On Hilfer-Prabhakar derivatives Formable integral transform and its applications to fractional differential equations

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Abstract

In this paper, we will derive the Formable integral transform of the Hilfer-Prabhakar and its regularized version of the Hilfer-Prabhakar fractional derivative. Then, we will use the Formable and Fourier transforms, which involve the three-parameter Mittag-Leffler function, to find the solution of some Cauchy type fractional differential equations with Hilfer-Prabhakar fractional derivatives.

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Abstract: In this paper, we will derive the Formable integral transform of the Hilfer-Prabhakar and its regularized version of the Hilfer-Prabhakar fractional derivative. Then, we will use the Formable and Fourier transforms, which involve the three-parameter Mittag-Leffler function, to find the solution of some Cauchy type fractional differential equations with Hilfer-Prabhakar fractional derivatives.

Keywords and phrases: Prabhakar integral, Hilfer-Prabhakar derivatives, Formable integral transform, Fourier transform, Mittage-Leffler functions.

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1 Introduction

Fractional Calculus is a rapidly growing field of research in recent years due to its wide range of applications and interdisciplinary approach. Applications of fractional calculus can be found in almost all sciences, and it is applied to real-world problems. Fractional differential equations are a field of mathematics that have a lot of applications in science and technology, as they are used in mathematical modeling (Mainardi 2022, Magin 2010, Ionescu 2017, Bagley 1986, Carpinteri 2014) and can help find solutions to physical and engineering problems, such as heat or sound propagation, fluid flow, elasticity, electronics, and other areas of science and technology (Mainardi 2010, Miller 1993, Oldham 1974, Prabhakar 1971, Samkokilbas 1993, Agrawal 2008).

An integral transform is a specific type of mathematical operator. Each integral transform has an associated inverse integral transform, which maps the original domain back to itself by passing through another domain. Integral transforms are very useful in solving differential equations. Solving fractional differential equations can be a challenging task. The goal is to extend integral transforms to solve fractional differential equations.

Many researchers have studied Hilfer-Prabhakar fractional derivatives, which have gained popularity in modeling and other fields due to their special properties, by using a combination of various integral transforms such as Laplace, Fourier, Sumudu, Shehu, Elzaki and others. The Laplace transform of the Hilfer-Prabhakar and its regularized version is studied in **[6]**, where the authors applied these results to classical equations of mathematical physics such as heat and free electron laser equations. **Panchal et al. [8]** applied the Sumudu transform to some non-homogeneous Cauchy type problems, **Yudhveer et al. [9]** applied the Elzaki transform to Hilfer-Prabhakar fractional derivatives and used these results to solve free electron laser type integro-differential equations, **Belgacem et al. [10]** applied the Shehu transform to Prabhakar and Hilfer-Prabhakar derivatives and used it to find solutions of some fractional differential equations. Similarly, the Formable transform has deeper connections with Laplace, Elzaki, Sumudu and Shehu transforms. In 2021, **Saadeh et al. [17]** discovered a new integral transform called the Formable transform, with the main purpose of solving ordinary and partial differential equations using this transform.

The main goal of this study is to use the Formable transform to solve fractional differential equations. In this paper, we apply the Formable transform to Prabhakar integral, Prabhakar derivatives, Hilfer-Prabhakar derivative and their regularized versions. We then use these results to solve some Cauchy type fractional differential equations involving Hilfer-Prabhakar fractional derivative presented in terms of Mittag-Leffler function.

2 Definitions and preliminaries

Definition 2.1. *([17]) The Formable integral transform denoted by $\mathcal{B}(r,v)$ for the function $\xi(t)$ which is given as:*

$$\begin{aligned}\mathcal{R}[\xi(t)] = \mathcal{B}(r,v) &= r\int_0^{\infty}\xi(vt)exp(-rt)dt \\ &= \frac{r}{v}\int_0^{\infty}exp\left(\frac{-rt}{v}\right)\xi(t)dt, r\in(\lambda_1,\lambda_2),\end{aligned} \tag{2.1}$$

this is a statement about the set of functions on which the problem or equation is defined,

$$\mathcal{W} = \left\{ \xi(t)\ s.t\ \exists\ N, 0 < \lambda_1, \lambda_2, 0 < k, |\xi(t)| \leq N\ e^{\left(\frac{t}{\lambda_j}\right)}, if\ t \in (-1)^j \times [0, \infty) \right\},$$

The integral transform (2.1) is defined for all values of $\xi(t)$ *that are greater than k.*

The name Formable indicates the flexibility to solve ordinary as well as partial differential equations.

