Linear constraints for ensuring k -hop connectivity using Mixed-Integer Programming for Multi-Agent Systems

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Abstract

This note concerns the problem of k-hop connectivity in a network of mobile agents, which is achieved if any pair of agents can communicate with each other through a link of k-1 or fewer intermediate nodes. We propose linear constraints involving binary optimization variables to ensure k-hop connectivity. Such constraints are then integrated into a Mixed-Integer Linear Programming (MILP) trajectory planning model. Simulation results illustrate the application of the proposed method and the effect of varying k in the context of a mission involving the visitation of multiple targets.

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TECHNICAL NOTE

Linear constraints for ensuring k-hop connectivity using Mixed-Integer Programming for Multi-Agent Systems

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Summary

This note concerns the problem of k-hop connectivity in a network of mobile agents, which is achieved if any pair of agents can communicate with each other through a link of $k-1$ or fewer intermediate nodes. We propose linear constraints involving binary optimization variables to ensure k-hop connectivity. Such constraints are then integrated into a Mixed-Integer Linear Programming (MILP) trajectory planning model. Simulation results illustrate the application of the proposed method and the effect of varying k in the context of a mission involving the visitation of multiple targets.

KEYWORDS:
Cooperative multiple agents, Connectivity, Hops, Mixed-Integer Linear Programming.

1 | INTRODUCTION

Multi-Agent Systems (MAS) are groups of autonomous entities collaborating to solve a problem. They offer enhanced value in many situations due to their capability to address complex objectives by dividing them into simpler tasks that can be handily completed[1]. There is a growing interest in the use of MAS formations in diverse domains, ranging from aquatic to aerospace applications[2,3,4], demonstrating the intent for the ubiquitous use of collaborating agents. There are many use cases where MAS have been demonstrated to be necessary or advantageous, such as the inspection of underground facilities[5], smart agriculture[6], and autonomous exploration[7].

Trajectory planning algorithms are fundamental to the operation of MAS[1], as they provide the required coordination that enables agents to efficiently complete tasks while retaining important features such as inter-agent communication. Indeed, the capacity to exchange information between agents is a common requirement for MAS. In Grøtli[8] and Oubbati *et al.*[9], for example, the agents act as data relays, and reliable communication allows the desired flow of data within the network. Distributed control formulations, in which the agents must often share information about their present states or objectives, also require communication capacities[10,11].

Graphs are often employed to represent communication networks in the design of trajectory planning algorithms for MAS[12]. Some of their properties[13] are directly correlated to desired characteristics for these systems; for example, the connectivity property implies the capacity of the agents to communicate among themselves[14]. Thus, one may leverage the results in the rich field of graph theory to derive topological conditions under which the agents in a MAS could communicate with one another.

Networks with a limited diameter are also desirable since the communication between agents connected through many relays (hops) requires more point-to-point transmissions. Each of these transmissions is subjected to potential failures and delays,

[0]**Abbreviations:** MAS, multi-agent system; MILP, mixed-integer linear programming; MPC, model predictive control.

decreasing the reliability and efficiency of the network [15]. This could be an issue when technologies such as Visual Light Communication (VLC) are employed [5]. The intricate communication mechanism required by this technology generates sizeable delays at each transmission. Thus, by limiting the maximum diameter of the graph and, consequently, the number of transmissions in the corresponding VLC-based network, one may prevent the detrimental effects of such delays in the communications within the network. Applications involving buffered flow networks may also benefit from network architectures with limited diameters. In Grøtli [8], a group of UAVs with limited data storage capacity are tasked with the surveillance of an environment. Data must be constantly collected and transmitted to a base without overflowing the storage buffers. A network with fewer relays can improve the velocity and reliability of data transmission.

Connectivity constraints and the maximum diameter problem have been addressed using control and trajectory planning algorithms in the literature. Zavlanos and Pappas [16] employ graph-theoretic properties and invariance to encode a connectivity maintenance constraint within a maximum of k hops, using centralized nonlinear optimization, to generate control reference signals in a leader-follower framework. Mondal *et al.* [17] guarantee connectivity by imposing that the second smallest eigenvalue of the Laplacian matrix (Fiedler value) is always greater than zero. The proposed formation control technique uses multiple member potential functions to assert collision avoidance, connectivity maintenance, formation behavior control, and trajectory tracking. Kantaros *et al.* [18] present a survey of communication modeling and control design for the connectivity assurance of multi-robot systems, in which the assumptions and qualities of each approach are highlighted.

