The characteristic polynomial in calculation of exponential and elementary functions in Clifford algebras

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December 14, 2022

Abstract

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Received: 00 Month 0000 | Revised: 00 Month 0000 | Accepted: 00 Month 0000

DOI: xxx/xxxx

RESEARCH ARTICLE

The characteristic polynomial in calculation of exponential and elementary functions in Clifford algebras

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Summary

Formulas to calculate multivector exponentials in a basis-free representation and orthonormal basis are presented for an arbitrary Clifford geometric algebra $Cl_{p,q}$. The formulas are based on the analysis of roots of characteristic polynomial of a multivector. Elaborate examples how to use the formulas in practice are presented. The results are generalised to arbitrary functions of multivector and may be useful in the quantum circuits or in the problems of analysis of evolution of the entangled quantum states.

KEYWORDS:
Clifford (geometric) algebra, exponential of Clifford number, function of multivector, computer-aided theory

1 | INTRODUCTION

Mathematical models of physical, economical, biological, etc processes often require computation of matrix functions. Since in many cases the matrices can be replaced by multivectors (MV), the exponential of MV [1,2,3,4] in geometric (Clifford) algebras has a wide range of applications as well. In this article we will focus on how to compute the exponential of MV, though the final formula will apply to arbitrary function of MV.

The exponential of matrix can be computed by a number of different ways [5,6,7,8,9]. The review article [10] presents twenty methods[1] related to the approximate (finite precision) methods only. According to [10], our approach in the present paper can be identified as METHOD 8 and falls into the class of *polynomial methods*, except that here we provide explicit and exact formulas for the basis expansion coefficients instead of recursive approximation. The polynomial methods [10] are known to have $O(d^4)$ complexity if classical matrix multiplication is used and, therefore, they are prohibitively expensive except for small matrix dimensions d. As far as the exact (closed form) formulas for exponentials and other functions are concerned, the most of the works deal either with low dimensional cases [9,7,6] (dimensions 5 and 6 are already causing problems [8]), with matrices that are representations of some Lie groups [11], or alternatively have some other special symmetries [5].

Some modern computer algebra systems have internal commands to compute functions of matrices. For example, *Mathematica* (version 9 and higher) has [12] the function **MatrixFunction[ ]** (and also more specialized **MatrixExp[ ]**), that allows to compute arbitrary function of matrix. For matrices with exact entries it utilizes the decomposition of matrix into Jordan canonical form. The proof that any square matrix can be brought into this form is rather complicated [13](Chs. 5-6), and [14](Sec. 4,p. 49-54 and 235). As far as we know, the implementation of the Jordan decomposition relies on iterative algorithm that produces the required basis step by step.

[0]**Abbreviations:** MV, multivector; GA, geometric (Clifford) algebra; 3D, three dimensional vector space

[1]The article is named "Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later". One more method was added in the revised version of the original article (published in 1978), however, authors wanted to preserve the article title. The next article update is planned in 2028.

In the context of geometric algebra (GA) there often is a need to compute a rotor, which is an exponential of GA bivector. The simplest half-angle rotors are related to trigonometric and hyperbolic functions. The GA exponential of an arbitrary bivector can be computed using the method of invariant decomposition[15], where the bivector is decomposed into commuting orthogonal 2-blades, exponentiation of which are more or less straightforward. For low dimensional cases other decomposition techniques can be applied as well[16,17].

When dealing with exponentials of pure bivector $\mathcal{A}$ one should always keep in mind that, strictly speaking, they do not form a group. For example, there are elements of $Spin_+(2,2)$ group that can't be written in the exponential form $\pm e^{\mathcal{A}}$. Also $SO_+(1,3)$ contains elements, that are not exponentials of bivectors[2] (p. 224).

For $n \leq 3$, explicit formulas for computation of general exponentials[18,19,20] and all of square roots[21,22] are known. Simple formulas for low dimensional algebras are faster and easier to implement.

In this paper, explicit formulas that allow to calculate the exponential and any other function of MV argument in an arbitrary $Cl_{p,q}$ are presented. Though the strict proof is given only in the case of exponential and diagonalizable MV, we have verified the formula for many other functions and established that it can handle a non-diagonalizable MV as a special limiting case.

In Section 3 the methods to generate characteristic polynomials in $Cl_{p,q}$ algebras characterized by arbitrary signature $\{p, q\}$ and vector space dimension $n = p + q$ are discussed. The method of calculation of the exponential is presented in Section 4. In Section 5 we demonstrate that the obtained GA exponentials may be applied to calculate the elementary and special GA functions.

2 | NOTATION

Below, the notation used in the paper is described briefly. For those readers who are unfamiliar with Clifford geometric algebras we recommend an excellent textbook by Lounesto[2].

In the orthonormalized basis used here the geometric product of basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ satisfy[2] the anti-commutation relation, $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = \pm 2\delta_{ij}$. The number of subscripts indicates the grade. For a mixed signature $Cl_{p,q}$ algebra the squares of basis vectors, correspondingly, are $\mathbf{e}_i^2 = +1$ and $\mathbf{e}_j^2 = -1$, where $i = 1, 2, \dots, p$ and $j = p+1, p+2, \dots, p+q$. The sum $n = p + q$ is the dimension of the vector space. The general MV is expressed as

$$\mathsf{A} = a_0 + \sum_i a_i\mathbf{e}_i + \sum_{i<j} a_{ij}\mathbf{e}_{ij} + \cdots + a_{1\dots n}\mathbf{e}_{1\dots n} = a_0 + \sum_J^{2^n-1} a_J\mathbf{e}_J, \tag{1}$$

where $a_i, a_{ij\dots}$ are the real coefficients. The ordered set of indices will be denoted by a single capital letter J referred to as a multi-index. Note, that in the multi-index representation the scalar is deliberately excluded from summation as indicated by the upper range $2^n - 1$ in the sum in the last expression. The convention is useful since the separated scalar term often enables to rewrite final formulas in a simpler form. The basis elements $\mathbf{e}_{ij\dots}$ are always assumed to be listed in the reverse degree lexicographic order. For example, when $p + q = 3$ then the basis elements are listed in the order $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123} \equiv I\}$, i.e., both the number of indices and their values always increases from left to right.

There are three well- known main involutions: the reversion (e.g., $\widetilde{\mathbf{e}_{12}} = \mathbf{e}_{21} = -\mathbf{e}_{12}$), the grade inverse (e.g., $\widehat{\mathbf{e}_{123}} = -\mathbf{e}_{123}$) and the Clifford conjugation ($\widetilde{\widehat{\mathbf{e}_{123}}} = -\mathbf{e}_{321} = \mathbf{e}_{123}$). We shall also take advantage of non-zero grade negation (see Table 1), which is denoted by overline $\overline{\mathsf{A}}$. It is an operation that changes signs of all grades, except the scalar, to opposite, i.e. $\overline{\mathsf{A}} = a_0 - \sum_J^{2^n-1} a_J\mathbf{e}_J$. The last involution we shall need is the well known Hermitian conjugation operation denoted as $\mathsf{A}^\dagger$. The MV Hermitian conjugation expressed for basis elements $\mathbf{e}_J$ in both real and complex GAs can be written as[3,23]

$$\mathsf{A}^\dagger = a_0^* + a_1^*\mathbf{e}_1^{-1} + \cdots + a_{12}^*\mathbf{e}_{12}^{-1} + \cdots + a_{123}^*\mathbf{e}_{123}^{-1}\cdots = a_0^* + \sum_J a_J^*\mathbf{e}_J^{-1}, \tag{2}$$

where a_J^* is the complex conjugated J-th coefficient and $\mathbf{e}_J^{-1}$ denotes inverse[2] basis element, $\mathbf{e}_J^{-1}\mathbf{e}_J = 1$. For each multi-index that represents the basis vector with $\mathbf{e}_J^2 = +1$ the Hermitian conjugation does nothing and changes signs if $\mathbf{e}_J^2 = -1$. Therefore, the basis elements $\mathbf{e}_J$ and $\mathbf{e}_J^\dagger$ can differ by a sign only.

[2] There is a simple trick to find $\mathbf{e}_J^{-1}$. Formally raise all indices and then lower them down but now taking into account the considered algebra signature $\{p, q\}$. Finally, apply the reversion. For example, in $Cl_{0,3}$ we have $\mathbf{e}_{123} \to \mathbf{e}^{123} \to -\mathbf{e}_{123} \to -\widetilde{\mathbf{e}_{123}} \to \mathbf{e}_{123}$. Thus, $\mathbf{e}_{123}^\dagger = \mathbf{e}_{123}$.

$Cl_{p,q}$	$\mathrm{Det}(\mathsf{A})$
$p+q=1,2$	$\mathsf{A}\bar{\mathsf{A}}$
$p+q=3,4$	$\frac{1}{3}\big(\mathsf{A}\mathsf{A}\overline{\mathsf{A}\mathsf{A}}+2\mathsf{A}\overline{\bar{\mathsf{A}}\overline{\bar{\mathsf{A}}\bar{\mathsf{A}}}}\big)$
$p+q=5,6$	$\frac{1}{3}\big(\mathsf{H}\mathsf{H}\overline{\mathsf{H}\mathsf{H}}+2\mathsf{H}\overline{\bar{\mathsf{H}}\overline{\bar{\mathsf{H}}\bar{\mathsf{H}}}}\big)$ with $\mathsf{H}=\mathsf{A}\widetilde{\bar{\mathsf{A}}}$

TABLE 1 Optimized expressions for determinant of MV A in low dimensional GAs, $n \leq 6$. The overbar denotes a negation of all grades except of the scalar, $\overline{\mathsf{A}} := 2\langle\mathsf{A}\rangle_0 - \mathsf{A}$.

3 | MV CHARACTERISTIC POLYNOMIAL AND EQUATION

The algorithm to calculate the exponential and associated functions presented below is based on a characteristic polynomial. Characteristic polynomial of MV can be computed by a number of ways, such as recursive Faddeev-LeVerrier algorithm, utilizing explicit MV determinant formulas, or the methods related to Bell polynomials[24,25,26] to mention few. In this section the listed methods are briefly summarized.

Every MV $\mathsf{A} \in Cl_{p,q}$ has a characteristic polynomial $\chi_{\mathsf{A}}(\lambda)$ of degree d in $\mathbb{R}$, where $d = 2^{\lceil \frac{n}{2} \rceil}$ is the integer, $n = p+q$. In particular, $d = 2^{n/2}$ if n is even and $d = 2^{(n+1)/2}$ if n is odd. The integer d may be also interpreted as a dimension of real or complex matrix representation[3] of Clifford algebra in the 8-fold periodicity table[2].