The inverse Formable transform of the function $\xi(t)$ is represented as follows;

$$\mathcal{R}^{-1}[\mathcal{B}(r, v)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{r} exp\left(\frac{rt}{v}\right) \mathcal{B}(r, v) dr \tag{2.2}$$

Proposition 2.1. *([17]) If* $\mathbb{F}(r, v)$ *and* $\mathbb{G}(r, v)$ *are the Formable transforms of the functions* $\xi(t)$ *and* $\chi(t)$ *respectively, then the Formable transform of their convolution is given as.*

$$\mathcal{R}[\xi(t) * \chi(t)), r] = \frac{v}{r}\mathbb{F}(r, v)\mathbb{G}(r, v), \tag{2.3}$$

or equivalently,

$$\mathcal{R}^{-1}\left[\frac{v}{r}\mathbb{F}(r, v)\mathbb{G}(r, v), t\right] = (\xi(t) * \chi(t)), \tag{2.4}$$

where

$$\xi(t)*\chi(t) = \int_0^\infty \xi(v)\chi(t - v)dv, \tag{2.5}$$

• **Formable-Sumudu duality:** Let $G(v)$ be the Sumudu transform of g(t) then

$$\mathcal{B}(1, v) = G(v) \tag{2.6}$$

• **Formable-Shehu duality:** Let $V(r, v)$ be the Shehu transform of g(t) then

$$\mathcal{B}(r, v) = \frac{r}{v}V(r, v) \tag{2.7}$$

Definition 2.2. *([2]) The Reimann Liouville integral operator of order* $\varrho > 0$ *of a function* $\xi(t)$ *is*

$${}_0\Im_t^\varrho \xi(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t - v)^{\varrho-1}\xi(v)dv, \varrho \in \mathbb{C}\ and\ t > 0. \tag{2.8}$$

Definition 2.3. *([2]) The Reimann Liouville Fraction derivative of order* $\varrho > 0$ *of a* *function $\xi(t)$ is*

$$ {}_0\mathbb{D}_t^{\varrho}\xi(t) = \frac{1}{\Gamma(n-\varrho)}\frac{d^n}{dt^n}\int_0^t (t-\upsilon)^{n-\varrho-1}\xi(\upsilon)d\upsilon, \;\; n-1<\varrho<n, \; n\in\mathbb{N}. \tag{2.9} $$

Definition 2.4. *([2]) Caputo frectional derivative of order $\varrho > 0$ of a function $\xi(t)$ is*

$$ ()_0^C\mathbb{D}_t^{\varrho}\xi(t) = \frac{1}{\Gamma(n-\varrho)}\int_0^t (t-\upsilon)^{n-\varrho-1}\xi^{(n)}(\upsilon)d\upsilon, \;\; n-1<\varrho<n, \;\; n\in\mathbb{N} \tag{2.10} $$

Definition 2.5. *([1]) For $0 < \varrho \leq 1$, and $0 \leq \rho \leq 1$, the Hilfer fractional derivative of order ϱ and ρ of a function $\xi(t)$ is*

$$ {}_0\mathbb{D}_t^{\varrho,\rho}\xi(t) = \left({}_0\Im_t^{\rho(1-\varrho)}\frac{d}{dt}({}_0\Im_t^{(1-\varrho)(1-\rho)}\xi(t))\right) \tag{2.11} $$

Definition 2.6. *([19, 20]) Weyl fractional differential operator of order $\varrho > 0$ of a function*
$xi(t)$ is defined as

$$ {}_{-\infty}\mathbb{D}_t^{\varrho}\xi(t) = \frac{1}{\Gamma(n-\varrho)}\frac{d^n}{dt^n}\int_{-\infty}^t (t-\upsilon)^{n-\varrho-1}\xi(\upsilon)d\upsilon, \;\; n-1<\varrho<n, \;\; n\in\mathbb{N} \tag{2.12} $$

The modified Fourier transform of the operator (2.12) given by Metzler and Klafter in [21]

$$ F\left\{{}_{-\infty}\mathbb{D}_t^{\varrho}\xi(x)\right\} = -k^{\varrho}\xi^*(k), \tag{2.13} $$

where $\xi^*(k)$ is the Fourier transform of $\xi(x)$

Definition 2.7. *([22]) Let $\xi(x)$ be a piecewise continuous function defined on $(-\infty,\infty)$ in each partial interval and absolutely integrable in $(-\infty,\infty)$ then Fourier transform is defined by the integral equation is*

$$ F[\xi(x),k] = \xi^*(k) = \int_{-\infty}^{\infty}\xi(x)exp(ikx)dx, \tag{2.14} $$

and inverse of Fourier is

$$ \xi^{-1}[\xi^*(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\xi(k)exp(-ikx)dk, \tag{2.15} $$

Definition 2.8. *([3]) For $\upsilon,\varrho\in\mathbb{C}$, $\;Re(\varrho)>0$, one parameter Mittag-Leffler function is* *given by*

$$E_{\varrho}(\upsilon)=\sum_{k=0}^{\infty}\frac{(\upsilon)^{k}}{\Gamma(\varrho k+1)}. \tag{2.16}$$

For $\upsilon,\varrho,\rho\in\mathbb{C},\ \ Re(\varrho)>0$, two parameter Mittag-Leffler function is given by

$$E_{\varrho,\rho}(\upsilon)=\sum_{k=0}^{\infty}\frac{(\upsilon)^{k}}{\Gamma(\varrho k+\rho)}, \tag{2.17}$$

Definition 2.9. *([3]) For $\upsilon,\varrho,\rho,\gamma\in\mathbb{C}$, $\varrho>0$, three parameter **Mittag-Leffler function**, also called Prabhakar function is given by*

$$E_{\varrho,\rho}^{\gamma}(\upsilon)=\sum_{k=0}^{\infty}\frac{(\gamma)_{k}}{\Gamma(\varrho k+\rho)}\frac{(\upsilon)^{k}}{k!}, \tag{2.18}$$

for applications purpose we will use further generalization of (2.18), which is given by

$$e_{\varrho,\rho,\omega}^{\gamma}=t^{\rho-1}E_{\varrho,\rho}^{\gamma}(\omega t^{\varrho}) \tag{2.19}$$

where $\omega\in\mathbb{C}$ is a parameter and $t>0$ the independent real variable.