Some works propose methods for limiting the network diameter in the context of MAS. Stump *et al.* [19] present a scheme that considers the k connectivity matrix to limit the number of hops in a communication network composed by a team of mobile robots. The authors were mainly concerned with robustness to disconnection. Katheri *et al.* [14] also bound the number of hops in the network. Their method only allows a disconnection in the network if an alternative k-hop path exists between the nodes.

In Afonso *et al.* [20], a Mixed-Integer Linear Programming (MILP) encoding was proposed for the design of a trajectory planning and decision-making system for a MAS under connectivity constraints. Two kinds of conditions for connectivity were derived from results pertaining to spanning tree graphs: one that was necessary and sufficient and a second one that was only sufficient. A numerical study showed that the latter was more advantageous in terms of the cost attained under computation time limitations. Some advantages of the approach presented in Afonso *et al.* [20] were: the treatment of collision avoidance with obstacles and between the agents and task assignment performed within the same MILP solved for trajectory planning, besides the connectivity constraint. An extension was presented in Caregnato-Neto *et al.* [21], where the connectivity constraints were modified to ensure that each vertex in the communication graph had a degree of at least two for redundancy of the links. Experimental validation was carried out in a receding horizon scheme for real-time trajectory planning and decision-making using a MAS comprised of differential drive robots, showing that approaches based on Mixed-Integer Programming (MIP) are amenable for deployment in real MAS. Despite the success of the application of MIP in this context, limiting the diameter of the communication graph in such formulations remains an open issue.

The present note expands Afonso *et al.* [20] by proposing a MILP-based trajectory planning algorithm that is able to guarantee the connectivity of the communication network of a MAS and bound its maximum allowed diameter to a desirable value. We present the corresponding necessary and sufficient conditions taken from elementary graph theory and then propose equivalent constraints that are used to encode such conditions in a MILP trajectory planning model. The results are evaluated in the context of a mission involving multiple targets to be visited, where the trade-off between the maximum allowed diameter of the network and the overall performance of the MAS is discussed.

The remainder of this note is organized as follows. Section 2 contains the relevant definitions, as well as results from graph theory and the demonstration of fundamental theorems that support the MILP trajectory planning model presented in Section 3. Section 4 provides simulation results illustrating the effectiveness of the proposed approach in a target visitation problem. Section 5 offers final remarks and ideas for future work.

1.1 | Notation and preliminaries

$\mathbb{N}$	Set of natural numbers $\{1, 2, ...\}$
$N_a \in \mathbb{N}$	Number of agents
$n_r \in \mathbb{N}$	Dimension of the position vector
$G = (V, E)$	Graph with vertices V and edges E
V	Set of vertices $\{i\}$, $i = 1, 2, \dots, N_a$
E	Set of edges $\{(i, j)\}$, $i = 1, 2, \dots, N_a$, $j = 1, 2, \dots, N_a$
$A \in \{0, 1\}^{N_a \times N_a}$	Adjacency matrix
$[H]_{ij}$	Entry in row i and column j of matrix H
$\mathbf{1}_q$	Column vector with all q entries equal to 1
1	Logical `true`
0	Logical `false`
$\wedge$	`and` logical operator
$\vee$	`or` logical operator
$\bigvee_{\ell=1}^{n} C_\ell$	$C_1 \vee C_2 \vee \cdots \vee C_n$

From the perspective of communication between agents, it is important to define what is meant by k-hop connectivity.

Definition 1. A network is k-hop connected if a message from any agent i to any agent j must pass through at most $k - 1$ agents between i and j.

Remark 1. With this definition and considering the communication network modeled as a graph $G = (V, E)$, where the vertices V are the agents and the edges E are communication links between them, Definition 1 can be interpreted as an upper bound on the diameter of the graph, as will be discussed in Section 2.

2 | POWERS OF THE ADJACENCY MATRIX, BINARIZATION AND K-HOP CONNECTIVITY

We define in the following a *path* and its *length*. The path will represent the sequence of all agents the message flows through, from the source towards the sink, including the intermediate relays. The notion of the length of a path deals with the number of agents involved in the path.

Definition 2. A *path* in a graph $G = (V, E)$ is a sequence $(v_\ell)_{\ell=1}^n$ of vertices $v_\ell \in V$, $\ell = 1, 2, \dots, n$, $v_i \neq v_j$ for $i \neq j$, such that every edge $(v_\ell, v_{\ell+1}) \in E$, $\ell = 1, 2, \dots n - 1$.

