computed the infinite time ruin probability by applying the weak law of large numbers.

Chen et al. (2020) applied the block trigonometric exponential neural network to find the approximate solutions of the ruin probability in the classical risk model. Lu, et al. (2020) studied a numerical method based on Legendre polynomials and extreme learning machine algorithm to solve the ruin probabilities in the classical risk model. Cheung and Zhang (2021) considered a renewal insurance risk model for a large class of interclaim time distributions (including a combination of exponentials), and developed an approximation for the ruin probability using Laguerre series expansion as a function of the initial surplus level.

Let us start by describing the risk model of insurance company that we will work with in this paper. We denote by $R(t)$ the risk reserve process of an insurance portfolio at time t:

$$R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k, \qquad t \geq 0, \tag{1.1}$$

where the explanation of notations are given in Table 1.

Table 1. The explanation and role of each term appearing in the risk model

Notation	Role of notation
u	Positive initial reserve
c	Premium income rate
$\{Y_k, k = 1, 2, \ldots\}$	Successive claim amounts with parameter ζ
$\{T_k, k = 1, 2, \ldots\}$	Sequence of independent inter-arrival times with parameter κ
$S(t) = \sum_{k=1}^{N(t)} Y_k$	The total amount up to time t which is a compound Poisson process
$\{N(t), t \geq 0\}$	A Poisson process, which counts the claim occurrences until time t with parameter β
$S(t) - ct$	The claim surplus process

The random variable $\{N(t), t \geq 0\}$ is generated by the sequence $\{T_k, k = 1, 2, \ldots\}$ and $\{Y_k, k = 1, 2, \ldots\}$ are represented by non-negative independent identically distributed random variables. We consider model (1.1) for light tailed claim amount distributions. In addition, sequences $\{T_k, k = 1, 2, \ldots\}$, and $\{Y_k, k = 1, 2, \ldots\}$ are supposed to be independent.

Definition (Light tailed distribution). A random variable Y is said to be light tailed if there exists some finite exponential moment, i.e. $E(e^{sY}) < \infty$ for some $s > 0$.

In fact, the existence of moment generating function is arguably the most popular method for classifying heavy tail versus light tail within the community of academic actuaries.

In many cases, in the insurance risk models, the sizes of the claims have light tailed distributions and significant works have been done on analogous problems for the light tailed cases. It has become clear that the insurance risk models employing light tailed distributions don't have a great tendency for extremal sizes.

Let $\phi(u, s)$ be the probability of ruin before time s, $0 \leq t \leq s$:

$$\phi(u, s) = P\left(\min_{0 \leq t \leq s} R(t) < 0 \middle| R(0) = u \right), \qquad u, t \geq 0. \tag{1.2}$$

It is clear that, (1.2) can be written as

$$\phi(u, s) = P\left(\max_{1 \leq n \leq N(s)} \left(\sum_{k=1}^{n} (Y_k - cT_k) \right) > u \right) \qquad u, t \geq 0.$$

The infinite time ruin probability is defined by

$$\phi(u) = P\left(\inf_{t \geq 0} R(t) < 0 \middle| R(0) = u \right), \qquad u, t \geq 0. \tag{1.3}$$

Also, for $u, t \geq 0$, (1.3) can be written as

$$\phi(u) = P\left(\sup_{n\geq 1}\left(\sum_{k=1}^{n}(Y_k - cT_k)\right) > u\right).$$

As $s \to \infty$, $\phi(u,s) \sim \phi(u)$. Sometimes, because of the structure of insurance risk model computing the exact value of ruin probability is not possible. There are different methods to evaluate the ruin probability, see, e.g., Ma and Sun (2003) and Pergamenshchikov and Zeitouny (2006).

Remark 1. Ma and Sun (2003) studied the classical ruin problem for a mixed insurance-finance model in which the risk reserve is connected to a financial market. In fact, they extended the classical Cramer–Lundberg model and used the exponential martingales to derive two main theorems for the Lundberg–type bounds. Some examples are presented to show the versatility of their method.

Pergamenshchikov and Zeitouny (2006) considered the ruin problem for an insurance company for which the premium rate is specified by a bounded non-negative random function c_t and derived exact upper and lower bounds for the ruin probability and in the case of exponential premium rate, i.e. $c_t = e^{\gamma t}$, with $\gamma \leq 0$.

In the present paper, we try to find some inequalities on the infinite time ruin probability with some assumptions to allow analysis of the model in more realistic cases of insurance.

For the first inter-arrival time T_1 and first successive claim amount Y_1, if the assumption $E(Y_1 - cT_1) < 0$ holds and for some positive w, $E(e^{wY_1}) < \infty$, then for all positive values of u, the assertion

$$\phi(u) \leq e^{-Ru}, \tag{1.4}$$

holds, where the unique and positive constant R is computed from the equation $E(e^{R(Y_1 - cT_1)}) = 1$. The proof of equation (1.4) can be found for example in Embrechts and Veraverbeke (1982) and Asmussen and Albrecher (2010).

Note that, to prove the equation (1.4) it is sufficient that for all $N \in \mathrm{N}$,

$$\hat{\phi}(u,N) = P\left(\max_{1\leq n\leq N}\left(\sum_{k=1}^{n}(Y_k - cT_k)\right) > u\right) \leq e^{-Ru}. \tag{1.5}$$

Mikosch (2009) showed that if $N \to \infty$, $\hat{\phi}(u,N) \sim \phi(u)$. The value of $E(Y_1 - cT_1)$ must be negative in equation (1.4), otherwise the ruin occurs with probability 1. Unfortunately, computing the exact value of ruin probabilities can only be for a few special cases of the statistical distributions of claim amounts and claim occurrences times. Thus, finding a suitable approximation, especially in the infinite time case is really important from a practical point of view.

In the present paper, we restrict ourselves to the collective insurance risk model where we make the additional assumptions with some mathematical restrictions on the model to find some inequalities on the infinite time ruin probability. These restrictions would allow analysis of the model in more realistic cases of insurance and the procedure has resulted very useful to obtain new and adequate estimate for the infinite time ruin probability. The given Theorems in the present paper are different from the inequality (1.4). The assumption $E(Y_1 - cT_1) < 0$ is necessary to hold the inequality (1.4) but in this paper, we suppose that for all of the sequence of independent random variables $\{Y_1 - cT_1, Y_2 - cT_2, \ldots\}$ the assumption $E(Y_k - cT_k) < 0$, $k \in N$, holds.

The main purpose of this paper is to find easily verifiable conditions and possible mathematical restrictions in the classical insurance surplus model, so that we obtain the optimal estimation on the infinite time ruin probability for light tailed distributions of claim amounts.