The characteristic polynomial[27,24,25,26] is defined by

$$\chi_{\mathsf{A}}(\lambda) = -\operatorname{Det}(\lambda - \mathsf{A}) = \sum_{k=0}^{d} C_{(d-k)}(\mathsf{A})\,\lambda^k. \tag{3}$$

The variable in the characteristic polynomial will be denoted by λ and the roots of the equation $\chi_{\mathsf{A}}(\lambda) = 0$ (which is called the characteristic equation) by λ_i, respectively. For real GAs the coefficients $C_{(k)} \equiv C_{(k)}(\mathsf{A})$ are real. They depend on a selected GA and MV A. The coefficient at the highest power of λ is always assumed to be $C_{(0)} = -1$. The coefficient $C_{(1)}(\mathsf{A})$ represents MV trace, $C_{(1)}(\mathsf{A}) = \mathrm{Tr}(\mathsf{A}) = d\,\langle\mathsf{A}\rangle_0$, where $\langle\mathsf{A}\rangle_0$ is the scalar part of MV in (1), i.e. $\langle\mathsf{A}\rangle_0 = a_0$. The coefficient $C_{(d)}(\mathsf{A})$ is related to MV determinant $\operatorname{Det}\mathsf{A} = -C_{(d)}(\mathsf{A})$.

Table 1 shows how the MV determinant (a real number) can be calculated in the low dimensional ($n \leq 6$) GAs. This table may be used to find the other coefficients $C_{(k)}(\mathsf{A})$ in the characteristic polynomial (3). For a concrete algebra it is enough to replace the products of A's in the Table 1 by products of $(\lambda - \mathsf{A})$.

Example 1. *Determinant of quaternion.* In case of Hamilton quaternion (algebra $Cl_{0,2}$) we have $\mathsf{A} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_{12}\mathbf{e}_{12}$ and $\overline{\mathsf{A}} = a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_{12}\mathbf{e}_{12}$, then from Table 1 we find

$$\chi_{\mathsf{A}}(\lambda) = -\operatorname{Det}(\lambda - \mathsf{A}) = -(\lambda - \mathsf{A})(\lambda - \overline{\mathsf{A}}) = -(a_0^2 + a_1^2 + a_2^2 + a_{12}^2) + 2a_0\lambda - \lambda^2. \tag{4}$$

Thus, $C_{(0)} = -1$, $C_{(1)} = 2a_0 = \operatorname{Tr}\mathsf{A}$ and $C_{(2)} = -(a_0^2 + a_1^2 + a_2^2 + a_{12}^2) = -\operatorname{Det}\mathsf{A}$. The latter is in accord with that calculated from quaternion matrix $\mathbb{H} \cong \mathbb{C}(2)$ representation: $\mathsf{A}=\begin{pmatrix} a_0+\mathrm{i}a_1 & a_2+\mathrm{i}a_{12} \\ -a_2+\mathrm{i}a_{12} & a_0-\mathrm{i}a_1 \end{pmatrix}$.

From **Example 1** it is easy to notice that Table 1 can be used to find explicit expressions of other coefficients $C_{(k)}(\mathsf{A})$ of polynomial (3) as well. To this end it is enough to replace $\mathrm{Det}(\mathsf{A})$ in the table by $\mathrm{Det}(\lambda - \mathsf{A})$ and then recursively differentiate with respect to λ a proper number of times,

$$C_{(k-1)}(\mathsf{A}) = -\frac{1}{d-(k-1)}\left.\frac{\partial C_{(k)}(\lambda - \mathsf{A})}{\partial\lambda}\right|_{\lambda=0} \qquad k = d, \ldots, 1\,, \tag{5}$$

which is a straightforward method to obtain the coefficient at λ^{d-k} of any polynomial.

[3]The dimensions of matrices given by periodicity table are $d' = \begin{cases} 2^{n/2}, & \text{if } 0 \equiv (p-q) \mod 8 \text{ or } 2 \equiv (p-q) \mod 8 \\ 2^{(n-1)/2} + 2^{(n-1)/2}, & \text{if } 1 \equiv (p-q) \mod 8 \\ 2^{(n-1)/2} & \text{if } 3 \equiv (p-q) \mod 8 \text{ or } 7 \equiv (p-q) \mod 8 \\ 2^{(n-2)/2} & \text{if } 4 \equiv (p-q) \mod 8 \text{ or } 6 \equiv (p-q) \mod 8 \\ 2^{(n-3)/2} + 2^{(n-3)/2} & \text{if } 5 \equiv (p-q) \mod 8 \end{cases}$, where the first two entries represents dimension of real matrices, the last two represent dimension of quaternionic matrices and the middle one is the dimension of complex matrix. In general $d' \neq d$ and, therefore, for computation purposes we replace quaternionic matrices by complex matrices of double dimension using the isomorphism $\mathbb{H} \cong \mathbb{C}(2)$.

If the explicit formula for the determinant of MV is unknown then the Faddeev-Leverrier method and its modifications[24,28,27,29] allow the coefficients $C_{(k)}(\mathsf{A})$ in the polynomial (3) to be calculated recursively, beginning from $C_{(1)}(\mathsf{A})$ and ending at $C_{(d)}(\mathsf{A})$. The computation starts by setting $\mathsf{A}_{(1)} = \mathsf{A}$. In the next step the coefficient $C_{(k)}(\mathsf{A}) = \frac{d}{k}\langle \mathsf{A}_{(k)}\rangle_0$ and the new MV $\mathsf{A}_{(k+1)} = \mathsf{A}\big(\mathsf{A}_{(k)} - C_{(k)}\big)$ are computed:

$$\begin{aligned} \mathsf{A}_{(1)} = \mathsf{A} &\rightarrow C_{(1)}(\mathsf{A}) = \tfrac{d}{1}\langle \mathsf{A}_{(1)}\rangle_0,\\ \mathsf{A}_{(2)} = \mathsf{A}\big(\mathsf{A}_{(1)} - C_{(1)}\big) &\rightarrow C_{(2)}(\mathsf{A}) = \tfrac{d}{2}\langle \mathsf{A}_{(2)}\rangle_0,\\ &\vdots\\ \mathsf{A}_{(d)} = \mathsf{A}\big(\mathsf{A}_{(d-1)} - C_{(d-1)}\big) &\rightarrow C_{(d)}(\mathsf{A}) = \tfrac{d}{d}\langle \mathsf{A}_{(d-1)}\rangle_0. \end{aligned} \tag{6}$$

The determinant of MV then is $\mathrm{Det}(\mathsf{A}) = \mathsf{A}_{(d)} = -C_{(d)} = \mathsf{A}\big(\mathsf{A}_{(d-1)} - C_{(d-1)}\big)$. If we extend the computation by one step more we will find the identity $\mathsf{A}_{(d+1)} = \mathsf{A}\big(\mathsf{A}_{(d)} - C_{(d)}\big) = 0$ is satisfied. The identity plays a key role in the proof of our main result. This algorithm if adapted to GA allows to compute characteristic polynomials for MV of arbitrary algebra $Cl_{p,q}$.

In alternative recursive method[24] instead of[4], one starts from $C'_{(0)}(\mathsf{A}) = 1$ (instead of $C_{(0)}(\mathsf{A}) = -1$) and initial MV $\mathsf{B}_0 = 1$, and uses the following iterative procedure,

$$C'_{(k)} = -\mathrm{Tr}(\mathsf{A}\mathsf{B}_{k-1})/k, \qquad \mathsf{B}_k = \mathsf{A}\mathsf{B}_{k-1} + C'_{(k)}, \quad k = 1, 2, \cdots, d. \tag{7}$$

The trace may be calculated after multiplication of MVs and taking the scalar part of the result, $\mathrm{Tr}(\mathsf{A}\mathsf{B}_{k-1}) = d\,\langle \mathsf{A}\mathsf{B}_{k-1}\rangle_0$, or using the trace formula for products of MVs[25,26].

The coefficients of characteristic polynomial also can be found explicitly from complete Bell polynomials. In this approach a set of scalars is used[25,26],

$$S_{(k)}(\mathsf{A}) := (-1)^{k-1} d(k-1)!\langle \mathsf{A}^k\rangle_0, \qquad k = 1, \ldots, d, \tag{8}$$

where $\langle \mathsf{A}^k\rangle_0$ is the scalar part of MV raised to k power. The needed coefficients are given by

$$C_{(0)}(\mathsf{A}) = -1; \qquad C_{(k)}(\mathsf{A}) = \frac{(-1)^{k+1}}{k!} B_k(S_{(1)}(\mathsf{A}), S_{(2)}(\mathsf{A}), S_{(3)}(\mathsf{A}), \ldots, S_{(k)}(\mathsf{A})), \qquad k = 1, \ldots, d, \tag{9}$$

where $B_k(x_1, \ldots, x_k)$ are the complete Bell polynomials. The first Bell polynomials[5] are defined by relations

$$\begin{aligned} B_0 = 1, \quad B_1(x_1) &= B_0 x_1 = x_1,\\ B_2(x_1, x_2) &= B_1 x_1 + B_0 x_2 = x_1^2 + x_2,\\ B_3(x_1, x_2, x_3) &= B_2 x_1 + \tbinom{2}{1} B_1 x_2 + B_0 x_4 = x_1^3 + 3x_1 x_2 + x_3,\\ B_4(x_1, x_2, x_3, x_4) &= B_3 x_1 + \tbinom{3}{1} B_2 x_2 + \tbinom{3}{2} B_1 x_3 + B_0 x_1 = x_1^4 + 6x_1^2 x_2 + 4x_1 x_3 + 3x_2^2 + x_4,\\ B_5(x_1, x_2, x_3, x_4, x_5) &= B_4 x_1 + \tbinom{4}{1} B_3 x_2 + \tbinom{4}{2} B_2 x_3 + \tbinom{4}{3} B_1 x_4 + B_0 x_5 = x_1^4 + 6x_1^2 x_2 + 4x_1 x_3 + 3x_2^2 + x_4, \end{aligned} \tag{10}$$

where $\binom{n}{r}$ is the binomial coefficient. This sequence can be easily extended to higher orders. The complete Bell polynomials can also be represented in a form of matrix determinant[30].

The coefficients of characteristic equation satisfy the following properties

$$\frac{\partial C_{(k)}(t\mathsf{A})}{\partial t} = k t^{k-1} C_{(k)}(t\mathsf{A}), \qquad \frac{\partial C_{(1)}(t\mathsf{A}^k)}{\partial t} = k t^{k-1} C_{(1)}(t\mathsf{A}^k), \tag{11}$$

where t is a scalar parameter. We will utilize them when proving the exponential formula.

[4]This means that all characteristic coefficients computed with this formula are of opposite sign compared to (6), i.e. $C'_{(k)} = -C_{(k)}$.

[5]*Mathematica* (v.8 and higher) already has function for partial Bell polynomials **BellY[ ]**. The Bell Complete Polynomial then can be computed as **BellCP[x_List]:= Sum[BellY[Length[x], k, x], {k,1,Length[x]}]**, where **x_List** is a list of variables x_i.