Definition 3. The *length* L of a path $(v_\ell)_{\ell=1}^n$ is the number of edges $(v_\ell, v_{\ell+1}) \in E$, $\ell = 1, 2, \dots n - 1$ associated to this path. Therefore, $L((v_\ell)_{\ell=1}^n) = n - 1$.

We define both the connectivity and its particular case k-hop connectivity. The connectivity ensures the possibility of sending messages between any two agents in the MAS, even if other agents may have to be used as relays.

Definition 4. A graph $G = (V, E)$ is *connected* if there is a path from any vertex $i \in V$ to any other vertex $j \in V$.

However, for purposes of delay in communications, it may be convenient to characterize the worst-case minimal number of agents that a message may have to go through to reach its destination. This is reflected by the concept of k-hop connectivity in Definition 1, as stated in Remark 1. The value k dictates the maximum allowed diameter that graph G may assume at any instant.

Definition 5. A graph G has diameter k if and only if there is a path $(v_\ell)_{\ell=1}^n$ from any vertex $v_1 \in V$ to any other vertex $v_n \in V$ such that $L((v_\ell)_{\ell=1}^n) = n - 1 \leq k$.

The tool that we will use to ensure the k-hop connectivity via optimization involves the graph's adjacency matrix and its powers.

Definition 6. The matrix $A \in \{0,1\}^{N_a \times N_a}$ with

$$[A]_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E, \\ 0, & \text{if } (i,j) \notin E, \end{cases} \tag{1}$$

is the adjacency matrix of the graph G.

Lemma 1. The k-th power $A^k \in \mathbb{N}^{N_a \times N_a}$ of the adjacency matrix A is such that $[A^k]_{ij}$ denotes the number of paths $(v_\ell)_{\ell=1}^{k+1}$ between $v_1 = i$ and $v_{k+1} = j$ with length k.

Proof. See Theorem 10.1 in Chartrand[22]. □

For our purposes, it suffices to determine that there exists at least one path with a length less than or equal to k between any pair of vertices. Therefore, the total number of paths loses importance, and one may use a binary version of the adjacency matrix's k-th power B^k, as given in Definition 7. Moreover, it is interesting to determine B^k recursively using only logical operators, eliminating the need for matrix multiplication, as shown in Lemma 2.

Definition 7. The matrices of binary entries $B^k \in \{0,1\}^{N_a \times N_a}$ are defined for $k \in \mathbb{N}$ as

$$[B^k]_{ij} = \begin{cases} 1, & \text{if } [A^k]_{ij} > 0, \\ 0, & \text{if } [A^k]_{ij} = 0. \end{cases} \tag{2}$$

Lemma 2. $[B^k]_{ij}$ may be written recursively using logical operators as

$$[B^{k+1}]_{ij} = \bigvee_{\ell=1}^{N_a} \left([B^k]_{i\ell} \wedge [B]_{\ell j} \right), k \in \mathbb{N}, \tag{3}$$

$$B^1 = B = A. \tag{4}$$

Proof. From matrix product rules:

$$[A^{k+1}]_{ij} = \sum_{\ell=1}^{N_a} [A^k]_{i\ell}[A]_{\ell j}. \tag{5}$$

In view of the non-negativity of both $[A^k]_{ij}$ and $[A]_{ij}$, it follows that

$$[A^{k+1}]_{ij} > 0 \Leftrightarrow ([A^k]_{i\ell} > 0) \wedge ([A]_{\ell j} > 0) \text{ for some } \ell \in \{1, 2, \dots, N_a\}. \tag{6}$$

The clause $([A^k]_{i\ell} > 0)$ can be replaced with $[B^k]_{i\ell}$ in (6) in light of (2). Since $B = A$, the clause $([A]_{\ell j} > 0)$ can be replaced with $[B]_{\ell j}$, yielding

$$[A^{k+1}]_{ij} > 0 \Leftrightarrow [B^k]_{i\ell} \wedge [B]_{\ell j} \text{ for some } \ell \in \{1, 2, \dots, N_a\}. \tag{7}$$

On the other hand,

$$[B^k]_{i\ell} \wedge [B]_{\ell j} \text{ for some } \ell \in \{1, 2, \dots, N_a\} \Leftrightarrow \bigvee_{\ell=1}^{N_a} \left([B^k]_{i\ell} \wedge [B]_{\ell j} \right). \tag{8}$$