The remaining part of this paper is organized as follows. In Section 2, we give the main results on the computation of the infinite time ruin probability in the collective insurance risk model depending on the amount of initial reserve, distribution of nonnegative inter-arrival times and successive claim amounts in the presence of positive constant premium income rate with some restrictions on the mathematical functions of random variables of model. In Section 3, we demonstrate some examples, which show the applicability of the theorems. Moreover, conclusions are given in Section 4.

2. Some inequalities on the infinite time ruin probability

In this section, we assume that there exist some mathematical restrictions on the sequences of independent inter-arrival times and independent successive claim amounts. The main results are associated to the infinite time ruin probability in the collective insurance risk model with these restrictions.

Theorem 1. For the classical insurance surplus model (1.1) with positive premium income rate c, if the following restrictions hold:

i) $E(Y_k - cT_k) < 0, \qquad k \in N$

ii) $\sup_{k\in N} E(e^{\lambda Y_k}) < \infty$, for some $\lambda > 0$,

iii) $\sup_{k\in N} E\big(T_k I(T_k > t)\big) = 0$,

where $I(.)$ is the indicator function, then there is a positive constant δ, such that for all positive values of u, the infinite time ruin probability is less or equal than $e^{-\delta u}$.

Proof. Consider the ruin probability (1.5), then

$$\hat{\phi}(u,N) = P\left(\bigcup_{n=1}^{N}\left(\sum_{k=1}^{n}(Y_k - cT_k) > u\right)\right).$$

To prove theorem 1, it is sufficient to prove that for some positive δ, all $u \ge 0$ and for an arbitrary collection of different values $\{k_1, k_2, \ldots, k_N\}$, the following inequality

$$\hat{\phi}(u,k_1,k_2,\ldots,k_N) = P\left(\max_{1\le j\le N}\left(\sum_{i=1}^{j}(Y_{k_i} - cT_{k_i})\right) > u\right) \le e^{-\delta u}, \tag{2.1}$$

holds. This inequality will be proved using the following method.

For all $k \in \mathrm{N}$, putting $\tau_k = Y_k - cT_k$. If $N = 1$, then for all $u \ge 0$, for all r in the interval $(0,\lambda]$ and for $k_1 \in \mathrm{N}$, using the Chebyshev's inequality, we have

$$\hat{\phi}(u,k_1) = P\big(\tau_{k_1} > u\big) = P\big(e^{r\tau_{k_1}} > e^{ru}\big) \le e^{-ru}E\big(e^{r\tau_{k_1}}\big). \tag{2.2}$$

Also for r in the interval $(0,\lambda]$ and $t > 0$, we have

$$\begin{aligned} E\big(e^{r\tau_{k_1}}\big) &= 1 + rE(\tau_k) + E\big(\big(e^{r\tau_{k_1}} - 1\big)I(\tau_{k_1} < -t)\big) - rE\big(\tau_{k_1}\, I(\tau_{k_1} < -t)\big) \\ &\quad + E\big(\big(e^{r\tau_{k_1}} - r\tau_{k_1} - 1\big)I(-t \le \tau_{k_1} \le 0)\big) \\ &\quad + E\big(\big(e^{r\tau_{k_1}} - r\tau_{k_1} - 1\big)I(\tau_{k_1} > 0)\big). \end{aligned} \tag{2.3}$$

In order to estimate the right side of (2.3), we use the following well-known inequalities:

$$\left|e^x-1\right|\le|x| \quad ; \quad \left|e^x-x-1\right|\le\frac{x^2}{2}, \quad x\le 0$$
$$\left|e^x-x-1\right|\le\frac{x^2e^x}{2} \quad ; \quad x\ge 0 \tag{2.4}$$

Using these inequalities, we derive

$$E\left(e^{r\tau_{k_1}}\right)\le 1+rE(\tau_k)+2rE\left(\left|\tau_{k_1}\right|I(\tau_{k_1}<-t)\right)$$
$$+\frac{r^2}{2}E\left(\tau_{k_1}^2\, I(-t\le\tau_{k_1}\le 0)\right) \tag{2.5}$$
$$+\frac{r^2}{2}E\left(\tau_{k_1}^2 e^{r\tau_{k_1}}\, I(\tau_{k_1}>0)\right),$$

for r in the interval $(0,\lambda]$ and $t>0$.

Now, we simplify the right hand side of (2.5). We observe that

$$E\left(\left|\tau_{k_1}\right|I(\tau_{k_1}<-t)\right)=E\left((cT_{k_1}-Y_{k_1})\,I(cT_{k_1}>t+Y_{k_1})\right)$$
$$\le cE\left(\tau_{k_1}\, I(\tau_{k_1}>\frac{t}{c})\right),$$

also for given the restriction (ii) of the theorem 1, $E\left(\tau_{k_1}^2\, I(-t\le\tau_{k_1}\le 0)\right)\le t^2$ and $E\left(e^{r\tau_{k_1}}\right)\le\sup_{k\in N}E(e^{\lambda Y_k})=c_1(\lambda)$. Thus, using the last estimate we derive that if $r\in\left(0,\frac{\lambda}{2}\right]$, then

$$E\left(\tau_{k_1}^2 e^{r\tau_{k_1}}\, I(\tau_{k_1}>0)\right)\le c_2(\lambda)E\left(e^{\frac{r\tau_{k_1}}{2}}e^{\frac{r\tau_{k_1}}{2}}\, I(\tau_{k_1}>0)\right)$$
$$\le c_2(\lambda)E\left(e^{r\tau_{k_1}}\right)\le c_1(\lambda)c_2(\lambda)=c_3(\lambda),$$

where $c_2(\lambda)$ is a constant from the inequality $x^2\le c_2(\lambda)e^{\lambda\frac{x}{2}}$ for $x\ge 0$. Substituting these results into (2.5), then for all $r\in\left(0,\frac{\lambda}{2}\right]$ and $t>0$, we have

$$E\left(e^{r\tau_{k_1}}\right)\le 1+r\left(\sup_{k\in N}E(\tau_k)+2c\sup_{k\in N}\left(T_kI(\tau_k>\frac{t}{c})\right)+\frac{rt^2}{2}+\frac{rc_3(\lambda)}{2}\right).$$

Let $t=\frac{1}{\sqrt[4]{r}}$, then the inequality

$$E\left(e^{r\tau_{k_1}}\right)\le 1+r\left(\sup_{k\in \mathrm{N}}E(\tau_k)+2c\sup_{k\in \mathrm{N}}\left(T_kI\left(\tau_k>\frac{1}{(c\sqrt[4]{r})}\right)\right)+\frac{\sqrt{r}}{2}+\frac{rc_3(\lambda)}{2}\right), \tag{2.6}$$

holds for all $r\in\left(0,\frac{\lambda}{2}\right]$. Using the inequality (2.6) and also restrictions (i) and (iii), imply that

$$E\left(e^{\delta\tau_{k_1}}\right)\le 1, \tag{2.7}$$

for $\delta\in r\in\left(0,\frac{\lambda}{2}\right]$ and $k_1\in\mathrm{N}$.