Definition 2.10. *([3]) Let $\xi\in L^{1}[0,1]; 0<t<b<\infty$ and the **Prabhakar fractional integral** can be written in the form*

$$\begin{aligned}\Im_{\varrho,\rho,\omega,0^{+}}^{\gamma}\xi(t)&=\int_{0}^{t}(t-\upsilon)^{\rho-1}E_{\varrho,\rho}^{\gamma}(\omega(t-\upsilon)^{\varrho})\xi(\upsilon)d\upsilon\\&=(\xi*e_{\varrho,\rho,\omega}^{\gamma})(t)\end{aligned} \tag{2.20}$$

()

$where\ \ \varrho,\rho,\gamma,\omega\in\mathbb{C}\ \ and\ \ \varrho,\rho>0,$

Definition 2.11. *([3]) Let $\xi\in L^{1}[0,1]; 0<t<b<\infty$ and $\xi*e_{\varrho,\rho,\omega}^{\gamma}\in Sv^{n,1}[0,b], n=\lceil\rho\rceil$. The **Prabhakar fractional derivative** can be written in the form*

$$\mathbb{D}_{\varrho,\rho,\omega,0^{+}}^{\gamma}\xi(t)=\frac{d^{n}}{dt^{n}}\Im_{\varrho,n-\rho,\omega,0^{+}}^{-\gamma}\xi(t), \tag{2.21}$$

where $Sv^{n,1}[a,b]$ is the Sobolev Space, $\varrho,\rho,\gamma,\omega\in\mathbb{C}$ with $Re(\varrho),Re(\rho)>0$.

The Reimann Liouville Fractional Derivative in (2.9) can be written in the form

$$\mathbb{D}_{\varrho,\rho,\omega,0^{+}}^{\gamma}\xi(t)=\mathbb{D}_{0^{+}}^{\rho+\epsilon}\Im_{\varrho,\epsilon,\omega,0^{+}}^{-\gamma}\xi(t), \tag{2.22}$$

Definition 2.12. *([6]) Let $\xi\in AC[0,b], 0<t<b<\infty$, and $n=\lceil\rho\rceil$. The **regularized***

Prabhakar fractional derivative *is given by*

$$ {}^{C}\mathbb{D}^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t) = \Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\frac{d^n}{dt^n}\xi(t) \tag{2.23} $$

where $\varrho,\rho,\gamma,\omega \in \mathbb{C}$ *with* $Re(\varrho), Re(\rho) > 0$.

Definition 2.13. *([6, 7]) Let* $\xi \in L^1[a,b], \rho \in (0,1), \nu \in [0,1], 0 < b < t \leq \infty,\ \xi * e^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega}(.) \in AC^1[a,b]$. *The* ***Hilfer-Prabhakar fractional derivative*** *is given by*

$$ \mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t) = \left(\Im^{-\gamma\nu}_{\varrho,\nu(1-\rho),\omega,0^+}\frac{d}{dt}(\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi)\right)(t) \tag{2.24} $$

Definition 2.14. *([7]) Let* $\xi \in L^1[a,b], \rho \in (0,1), \nu \in [0,1], 0 < b < t \leq \infty$. *The* ***regularized Hilfer-Prabhakar fractional derivative*** *of* $\xi(t)$ *is given by*

$$ \begin{aligned} {}^{C}\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t) &= \left(\Im^{-\gamma\nu}_{\varrho,\nu(1-\rho),\omega,0^+}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\frac{d}{dt}\xi\right)(t) \\ &= \Im^{-\gamma}_{\varrho,1-\rho,\omega,0^+}\frac{d}{dt}\xi(t) \end{aligned} \tag{2.25} $$

Theorem 2.1. *([17]) Suppose* $\mathcal{B}(r,v)$ *is the Formable transform of* $\xi(t)$*, then the Formable transform of* n^{th} *derivative* $\xi^{(n)}(t)$ *is denoted by* $\mathcal{B}_n(r,v)$ *and*

$$ \mathcal{R}_n(r,v) = \mathcal{R}[\xi^{(n)}(t)] = \left(\frac{r}{v}\right)^n \mathcal{B}(r,v) - \sum_{k=0}^{n-1}\left(\frac{r}{v}\right)^{n-k}\xi^{(k)}(0), n \geq 0 \tag{2.26} $$

or equivalently,

$$ \mathcal{R}_n(r,v) = \mathcal{R}[\xi^{(n)}(t)] = \left(\frac{v}{r}\right)^{-n} \mathcal{B}(r,v) - \sum_{k=0}^{n-1}\left(\frac{v}{r}\right)^{k-n}\xi^{(k)}(0) \tag{2.27} $$