Therefore, by using (8) one may rewrite the right-hand-side of (7) as

$$[A^{k+1}]_{ij} > 0 \Leftrightarrow \bigvee_{\ell=1}^{N_a} \left([B^k]_{i\ell} \wedge [B]_{\ell j} \right). \tag{9}$$

In view of (2), the clause on the left-hand-side of (9) can be replaced with $[B^{k+1}]_{ij}$, yielding

$$[B^{k+1}]_{ij} \Leftrightarrow \bigvee_{\ell=1}^{N_a} \left([B^k]_{i\ell} \wedge [B]_{\ell j} \right). \tag{10}$$

Finally, the equivalence may be replaced with

$$[B^{k+1}]_{ij} = \bigvee_{\ell=1}^{N_a} \left([B^k]_{i\ell} \wedge [B]_{\ell j} \right). \tag{11}$$

□

As stated in Lemma 3, it suffices to check B^k to determine whether there exists a path of length k between any pair of nodes of G as stated in Lemma 3 below.

Lemma 3. There is a path of length k between vertices i and j if and only if $[B^k]_{ij}$.

Proof. From Lemma 1, there is a path between i and j with k hops if and only if $[A^k]_{ij} > 0$. On the other hand, from (2), $[A^k]_{ij} > 0 \Leftrightarrow [B^k]_{ij}$. □

Lemma 4. There is a path between i and j with length at most k if and only if $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij}$.

Proof. Follows directly from the use of Lemma 3 for paths of length equal to $1, 2, \ldots, k$. □

Moreover, it is possible to check the k-hop connectivity of the graph G given in Definition 5 as well as the connectivity defined in Definition 4, as shown in Theorem 1 and Corollary 1.

Theorem 1. There exists a path of length at most k between any pair of vertices if and only if $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij}$ for all $i = 1, 2, \ldots, N_a$, $j = 1, 2, \ldots, N_a$.

Proof. Follows directly from the use of Lemma 4 for every $i = 1, 2, \ldots, N_a$, $j = 1, 2, \ldots, N_a$. □

Corollary 1. Given $k \in \mathbb{N}$, if $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij}$ for all $i = 1, 2, \ldots, N_a$, $j = 1, 2, \ldots, N_a$, then G is connected.

Proof. From Theorem 1, for every pair of nodes, there is a path between them, which means that the graph is connected according to Definition 4. □

3 | IMPLEMENTATION WITH LINEAR CONSTRAINTS ON BINARY VARIABLES

For usage of the binarized version of the k-th power B^k of the adjacency matrix in a Mixed-Integer Linear/Quadratic Programming framework, it is fundamental to recast the logical operations in the form of linear constraints on binary variables.

Given the adjacency matrix A of the graph G, we show how the recursion in (3) can be implemented using linear inequalities.

Lemma 5. The following linear inequalities involving matrices of auxiliary binary variables $[B^{k+1}_{\text{aux}}]_{ij\ell} \in \{0,1\}^{N_a \times N_a \times N_a}$ ensure that statement (3) holds

$$[B^{k+1}]_{ij} \geq [B^k]_{i\ell} + [B]_{\ell j} - 1.5, \forall \ell \in \{1, 2, \ldots, N_a\},\ k \in \mathbb{N}, \tag{12}$$

$$2[B^{k+1}_{\text{aux}}]_{ij\ell} \leq [B^k]_{i\ell} + [B]_{\ell j}, \forall \ell \in \{1, 2, \ldots, N_a\},\ k \in \mathbb{N}, \tag{13}$$

$$[B^{k+1}]_{ij} \leq \sum_{\ell=1}^{N_a} [B^{k+1}_{\text{aux}}]_{ij\ell},\ k \in \mathbb{N}. \tag{14}$$

Proof. Recall from the proof of Lemma 2 that the recursion in (3) is equivalent to

$$[B^{k+1}]_{ij} \Leftrightarrow [B^k]_{i\ell} \wedge [B]_{\ell j} \text{ for some } \ell \in \{1, 2, \ldots, N_a\}. \tag{15}$$

We shall now prove that (12)–(14) ensure that (15) holds.