According to the inequality (2.2), for the same positive δ, $k_1 \in \mathrm{N}$ and for all $u \geq 0$, we have

$$\hat{\phi}(u,k_1) = P\left(\tau_{k_1} > u\right) \leq e^{\delta u},$$

and we result that the inequality (2.1) is correct for $N=1$. If the inequality (2.1) is correct for $N = w \geq 1$, i.e., for the above positive δ, for all positive values of u and an arbitrary collection of different values $\{k_1, k_2, \ldots, k_w\}$, we have

$$\hat{\phi}(u,k_1,k_2,\ldots,k_w) = P\left(\max_{1\le j\le w}\left(\sum_{i=1}^{j}\tau_k\right) > u\right) \leq e^{-\delta u}. \tag{2.8}$$

Then we must show that the inequality (2.1) is correct for $N = w+1$. According to the inequality (2.7) and (2.8), for an arbitrary collection of different values $\{k_1, k_2, \ldots, k_{w+1}\}$ and for all positive values of u, we get that

$$\hat{\phi}(u,k_1,k_2,\ldots,k_{w+1}) = P\left(\max_{1\le j\le w+1}\left(\sum_{i=1}^{j}\tau_k\right) > u\right) = P\left(\max\left(\tau_{k_1}, \max_{2\le j\le w+1}\left(\tau_{k_1} + \sum_{i=2}^{j}\tau_{k_i}\right)\right) > u\right)$$

$$= P\left(\tau_{k_1} > u\right) + P\left(\tau_{k_1} + \max_{2\le j\le w+1}\left(\sum_{i=2}^{j}\tau_{k_i}\right) > u, \tau_{k_1} \le u\right)$$

$$= \int_{(u,\infty)} dF_{\tau_{k_1}}(x) + \int_{(-\infty,u]}\left(\max_{2\le j\le w+1}\left(\sum_{i=2}^{j}\tau_{k_i}\right) > u - x\right) dF_{\tau_{k_1}}(x)$$

$$\le \int_{(u,\infty)} e^{-\delta(u-x)} dF_{\tau_{k_1}}(x) + \int_{(-\infty,u]} e^{-\delta(u-x)}\, dF_{\tau_{k_1}}(x)$$

$$= e^{-\delta u} E\left(e^{\delta\tau_{k_1}}\right) \le e^{-\delta u},$$

and this completes the proof. □

Theorem 2. For the classical insurance surplus model (1.1), if for some constants $\tau > 0$, $\lambda > 0$, $\vartheta \geq 0$, $\varsigma > 0$ and $\eta \geq 0$, the following restrictions hold:

i) $\sup_{k\in N} E(Y_k - cT_k) < -\tau$,

ii) $\sup_{k\in N} E\left(e^{\lambda(Y_k - cT_k)} I(Y_k - cT_k > 0)\right) \le \vartheta$,

iii) $\sup_{k\in N} E\left(T_k I(T_k > \frac{\varsigma}{c})\right) \le \eta$,

and for some $\Delta = \Delta(\tau,\vartheta,\lambda,\varsigma,\eta,c) \in \left(0, \frac{1}{2}\right]$,

$$2c\eta + \frac{\Delta\lambda\varsigma^2}{2} + \frac{2\Delta\vartheta}{\lambda} - \tau \le 0, \tag{2.9}$$

then for all positive values of u, the inequality $\phi(u) \le e^{-\Delta\lambda u}$ holds.

Proof. First, consider that for an arbitrary collection of different values $k_1, k_2, \ldots, k_N \in \mathrm{N}$, the following inequality is completely clear

$$\hat{\phi}(u,N) \le \sup_{k_1,k_2,\ldots,k_N} \hat{\phi}(u,k_1,k_2,\ldots,k_N).$$

Therefore, to prove theorem 3, it is sufficient to prove that for all $\Delta > 0$, $u \geq 0$, $N \in \mathrm{N}$, the inequality

$$\hat{\phi}(u,k_1,k_2,\ldots,k_N) \le e^{-\Delta\lambda u}, \tag{2.10}$$

holds for an arbitrary collection of different values $\{k_1, k_2, \ldots, k_N\}$.
Proof of theorem 2 is similar to the proof of Theorem 1. Again, let for all $k \in \mathrm{N}$, $\tau_k = Y_k - cT_k$. If $N = 1$, again consider (2.2) and (2.3) for $k_1 \in \mathrm{N}$, r in the interval $(0, \lambda]$ and $t = \varsigma$. Using the mathematical inequalities (2.4), we get that

$$\begin{aligned} E\left(e^{r\tau_{k_1}}\right) \le 1 + rE(\tau_k) + 2rE\left(\left|\tau_{k_1}\right| I(\tau_{k_1} < -\varsigma)\right) \\ + \frac{r^2}{2} E\left(\tau_{k_1}^2 I(-\varsigma \le \tau_{k_1} \le 0)\right) \\ + \frac{r^2}{2} E\left(\tau_{k_1}^2 e^{r\tau_{k_1}} I(\tau_{k_1} > 0)\right), \end{aligned} \tag{2.11}$$

for any $k_1 \in \mathrm{N}$. On the other hand

$$\begin{aligned} E\left(\left|\tau_{k_1}\right| I(\tau_{k_1} < -\varsigma)\right) &= E\left((cT_{k_1} - Y_{k_1}) I(cT_{k_1} > \varsigma + Y_{k_1})\right) \\ &\le cE\left(\tau_{k_1} I(\tau_{k_1} > \frac{\varsigma}{c})\right), \end{aligned} \tag{2.12}$$

and for any positive value ς, we have

$$E\left(\tau_{k_1}^2 I(-\varsigma \le \tau_{k_1} \le 0)\right) \le \varsigma^2. \tag{2.13}$$

In addition, using restriction (ii) of the theorem and the inequality $x^2 \le e^x$ for $x \ge 0$, we get that

$$\begin{aligned} E\left(\tau_{k_1}^2 e^{r\tau_{k_1}} I(\tau_{k_1} > 0)\right) &\le \frac{4}{\lambda^2} E\left(\left(\frac{\lambda\tau_k}{2}\right)^2 e^{r\tau_{k_1}} I(\tau_{k_1} > 0)\right) \\ &\le \frac{4}{\lambda^2} E\left(e^{\frac{r\tau_{k_1}}{2}} e^{r\tau_{k_1}} I(\tau_{k_1} > 0)\right) \\ &\le \frac{4}{\lambda^2} E\left(e^{r\tau_{k_1}} I(\tau_{k_1} > 0)\right) \le \frac{4\vartheta}{\lambda^2}, \end{aligned} \tag{2.14}$$

for $r \in \left(0, \frac{\lambda}{2}\right]$.