Definition 2.15. *([18]) For* $0 < \varrho < 1$ *and* $\omega \in \mathbb{C}$ *such that* $Re(\varrho) > 0,\ Re(\rho) > 0),\ Re(\gamma) > 0$. *The Shehu transform of Mittage-Leffler function* $t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})$ *is given by*

$$ SH\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})\right](r,v) = \left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma} \tag{2.28} $$

Lemma 2.2. *Let* $0 < \varrho < 1$ *and* $\omega \in \mathbb{C}$ *such that* $Re(\varrho) > 0,\ Re(\rho) > 0),\ Re(\gamma) > 0$. *The Formable transform of Mittage-Leffler type function* $t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})$*, is given by*

$$ \mathcal{R}\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})\right](r,v) = \left(\frac{r}{v}\right)^{1-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma}, \tag{2.29} $$

Proof. Using equation (2.28) and the duality of Formable-Shahu transform (2.7) we got the desired result

$$\mathcal{R}\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})\right](r,v) = \left(\frac{v}{r}\right)^{\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma}$$

□

Lemma 2.3. *Let the Formable transform of the function $\xi(t)$ be $\mathcal{B}(r,v)$ then the Formable transform of* ***Prabhakar fractional integral*** *of $\xi(t)$ by using (2.3), (2.29) is given as:*

$$\mathcal{R}[\Im^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)] = \left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma}, \tag{2.30}$$

Proof.

$$\begin{aligned}\mathcal{R}[\Im^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)](r,v) &= \mathcal{R}\left[\int_0^t (t-v)^{\rho-1}E^{\gamma}_{\varrho,\rho}[\omega(t-v)^{\varrho}]\xi(v)dv, s\right](r,v)\\ &= \mathcal{R}\left[(\xi * e^{\gamma}_{\varrho,\rho,\omega})(t)\right](r,v)\\ &= \frac{v}{r}.\mathcal{R}\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})\right]\times\mathcal{R}[\xi(t)]\\ &= \left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma}\mathcal{B}(r,v),\end{aligned}$$

□

3 Main results

Theorem 3.1. ***[Formable transform of Prabhakar derivative]:****The Formable transform of the Prabhakar fractional derivative is represented as follows*

$$\mathcal{R}[\mathbb{D}^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)] = \left(\frac{r}{v}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v) - \sum_{k=0}^{m-1}\left(\frac{r}{v}\right)^{n-k}\mathbb{D}^{\gamma}_{\varrho,k-n+\rho,\omega,0^+}\xi(t)|_{t=0} \tag{3.1}$$

Proof. If the Formable transform of $\xi(t)$ is represented by $\mathcal{B}(r,v)$, by applying the Formable transform to the Prabhakar fractional derivative (2.21) w.r.t variable t and using equations

(2.26) and convolution (2.3), we get the following result

$$
\begin{aligned}
&\mathcal{R}[\mathbb{D}^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)](r,v)\\
&= \mathcal{R}\left[\frac{d^n}{dt^n}\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\xi(t)\right](r,v)\\
&= \mathcal{R}\left[\frac{d^n}{dt^n}g(t)\right](r,v), \quad where \quad g(t)=\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\xi(t)\\
&= \left(\frac{r}{v}\right)^{n}\mathcal{R}[g(t)](r,v)-\sum_{k=0}^{n-1}\left(\frac{r}{v}\right)^{n-k}g^{(k)}(0), \quad g^{(k)}(0)=\frac{d^k}{dt^k}\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\xi(0)\\
&= \left(\frac{r}{v}\right)^{n}\mathcal{R}\left[(\xi * e^{\gamma}_{\varrho,(n-\rho),\omega})(t)\right](r,v)-\sum_{k=0}^{n-1}\left(\frac{r}{v}\right)^{n-k}g^{(k)}(0)\\
&= \left(\frac{r}{v}\right)^{n}\left(\frac{v}{r}\right)^{n-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{R}[\xi(t)]-\sum_{k=0}^{n-1}\left(\frac{r}{v}\right)^{n-k}\frac{d^k}{dt^k}\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\xi(t)\big|_{t=0}\\
&= \left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)-\sum_{k=0}^{n-1}\left(\frac{v}{r}\right)^{k-n}\left[\mathbb{D}^{\gamma}_{\varrho,k-n+\rho,\omega,0^+}\xi(t)\right]_{t=0}
\end{aligned}
$$

□

Theorem 3.2. ***[Formable transform of regularised Prabhakar derivative]:*** *The Formable transform of regularised Prabhakar fractional derivative is expressed as*

$$
\mathcal{R}[{}^{C}\mathbb{D}^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)]=\left(\frac{r}{v}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)-\sum_{k=0}^{n-1}\left(\frac{r}{v}\right)^{\rho-k}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\xi^{(k)}(0^+) \tag{3.2}
$$