Firstly, assume that (12)–(14) hold and that $[B^{k+1}]_{ij} = 1$. Therefore, it follows from (14) that

$$\sum_{\ell=1}^{N_a} [B^{k+1}_{\text{aux}}]_{ij\ell} \geq 1,\ k \in \mathbb{N}, \tag{16}$$

which is satisfied if and only if $[B^{k+1}_{\text{aux}}]_{ij\ell} = 1$ for some $\ell \in \{1, 2, \ldots, N_a\}$. In turn, $[B^{k+1}_{\text{aux}}]_{ij\ell} = 1$ implies that $[B^k]_{i\ell} = [B]_{\ell j} = 1$ in light of (13).

Now, assume that (12)–(14) hold and that $[B^k]_{i\ell} \wedge [B]_{\ell j}$ for some $\ell \in \{1, 2, \ldots, N_a\}$, Then, $[B^k]_{i\ell} = [B]_{\ell j} = 1$. From (12) it follows that

$$[B^{k+1}]_{ij} \geq [B^k]_{i\ell} + [B]_{\ell j} - 1.5 = 1 + 1 - 1.5 = 0.5,\ k \in \mathbb{N}, \tag{17}$$

with the only solution as $[B^{k+1}]_{ij} = 1$. From (14) it follows that

$$\sum_{\ell=1}^{N_a} [B_{\text{aux}}^{k+1}]_{ij\ell} \geq 1,\ k \in \mathbb{N}, \tag{18}$$

which is satisfied if and only if $[B_{\text{aux}}^{k+1}]_{ij\ell} = 1$ for some $\ell \in \{1, 2, \ldots, N_a\}$.

□

The following Lemma shows that linear constraints over the elements of the binarized version B^n of the n-th power of the adjacency matrix A may be used to impose k-hop connectivity.

Lemma 6. The constraints

$$\sum_{n=1}^{k} [B^n]_{ij} \geq 1,\ i = 1, 2, \ldots, N_a, j = 1, 2, \ldots, N_a, \tag{19}$$

are equivalent to $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij}$ for all $i = 1, 2, \ldots, N_a$, $j = 1, 2, \ldots, N_a$.

Proof. ($\Rightarrow$) $\sum_{n=1}^{k} [B^n]_{ij} \geq 1$ implies that $[B^n]_{ij} = 1$ for some $n \in \{1, 2, \ldots, k\}$, which in turn implies $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij} = 1$.

($\Leftarrow$) $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij} = 1$ implies that $[B^n]_{ij} = 1$ for some $n \in \{1, 2, \ldots, k\}$, which in turn implies $\sum_{n=1}^{k} [B^n]_{ij} \geq 1$ in view of the non-negativity of each $[B^n]_{ij}$. □

We are now in place to enunciate the following Theorem stating that satisfaction of the linear constraints developed so far implies k-hop connectivity.

Theorem 2. For a given $k \in \mathbb{N}$, satisfaction of the linear constraints (12), (13), (14), and (19) implies that the underlying graph is connected and there exists a path of length at most k between every pair of nodes.

Proof. From Lemma 5 the solutions satisfying inequalities (12), (13), and (14) result in (3). Moreover, by Lemma 6, the inequalities (19) are equivalent to $[B^1]_{ij} \vee [B^2]_{ij} \vee \cdots \vee [B^k]_{ij}$. The conclusion follows from Theorem 1 and Corollary 1. □

3.1 | Additional conditions for connectivity

Physical conditions for connectivity, such as the proximity conditions used in Afonso *et al.*[20] can be imposed using $[B^1]_{ij}$ in place of the binary variables in that paper. Therefore, consider a polytope $\mathcal{P} = \{\mathbf{r} \in \mathbb{R}^{n_r} : P^{con}\mathbf{r} \leq \mathbf{p}^{con}\}$ centered at the origin with q facets, with constant $P^{con} \in \mathbb{R}^{q \times n_r}$ and $\mathbf{p}^{con} \in \mathbb{R}^q$. Let $\mathbf{r}_i$ denote the position of agent i, associated with node i in the communication graph and $M \in \mathbb{R}$ be a "big-M" constant[23]. Then

$$P^{con}(\mathbf{r}_i - \mathbf{r}_j) \leq \mathbf{p}^{con} + M(1 - [B^1]_{ij})\mathbf{1}_q \tag{20}$$

imposes a proximity condition for connectivity, because $[B^1]_{ij} = 1$ is only feasible if $P^{con}(\mathbf{r}_i - \mathbf{r}_j) \leq \mathbf{p}^{con}$, i.e., the polytope $\mathcal{P}$ centered at the position $\mathbf{r}_j$ of agent j contains the position $\mathbf{r}_i$ of agent i. The edge (i, j) can only exist in the graph if the associated positions satisfy the communication range constraint described by $\mathcal{P}$. Alternatively, for M large enough such that $P^{con}(\mathbf{r}_i - \mathbf{r}_j) \leq \mathbf{p}^{con} + M\mathbf{1}_q$ for all achievable $\mathbf{r}_i$ and $\mathbf{r}_j$, then $[B^1]_{ij} = 0$ trivially satisfies (20), relaxing the constraint.