Substituting the results (2.12), (2.13) and (2.14) into (2.10), then for r in the interval $\left(0, \frac{\lambda}{2}\right]$ and $k_1 \in \mathrm{N}$, we get that

$$\begin{aligned} E\left(e^{r\tau_{k_1}}\right) &\le 1 + r\left(\sup_{k \in N} E(\tau_k) + 2c \sup_{k \in N}\left(T_k I(\tau_k > \frac{t}{c})\right) + \frac{r\varsigma^2}{2} + \frac{2r\vartheta}{\lambda^2}\right) \\ &\le 1 + r\left(-\tau + 2c\varepsilon + \frac{r\varsigma^2}{2} + \frac{2r\vartheta}{\lambda^2}\right). \end{aligned}$$

For Δ in the interval $\left(0, \frac{1}{2}\right]$, let $r = \Delta\lambda$. Then using the restriction (iv), we have that $E\left(e^{r\tau_{k_1}}\right) \le 1$, for any $k_1 \in \mathrm{N}$. This together with inequality (2.2) for any $k_1 \in \mathrm{N}$ and all positive value of u, implies that

$$\hat{\phi}(u,k_1)=P\left(\tau_{k_1}>u\right)\le e^{-\Delta\lambda u}.$$

Therefore, as similar approach to the proof of Theorem 1, the inequality (2.10) is proved. □

Remark 2. If in the insurance risk model, for any $k\in N$, the assumption $E(Y_k-cT_k)<0$ holds together with two natural related assumptions (see Theorem 1), then for some positive δ, the infinite time ruin probability is less or equal than $e^{-\delta u}$. But if we know all restrictions and assumptions in the insurance risk model (see Theorem 2), then using a different approach, we compute the inequality $\phi(u)\le e^{-\Delta\lambda u}$ on the infinite time ruin probability.

Theorem 3. For the classical insurance surplus model (1.1), if the restrictions given in Theorem 1 hold, then for all r in the interval $(0,R]$, the inequality

$$\sup_{k\in N}E(e^{r(Y_k-cT_k)})\le 1,$$

holds, and for all positive values of u, $\phi(u)\le \inf_{r\in(0,R]}\left(e^{-ru}\sup_{k\in N}E(e^{r(Y_k-cT_k)})\right)$.

Proof. As before, let for all $k\in\mathrm{N}$, $\tau_k=Y_k-cT_k$. According to the given restriction in the theorem and using (2.6), there exist r in the interval $(0,R]$, such that $\sup_{k\in\mathrm{N}}E\left(e^{r\tau_{k_1}}\right)\le 1$ and this is proof of the first part of the theorem.

To prove the second part, consider that according to the inequality (2.2), for $u\ge 0$, r in the interval $(0,R]$ and $k_1\in\mathrm{N}$, we have

$$\hat{\phi}(u,k_1)=P\left(\tau_{k_1}>u\right)\le e^{-ru}\sup_{k\in\mathrm{N}}E\left(e^{r\tau_{k_1}}\right).$$

Using the given approach for the proof of Theorem 3, we derive

$$\hat{\phi}(u,N)\le e^{-ru}\sup_{k\in\mathrm{N}}E\left(e^{r\tau_{k_1}}\right).$$

Then it is clear that, for $u\ge 0$ and $r\in(0,R]$, the infinite time ruin probability is less or equal than $e^{-ru}\sup_{k\in\mathrm{N}}E\left(e^{r\tau_{k_1}}\right)$, and this completes the proof of the second part of the theorem. □

As an intuitively explanation the obtained results in Theorem 3 is useful for computing the optimal estimation of ruin probability. For example, if $\sup_{k\in N}E(e^{r(Y_k-cT_k)})\le\hat{b}\le 1$, for $r\in(0,R]$, then the estimate of Theorem 3, implies that for $u\ge 0$, $\phi(u)\le\hat{b}e^{-Ru}$.

3. Some numerical examples

In this section, to show the applicability of theorems some numerical examples are presented. In example 1, we study the discrete risk model with given the probability density functions of claim amounts and continuous risk model of insurance company. In example 2, we investigate the insurance risk model with Gamma and exponential distributions of claim amounts and inter-arrival times, respectively. In example 3, the dataset covering losses resulting from different car claims events which is provided by Iran insurance company that occurred between 2005 and 2021.

Example 1. Suppose that in a classical insurance surplus model the premium income rate is equal to one, the sequence of independent inter-arrival times $\{T_k,k=1,2,\ldots\}$ are equal to 1, and a sequence of independent successive claim amounts

$\{Y_k, k=1,2,\ldots\}$ are such that for all $z\in \mathrm{N}$ and $k\in\{1,2,3,4,5\}$, the two random variables Y_{5z+k} and Y_k have the same distribution function and the probability density functions are given in Table 2.

Table 2. Probability density functions of claim amounts

Y_k	0	1	k
$P(Y_k=y_k)$	$1-\frac{1}{3k}-\frac{1}{3k^2}$	$\frac{1}{3k}$	$\frac{1}{3k^2}$

First of all, according to Theorem 2 for all $\lambda\in(0,0.61]$, we obtain:

$$\sup_{k\in N} E(Y_k-cT_k)=\sup_{k\in N}\left(\frac{2}{3k}-1\right)=-\frac{1}{3},$$

$$\sup_{k\in N} E\left(e^{\lambda(Y_k-cT_k)}I(Y_k-cT_k>0)\right)=\max\left\{0,\frac{e^{\lambda}}{12},\frac{e^{2\lambda}}{27},\frac{e^{3\lambda}}{48},\frac{e^{4\lambda}}{75}\right\}=\frac{e^{\lambda}}{12},$$

and $\sup_{k\in N} E\left(T_k I(T_k>1)\right)=0$. According to the obtained estimates, we get that conditions of Theorem 2 hold with parameters in Table 3.