Since provided below formulas contain the sums over roots of characteristic polynomial, it is worth to remind generalized Viète's formulas that relate coefficients of characteristic polynomial to specific sums over the roots r_i:

$$r_1 + r_2 + \cdots + r_{d-1} + r_d = (-1)^1 \frac{C_{(1)}}{C_{(0)}} \tag{12}$$

$$\left(r_1 r_2 + r_1 r_3 + \cdots + r_1 r_d\right) + \left(r_2 r_3 + r_2 r_4 + \cdots + r_2 r_d\right) + \cdots + r_{d-1} r_d = (-1)^2 \frac{C_{(2)}}{C_{(0)}} \tag{13}$$

$$\vdots$$

$$r_1 r_2 \ldots r_d = (-1)^d \frac{C_{(d)}}{C_{(0)}}. \tag{14}$$

The other interesting identity[27], which is important for integral representation of functions is

$$\operatorname{Tr} \mathcal{L}\left(\mathrm{e}^{t\mathsf{A}}\right) = \frac{\chi'_{\mathsf{A}}(\lambda)}{\chi_{\mathsf{A}}(\lambda)}, \tag{15}$$

where $\mathcal{L}$ denotes Laplace transform $\mathcal{L}\left(\mathrm{e}^{t\mathsf{A}}\right) = \left(\lambda - \mathsf{A}\right)^{-1}$ of MV A and $\chi'_{\mathsf{A}}(\lambda)$ is a derivative of the characteristic polynomial $\chi_{\mathsf{A}}(\lambda)$ (see (3)) with respect to polynomial variable λ.

In matrix theory very important polynomial is a minimal polynomial $\mu_A(\lambda)$. It establishes the conditions of diagonalizability of matrix A. Similar polynomial $\mu_{\mathsf{A}}(\lambda)$ may be defined for MV. In particular, it is well-known that matrix is diagonalizable (aka nondefective) if and only if the minimal polynomial of the matrix does not have multiple (repeated) roots, i.e., when the minimal polynomial is a product of distinct linear factors. It is also well-known that the minimal polynomial divides characteristic polynomial. If roots of characteristic equation are all different, then matrix/MV is diagonalizable.

The converse statement is not true. The MV, characteristic polynomial of which has multiple roots, may be diagonalizable. It is also established[31] that almost all matrices over the complex numbers $\mathbb{C}$ are diagonalizable, i.e., the set of complex $d \times d$ matrices considered as a subset of $\mathbb{C}^{d\times d}$ that are not diagonalizable over $\mathbb{C}$ has the Lebesgue measure zero (with respect to the Zariski topology). An algorithm how to compute minimal polynomial of MV without doing a reference to matrix representation of the MV is given in Appendix 7.

4 | MV EXPONENTIALS IN $Cl_{p,q}$ ALGEBRA

This section presents main results of the article.

Theorem 1 (MV exponential in basis-free form)**.** The exponential of multivector $\mathsf{A} = a_0 + \sum_J^{2^n-1} a_J \mathbf{e}_J$ in $Cl_{p,q}$ GA is the multivector given by

$$\exp(\mathsf{A}) = \sum_{i=1}^{d} \exp(\lambda_i)\, \beta(\lambda_i) \sum_{m=0}^{d-1} \left(\sum_{k=0}^{d-m-1} \lambda_i^k C_{(d-k-m-1)}(\mathsf{A}) \right) \mathsf{A}^m, \quad \text{with} \quad \beta(\lambda_i) = \frac{1}{\sum_{j=0}^{d-1} (j+1)\, C_{(d-j-1)}(\mathsf{A})\, \lambda_i^j}\,. \tag{16}$$

Here λ_i and λ_i^j denotes, respectively, the root of a characteristic equation and the root raised to power j. The first sum is over all roots λ_i of characteristic equation $\chi_{\mathsf{A}}(\lambda) = 0$, where $\chi_{\mathsf{A}}(\lambda) = \sum_{i=0}^{d} C_{(d-i)}(\mathsf{A})\, \lambda^i$ is the characteristic polynomial of MV A. $C_{(d-i)}(\mathsf{A})$ denotes the coefficient at variable λ raised to power i.

The formula has some similarity with expression for exponential of a square matrix presented in[7].

Proof. We will prove basis-free formula (16) by verifying the defining equation of exponential function

$$\left.\frac{\partial \exp(\mathsf{A}t)}{\partial t}\right|_{t=1} = \mathsf{A} \exp(\mathsf{A}) = \exp(\mathsf{A})\mathsf{A}, \tag{17}$$

where A is independent of a scalar parameter t.

First, using properties of characteristic coefficients in (11) and noting that the replacement $\mathsf{A} \to \mathsf{A}t$ implies $\lambda_i \to \lambda_i t$, we perform differentiation $\left.\frac{\partial \exp(\mathsf{A}t)}{\partial t}\right|_{t=1}$ and obtain that $\exp(\lambda_i)$ in the right hand side of (16) is replaced by $\lambda_i \exp(\lambda_i)$:

$$\left.\frac{\partial \exp(\mathsf{A}t)}{\partial t}\right|_{t=1} = \sum_{i=1}^{d} \lambda_i \exp(\lambda_i) \beta(\lambda_i) \sum_{m=0}^{d-1} \left(\sum_{k=0}^{d-m-1} \lambda_i^k C_{(d-k-m-1)}(\mathsf{A}) \right) \mathsf{A}^m. \tag{18}$$

The weight factor $\beta(\lambda_i)$ further plays no role in the proof and will be ignored. Next, we multiply the basis-free formula (16) by A

$$\mathsf{A}\exp(\mathsf{A}) = \sum_{i=1}^{d} \exp(\lambda_i)\,\beta(\lambda_i) \sum_{m=0}^{d-1} \Bigg(\sum_{k=0}^{d-m-1} \lambda_i^k C_{(d-k-m-1)}(\mathsf{A}) \Bigg) \mathsf{A}^{m+1}\,, \tag{19}$$

and subtract the second equation from the first for each fixed root λ_i, i.e. temporary ignore the summation over roots,

$$\begin{aligned} \left(\frac{\partial \exp(\mathsf{A}t)}{\partial t}\bigg|_{t=1} - \mathsf{A}\exp(\mathsf{A}) \right)\Bigg|_{\lambda_i} &= \exp(\lambda_i)\,\beta(\lambda_i)\Big(\sum_{k=1}^{d} \lambda_i^k C_{(d-k)}(\mathsf{A}) - \mathsf{A}^k C_{(d-k)}(\mathsf{A}) \Big) \\ &= \exp(\lambda_i)\,\beta(\lambda_i)\Big(\big(\lambda_i^d - \mathsf{A}^d\big) C_{(0)}(\mathsf{A}) + \cdots + \big(\lambda_i - \mathsf{A}\big) C_{(d-1)}(\mathsf{A}) \Big)\,. \end{aligned} \tag{20}$$

Using the Cayley-Hamilton relation for A, which follow from (6),

$$\sum_{k=0}^{d} \mathsf{A}^k C_{(d-k)}(\mathsf{A}) = \mathsf{A}^d C_{(0)}(\mathsf{A}) + \mathsf{A}^{d-1} C_{(1)}(\mathsf{A}) + \cdots + C_{(d)}(\mathsf{A}) = 0,$$

and the same relation for λ_i^d, we solve for the highest powers A^d and λ_i^d, and substitute them into the difference formula (20). As a result, after expansion we obtain zero. Since the identity holds for each of roots λ_i, it is true for a sum over roots as well. □

Corollary 1. The exponential formula (16) can be rewritten in the form which is more suitable for implementation. In particular, the scalar part can be summed up in a closed form that yields a sum of exponentials of eigenvalues divided by d,

$$\exp(\mathsf{A}) = \sum_{i=1}^{d} \exp(\lambda_i)\left(\frac{1}{d} + \beta(\lambda_i) \sum_{m=0}^{d-2} \Bigg(\sum_{k=0}^{d-m-2} \lambda_i^k C_{(d-k-m-2)}(\mathsf{A}) \Bigg) \langle \mathsf{A}^{m+1} \rangle_{-0} \right) = \sum_{i=1}^{d} \exp\big(\lambda_i\big)\Big(\frac{1}{d} + \beta(\lambda_i)\mathsf{B}(\lambda_i) \Big)\,, \tag{21}$$

where the expression $\langle \mathsf{A}^{m+1} \rangle_{-0} \equiv \frac{1}{2}\big(\mathsf{A}^{m+1} - \overline{\mathsf{A}^{m+1}}\big)$ indicates that all grades of multivector A^{m+1} are included except the grade-0. The proof follows after simple algebraic manipulations.

Since the roots of a characteristic equation in general are complex numbers, the individual terms in the sums, strictly speaking, are complex. However, the coefficients at basis elements of the final result always simplify to real numbers for real GA (see subsection 4.1).. Let's demonstrate how to apply formula (21) to the diagonalizable MV the characteristic polynomial of which has multiple (repeated) roots.

Example 2. *Exponential of MV in $Cl_{4,0}$ for multiple and zero eigenvalues.* Let's compute the exponential of $\mathsf{A} = -4-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-2\sqrt{3}\,\mathbf{e}_{1234}$ by the basis-free formula (21). Using Table 1 one can verify that $\mathrm{Det}(\mathsf{A}) = 0$. For algebra $Cl_{4,0}$ we have $d = 4$. The characteristic polynomial is $\chi_{\mathsf{A}}(\lambda) = C_{(4)}(\mathsf{A})+C_{(3)}(\mathsf{A})\lambda+C_{(2)}(\mathsf{A})\lambda^2+C_{(1)}(\mathsf{A})\lambda^3+C_{(0)}(\mathsf{A})\lambda^4 = -64\lambda^2-16\lambda^3-\lambda^4 = -\lambda^2(8+\lambda)^2$. The roots are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -8, \lambda_4 = -8$. Because multiple roots appear, we compute the minimal polynomial of MV A $\mu_{\mathsf{A}}(\lambda) = \lambda(8 + \lambda)$ (see Appendix 7). Since $\mu_{\mathsf{A}}(\lambda)$ consists of linear factors only, the MV is diagonalizable, and the formula for $\mu_{\mathsf{A}}(\lambda)$ can be applied without modification. It is also easy to verify that the minimal polynomial divides the characteristic polynomial: $\chi_{\mathsf{A}}(\lambda)/\mu_{\mathsf{A}}(\lambda) = \frac{-\lambda^2(8+\lambda)^2}{\lambda(8+\lambda)} = -\lambda(8 + \lambda)$. This confirms the property that non-repeating roots of a characteristic polynomial are sufficient criterion of MV diagonalizability. Then, we have

$$\begin{aligned} \beta(\lambda_i)\mathsf{B}(\lambda_i) &= \frac{1}{\sum_{j=0}^{d-1}(j+1)\,C_{(d-j-1)}(\mathsf{A})\,\lambda_i^j} \sum_{m=0}^{d-2} \sum_{k=0}^{d-m-2} \lambda_i^k C_{(d-k-m-2)}(\mathsf{A})\,\langle \mathsf{A}^{m+1} \rangle_{-0} \\ &= \frac{8+\lambda_i}{4\lambda_i(4+\lambda_i)}\langle \mathsf{A} \rangle_{-0} + \frac{16+\lambda_i}{4\lambda_i(4+\lambda_i)(8+\lambda_i)}\langle \mathsf{A}^2 \rangle_{-0} + \frac{1}{4\lambda_i(4+\lambda_i)(8+\lambda_i)}\langle \mathsf{A}^3 \rangle_{-0} \\ &= -\frac{1}{\lambda_i+4} - \frac{1}{4(\lambda_i+4)}\mathbf{e}_1 - \frac{1}{4(\lambda_i+4)}\mathbf{e}_2 - \frac{1}{4(\lambda_i+4)}\mathbf{e}_3 - \frac{1}{4(\lambda_i+4)}\mathbf{e}_4 - \frac{\sqrt{3}}{2\lambda_i+8}\mathbf{e}_{1234}. \end{aligned} \tag{22}$$