3.2 | Complete optimization problem for the MAS

In this subsection, we present the complete optimization problem that yields the planned trajectory for the MAS considering k-hop connectivity. Except for the connectivity constraints developed in the present note, the other parts of the formulation are inherited from Afonso *et al.*[20]. We aim to show that the connectivity constraints in Afonso *et al.*[20] can be seamlessly replaced by the ones we propose. For conciseness, we consider a control task without obstacle and inter-agent collision avoidance to focus on the main contribution of the current note, namely, k-hop connectivity of the network while visiting target sets. We emphasize that this choice is only made to focus the discussion on the novel aspects of the contribution and does not represent any limitation regarding the integration of collision avoidance into this framework.

The task considered is enunciated as: given N_t polytopic target sets $\mathcal{T}_m = \left\{\mathbf{r} \in \mathbb{R}^{n_r} : P_m^{\text{target}}\mathbf{r} \leq \mathbf{p}_m^{\text{target}}\right\}$, $m \in \{1, 2, \ldots, N_t\}$, a maximum network diameter $k \in \{1, 2, ..., N_a - 1\}$, and a time limit $T \in \mathbb{N}$, the group of N_a agents should visit all N_t target sets within T time steps, with each agent subject to upper and lower bounds on their states $\mathbf{x}_{\ell,t}$ and controls $\mathbf{u}_{\ell,t}$, $\forall \ell \in \{1, 2, \ldots, N_a\}$, $\forall t \in \{1, 2, \ldots, T^*\}$, where $T^* \in \mathbb{N}$ is an optimization variable and $T^* \leq T$. The network must be k-hop connected at all times $t \in \{1, 2, \ldots, T^*\}$.

All agents are assumed to have linear dynamics. Without loss of generality, we assume that all N_a agents have the same dynamic model:

$$\mathbf{x}_{\ell,t+1} = F\mathbf{x}_{\ell,t} + G\mathbf{u}_{\ell,t}, \forall \ell \in \{1, 2, \ldots, N_a\}, \forall t \in \mathbb{N} \cup \{0\}. \tag{21}$$

The vector state of the agents is supposed to contain the position vector $\mathbf{r}$ as part of the state components.

To accomplish this task, a Mixed-Integer Linear Program (MILP) is formulated involving a cost J composed of a weighted sum of two components, namely

$$J = \sum_{j=0}^{T} j b_j^{\text{hor}} + \gamma J_C, \tag{22}$$

where J_C is the control cost, i.e., a measure of the total fuel spent by the agents

$$J_C = \sum_{\ell=1}^{N_a} \sum_{t=0}^{T-1} \|\mathbf{u}_{\ell,t}\|_1 \tag{23}$$

and $\sum_{j=0}^{T} j b_j^{\text{hor}} = T^*$, $b_j^{\text{hor}} \in \{0, 1\}$, $j \in \{0, 1, \ldots, T\}$, corresponds to the time to accomplish the task, upper-bounded by a given time limit T. The constant $\gamma \in \mathbb{R}$ is chosen to tune the compromise between time minimization and fuel expense. Besides the auxiliary binary variables b_j^{hor} used to handle the variable horizon, $b_{\ell,m,t}^{\text{target}} \in \{0, 1\}$, $\ell \in \{1, 2, \ldots, N_a\}$, $m \in \{1, 2, \ldots, N_t\}$, $t \in \{1, 2, \ldots, T\}$ are used to deal with the visitation of the target sets.