Table 3. The values of parameters in Theorem 2

Parameter	τ	λ	ϑ	ς	η
Value	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{e^{\frac{3}{5}}}{12}$	1	0

Substituting the obtained constants into inequality (2.9), we derive the parameter Δ in the interval $(0,0.41]$. If we choose $\Delta=0.3$, then from Theorem 2, we have that $\phi(u)\le e^{-\frac{9}{50}u}=p_1$, for all $u\ge 0$.

Now, according to the Theorem 3, we need to compute the term $\sup_{k\in N} E(e^{r(Y_k-cT_k)})$.

First, consider that $k\in\{1,2,3,4,5\}$ and λ in the interval $\left(0,\frac{4}{5}\right]$. The values of $E(e^{\lambda Y_k})=\sum_{k=1}^{5} e^{\lambda y_k}P(Y_k=y_k)$ are given in Table 4.

Table 4. The values of $E(e^{\lambda Y_k})$ for $k\in\{1,2,3,4,5\}$

k	1	2	3	4	5
$E(e^{\lambda Y_k})$	$\frac{2e^{\lambda}+1}{3}$	$\frac{2e^{\lambda}+e^{2\lambda}+9}{12}$	$\frac{3e^{\lambda}+e^{3\lambda}+23}{27}$	$\frac{4e^{\lambda}+e^{4\lambda}+43}{12}$	$\frac{5e^{\lambda}+e^{5\lambda}+69}{75}$

Therefore, for all values of $k\in\{1,2,3,4,5\}$, $\sup_k E(e^{\lambda Y_k})=\frac{2e^{\lambda}+1}{3}$. Also, for positive value r, the values of $E(e^{r(Y_k-cT_k)})$ are given in Table 5.

Table 5. The values of $E(e^{r(Y_k-cT_k)})$ for $k\in\{1,2,3,4,5\}$

k	1	2	3	4	5
$E(e^{r(Y_k-cT_k)})$	$\frac{2e^{r}+1}{3e^{r}}$	$\frac{2e^{r}+e^{2r}+9}{12e^{r}}$	$\frac{3e^{r}+e^{3r}+23}{27e^{r}}$	$\frac{4e^{r}+e^{4r}+43}{12e^{r}}$	$\frac{5e^{r}+e^{5r}+69}{75e^{r}}$

For all r in the interval $\left(0, \frac{45}{50}\right]$, and $u \geq 0$;

$$\phi(u) \leq \inf_{r \in \left(0, \frac{45}{50}\right]} \left(e^{-ru} \sup_{k \in N} E(e^{r(Y_k - cT_k)}) \right) = e^{-\frac{45}{50}u} = p_2.$$

Now to compute the values of infinite time ruin probability $\phi(u)$, we have performed the Monte Carlo simulation method. As we said before for amount of initial reserve $u \geq 0$, as $N \to \infty$, $\hat{\phi}(u, N) \sim \phi(u)$. To estimate these values, we consider the amount of initial $u \in \{0, 1, 2, \ldots, 15\}$ and $N = 1000$ samples. For each amount of initial with 1000 random claims in the discrete risk model, the value of $\hat{\phi}(u, N)$ is computed. We repeat this process 10000 times and calculate how many times on average they fall below zero in order to get values of infinite time ruin probability $\phi(u)$. The values of ruin probabilities are given in Table 6. Also, the results are depicted in Figures 1 and 2. According to the obtained values of ruin probability, it is observed that $\phi(u)$ decreases as the initial reserve increases.

Table 6. The values of ruin probabilities

u	$\phi(u)$	p_1	p_2
0	0.15953	1	1
1	0.13784	0.83527	0.40657
2	0.07520	0.69767	0.16529
3	0.01036	0.58274	0.06720
4	0.00732	0.48675	0.02732
5	0.00281	0.40657	0.01111
6	0.000941	0.22959	0.00451
7	0.000485	0.28365	0.00183
8	0.000106	0.23692	0.00074
9	0.000075	0.19789	0.00030
10	0.000031	0.16529	0.00012
11	0.0000056	0.13806	5.0177e-5
12	0.0000022	0.11532	2.0399e-5
13	0.0000007	0.09632	8.2938e-6
14	0.0000003	0.08045	3.3720e-6
15	0.0000001	0.06720	1.3709e-6

Figure 1. Ruin probabilities $\phi(u)$, p_1 and p_2

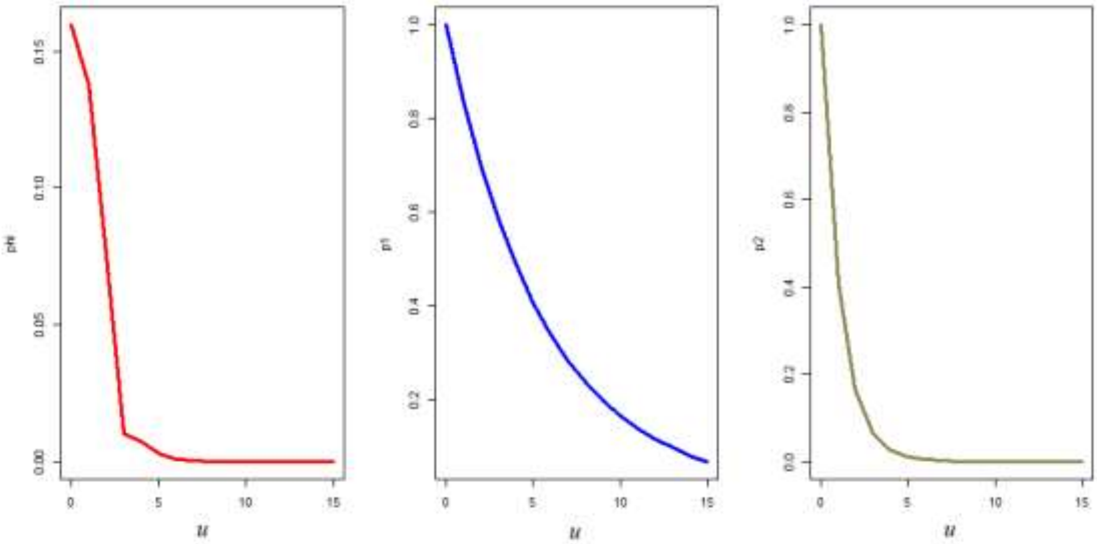
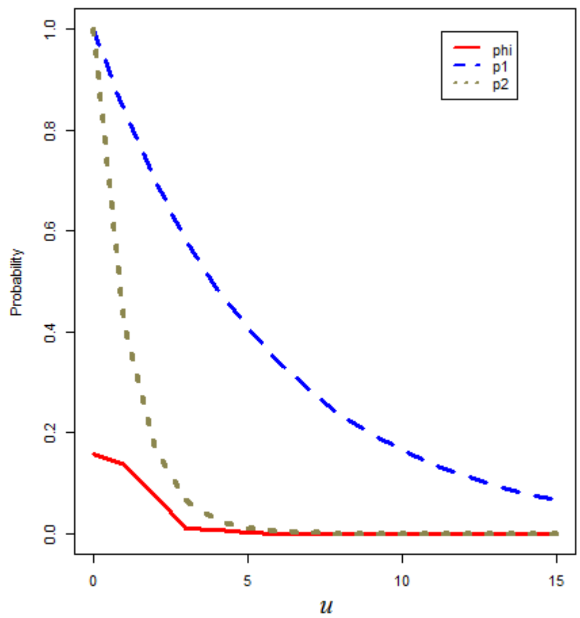