From the middle line in Eq. (22) one may suppose that the sum over roots would yield division by zero due to zero denominators. The last line, however, demonstrates that this is not the case, since after collecting terms at basis elements we see that all potential zeroes in the denominators cancelled out. Unfortunately, this would not occur if the MV were non-diagonalizable. Lastly, after performing summation $\sum_{i=1}^{d} \exp(\lambda_i)\big(\frac{1}{d}+\beta(\lambda_i)\mathsf{B}(\lambda_i)\big)$ over the complete set of roots $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{0, 0, -8, -8\}$ with exponent weight factor $\exp(\lambda_i)$, which can be replaced by any other function or transformation (see Section 5) we obtain

$$\exp(\mathsf{A}) = \frac{1+\mathrm{e}^8}{2\mathrm{e}^8} + \frac{1-\mathrm{e}^8}{8\mathrm{e}^8}\big(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 - 2\sqrt{3}\mathbf{e}_{1234}\big).$$

The main formula (16) can be rewritten in coordinate form as well.

Corollary 2 (Exponential in the coordinate form)**.** The exponential of the multivector $\mathsf{A} = a_0 + \sum_J^{2^n-1} a_J \mathbf{e}_J$ is the MV

$$\exp(\mathsf{A}) = \frac{1}{d}\sum_{i=1}^{d}\exp(\lambda_i)\left(1+\sum_{J}^{2^n-1}\mathbf{e}_J\,\frac{\sum_{m=0}^{d-2}\lambda_i^m\sum_{k=0}^{d-m-2}C_{(k)}(\mathsf{A})\,C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^{d-k-m-1})}{\sum_{k=0}^{d-1}(k+1)\,C_{(d-k-1)}(\mathsf{A})\,\lambda_i^k}\right) = \frac{1}{d}\sum_{i=1}^{d}\exp\big(\lambda_i\big)\Big(1+\sum_{J}^{2^n-1}\mathbf{e}_J\,b_J(\lambda_i)\Big). \quad (23)$$

Here λ_i and λ_i^j denotes, respectively, the root of a characteristic equation and the root raised to power j. $C_{(d-i)}(\mathsf{A})$ is the coefficient at λ^i. The symbol $C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^k) = d\,\langle\mathbf{e}_J^\dagger\mathsf{A}^k\rangle_0$ denotes the first coefficient (the coefficient at λ^{d-1}) in the characteristic polynomial that consists of geometric product of the hermitian conjugate basis element $\mathbf{e}_J^\dagger$ and k-th power of initial MV: $\mathbf{e}_J^\dagger\mathsf{A}^k = \mathbf{e}_J^\dagger\,\underbrace{\mathsf{A}\mathsf{A}\cdots\mathsf{A}}_{k\text{ terms}}$.

Proof. The coordinate formula is easy to prove by noting that projection coefficient onto basis element $\mathbf{e}_J$ can be written as $\mathrm{Tr}\big(\mathbf{e}_J^\dagger\mathsf{A}^r\big) = C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^r)$. □

Example 3. *Exponential of generic MV in $Cl_{0,3}$ with all different roots.* Let's compute the exponential of $\mathsf{A} = 8 - 6\mathbf{e}_2 - 9\mathbf{e}_3 + 5\mathbf{e}_{12} - 5\mathbf{e}_{13} + 6\mathbf{e}_{23} - 4\mathbf{e}_{123}$ in coordinate form, Eq. (23). Computation of coefficients of the characteristic polynomial $\chi_\mathsf{A}(\lambda) = C_{(4)}(\mathsf{A}) + C_{(3)}(\mathsf{A})\lambda + C_{(2)}(\mathsf{A})\lambda^2 + C_{(1)}(\mathsf{A})\lambda^3 + C_{(0)}(\mathsf{A})\lambda^4$ for MV A when $d = 4$. yields $C_{(0)}(\mathsf{A}) = -1$, $C_{(1)}(\mathsf{A}) = 32$, $C_{(2)}(\mathsf{A}) = -758$, $C_{(3)}(\mathsf{A}) = 10432$, $C_{(4)}(\mathsf{A}) = -72693$. The characteristic equation $\chi_\mathsf{A}(\lambda) = 0$ then becomes $-72693 + 10432\lambda - 758\lambda^2 + 32\lambda^3 - \lambda^4 = 0$, which has four different roots $\lambda_1 = 12 - \mathrm{i}\sqrt{53}$, $\lambda_2 = 12 + \mathrm{i}\sqrt{53}$, $\lambda_3 = 4 - \mathrm{i}\sqrt{353}$, $\lambda_4 = 4 + \mathrm{i}\sqrt{353}$. For every multi-index J and each root λ_i we have to compute the coefficients

$$b_J(\lambda_i) = \frac{-\lambda_i^2 C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}) + \lambda_i\big(32C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}) - C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^2)\big) - 758C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}) + 32C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^2) - C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^3)}{-4\lambda_i^3 + 96\lambda_i^2 - 1516\lambda_i + 10432},$$

where we still have to substitute the coefficients, different for each multi-index J:. $C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^k)$

$C_{(1)}(\mathbf{e}_J^\dagger\mathsf{A}^k)$	$\mathbf{e}_1^\dagger$	$\mathbf{e}_2^\dagger$	$\mathbf{e}_3^\dagger$	$\mathbf{e}_{12}^\dagger$	$\mathbf{e}_{13}^\dagger$	$\mathbf{e}_{23}^\dagger$	$\mathbf{e}_{123}^\dagger$
$k=1$	0	−24	−36	20	−20	24	−16
$k=2$	192	−224	−416	32	−128	384	−856
$k=3$	8208	5952	5508	−11572	7468	888	−7984

The Hermite conjugate elements are $\mathbf{e}_J^\dagger = \{-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_{12}, -\mathbf{e}_{13}, -\mathbf{e}_{23}, \mathbf{e}_{123}\}$. After substituting all computed quantities into (23) we finally get, where $\alpha = \sqrt{53}$ and $\beta = \sqrt{353}$,

$$\begin{aligned}\exp(\mathsf{A}) =& \frac{1}{2}\mathrm{e}^4\left(\mathrm{e}^8\big(\cos(\alpha)+\cos(\beta)\big)\right) + \left(\frac{3}{\alpha}\mathrm{e}^{12}\sin(\alpha) - \frac{3}{\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_1\\ &+\left(\frac{-1}{2\alpha}\mathrm{e}^{12}\sin(\alpha) - \frac{11}{2\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_2 + \left(-\frac{2}{\alpha}\mathrm{e}^{12}\sin(\alpha) - \frac{7}{\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_3\\ &+\left(-\frac{2}{\alpha}\mathrm{e}^{12}\sin(\alpha) + \frac{7}{\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_{12} + \left(\frac{1}{2\alpha}\mathrm{e}^{12}\sin(\alpha) - \frac{11}{2\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_{13}\\ &+\left(\frac{3}{\alpha}\mathrm{e}^{12}\sin(\alpha) + \frac{3}{\beta}\mathrm{e}^4\sin(\beta)\right)\mathbf{e}_{23} + \frac{1}{2}\mathrm{e}^4\left(\cos(\beta) - \mathrm{e}^8\cos(\alpha)\right)\mathbf{e}_{123}.\end{aligned} \quad (24)$$

which after simplification coincides with our earlier result[18].

Example 4. *Exponential of generic MV in $Cl_{4,2}$ with different roots.* Let's compute the exponential of $\mathsf{A} = 2 + 3\mathbf{e}_4 + 3\mathbf{e}_{26} + \mathbf{e}_{1345} - 2\mathbf{e}_{12456} + 3\mathbf{e}_{123456}$ using formula (23). In this case $d = 8$ and $\chi_\mathsf{A}(\lambda) = C_{(8)}(\mathsf{A}) + C_{(7)}(\mathsf{A})\lambda + C_{(6)}(\mathsf{A})\lambda^2 + C_{(5)}(\mathsf{A})\lambda^3 + C_{(4)}(\mathsf{A})\lambda^4 + C_{(3)}(\mathsf{A})\lambda^5 + C_{(2)}(\mathsf{A})\lambda^6 + C_{(1)}(\mathsf{A})\lambda^7 + C_{(0)}(\mathsf{A})\lambda^8$. The coefficients of characteristic polynomial $\chi_\mathsf{A}(\lambda)$ are $C_{(0)}(\mathsf{A}) = -1$, $C_{(1)}(\mathsf{A}) = 16$, $C_{(2)}(\mathsf{A}) = -64$, $C_{(3)}(\mathsf{A}) = 16$, $C_{(4)}(\mathsf{A}) = 32$, $C_{(5)}(\mathsf{A}) = -1280$, $C_{(6)}(\mathsf{A}) = 20672$, $C_{(7)}(\mathsf{A}) = -42752$, $C_{(8)}(\mathsf{A}) = 14336$. The characteristic equation $\chi_\mathsf{A}(\lambda) = 0$ is $14336 - 42752\lambda + 20672\lambda^2 - 1280\lambda^3 + 32\lambda^4 + 16\lambda^5 - 64\lambda^6 + 16\lambda^7 - \lambda^8 = 0$. It has eight different roots $\lambda_1 = -4$, $\lambda_2 = 2$, $\lambda_3 = 5 - \mathrm{i}\sqrt{3}$, $\lambda_4 = 5 + \mathrm{i}\sqrt{3}$, $\lambda_5 = -1 - \mathrm{i}\sqrt{15}$, $\lambda_6 = -1 + \mathrm{i}\sqrt{15}$, $\lambda_7 = 5 - \sqrt{21}$, $\lambda_8 = 5 + \sqrt{21}$.