The following optimization problem, when feasible, yields a solution that complies with the requirements of the control task described so far in the present subsection.

$$\mathbb{P}[(\mathbf{x}_{\ell,0})_{\ell=1}^{N_a}] = \min_{\mathbf{x}_{\ell,t+1}, \mathbf{u}_{\ell,t}, b_t^{\text{hor}}, b_{\ell,m,t}^{\text{target}}, B_t, B_t^2, \ldots, B_t^k, B_{\text{aux},t}, B_{\text{aux},t}^2, \ldots, B_{\text{aux},t}^k} J, \tag{24}$$

subject to

$$\mathbf{x}_{\ell,t+1} \leq F\mathbf{x}_{\ell,t} + G\mathbf{u}_{\ell,t} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},$$
$$\forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25a}$$

$$-\mathbf{x}_{\ell,t+1} \leq -F\mathbf{x}_{\ell,t} - G\mathbf{u}_{\ell,t} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},$$
$$\forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25b}$$

$$-\mathbf{x}_{\ell,t+1} \leq -\mathbf{x}_{min} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},\ \forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25c}$$

$$\mathbf{x}_{\ell,t+1} \leq \mathbf{x}_{max} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},\ \forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25d}$$

$$-\mathbf{u}_{\ell,t} \leq -\mathbf{u}_{min} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},\ \forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25e}$$

$$\mathbf{u}_{\ell,t} \leq \mathbf{u}_{max} + M \sum_{i=0}^{t-1} b_i^{\text{hor}},\ \forall \ell \in \{1, 2, \ldots, N_a\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25f}$$

$$P_m^{\text{target}} \mathbf{r}_{\ell,t+1} \leq \mathbf{p}_m^{\text{target}} + M(1 - b_{\ell,m,t}^{\text{target}}) \mathbf{1}_{N_{st}},$$
$$\forall \ell \in \{1, 2, \ldots, N_a\}, \forall m \in \{1, 2, \ldots, N_t\},\ \forall t \in \{0, 1, \ldots, T\}, \tag{25g}$$

$$\sum_{t=0}^{T} b_t^{\text{hor}} = 1, \tag{25h}$$

$$\sum_{\ell=1}^{N_a}\sum_{t=0}^{T} b_{\ell,m,t}^{\text{target}} = 1, \forall m \in \{1,2,\ldots,N_t\}, \tag{25i}$$

$$\sum_{t=0}^{T} t(b_{\ell,m,t}^{\text{target}} - b_t^{\text{hor}}) \le 0,\ \forall \ell \in \{1,2,\ldots,N_a\},\ \forall m \in \{1,2,\ldots,N_t\}, \tag{25j}$$

$$P^{con}(\mathbf{r}_{i,t+1} - \mathbf{r}_{j,t+1}) \le \mathbf{p}^{con} + M(1 - [B_t^1]_{ij})\mathbf{1}_q + M\sum_{r=0}^{t-1} b_r^{\text{hor}},$$
$$\forall i \in \{1,2,\ldots,N_a-1\},\ \forall j > i, \forall t \in \{0,1,\ldots,T\}, \tag{25k}$$

$$[B_t^{n+1}]_{ij} \ge [B_t^n]_{i\ell} + [B_t]_{\ell j} - 1.5,$$
$$\forall n \in \{1,2,\ldots,k-1\}, \forall i,j,\ell \in \{1,2,\ldots,N_a\}, \forall t \in \{0,1,\ldots,T\}, \tag{25l}$$

$$2[B_{\text{aux},t}^{n+1}]_{ij\ell} \le [B_t^n]_{i\ell} + [B_t]_{\ell j},$$
$$\forall n \in \{1,2,\ldots,k-1\}, \forall i,j,\ell \in \{1,2,\ldots,N_a\}, \forall t \in \{0,1,\ldots,T\}, \tag{25m}$$

$$[B_t^{n+1}]_{ij} \le \sum_{\ell=1}^{N_a}[B_{\text{aux},t}^{n+1}]_{ij\ell},$$
$$\forall n \in \{1,2,\ldots,k-1\}, \forall i,j \in \{1,2,\ldots,N_a\}, \forall t \in \{0,1,\ldots,T\}, \tag{25n}$$

$$\sum_{n=1}^{k}[B_t^n]_{ij} \ge 1, \forall i,j \in \{1,2,\ldots,N_a\}, \forall t \in \{0,1,\ldots,T\}. \tag{25o}$$