Proof. If the Formable transform of $\xi(t)$ is represented by $\mathcal{B}(r,v)$, by applying the Formable transform to the regularized Prabhakar fractional derivative (2.23) w.r.t variable t, and using equations (2.30), (2.26), and the convolution (2.3) of the Formable transform, we get the following result

$$\begin{aligned}
&\mathcal{R}[{}^{C}\mathbb{D}^{\gamma}_{\varrho,\rho,\omega,0^+}\xi(t)](r,v)\\
&\quad=\mathcal{R}\left[\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}\frac{d^n}{dt^n}\xi(t)\right](r,v)\\
&\quad=\mathcal{R}\left[\Im^{-\gamma}_{\varrho,n-\rho,\omega,0^+}h(t)\right](r,v),\quad where\quad h(t)=\frac{d^n}{dt^n}\xi(t)\\
&\quad=\left(\frac{v}{r}\right)^{n-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{R}[h(t)](r,v)\\
&\quad=\left(\frac{v}{r}\right)^{n-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[\left(\frac{v}{r}\right)^{-n}\mathcal{R}[\xi(t)]-\sum_{k=0}^{m-1}\left(\frac{v}{r}\right)^{k-n}\xi^{(k)}(0)\right]\\
&\quad=\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)-\sum_{k=0}^{n-1}\left(\frac{v}{r}\right)^{k-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\xi^{(k)}(0^+)
\end{aligned}$$

□

Theorem 3.3. ***[Formable transform of Hilfer-Prabhakar derivative]:*** *The Formable transform to the Hilfer-Prabhakar fractional derivative is represented as follows*

$$\begin{aligned}
\mathcal{R}[\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t)]=&\left(\frac{r}{v}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)\\
&-\left(\frac{r}{v}\right)^{\nu(\rho-1)+1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}\ \Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(t)|_{t=0^+}
\end{aligned}\tag{3.3}$$

Proof. If the Formable transform of $\xi(t)$ is represented by $\mathcal{B}(r,v)$, by applying the Formable transform to the Hilfer-Prabhakar fractional derivative (2.24) w.r.t variable t, and using equations (2.30), (2.26), we get the following result

$$
\begin{aligned}
&\mathcal{R}[\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t)](r,v)\\
&\quad= \mathcal{R}\left[\left(\Im^{-\gamma\nu}_{\varrho,\nu(1-\rho),\omega,0^+}\frac{d}{dt}(\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi)\right)(t)\right](r,v)\\
&\quad= \mathcal{R}\left[\Im^{-\gamma\nu}_{\varrho,\nu(1-\rho),\omega,0^+}k(t)\right](r,v), \;\; where\;\; k(t)=\frac{d}{dt}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(t)\\
&\quad= \left(\frac{v}{r}\right)^{\nu(1-\rho)}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}\\
&\quad\times\left[\left(\frac{v}{r}\right)^{-1}\mathcal{R}[\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(t)](r,v)-\left(\frac{v}{r}\right)^{-1}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(0^+)\right]\\
&\quad= \left(\frac{v}{r}\right)^{\nu(1-\rho)}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}\\
&\quad\times\left[\left(\frac{v}{r}\right)^{(1-\nu)(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma(1-\nu)}\mathcal{R}[\xi(t)]-\left(\frac{v}{r}\right)^{-1}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(0^+)\right]\\
&\quad= \left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)\\
&\quad-\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(t)|_{t=0^+}
\end{aligned}
$$

□

Theorem 3.4. ***[Formable transform of regularized Hilfer-Prabhakar derivative]:*** *The Formable transform to the regularized Hilfer-Prabhakar fractional derivative is represented as follows*

$$\mathcal{R}[{}^{C}\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t)] = \left(\frac{r}{v}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)-\left(\frac{r}{v}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\xi(0^+) \quad (3.4)$$

Proof. If the Formable transform of $\xi(t)$ is denoted by $\mathcal{B}(r,v)$, by applying the Formable transform to the regularized Hilfer-Prabhakar fractional derivative (2.25) w.r.t variable t, and using the equations (2.30), (2.26) and convolution (2.3) of Formable transform, we get the following result

$$
\begin{aligned}
\mathcal{R}[{}^{C}\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(t)](r,v) &= \mathcal{R}\left[\Im^{-\gamma\nu}_{\varrho,\nu(1-\rho),\omega,0^+}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\frac{d}{dt}\xi(t)\right](r,v)\\
&= \mathcal{R}\left[\Im^{-\gamma}_{\varrho,1-\rho,\omega,0^+}\frac{d}{dt}\xi(t)\right](r,v)\\
&= \mathcal{R}\left[\Im^{-\gamma}_{\varrho,1-\rho,\omega,0^+}z(t)\right](r,v), \ \ z(t)=\frac{d}{dt}\xi(t)\\
&= \left(\frac{v}{r}\right)^{1-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{R}[z(t)](r,v)\\
&= \left(\frac{v}{r}\right)^{1-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[\left(\frac{v}{r}\right)^{-1}\mathcal{R}[\xi(t)]-\left(\frac{v}{r}\right)^{-1}f(0^+)\right]\\
&= \left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\mathcal{B}(r,v)-\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\xi(0^+)
\end{aligned}
$$

□

4 Applications

In this section, we will use the Formable transform of Hilfer-Prabhakar and regularized Hilfer-Prabhakar fractional derivative to find solutions of some Cauchy type fractional differential equations.