Then, for every multi-index J and each root λ_i we have to compute the coefficients

$$
\begin{aligned}
&b_J(\lambda_i) = \\
&\Big(C_0(A)C_1(\mathbf{e}_J^\dagger A^1)\lambda_i^6 + \big(C_1(A)C_1(\mathbf{e}_J^\dagger A^1) + C_0(A)C_1(\mathbf{e}_J^\dagger A^2)\big)\lambda_i^5 + \big(C_2(A)C_1(\mathbf{e}_J^\dagger A^1) + C_1(A)C_1(\mathbf{e}_J^\dagger A^2) + C_0(A)C_1(\mathbf{e}_J^\dagger A^3)\big)\lambda_i^4 \\
&\quad + \big(C_3(A)C_1(\mathbf{e}_J^\dagger A^1) + C_2(A)C_1(\mathbf{e}_J^\dagger A^2) + C_1(A)C_1(\mathbf{e}_J^\dagger A^3) + C_0(A)C_1(\mathbf{e}_J^\dagger A^4)\big)\lambda_i^3 \\
&\quad + \big(C_4(A)C_1(\mathbf{e}_J^\dagger A^1) + C_3(A)C_1(\mathbf{e}_J^\dagger A^2) + C_2(A)C_1(\mathbf{e}_J^\dagger A^3) + C_1(A)C_1(\mathbf{e}_J^\dagger A^4) + C_0(A)C_1(\mathbf{e}_J^\dagger A^5)\big)\lambda_i^2 \\
&\quad + \big(C_5(A)C_1(\mathbf{e}_J^\dagger A^1) + C_4(A)C_1(\mathbf{e}_J^\dagger A^2) + C_3(A)C_1(\mathbf{e}_J^\dagger A^3) + C_2(A)C_1(\mathbf{e}_J^\dagger A^4) + C_1(A)C_1(\mathbf{e}_J^\dagger A^5) + C_0(A)C_1(\mathbf{e}_J^\dagger A^6)\big)\lambda_i \\
&\quad + C_6(A)C_1(\mathbf{e}_J^\dagger A^1) + C_5(A)C_1(\mathbf{e}_J^\dagger A^2) + C_4(A)C_1(\mathbf{e}_J^\dagger A^3) + C_3(A)C_1(\mathbf{e}_J^\dagger A^4) + C_2(A)C_1(\mathbf{e}_J^\dagger A^5) + C_1(A)C_1(\mathbf{e}_J^\dagger A^6) \\
&\qquad + C_0(A)C_1(\mathbf{e}_J^\dagger A^7)\Big)\Big/\Big(8\lambda_i^7 C_0(A) + 7\lambda_i^6 C_1(A) + 6\lambda_i^5 C_2(A) + 5\lambda_i^4 C_3(A) + 4\lambda_i^3 C_4(A) + 3\lambda_i^2 C_5(A) + 2\lambda_i C_6(A) + C_7(A)\Big).
\end{aligned}
$$

Here the coefficients $C_{(1)}(\mathbf{e}_J^\dagger \mathsf{A}^k)$ have the following values

k	$\mathbf{e}_4^\dagger$	$\mathbf{e}_{15}^\dagger$	$\mathbf{e}_{26}^\dagger$	$\mathbf{e}_{34}^\dagger$	$\mathbf{e}_{145}^\dagger$	$\mathbf{e}_{246}^\dagger$	$\mathbf{e}_{1256}^\dagger$	$\mathbf{e}_{1345}^\dagger$	$\mathbf{e}_{2346}^\dagger$	$\mathbf{e}_{12456}^\dagger$	$\mathbf{e}_{123456}^\dagger$
1	24	0	24	0	0	0	0	8	0	−16	24
2	96	0	144	0	−96	−144	−96	−112	0	−64	48
3	1200	864	1008	−288	−672	−1008	−576	−672	96	−960	672
4	9792	8064	8256	−1152	−8832	−10368	−8064	−5312	−2688	−7808	5568
5	94848	80640	82944	−26496	−81792	−91008	−82560	−42752	−24960	−84992	46848
6	859008	787968	752256	−294912	−826368	−876672	−797184	−397824	−288768	−817152	370176
7	8221440	7628544	7243008	−3059712	−7972608	−8163072	−7531776	−3403264	−3028992	−8024320	3460608

In the Table, coefficients that are not listed must be equated to zero. The Hermitian conjugate basis elements in the inverse degree lexicographical ordering are

$$
\begin{aligned}
&\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, -\mathbf{e}_5, -\mathbf{e}_6, -\mathbf{e}_{12}, -\mathbf{e}_{13}, -\mathbf{e}_{14}, \mathbf{e}_{15}, \mathbf{e}_{16}, -\mathbf{e}_{23}, -\mathbf{e}_{24}, \mathbf{e}_{25}, \mathbf{e}_{26}, -\mathbf{e}_{34}, \mathbf{e}_{35}, \mathbf{e}_{36}, \mathbf{e}_{45}, \mathbf{e}_{46}, -\mathbf{e}_{56}, -\mathbf{e}_{123}, -\mathbf{e}_{124}, \mathbf{e}_{125}, \mathbf{e}_{126}, -\mathbf{e}_{134}, \\
&\mathbf{e}_{135}, \mathbf{e}_{136}, \mathbf{e}_{145}, \mathbf{e}_{146}, -\mathbf{e}_{156}, -\mathbf{e}_{234}, \mathbf{e}_{235}, \mathbf{e}_{236}, \mathbf{e}_{245}, \mathbf{e}_{246}, -\mathbf{e}_{256}, \mathbf{e}_{345}, \mathbf{e}_{346}, -\mathbf{e}_{356}, -\mathbf{e}_{456}, \mathbf{e}_{1234}, -\mathbf{e}_{1235}, -\mathbf{e}_{1236}, -\mathbf{e}_{1245}, -\mathbf{e}_{1246}, \mathbf{e}_{1256}, \\
&-\mathbf{e}_{1345}, -\mathbf{e}_{1346}, \mathbf{e}_{1356}, \mathbf{e}_{1456}, -\mathbf{e}_{2345}, -\mathbf{e}_{2346}, \mathbf{e}_{2356}, \mathbf{e}_{2456}, \mathbf{e}_{3456}, -\mathbf{e}_{12345}, -\mathbf{e}_{12346}, \mathbf{e}_{12356}, \mathbf{e}_{12456}, \mathbf{e}_{13456}, \mathbf{e}_{23456}, -\mathbf{e}_{123456}\}.
\end{aligned}
$$

Substituting all quantities into (23) after simplification we get

$$
\begin{aligned}
\exp(\mathsf{A}) = {} & \frac{1+e^6+2e^3\cos\sqrt{15}+2e^9\left(\cos\sqrt{3}+\cosh\sqrt{21}\right)}{8e^4} + \frac{-175+175e^6+14\sqrt{15}e^3\sin\sqrt{15}+10\sqrt{3}e^9\left(7\sin\sqrt{3}+5\sqrt{7}\sinh\sqrt{21}\right)}{840e^4}\mathbf{e}_4 \\
& - \frac{1+e^6-2e^3\cos\sqrt{15}+2e^9\left(\cos\sqrt{3}-\cosh\sqrt{21}\right)}{8e^4}\mathbf{e}_{15} - \frac{1+e^6+2e^3\cos\sqrt{15}-2e^9\left(\cos\sqrt{3}+\cosh\sqrt{21}\right)}{8e^4}\mathbf{e}_{26} \\
& + \frac{35-35e^6+14\sqrt{15}e^3\sin\sqrt{15}+5\sqrt{3}e^9\left(7\sin\sqrt{3}-\sqrt{7}\sinh\sqrt{21}\right)}{210e^4}\mathbf{e}_{34} + \frac{-175+175e^6-14\sqrt{15}e^3\sin\sqrt{15}+10\sqrt{3}e^9\left(7\sin\sqrt{3}-5\sqrt{7}\sinh\sqrt{21}\right)}{840e^4}\mathbf{e}_{145} \\
& + \frac{-175+175e^6+14\sqrt{15}e^3\sin\sqrt{15}-10\sqrt{3}e^9\left(7\sin\sqrt{3}+5\sqrt{7}\sinh\sqrt{21}\right)}{840e^4}\mathbf{e}_{246} - \frac{1+e^6-2e^3\cos\sqrt{15}+2e^9\left(\cosh\sqrt{21}-\cos\sqrt{3}\right)}{8e^4}\mathbf{e}_{1256} \\
& + \frac{-35+35e^6+14\sqrt{15}e^3\sin\sqrt{15}-5\sqrt{3}e^9\left(7\sin\sqrt{3}+\sqrt{7}\sinh\sqrt{21}\right)}{210e^4}\mathbf{e}_{1345} + \frac{-35+35e^6-14\sqrt{15}e^3\sin\sqrt{15}+5\sqrt{3}e^9\left(7\sin\sqrt{3}-\sqrt{7}\sinh\sqrt{21}\right)}{210e^4}\mathbf{e}_{2346} \\
& + \frac{175-175e^6+14\sqrt{15}e^3\sin\sqrt{15}+10\sqrt{3}e^9\left(7\sin\sqrt{3}-5\sqrt{7}\sinh\sqrt{21}\right)}{840e^4}\mathbf{e}_{12456} + \frac{-35+35e^6+14\sqrt{15}e^3\sin\sqrt{15}+5\sqrt{3}e^9\left(7\sin\sqrt{3}+\sqrt{7}\sinh\sqrt{21}\right)}{210e^4}\mathbf{e}_{123456}.
\end{aligned}
$$

Note that the coefficients at basis elements include trigonometric and hyperbolic functions.

All previous examples illustrated that both coordinate and basis-free formulas allow to compute exponential for diagonalizable MV. Below, the last example is intended to demonstrate that the formula yields a correct limiting case for non-diagonalizsble MVs as well.