Figure 2. Ruin probabilities $\phi(u)$, p_1 and p_2



Example 2. Suppose that in a risk model of insurance company the premium income rate is equal to 1.1, a sequence of independent inter-arrival times $\{T_k, k=1,2,\ldots\}$ are distributed according to the following density function:

$$f_{T_k}(t) = \frac{k^k}{\Gamma(k)} t^{k-1} e^{-kt},$$

and a sequence of independent successive claim amounts $\{Y_k, k=1,2,\ldots\}$ are distributed according to the Exponential density function:

$$f_{Y_k}(t) = (\cos k + 3)\exp\left(-(\cos k + 3)t\right), \qquad t > 0.$$

Then

$$\sup_{k \in N} E(Y_k - cT_k) = \sup_{k \in N}\left(\frac{1}{\cos k + 3} - \frac{11}{10}\right) \le -\frac{3}{5},$$

$$\sup_{k\in N} E\big(T_k I(T_k>4)\big)=\sup_{k\in N}\int_4^{\infty} t\,\frac{k^k t^{k-1}e^{-kt}}{\Gamma(k)}dt=\frac{5}{e^4},$$

and the distribution function of random variable $(Y_k - cT_k)$ is given by:

$$P(Y_k - cT_k \le t)=\int_{-\infty}^{\infty} P\big(Y_k - 1.1y \le x\big)dP\big(T_k \le y\big)$$

$$=\frac{k^k}{\Gamma(k)}\int_{\max\left\{0,-\frac{10t}{11}\right\}}^{\infty}\left(1-e^{-(\cos k+3)(t+1.1y)}\right)y^{k-1}e^{-ky}\,dy.$$

The random variable $(Y_k - cT_k)$ has the following density:

$$f_{Y_k-cT_k}(t)=\frac{(\cos k+3)k^k}{\Gamma(k)}e^{-(\cos k+3)t}\int_0^{\infty} y^{k-1}e^{-(k+3.3+1.1(\cos k))y}\,dy.$$

Therefore for all λ in the interval $(0,2)$; the following assertion holds:

$$\sup_{k\in N} E\Big(e^{\lambda(Y_k-cT_k)}I(Y_k-cT_k>0)\Big)=\sup_{k\in N}\int_0^{\infty} e^{\lambda t}\,f_{Y_k-cT_k}(t)\,dt$$

$$=\sup_{k\in N}\left(\frac{(\cos k+3)k^k}{\Gamma(k)}\left(\int_0^{\infty} y^{k-1}e^{-(k+3.3+1.1(\cos k))y}\,dy\right)\left(\int_0^{\infty} e^{(\lambda-3-\cos k)t}dt\right)\right)$$

$$\le\frac{5}{8(2-\lambda)}.$$

The parameters and their values are given in Table 7.

Table 7. The values of parameters in Theorem 2

Parameter	τ	λ	ϑ	ς	η
Value	$\frac{3}{5}$	$\frac{13}{10}$	$\frac{25}{28}$	$\frac{44}{10}$	$\frac{5}{e^4}$

The same as example 1, substituting the obtained constants into inequality (2.9), we obtain $\Delta\in(0,0.0285]$. If we choose $\Delta=0.025$, then from Theorem 2, for all $u\ge 0$, we get that $\phi(u)\le e^{-\frac{13}{400}u}=p_{11}$.

Now, according to the Theorem 3, we need to compute the term $\sup_{k\in N} E(e^{r(Y_k-cT_k)})$. For all λ in the interval $(0,2)$; $\sup_{k\in N} E(e^{\lambda Y_k})=\sup_{k\in N}\left(\frac{(\cos k+3)}{(3-\lambda+\cos k)}\right)\le\frac{2}{2-\lambda}$. Also, for r in the interval $(0,1.2)$, we have

$$\sup_{k\in N} E(e^{r(Y_k-cT_k)})=\sup_{k\in N}\left(\frac{(\cos k+3)}{(3-r+\cos k)}\frac{1}{(1+1.1r)^k}\right)\le 1.$$

Then for all positive values of u, we derive

$$\phi(u)\le\inf_{r\in(0,1.2)}\left(e^{-ru}\sup_{k\in N} E(e^{r(Y_k-cT_k)})\right)=e^{-1.2u}=p_{22}.$$

After all these calculations, we again apply the implementation of Monte Carlo method in order to estimate the values of ruin probability $\phi(u)$ as in the previous example. We also analyze the same way, where the amount of initial reserve $u\in\{0,1,2,\ldots,15\}$ and $N=1000$. Finally, with repetition the process, the infinite time ruin probabilities are computed. The values of ruin probabilities are given in Table 8. Also, the results are depicted in Figures 3 and 4.

Table 8. The values of ruin probabilities

u	$\phi(u)$	p_{11}	p_{22}
0	0.19115	1	1
1	0.10844	0.96802	0.30119
2	0.03886	0.93706	0.09071
3	0.00825	0.90712	0.02732
4	0.00557	0.87909	0.00822
5	0.00151	0.85001	0.00247
6	0.00021	0.82283	0.00074
7	0.000075	0.79652	0.000224
8	0.0000083	0.77105	6.77287e-5
9	0.0000024	0.74639	2.03995e-5
10	0.0000012	0.72252	6.14421e-6
11	0.0000000	0.69942	1.85060e-6
12	0.0000000	0.67705	5.57390e-7
13	0.0000000	0.65540	1.67882e-7
14	0.0000000	0.63444	5.05653e-8
15	0.0000000	0.61416	1.52299e-8