Theorem 4.1. *The solution of the generalized Cauchy type problem for fractional advection dispersion equation*

$$\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(x,t) = -w\mathbb{D}_x\xi(x,t)+\vartheta\,\Delta^{\frac{\lambda}{2}}\xi(x,t) \tag{4.1}$$

subjects to below constraints

$$\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(x,0^+) = g(x), \ \ \omega, \ \gamma, \ x\in\mathbb{R}, \ \varrho>0, \tag{4.2}$$

$$\lim_{x\to\infty}\xi(x,t)=0, \ \ t\geq 0, \tag{4.3}$$

is given by

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})^n t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\omega t^{\varrho})dk \tag{4.4}$$

where $\Delta^{\frac{\lambda}{2}}$ *is the fractional generalized Laplace operator of order* λ, $\lambda\in(0,2)$, $\rho\in(0,1)$, $\nu\in[0,1]:\ x\in\mathbb{R},\ t\in\mathbb{R}^+,\ \gamma>0$ *Fourier transform of* $\Delta^{\frac{\lambda}{2}}$ *is* $-|k|^{\lambda}$ *discussed* *in [30]*

Proof. Applying the Fourier and Formable transforms on equation (4.1) by using the equations (3.3), (2.13). First we will use Fourier transform

$$\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi^*(x,t) = iwk\xi^*(k,t) - \vartheta|k|^\lambda\xi^*(k,t) \tag{4.5}$$

where $\xi^*(k,t)$ is the Fourier transform of $\xi(x,t)$ with respect to variable x, now applying the Formable transform on (4.5), we will get

$$\begin{aligned}\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^*(k,r,v) - \left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)\\ = iwk\overline{\xi}^*(k,r,v) - \vartheta|k|^\lambda\overline{\xi}^*(k,r,v)\end{aligned}$$

where $\overline{\xi}^*(k,r,v)$ is the Formable integral transform of $\xi^*(k,t)$ with respect to variable t, therefore, we have

$$\overline{\xi}^*(k,r,v)\left[\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma} + \vartheta|k|^\lambda - iwk\right] = \left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)$$

$$\overline{\xi}^*(k,r,v) = \frac{\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{\vartheta|k|^\lambda-iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]}, if\ \frac{\vartheta|k|^\lambda-iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}} < 1$$

$$\overline{\xi}^*(k,r,v) = \left(\frac{v}{r}\right)^{\nu(1-\rho)+\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}\left[\frac{-\vartheta|k|^\lambda+iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]^n$$

$$\overline{\xi}^*(k,r,v) = \sum_{n=0}^{\infty}(iwk-\vartheta|k|^\lambda)^n\left(\frac{v}{r}\right)^{\nu(1-\rho)+\rho+\rho n-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu-\gamma n-\gamma}g^*(k)$$

now, applying the inverse of Fourier $\xi^{-1}[\xi^*(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}\xi(k)dk$ and Formable $\mathcal{R}^{-1}[\left(\frac{v}{r}\right)^{\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma}] = t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\omega t^{\varrho})$ transforms and we will get the desired result

$$\xi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^\lambda)^n t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\omega t^{\varrho})dk$$

□

Remark 1: If we take $w = 0$ and $\vartheta = \frac{ih}{2m}$ in the abobe equation (4.1), the result will reach to one dimensional space time Schrodinger fractional equation for mass m and plank constant h.

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(-\frac{ih}{2m}|k|^{\lambda})t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\omega t^{\varrho})dk \tag{4.6}$$

Theorem 4.2. *The solution of the generalized Cauchy type problem for fractional advection dispersion equation*

$$^{C}\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(x,t)=-w\mathbb{D}_x\xi(x,t)+\vartheta\,\Delta^{\frac{\lambda}{2}}\xi(x,t) \tag{4.7}$$

subjects to below constraints

$$\xi(x,0^+)=g(x),\ x\in\mathbb{R}, \tag{4.8}$$

$$\lim_{|x|\to\infty}\xi(x,t)=0,\ \ t\geq 0, \tag{4.9}$$

is given by

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})t^{\rho n}E^{\gamma n}_{\varrho,\rho(n+1)}(\omega t^{\varrho})dk \tag{4.10}$$

where $\Delta^{\frac{\lambda}{2}}$ *is the fractional generalized Laplace operator of order* λ*,* $\lambda\in(0,2),\ \rho\in(0,1),\ \nu\in[0,1]:\ x\in\mathbb{R},\ t\in\mathbb{R}^+,\ \gamma>0$ *Fourier transform of* $\Delta^{\frac{\lambda}{2}}$ *is* $-|k|^{\lambda}$ *discussed in [30]*