Example 5. *Exponential of non-diagonalizable MV in $Cl_{3,0}$.* Let's find the exponential of $\mathsf{A} = -1 + 2\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3 - 2\mathbf{e}_{12} - 2\mathbf{e}_{13} + \mathbf{e}_{23} - \mathbf{e}_{123}$ with the help of base-free formula (21). For algebra $Cl_{3,0}$ we have $d = 4$. The minimal polynomial is $\mu_{\mathsf{A}}(\lambda) = -(2 + 2\lambda + \lambda^2)^2$ which coincides with characteristic polynomial $\chi_{\mathsf{A}}(\lambda) = -\operatorname{Det}(\lambda - \mathsf{A})$ and has multiple roots $\{-(1+\mathrm{i}), -(1+\mathrm{i}), -(1-\mathrm{i}), -(1-\mathrm{i})\}$. Now, if we proceed as in Example 2 then for some basis elements we will get division by zero. To avoid this, we will add a small element to MV, $\mathsf{A} + \varepsilon\mathbf{e}_1 = \mathsf{A}'$, and after exponentiation and simplification will compute a limiting value when $\varepsilon \to 0$. The infinitesimal element $\varepsilon\mathbf{e}_1$ may be replaced by any other, provided that it does not belong to algebra center. We find that $\chi_{\mathsf{A}'}(\lambda) = -\lambda^4 - 4\lambda^3 + (2(\varepsilon-4)\varepsilon - 8)\lambda^2 + (4(\varepsilon-6)\varepsilon - 8)\lambda - \varepsilon(\varepsilon((\varepsilon-8)\varepsilon+20)+8) - 4$, the limit of which is $\lim_{\varepsilon\to 0}\chi_{\mathsf{A}'}(\lambda)=\chi_{\mathsf{A}}(\lambda)$. If ε is included it has four (now different) roots $\lambda_1=-(1+i)-\sqrt{\varepsilon^2-(4+2i)\varepsilon}$, $\lambda_2=-(1+i)+\sqrt{\varepsilon^2-(4+2i)\varepsilon}$, $\lambda_3=-(1-i)-\sqrt{\varepsilon^2-(4-2i)\varepsilon}$, $\lambda_4=-(1-i)+\sqrt{\varepsilon^2-(4-2i)\varepsilon}$ which in the limit $\varepsilon\to 0$ return back to multiple roots. Since the roots with ε included are different in calculation of $\beta(\lambda_i)\mathsf{B}(\lambda_i)$ the division by zero disappears,

$$
\begin{aligned}
\beta(\lambda_i)\mathsf{B}(\lambda_i) =\ & \frac{\left(-2\varepsilon^2-8\varepsilon+\lambda_i^2+4\lambda_i+8\right)\langle\mathsf{A}'\rangle_{-0}}{4\left(-\varepsilon^2\lambda_i-\varepsilon^2-4\varepsilon\lambda_i-6\varepsilon+\lambda_i^3+3\lambda_i^2+4\lambda_i+2\right)}+\frac{(\lambda_i+4)\langle\mathsf{A}'^2\rangle_{-0}}{4\left(-\varepsilon^2\lambda_i-\varepsilon^2-4\varepsilon\lambda_i-6\varepsilon+\lambda_i^3+3\lambda_i^2+4\lambda_i+2\right)}\\
&+\frac{\langle\mathsf{A}'^3\rangle_{-0}}{4\left(-\varepsilon^2\lambda_i-\varepsilon^2-4\varepsilon\lambda_i-6\varepsilon+\lambda_i^3+3\lambda_i^2+4\lambda_i+2\right)}\\
=\ & \frac{1}{4}\Bigg(1+\frac{1}{\lambda_i^3+3\lambda_i^2+\left(4-\varepsilon(\varepsilon+4)\right)\lambda_i+2-\varepsilon(\varepsilon+6)}\bigg(\big((\varepsilon+2)\lambda_i^2+2(\varepsilon+3)\lambda_i-\varepsilon(10+\varepsilon(\varepsilon+6))+2\big)\mathbf{e}_1\\
&+\left(\lambda_i^2+6\lambda_i-\varepsilon(\varepsilon+8)+4\right)\mathbf{e}_2+2\left(\lambda_i^2-\varepsilon(\varepsilon+2)-2\right)\mathbf{e}_3+2\left(-\lambda_i^2-4\lambda_i+\varepsilon(\varepsilon+6)-2\right)\mathbf{e}_{12}\\
&+2\left(-\lambda_i^2-\lambda_i+\varepsilon(\varepsilon+3)+1\right)\mathbf{e}_{13}+\left(\lambda_i^2-2(\varepsilon+1)\lambda_i+(\varepsilon-2)\varepsilon-4\right)\mathbf{e}_{23}\\
&-\left(\lambda_i^2-2(\varepsilon-1)\lambda_i+\varepsilon(\varepsilon+2)+2\right)\mathbf{e}_{123}\bigg)\Bigg)\,.
\end{aligned}
$$

After summation over all roots $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ in $\sum_{i=1}^4\exp\left(\lambda_i\right)\left(\frac{1}{4}+\mathsf{B}(\lambda_i)\right)$, we collect terms at basis elements and finally compute the limit $\varepsilon\to 0$ for each of coefficients. Then, after simplification we get the following answer,

$$
\begin{aligned}
\exp(\mathsf{A}) =\ & \frac{1}{\mathrm{e}}\big(\cos(1)+(\sin(1)+2\cos(1))\mathbf{e}_1+(2\sin(1)+\cos(1))\mathbf{e}_2+2(\cos(1)-\sin(1))\mathbf{e}_3-2(\sin(1)+\cos(1))\mathbf{e}_{12}\\
&+(\sin(1)-2\cos(1))\mathbf{e}_{13}+(\cos(1)-2\sin(1))\mathbf{e}_{23}-\sin(1)\mathbf{e}_{123}\big)\,.
\end{aligned}
$$

It should be noted that computation of the limit is highly nontrivial task, especially when dealing with the roots of high degree polynomial equations. The primary purpose of Example 5 is to show that non-diagonalizable matrices/MVs represent some limiting case and the (symbolic) formula is able to take into account this case. To illustrate how complicated computation of exponential of non-diagonalizable matrix for higher dimensional Clifford algebras could be we have tested internal *Mathematica* command **MatrixExp[ ]** using $Cl_{4,2}$ algebra and non-diagonalizable MV $\mathsf{A}''=-1-\mathbf{e}_3+\mathbf{e}_6-\mathbf{e}_{12}-\mathbf{e}_{13}+\mathbf{e}_{15}-\mathbf{e}_{24}-\mathbf{e}_{25}+\mathbf{e}_{26}-\mathbf{e}_{34}-\mathbf{e}_{35}+\mathbf{e}_{36}-\mathbf{e}_{45}+\mathbf{e}_{56}+\mathbf{e}_{123}+\mathbf{e}_{124}+\mathbf{e}_{126}+\mathbf{e}_{134}+\mathbf{e}_{135}+\mathbf{e}_{136}+\mathbf{e}_{146}+\mathbf{e}_{234}-\mathbf{e}_{235}-\mathbf{e}_{236}-\mathbf{e}_{245}-\mathbf{e}_{246}-\mathbf{e}_{256}+\mathbf{e}_{456}-\mathbf{e}_{1236}+\mathbf{e}_{1245}-\mathbf{e}_{1246}+\mathbf{e}_{1256}-\mathbf{e}_{1345}-\mathbf{e}_{1346}-\mathbf{e}_{1356}+\mathbf{e}_{1456}-\mathbf{e}_{2346}-\mathbf{e}_{2356}+\mathbf{e}_{2456}+\mathbf{e}_{3456}+\mathbf{e}_{12345}-\mathbf{e}_{12346}+\mathbf{e}_{12356}$ that was converted to 8×8 real matrix representation. The respective MV has minimal polynomial $(\lambda-1)^2\left(\lambda^6+10\lambda^5+39\lambda^4+124\lambda^3+543\lambda^2-198\lambda-4743\right)$. *Mathematica* (version 13.0) has crashed after almost 48 hours of computation after all 96 GB of RAM was exhausted. This strongly contrasts with the exponentiation of diagonalizable matrix of the same $Cl_{4,2}$ algebra, where it took only a fraction of a second to complete the task.

4.1 | Making the answer real

Formulas (16) and (23) include summation over (in general complex valued) roots of characteristic polynomial, therefore, formally the result is a complex number. Here we are dealing with real Clifford algebras, consequently, a pure imaginary part in the final result must vanish. Below we will describe a procedure which enables to get rid of the imaginary part.

First we remind, that if the characteristic polynomial is made up of real coefficients, then the roots of the polynomial always come in complex conjugate pairs. Thus, for real valued function the summation over each of a complex root pair should produce real answer. Indeed, assuming that symbols a,b,c,d,g,h take values from the real numbers and computing the sum over a single complex conjugate root pair we come to the following relation for exponential function

$$\exp(a+\mathrm{i}b)\frac{c+\mathrm{i}d}{g+\mathrm{i}h}+\exp(a-\mathrm{i}b)\frac{c-\mathrm{i}d}{g-\mathrm{i}h}=\frac{2\mathrm{e}^a\big((cg+dh)\cos b+(ch-dg)\sin b\big)}{g^2+h^2}\,.$$

The right hand side of the identity now formally represents a real number as expected. The left hand side contains the terms which appear in (16) and (23) expressions after a pair of complex conjugate roots has been substituted. However, from symbolic computational point of view the issue is not so simple. In general, the roots of high degree ($d\geq 5$) polynomial equations cannot be solved in radicals and, therefore, in the symbolic packages they are represented as enumerated formal functions/algorithms of some irreducible polynomials. In *Mathematica* the formal solution is represented as **Root[P,k]**, where index $k\leq d$ simply enumerates the roots of polynomial P in a specific way. In order to obtain a real-valued answer, we have to know how to manipulate these formal objects algebraically. To that end there exist the algorithms which allow to rewrite the coefficients of irreducible polynomials P after they have been algebraically manipulated. The operation, however, appears to be nontrivial and time consuming. In *Mathematica* it is implemented by **RootReduce[ ]** command, which produces another **Root[P', k$'$]** object. Such a root reduction typically raises the order of the polynomial. From pure numerical point of view, of course, we may safely remove spurious complex part in the final answer to get real numerical value.

5 | ARBITRARY FUNCTION OF MV

The exponential formulas (16) and (23) appeared to be more universal than we have expected. In fact, they allow to compute any well-behaved function or transformation of MV if one replaces the exponential weight $\exp(\lambda_i)$ by any other arbitrary function f and allows to use complex numbers.

Conjecture 1 (Arbitrary function of MV, basis free form)**.** Arbitrary function $f(\mathsf{A})$ of multivector $\mathsf{A} = a_0 + \sum_J^{2^n-1} a_J \mathbf{e}_J$ in $Cl_{p,q}$ can be explicitly computed by

$$f(\mathsf{A}) = \sum_{i=1}^{d} f(\lambda_i)\, \beta(\lambda_i) \sum_{m=0}^{d-1} \left(\sum_{k=0}^{d-m-1} \lambda_i^k C_{(d-k-m-1)}(\mathsf{A}) \right) \mathsf{A}^m, \quad \text{with} \quad \beta(\lambda_i) = \frac{1}{\sum_{j=0}^{d-1} (j+1)\, C_{(d-j-1)}(\mathsf{A})\, \lambda_i^j}, \tag{25}$$

where λ_i and $C_k(\mathsf{A})$, respectively, denotes roots and coefficients of characteristic polynomial $\chi_{\mathsf{A}}(\lambda)$ already discussed in Theorem 1.

A more simple variant of (25) can be written for a power of MV,

$$\mathsf{A}^s = \sum_{i=1}^{d} \lambda_i^s\, \beta(\lambda_i) \sum_{m=0}^{d-1} \left(\sum_{k=0}^{d-m-1} \lambda_i^k C_{(d-k-m-1)}(\mathsf{A}) \right) \mathsf{A}^m, \tag{26}$$

where A^s denotes the MV raised to the natural power $s \in \mathbb{N}$, which we make an effort to understand deeper.[6] Therefore, below we restrict ourselves by demonstrating how the formula (25) can be used to compute values of various functions of MV argument (A).