Figure 3. Ruin probabilities $\phi(u)$, p_{11} and p_{22}

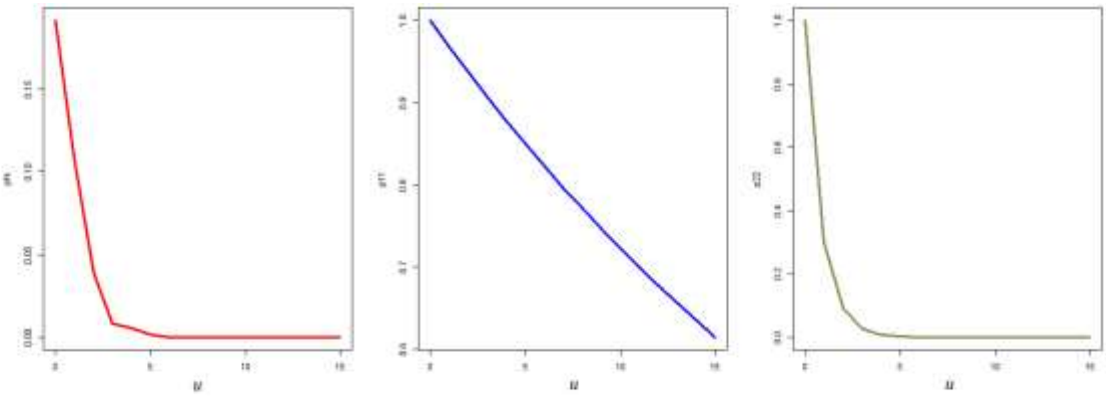
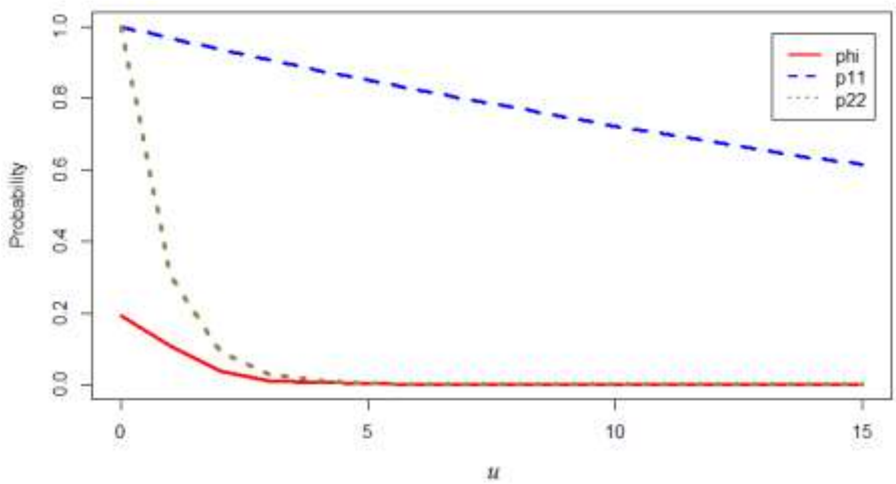


Figure 4. Ruin probabilities $\phi(u)$, p_{11} and p_{22}



Example 3 (Implementation of risk model results with empirical data).
In this example, we first explain in detail the empirical insurance data set used in the sequel. The data set is provided by one of the largest and oldest Iran insurance company which is owned by the Iranian government and claim occurrences from the class of legal expenses insurance, which refers to insurance protection covering the costs of a legal dispute. The data set we consider consists of different cars claims which occurred during the time period from 01 January 2005 to 31 December 2021. For each claim payment, only the day of the payment is recorded as any finer granularity is not of particular interest to the insurance company. As the process is aggregated over multiple clients and claim occurrences, on some days there are multiple arrivals with the same timestamp (day). Based on the obtained information from the insurance company, the initial reserve u is assumed to be 567 million dollars. One challenge about working with the data set provided to us by the insurance company was its overall size with a high average number of claims per day for some of the big cities.
We shown that the use of a Poisson process for claim occurrences per year during (2005-2021) would be suitable and fit the two exponential distributions to the successive claim amounts and claim inter-arrival times $\{T_k, k=1,2,\ldots\}$. The goodness of fit test is done on the data at significance level $\alpha=0.05$. The $p-value=0.2512$ for the goodness of fit test is reported. The result indicates evidence for the null hypothesis that justifying application of the Poisson process.
Also, the number of claim occurrences per in the first per week (7 days) over the whole time period of days gives in Figure 5.

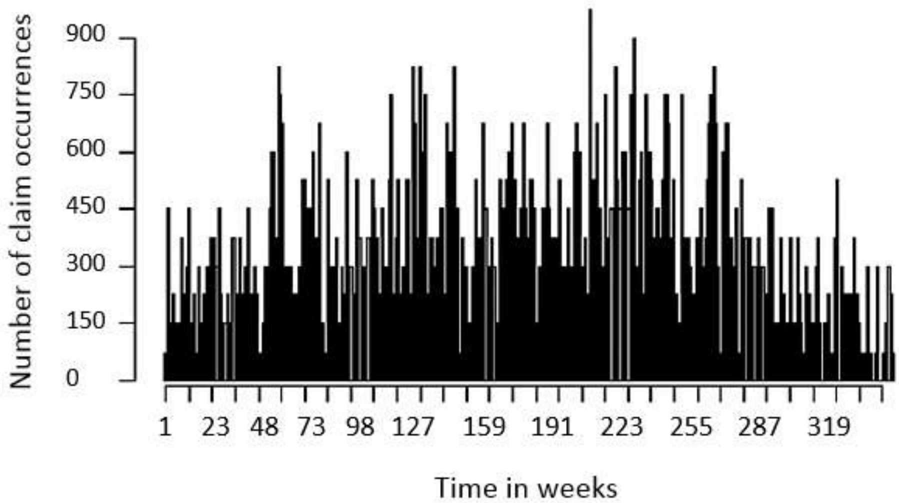


Figure 5. The number of claim occurrences per week

In the risk model (1.1) the premium income rate is constant but in reality this assumption rarely happens because of the annual inflation. Therefore, for this dataset, we set the mean of all premium income rates as $c=254.37$ dollars. Using the goodness of fit tests at significance level $\alpha=0.05$ we have obtained $p-values=0.3157$ and 0.2840, which show that the successive claim amounts and independent inter-arrival times being the exponential distributions with parameters ζ and κ, respectively. These parameters are estimated with maximum likelihood estimation using data set as $\hat{\zeta}=105.18$ and $\hat{\kappa}=4.60$.

We generate $N = 1000$ simulations of a risk process with u and c given above. The simulation results for the first 10 years are presented in Figure 6.