Proof. Applying the Fourier and Formable integral transforms on equation (4.7) by using the equations (3.4), (2.13), first we will aply the Fourier transform

$$^{C}\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi^*(k,t)=iwk\xi^*(k,t)-\vartheta|k|^{\lambda}\xi^*(k,t) \tag{4.11}$$

where $\xi^*(k,t)$ is the Fourier transform of $\xi(x,t)$ with respect to variable x. Now, we will apply the Formable integral transform on equation (4.11)

$$\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^*(k,r,v)-\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\xi^*(k,0)$$
$$=iwk\overline{\xi}^*(k,r,v)-\vartheta|k|^{\lambda}\overline{\xi}^*(k,r,v)$$

where $\overline{\xi}^*(k,r,v)$ is the Formable integral transform of $\xi^*(k,t)$ with respect to variable t, therefore

$$\overline{\xi}^*(k,r,v)\left[\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}+\vartheta|k|^{\lambda}-iwk\right]=\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}g^*(k)$$

$$\overline{\xi}^*(k,r,v)=\frac{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}g^*(k)}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\left[\frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]<1$$

$$\overline{\xi}^*(k,r,v)=\frac{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}g^*(k)}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\left[1+\frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]^{-1}$$

$$\overline{\xi}^*(k,r,v)=g^*(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})^n\left(\frac{v}{r}\right)^{\rho n}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma n},$$

applying the inverse of Formable and Fourier on both sides of the above equation, we will get the final result

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})t^{\rho n}E^{\gamma n}_{\varrho,\rho(n+1)}(\omega t^{\varrho})dk$$

□

Theorem 4.3. *The solution of generalized Cauchy type problem for fractional heat equation*

$$\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi(x,t)=M\frac{\partial^2}{\partial x^2}\xi(x,t), \tag{4.12}$$

$$\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(x,t)|_{t=0}=g(x), \tag{4.13}$$

$$\lim_{x\to\infty}\xi(x,t)=0,$$

$with\ \ \rho\in(0,1),\ \nu[0,1];\ \omega,x\in\mathbb{R};\ M,\varrho>0,\gamma\geq 0,\ \ is\ \ given\ \ by$

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}g(k)\sum_{n=0}^{\infty}(-Mk^2)^n t^{\rho(n+1)-\nu(\rho-1)-1}E^{\gamma(n+1-\nu)}_{\varrho,\rho(n+1)+\nu(1-\rho)}(\omega t^{\varrho})dk \tag{4.14}$$

Proof. Applying the Fourier and Formable transform on equation (4.12) by using equations (3.3), (4.13), (2.15), first we will apply the Fourier transform

$$\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+}\xi^*(x,t)=-Mk^2\xi^*(k,t) \tag{4.15}$$

where $\xi^*(k,t)$ is the Fourier transform of $\xi(x,t)$ with respect to variable x, now applying the Formable integral transform on equation (4.15)

$$\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^{*}(k,r,v)-\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^{+}}f^{*}(x,0)$$
$$=-Mk^{2}\overline{\xi}^{*}(k,r,v)$$

where $\overline{\xi}^{*}(k,r,v)$ is the Formable integral transform of $\xi^{*}(k,t)$ with respect to variable t, therefore we have

$$\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^{*}(k,r,v)-\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^{*}(k)=-Mk^{2}\overline{\xi}^{*}(k,r,v)$$
$$\overline{\xi}^{*}(k,r,v)\left[\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}+Mk^{2}\right]=\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^{*}(k)$$
$$\overline{\xi}^{*}(k,r,v)=\frac{\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^{*}(k)}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}+Mk^{2}}$$
$$\overline{\xi}^{*}(k,r,v)=\frac{\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}g^{*}(k)}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{Mk^{2}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\left(\frac{Mk^{2}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right)<1$$
$$\overline{\xi}^{*}(k,r,v)=\left(\frac{v}{r}\right)^{\nu(1-\rho)+\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu-\gamma}g^{*}(k)\sum_{n=0}^{\infty}\left[\frac{-Mk^{2}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]^{n}$$
$$\overline{\xi}^{*}(k,r,v)=\left(\frac{v}{r}\right)^{\nu(1-\rho)+\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu-\gamma}g^{*}(k)\sum_{n=0}^{\infty}(-Mk^{2})^{n}\left(\frac{v}{r}\right)^{\rho n}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\gamma n}$$
$$\overline{\xi}^{*}(k,r,v)=g^{*}(k)\sum_{n=0}^{\infty}(-Mk^{2})^{n}\left(\frac{v}{r}\right)^{\rho n+\nu(1-\rho)+\rho-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu-\gamma n-\gamma},$$

applying the inverse of Fourier and Formable transforms, we will get the desired result

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}g(k)\sum_{n=0}^{\infty}(-Mk^{2})^{n}t^{\rho(n+1)-\nu(\rho-1)-1}E^{\gamma(n+1-\nu)}_{\varrho,\rho(n+1)+\nu(1-\rho)}(\omega t^{\varrho})dk$$

□

Theorem 4.4. *The solution of Cauchy type fractional differential equation*

$$\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^{+}}\xi(t)=\lambda\Im^{\delta}_{\varrho,\rho,\omega,0^{+}}\xi(t)+y(t), \tag{4.16}$$

$$\left(\Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^{+}}\xi(t)\right)|_{t=0}=M \tag{4.17}$$

where $\xi(t)\in L_1[0,\infty):\rho\in(0,1),\nu\in[0,1]:\omega,\lambda\in\mathbb{C}:t,\varrho>0,K,\gamma,\delta\geq$