Example 6. We shall compute $\log(\mathsf{A})$, $\sinh(\mathsf{A})$, $\operatorname{arcsinh}(\mathsf{A})$, $(\mathsf{A})^{-1}$, $\sqrt{\mathsf{A}}$ and Bessel $J_0(\mathsf{A})$ GA functions for $Cl_{4,0}$ diagonalizable MV $\mathsf{A} = 1 + \mathbf{e}_1 + 3\mathbf{e}_{23} - \mathbf{e}_{24}$. This MV ensures that the computed answers are of manageable size. Its characteristic polynomial is $\chi_{\mathsf{A}}(\lambda) = -(10+\lambda^2)(14 - 4\lambda + \lambda^2)$, which is also minimal polynomial. The roots of characteristic polynomial are $\lambda_1 = -\mathrm{i}\sqrt{10}$, $\lambda_2 = \mathrm{i}\sqrt{10}$, $\lambda_3 = 2 - \mathrm{i}\sqrt{10}$, $\lambda_4 = 2 + \mathrm{i}\sqrt{10}$.

Replacement of f by log in (25) and summation over all roots (after slight rearrangement of *Mathematica* output) gives

$$\begin{aligned}\log \mathsf{A} =& \frac{1}{4}\log 140 + \frac{1}{4}\log(7/5)\,\mathbf{e}_1 + \frac{3}{4\sqrt{10}}\left(\pi + 2\arctan\sqrt{5/2}\right)\mathbf{e}_{23} - \frac{1}{4\sqrt{10}}\left(\pi + 2\arctan\sqrt{5/2}\right)\mathbf{e}_{24} \\ &- \frac{3}{4\sqrt{10}}\left(\pi - 2\arctan\sqrt{5/2}\right)\mathbf{e}_{123} + \frac{1}{4\sqrt{10}}\left(\pi - 2\arctan\sqrt{5/2}\right)\mathbf{e}_{124}\end{aligned} \tag{27}$$

It is easy to check that the exponentiation of $\log \mathsf{A}$ yields $\exp(\log(\mathsf{A})) = \mathsf{A}$, i.e., the natural logarithm of log function in Eq. (27) is formal inverse of exp.

Computation of hyperbolic and trigonometric functions and their inverses[7] is also straightforward. Since characteristic coefficients have been already found, the computation of remaining functions requires very little effort.

[6]Of course, since the square root and logarithm functions can be computed by (25), it is obvious that s is not restricted to $\mathbb{N}$ only. We carefully verified the identity by testing $s \in \mathbb{Z}$ and $s \in \mathbb{Q}$ cases. For example, after insertion of $s = -1$ we checked the formula for inverse A^{-1}, and after insertion of $s = 1/2$ we compared the result for a square root of MV (when it exist in real GA) with the known one. In particular, computation of the square root[21] with this formula produces a single root of MV, i.e. the root, which is connected to unity. When $s \in \mathbb{R}$ or $s \in \mathbb{C}$ the verification itself becomes a problem.

[7]It looks as if the complex numbers are inevitable in computing trigonometric functions in most of real Clifford algebras, except $Cl_{3,0}$ as well as few others[32].

Indeed, the computation of $\sinh(\mathsf{A})$, $\operatorname{arcsinh}(\mathsf{A})$, $(\mathsf{A})^{-1}$, $\sqrt{\mathsf{A}}$ and Bessel function $J_0(\mathsf{A})$ using (25) and subsequent simplification yields, respectively,

$$\begin{aligned}\operatorname{arcsinh}\mathsf{A} =& \frac{1}{4}\left(\operatorname{arcsinh}(2-\mathrm{i}\sqrt{10})+\operatorname{arcsinh}(2+\mathrm{i}\sqrt{10})\right)+\frac{1}{4}\left(\operatorname{arcsinh}(2-\mathrm{i}\sqrt{10})+\operatorname{arcsinh}(2+\mathrm{i}\sqrt{10})\right)\mathbf{e}_1\\ &+\frac{3}{4\sqrt{10}}\left(\pi+2\arcsin\sqrt{10}-\arcsin\left(2\sqrt{7\sqrt{185}-95}\right)\right)\mathbf{e}_{23}\\ &-\frac{1}{4\sqrt{10}}\left(2\arcsin\sqrt{10}+\mathrm{i}\left(\operatorname{arcsinh}(2-\mathrm{i}\sqrt{10})-\operatorname{arcsinh}(2+\mathrm{i}\sqrt{10})\right)\right)\mathbf{e}_{24}\\ &+\frac{3}{4\sqrt{10}}\left(2\arccos\sqrt{10}-\arcsin\left(2\sqrt{7\sqrt{185}-95}\right)\right)\mathbf{e}_{123}\\ &+\frac{1}{4\sqrt{10}}\left(\arcsin\left(2\sqrt{7\sqrt{185}-95}\right)-2\arccos\sqrt{10}\right)\mathbf{e}_{124}\,.\end{aligned}\tag{28}$$

$$\begin{aligned}\sinh\mathsf{A} =& \frac{1}{2}\sinh 2\,\cos\sqrt{10}+\frac{1}{2}\sinh 2\,\cos\sqrt{10}\mathbf{e}_1+\frac{3}{4\sqrt{10}\mathrm{e}^2}\sin\sqrt{10}\,(1+\mathrm{e}^2)^2\mathbf{e}_{23}-\frac{1}{4\sqrt{10}\mathrm{e}^2}\sin\sqrt{10}\,(1+\mathrm{e}^2)^2\mathbf{e}_{24}\\ &+\frac{3}{4\sqrt{10}\mathrm{e}^2}\sin\sqrt{10}\,(\mathrm{e}^2-1)^2\mathbf{e}_{123}-\frac{1}{4\sqrt{10}\mathrm{e}^2}\sin\sqrt{10}\,(\mathrm{e}^2-1)^2\mathbf{e}_{124}\,.\end{aligned}\tag{29}$$

It is easy to check that the identity $\sinh\big(\operatorname{arcsinh}(\mathsf{A})\big)=\mathsf{A}$ is satisfied. Also, series expansion of the GA functions with respect to MV may be used. The computation of inverse yields a simple output, and it is easy to check that $\mathsf{A}^{-1}\mathsf{A}=\mathsf{A}\mathsf{A}^{-1}=1$,

$$\mathsf{A}^{-1}=\frac{1}{14}+\frac{1}{14}\mathbf{e}_1-\frac{9}{35}\mathbf{e}_{23}+\frac{3}{35}\mathbf{e}_{24}+\frac{3}{70}\mathbf{e}_{123}-\frac{1}{70}\mathbf{e}_{124}\,.\tag{30}$$

The computation of square root gives more complex answer

$$\begin{aligned}\sqrt{\mathsf{A}} =& \frac{1}{2\sqrt{2}}\left(\sqrt[4]{10}+\sqrt{2+\sqrt{14}}\right)-\frac{1}{2\sqrt{2}}\left(\sqrt[4]{10}-\sqrt{2+\sqrt{14}}\right)\mathbf{e}_1+\frac{3}{2\,2^{3/4}\sqrt{5}}\left(\sqrt[4]{5}+\sqrt[4]{9-2\sqrt{14}}\right)\mathbf{e}_{23}\\ &-\frac{1}{2\,2^{3/4}\sqrt{5}}\left(\sqrt[4]{5}+\sqrt[4]{9-2\sqrt{14}}\right)\mathbf{e}_{24}-\frac{3}{4\sqrt{5}}\left(\sqrt[4]{10}-\sqrt{\sqrt{14}-2}\right)\mathbf{e}_{123}+\frac{\left(5^{3/4}\sqrt{7+\sqrt{14}}-5\sqrt[4]{7}\right)}{10\,2^{3/4}\sqrt{7+\sqrt{14}}}\mathbf{e}_{124}\,.\end{aligned}\tag{31}$$

Again, one readily tests that $\left(\sqrt{\mathsf{A}}\right)^2=\mathsf{A}$. One can also check the following property. If the initial MV is replaced by $\mathsf{A}'=1+\varepsilon\mathbf{e}_1+3\varepsilon\mathbf{e}_{23}-\varepsilon\mathbf{e}_{24}$ then the limit $\lim_{\varepsilon\to 0}\sqrt{\mathsf{A}'}$ is 1, i.e., the formula yields the root which is connected to unity. As a last example we compute Bessel $J_0(\mathsf{A})$ function,

$$\begin{aligned}J_0(\mathsf{A}) =& \frac{1}{4}J_0(2+\mathrm{i}\sqrt{10})+\frac{1}{4}J_0(2-\mathrm{i}\sqrt{10})+\frac{1}{2}I_0(\sqrt{10})+\left(\frac{1}{4}J_0(2-\mathrm{i}\sqrt{10})+\frac{1}{4}J_0(2+\mathrm{i}\sqrt{10})-\frac{1}{2}I_0(\sqrt{10})\right)\mathbf{e}_1\\ &+\frac{3\mathrm{i}}{4\sqrt{10}}\left(J_0(2-\mathrm{i}\sqrt{10})-J_0(2+\mathrm{i}\sqrt{10})\right)\mathbf{e}_{23}+\frac{\mathrm{i}}{4\sqrt{10}}\left(J_0(2+\mathrm{i}\sqrt{10})-J_0(2-\mathrm{i}\sqrt{10})\right)\mathbf{e}_{24}\\ &+\frac{3\mathrm{i}}{4\sqrt{10}}\left(J_0(2-\mathrm{i}\sqrt{10})-J_0(2+\mathrm{i}\sqrt{10})\right)\mathbf{e}_{123}+\frac{\mathrm{i}}{4\sqrt{10}}\left(J_0(2+\mathrm{i}\sqrt{10})-J_0(2-\mathrm{i}\sqrt{10})\right)\mathbf{e}_{124}\,,\end{aligned}\tag{32}$$

where $I_0(x)$ denotes modified Bessel function. Here we do not question where special functions of MV argument may find application in a practice.

6 | CONCLUSIONS AND PERSPECTIVE

The paper shows that in Clifford geometric algebras (GA) the exponential (or any other function) of a general multivector is associated with the characteristic polynomial of the multivector and may be expressed in terms of roots of respective characteristic equation. The main results of this paper are the formulas (16), (23) and (25), where real GA function or exponential are presented in an expanded basis-free form or in orthonormal coordinates. In higher dimensional algebras the coefficients at basis elements of the exponential function, in agreement with [18,19], include a mixture of trigonometric and hyperbolic functions. We implemented the general formula (25) in the *Mathematica* package[33]. Apart from explicit examples of GA functions presented in the article, we also were able to compute fractional powers of MV, many special functions available in *Mathematica*[34], in particular, **HermiteH[ ]**, **LaguerreL[ ]** (also with rational parameters), hypergeometric and many others, which are implemented in *Mathematica* for scalar argument.