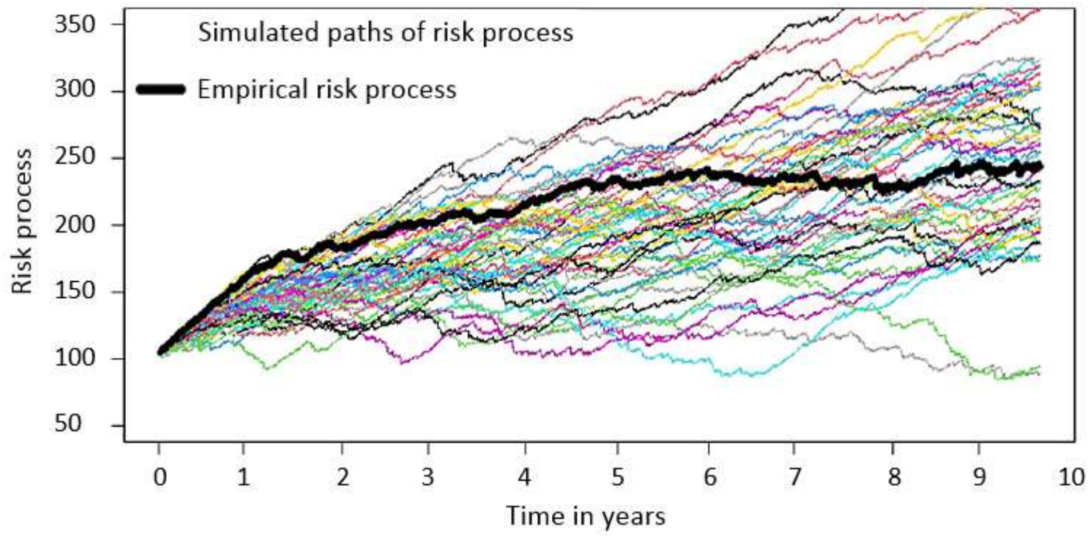


Figure 6. Plotting the empirical risk process with $u = 100$ and $c = 254.37$ for the first 10 years against the simulated paths of risk process for successive claim amounts and independent inter-arrival times distributed as exponential distributions with $\hat{\zeta} = 105.18$ and $\hat{\kappa} = 4.60$.

According to the obtained estimates, we get that conditions of Theorem 2 hold with parameters $\tau = 518$, $\lambda = \frac{17}{4}$, $\vartheta = \frac{2}{5}$, $\varsigma = \frac{19}{6}$ and $\eta = 1$. Substituting the obtained constants into inequality (2.9), we obtain $\Delta \in (0, 0.4307]$. Choosing $\Delta = 0.215$, then we derive the inequality $\phi(u) \le e^{-\frac{1}{27}u} = \phi_1(u)$ for all $u \ge 0$. For all λ in the interval $(0, 4)$ and $r \in (0, 1.5)$, the inequalities $\sup_{k \in N} E(e^{\lambda Y_k}) \le \frac{\lambda}{1+\lambda}$ and $\sup_{k \in N} E(e^{r(Y_k - cT_k)}) \le 1$ hold. Thus for all positive value of u, $\phi(u) \le e^{-1.5u} = \phi_2(u)$.

For our data set, we compute the infinite time ruin probability using two formulas. One of them is given in Rolski et al. (1999), where we compute the values of $\phi(u)$ by $\phi(u) = \left(1 - \frac{\nu}{\zeta}\right)e^{-\nu u}$, where ν is a solution of $\frac{\zeta}{\zeta - \nu} L_T(c\nu) = 1$, and $L_T(.)$ is the Laplace transform of claim inter-arrival times distribution and another formula is an approximation formula which is given in Choi et al. (2010) as $\phi_H(u)$, which the relative security loading $\theta = 10.125$. In Table 9, the values of ruin probabilities $\phi(u)$, $\phi_H(u)$, $\phi_1(u)$ and $\phi_2(u)$ are listed for various values of u. The values of $\phi_1(u)$ and $\phi_2(u)$ assure us that the ruin probabilities will be less than or equal to them. The values of $\phi_2(u)$ is the best estimate for ruin probabilities.

Table 9. The values of ruin probabilities

u	$\phi(u)$	$\phi_H(u)$	$\phi_1(u)$	$\phi_2(u)$
0	0.2504	0.3157	1	1
1	0.1916	0.2028	0.9636	0.2231
2	0.0325	0.0371	0.9286	0.0497
3	0.0233	0.0210	0.8948	0.0280
4	0.0015	0.0019	0.8680	0.00247
5	0.0002	0.0004	0.8309	0.00055
10	3.8624178e-28	4.3172364e-16	0.7386	3.0590231e-7
50	5.2394254e-52	8.0435422e-74	0.3057	3.9754497e-31
60	7.7208111e-87	10.51813502e-25	0.1084	8.1940126e-40
80	9.2344612e-95	0.0000000	0.0516	7.6676480e-53
90	4.34555134e-71	0.0000000	0.0356	2.34555134e-59
100	0.0000000	0.0000000	0.0246	7.17509597e-66
150	0.0000000	0.0000000	0.00386	1.92194773e-98

4. Conclusions

In this work, we investigated the infinite time ruin probability of a classical insurance risk model in the presence of positive constant interest rate with statistical and mathematical methods for all positive initial surplus when the assumption of net profit does not hold, but some other restrictions hold on the mathematical functions of random variables of model. Our goal is to make conditions leading to the risk model for computing some inequalities on the infinite time ruin probability. Theorem 1 confirms the existence of positive constant δ in the interval $\left(0, \frac{\lambda}{2}\right]$ for some $\lambda > 0$ to find the inequality on the infinite time ruin probability. Theorem 2 gives another inequality on the infinite time ruin probability depends on the values of $\Delta \in \left(0, \frac{1}{2}\right]$ when for some constants the inequality (2.9) holds. In order to get these constants all requirements for the model should have expressive form but because of the existence of large number of constants in the process it is difficult to obtain an optimal estimation. Theorem 3 computes the sharpest inequality for all δ in the interval in the interval $(0, R]$. Finally, to show the application of theorems, three examples presented. In example 1, the inter-arrival times are equal to one and discrete probability density functions are supposed for claim amounts. In example 2, inter-arrival times are distributed as Gamma distribution and claim amounts follow the Exponential distribution. According to the Theorems 2 and 3, the inequalities on the infinite time ruin probabilities computed and the results show that the optimal estimation can be obtain using Theorem 3. In example 3, we used the empirical insurance data set which is provided by Iran insurance company and using of Theorems 3 shows that optimal estimation holds on the infinite time ruin probability.

Declarations

Availability of data and materials

The data sets are generated or analysed during the current study.

Competing interests

I declare that they have no competing interests.

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Authors' contributions

I were a major contributor in writing the manuscript, read and approved the final manuscript.

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