$0, \ \ is \ \ given \ \ by$

$$\xi(t) = \sum_{n=0}^{\infty} \lambda^n \Im^{\gamma+n(\delta+\gamma)}_{\varrho,\rho(2n+1),\omega,0^+} y(t) + M \sum_{n=0}^{\infty} \lambda^n t^{\rho(2n+1)+\nu(1-\rho)-1} \times E^{\delta n+\gamma n+\gamma-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(2n+1)}(\omega t^{\varrho}) \tag{4.18}$$

Proof. Let $\mathcal{B}(r, v)$ be the Formable transform of $\xi(t)$, applying the Formable transform on both side of equation (4.16) and using (3.3), (4.17), (2.30) equations, then

$$\begin{aligned}
\mathcal{R}[\lambda \Im^{\delta}_{\varrho,\rho,\omega,0^+}\xi(t) + y(t)](r, v) &= \mathcal{R}\left[\lambda \Im^{\delta}_{\varrho,\rho,\omega,0^+}\xi(t)\right](r, v) + \mathcal{R}[y(t)](r, v) \\
&= \lambda \mathcal{R}[\xi(t) t^{\rho-1} E^{\delta}_{\varrho,\rho}(\omega t^{\varrho})](r, v) + \mathcal{R}[y(t)](r, v) \\
&= \lambda \left(\frac{v}{r}\right) \mathcal{B}(r, v) \left(\frac{v}{r}\right)^{\rho-1} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta} + \mathcal{R}[y(t)](r, v) \\
&= \lambda \left(\frac{v}{r}\right)^{\rho} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta} \mathcal{B}(r, v) + \mathcal{R}[y(t)](r, v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}\left[\mathbb{D}^{\gamma,\rho,\nu}_{\varrho,\omega,0^+} y(t)\right](r, v) &= \left(\frac{v}{r}\right)^{-\rho} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma} \mathcal{B}(r, v) \\
&- \left(\frac{v}{r}\right)^{\nu(1-\rho)-1} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu} \Im^{-\gamma(1-\nu)}_{\varrho,(1-\nu)(1-\rho),\omega,0^+}\xi(t)|_{t=0}, \\
&= \left(\frac{v}{r}\right)^{-\rho} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma} \mathcal{B}(r, v) - \left(\frac{v}{r}\right)^{\nu(1-\rho)-1} \left(1 - \omega \left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu} M
\end{aligned}$$

therefore

$$
\begin{aligned}
\mathcal{B}(r,v) &= \frac{\mathcal{R}[y(t)](r,v)+\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}M}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}\left[1-\frac{\lambda\left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\left[\frac{\lambda\left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]<1 \\
&= \frac{\mathcal{R}[y(t)](r,v)+\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}M}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\sum_{n=0}^{\infty}\left[\frac{\lambda\left(\frac{v}{r}\right)^{\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta}}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\right]^{n} \\
&= \frac{\mathcal{R}[y(t)](r,v)+\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}M}{\left(\frac{v}{r}\right)^{-\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma}}\sum_{n=0}^{\infty}\lambda^{n}\left(\frac{v}{r}\right)^{2\rho n}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta n-\gamma n} \\
&= \left(\mathcal{R}[y(t)](r,v)+\left(\frac{v}{r}\right)^{\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{\gamma\nu}M\right) \\
&\times\sum_{n=0}^{\infty}\lambda^{n}\left(\frac{v}{r}\right)^{2\rho n+\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta n-\gamma n-\gamma} \\
&= \mathcal{R}[y(t)](r,v)\sum_{n=0}^{\infty}\lambda^{n}\left(\frac{v}{r}\right)^{2\rho n+\rho}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta n-\gamma n-\gamma} \\
&+ M\sum_{n=0}^{\infty}\lambda^{n}\left(\frac{v}{r}\right)^{2\rho n+\rho+\nu(1-\rho)-1}\left(1-\omega\left(\frac{v}{r}\right)^{\varrho}\right)^{-\delta n-\gamma n-\gamma+\gamma\nu},
\end{aligned}
$$

by taking the inverse of Formable transform on both sides of the above equation, we will get the desired result

$$
\begin{aligned}
\xi(t) = \sum_{n=0}^{\infty}\Im^{\gamma+n(\delta+\gamma)}_{\varrho,\rho(2n+1),\omega,0^{+}}\lambda^{n}y(t)+\sum_{n=0}^{\infty}t^{\rho(2n+1)+\nu(1-\rho)-1}M\lambda^{n} \\
\times E^{\delta n+\gamma n+\gamma-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(2n+1)}(\omega t^{\varrho})
\end{aligned}
$$

□

5 Conclusion

In this study, we first applied the Formable transform to the Hilfer-Prabhakar fractional derivative and its regularized version. Next, we used these results to solve some Cauchy type fractional differential equations involving Hilfer-Prabhakar fractional derivatives by using the Formable and Fourier transforms, which involve the three parameter Mittag-Leffler function. The results indicate that this transform is very useful for solving fractional differential equations.

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