Recently we made a some progress on how to compute the functions of non-diagonalizable MVs, which do not require perturbation and evaluation of a limit. Implementation of this program, however, is rather involved and at the moment of writing is not yet complete. Moreover, to test properly how it works we had to learn how to generate non-diagonalizable MVs, which are rather rare (see Sec. 3), especially in GAs of higher dimension. Unfortunately, at the moment, we don't know how to perform this task completely in GA and we have to refer to matrix representations. In particular, we can generate a nontrivial Jordan form, which then is transformed by random non-singular matrix of 8-periodicity table. It should be noticed that for non-diagonalizable MV we cannot use the mentioned isomorphism $\mathbb{H} \cong \mathbb{C}(2)$ to double the dimension of the matrix, since in doing so the minimal polynomial gets spoiled. Also conversion to and from matrix representations for real GAs have some subtle points that have to be taken into account. In the end we would like to find a simpler and more straightforward criterion that would allow to determine whether the provided MV is nondefective, without involving costly and purely algorithmic computation of minimal polynomial. And, of course, our **Conjecture** 1 needs a full proof which, due to its general shape, looks (at least to us) like a highly nontrivial task.

7 | APPENDIX: MINIMAL POLYNOMIAL OF MV

A simple algorithm for computation of matrix minimal polynomial is given in[35]. It starts by constructing $d \times d$ matrix M and its powers $\{1, M, M^2, \dots\}$ and then converting each of matrix into vector of length $d \times d$. The algorithm then checks consequently the sublists $\{1\}$, $\{1, M\}$, $\{1, M, M^2\}$ etc until the vectors in a particular sublist are detected to be linearly dependent. Once linear dependence is established the algorithm returns the polynomial, where coefficients of linear combination are multiplied by corresponding variable λ^i.

In GA, the orthonormal basis elements $\mathbf{e}_J$ are already linearly independent, therefore it is enough to construct vectors made from coefficients of MV. Then, the algorithm starts searching when these coefficient vectors become linearly dependent.

A vector constructed of matrix representation of MV has $d^2 = \left(2^{\lceil \frac{n}{2} \rceil}\right)^2$ components. This is exactly the number of coefficients (2^n) in MV for Clifford algebras of even n and twice less than number of matrix elements $d \times d$ for odd n. The latter can be easily understood if one remembers that for odd n the matrix representation of Clifford algebra has block diagonal form. Therefore only a single block should suffice for the matrix algorithm. Below the Algorithm 1 describes how to compute minimal polynomial of MV without employing a matrix representation.

Algorithm 1 Minimal polynomial of MV

procedure MINIMALPOLY(A, x) ▷ Input is multivector $\mathsf{A} = a_0 + \sum_J^{2^n-1} a_J \mathbf{e}_J$ and polynomial variable x
  nullSpace={}; lastProduct=1; vectorList={} ▷ Initialization
  while nullSpace==={} **do** ▷ keep adding MV coefficient vectors to vectorList until null space becomes nonempty
    lastProduct← A∘lastProduct
    vectorList←AppendTo[vectorList, ToCoefficientList[lastProduct]]
    nullSpace←NullSpace[Transpose[vectorList]];
  end while
  return First[nullSpace] $\cdot \{x^0, x^1, x^2, \dots, x^{\text{Length[nullSpace]}-1}\}$ ▷ Construct minimal polynomial from nullspace coefficients and powers of input variable
end procedure

All functions in the Algorithm 1 code are internal *Mathematica* functions, except the symbol ∘ (geometric product) and ToCoefficientList[ ] function which is very simple. It takes a multivector A and construct a vector (list) from its coefficients, i.e. ToCoefficientList[$a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_I I$]→ $\{a_0, a_1, a_2, \dots, a_I\}$. A real job is done by *Mathematica* function **NullSpace[ ]**, which searches for linear dependency in the inserted list of vectors. The **NullSpace[ ]** function is a standard function of linear algebra library. If the list of vectors is found to be linearly dependent it outputs a set of weight factors of the linear combination for which the sum of vectors turns to zero, or an empty list otherwise. The **AppendTo[vectorList, newVector]** appends the **newVector** to the list of already checked vectors in **vectorList**.

Conflict of interest

The authors declare no potential conflict of interests.

SUPPORTING INFORMATION

There are no funders to report for this submission.

References

1. Gürlebeck K., Sprössig W.. *Quaternionic and Clifford Calculus for Physicists and Engineers*. Chichester, England: John Wiley and Sons; 1998. ISBN: 978-0-471-96200-7.

2. Lounesto P.. *Clifford Algebra and Spinors*. Cambridge: Cambridge University Press; 1997. ISBN-13: 978-0521599160.

3. Marchuk N. G., Shirokov D.S.. *Theory of Clifford Algebras and Spinors*. Moscow: Krasand; 2020. ISBN: 978-5-396-01014-7, in Russian.

4. Hanson A. J.. *Visualizing Quaternions*. Amsterdam: Elsevier; 2006. ISBN 13:978-0-12-088400-1.

5. Ramakrishna Viswanath. On the exponentials of some structured matrices. *Journal of Physics A*. 2004;37:11613-11627.

6. Fujii Kazuyuki. Exponentiation of certain Matrices related to the Four Level System by use of the Magic Matrix. *arXiv:math-ph/0508018v1*. 2007;.

7. Fujii Kazuyuki, Oike Hiroshi. How to Calculate the Exponential of Matrices. *Far East Journal of Mathematical Education*. 2012;9(1):39-55. P-ISSN: 0973-5631; arXiv:quant-ph/0604115v1.

8. Herzig Emily, Ramakrishna Viswanath, Dabkowski Mieczyslaw K.. Note on Reversion, Rotation and Exponentiation in Dimensions Five and Six. *Journal of Geometry and Symmetry in Physics*. 2014;35(none):61 – 101.

9. Zela F. De. Closed-Form Expressions for the Matrix Exponential. *Symmetry*. 2014;6:329-344. ISSN 2073-8994. doi:10.3390/sym6020329, www.mdpi.com/journal/symmetry.

10. Moler Cleve, Loan Charles Van. Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later. *SIAM Review*. 2003;45(1):3-49.

11. Ramakrishna Viswanath, Zhou Hong. On the Exponential of Matrices in su(4). *Journal of Physics A*. 2005;39:3021-3034. doi:10.1088/0305-4470/39/12/011,https://arxiv.org/abs/math-ph/0508018v1.

12. *Wolfram Research (2012), MatrixFunction, Wolfram Language function,*. https://reference.wolfram.com/language/ref/MatrixFunction.html; 2014.

13. H.W. Turnbull A.C. Aitken. *An Introduction to the Theory of Canonical Matrices*. London: Blackie and Sons; 1932.

14. Faddeeva V.N.. *Computational Methods of Linear Algebra*. New York: Dover; 1958.

15. Roelfs Martin, Keninck Steven De. Graded Symmetry Groups: Plane and Simple. *arXiv:math-phi/2107.03771v1*. 2021;:1-17.

16. Dorst L., Valkenburg R.. Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition. In: Dorst L., Lasenby L., eds. *Guide to Geometric Algebra in Practice*, London: Springer-Verlag 2011 (pp. 81-104). ISBN 978-0-85729-810-2. doi : 10.1007/978-0-85729-811-9_5.

17. Hitzer E.. On Factorization of Multivectors in Cl(3,0), Cl(1,2) and Cl(0,3), by Exponentials and Idempotents. *Complex Variables and Elliptic Equations.* 2021;:1-23.

18. Dargys A., Acus A.. Exponential of General Multivector (MV) in 3D Clifford Algebras. *Nonlinear Analysis: Modelling and Control.* 2022;27(1):179-197. https://doi.org/10.15388/namc.2022.27.24476.

19. Acus A., Dargys A.. Coordinate-free Exponentials of General Multivector in Cl(p,q) Algebras for p+q=3. *Math Meth Appl Sci.*. 2022;:1-13. doi:10.1002/mma.8529,.

20. Dargys A., Acus A.. Exponentials and Logarithms of Multivector in Low Dimensional n=p+q<3 Clifford Algebras. *Nonlinear Analysis: Modelling and Control.* 2022;27(6):1129-1149. https://doi.org/10.15388/namc.2022.27.29528.

21. Acus A., Dargys A.. Square Root of a Multivector of Clifford Algebras in 3D: A Game with Signs. *arXiv:math-phi/2003.06873.* 2020;:1-29.

22. Dargys A., Acus A.. Square Root of a Multivector in 3D Clifford Algebras. *Nonlinear Analysis: Modelling and Control.* 2020;25(3):301-320. https://doi.org/10.15388/namc.2020.25.16519.

23. Shirokov D. S.. Classification of Lie Algebras of Specific Type in Complexified Clifford Algebras. *Linear and Multilinear Algebra.* 2018;66(9):1870-1887. arXiv:math-ph/1704.03713v2, 2017; 1-18.

24. Helmstetter J.. Characteristic Polynomials in Clifford Algebras and in More General Algebras. *Adv. Appl. Clifford Algebras.* 2019;29(30). https://doi.org/10.1007/s00006-019-0944-5.

25. Shirokov D. S.. On Computing the Determinant, other Characteristic Polynomial Coefficients, and Inverses in Clifford Algebras of Arbitrary Dimension. *Comput. Appl. Math.*. 2021;40(173):1-29. doi.org/10.1007/s40314-021-01536-0.

26. Abdulkhaev K., Shirokov D.. On Explicit Formulas for Characteristic Polynomial Coefficients in Geometric Algebras. *Adv. Comp. Graphics. Lecture Notes in Computer Science.* 2021;13002:679-680. doi.org/10.1007/978-3-030-89029-2_50.

27. Shui-Hung Hou . Classroom note: A Simple Proof of the LeVerrier-Faddeev Characteristic Polynomial Algorithm. *SIAM Review.*. 1998;40(3):706-709.

28. Householder Alston S.. *The Theory of Matrices in Numerical Analysis*. New York, N. Y. 10014: Dover Publications, Inc; 1975.

29. Shirokov D. S.. On Computing the Determinant, other Characteristic Polynomial Coefficients, and Inverse in Clifford Algebras of Arbitrary Dimension. *Computational and Applied Mathematics.* 2021;40(5):29pp. doi: 10.1007/s40314-021-01536-0.

30. *Bell polynomials.* Wikipedia, https://en.wikipedia.org/wiki/Bell_polynomials; 2022.

31. *Diagonalizable matrix.* Wikipedia, https://en.wikipedia.org/wiki/Diagonalizable_matrix; 2022.

32. Chappell J. M., Iqbal A., Gunn L. J., Derek D.. Functions of Multivector Variables. *arXiv.* 2014;(1409.6252v.1):1-19.

33. Acus A., Dargys A.. *Geometric Algebra Mathematica package.* Download from https://github.com/ArturasAcus/GeometricAlgebra; 2022.

34. Inc. Wolfram Research,. Mathematica, Version 13.1 Champaign, IL, 2022; .

35. Rowland Todd, Weisstein Eric W.. *Matrix Minimal Polynomial.* From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/MatrixMinimalPolynomial.html; 2022.

How to cite this article: A. Acus, and A. Dargys (2022), The characteristic polynomial in calculation of exponential and elementary functions in Clifford algebras, *Math Meth Appl Sci.*, *2022;XX:x–xx.*.