Constraints (25a) and (25b) impose the linear dynamics presented in (21) for each agent, whereas (25c) and (25d) impose lower and upper bounds on the states, respectively, and (25e) and (25f) impose lower and upper bounds on controls, respectively. Visitation of the N_t polytopic targets is imposed by (25g) in conjunction with (25h) stating that the maneuver ends within at most T time steps, (25i) ensuring that all targets are visited, and (25j) guaranteeing that all targets are visited before the maneuver ends at T^*. The physical proximity constraints for connectivity in (20) are imposed by (25k) at every $t \in \{0,1,\ldots,T\}$ and relaxed for $t > T^*$. The k-hop connectivity constraints (12), (13), (14), and (19) are rewritten as (25l), (25m), (25n), and (25o) modified to include the time index t. The only necessary modification, in this case, is to define matrices B_t, B_t^{n+1} and $B_{\text{aux},t}^{n+1}$, $\forall n \in \{1,2,\ldots,k-1\}$, which correspond to B, B^{n+1} and B_{aux}^{n+1}, $\forall n \in \{1,2,\ldots,k-1\}$ at a given instant t and repeat the constraints for all $t \in \{0,1,\ldots,T\}$.

4 | SIMULATION EXAMPLES

To validate the proposed formulation, we present two examples. In the first one, we determine the minimal horizon T for feasibility of the optimization problem considering three different values for k. As for the second example, we fix T and observe the evolution of the incumbent solution as the computation time is increased, considering three different values for k.

Five agents ($N_a = 5$) are spatially distributed in $\mathbb{R}^2$. As in Afonso *et al.*[20], the state of each agent $\ell \in \{1,2,\ldots,5\}$ is $\mathbf{x}_\ell = \begin{bmatrix}\mathrm{x}_\ell & \mathrm{v}_{\mathrm{x}\ell} & \mathrm{y}_\ell & \mathrm{v}_{\mathrm{y}\ell}\end{bmatrix}^\top$, where x_ℓ and y_ℓ are the positions in the plane, i.e. $\mathbf{r}_\ell = \begin{bmatrix}\mathrm{x}_\ell & \mathrm{y}_\ell\end{bmatrix}^\top$, whereas $\mathrm{v}_{\mathrm{x}\ell}$ and $\mathrm{v}_{\mathrm{y}\ell}$ are the corresponding velocities. The controls are the accelerations $\mathrm{a}_{\mathrm{x}\ell}$ and $\mathrm{a}_{\mathrm{y}\ell}$ along each axis, thus $\mathbf{u}_\ell = \begin{bmatrix}\mathrm{a}_{\mathrm{x}\ell} & \mathrm{a}_{\mathrm{y}\ell}\end{bmatrix}^\top$. The dynamics representing a double integrator in each direction in discrete time with a sample time of one time unit is

$$\mathbf{x}_{\ell,t+1} = \begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\mathbf{x}_{\ell,t} + \begin{bmatrix}\frac{1}{2}&0\\1&0\\0&\frac{1}{2}\\0&1\end{bmatrix}\mathbf{u}_{\ell,t}, t \in \mathbb{N}\cup\{0\}. \tag{26}$$

The initial conditions and constraints are respectively described in Tables 1 and 2 .

The task is to visit two target sets at some $t \le T$, where T is the maximal time while keeping the network k-hop connected for some chosen k and enforcing the dynamical constraints in Table 1 . The targets are two squares ($N_t = 2$) described by $\mathcal{T}_1 = \{\mathbf{r} \in \mathbb{R}^2 : 1.00 \le \mathrm{x} \le 1.02,\ 0.95 \le \mathrm{y} \le 0.97\}$ and $\mathcal{T}_2 = \{\mathbf{r} \in \mathbb{R}^2 : 0.80 \le \mathrm{x} \le 0.82,\ 0.05 \le \mathrm{y} \le 0.07\}$.

The communication range is the same among all agents, represented by a regular polygon $\mathcal{P}$ with 20 vertices inscribed in a circle with radius $r_{\text{comm}} = 0.11$. The weight of the fuel in the cost function is $\gamma = 10$. For comparison, $k = 2,\ 3,\ 4$ is adopted in

TABLE 1 Initial states of each agent.

Agent ℓ	1	2	3	4	5
$\mathbf{x}_{\ell,0}$	$\begin{bmatrix}0.10\\0\\0.35\\0\end{bmatrix}$	$\begin{bmatrix}0.15\\0\\0.30\\0\end{bmatrix}$	$\begin{bmatrix}0.10\\0\\0.25\\0\end{bmatrix}$	$\begin{bmatrix}0.15\\0\\0.20\\0\end{bmatrix}$	$\begin{bmatrix}0.10\\0\\0.15\\0\end{bmatrix}$

TABLE 2 State and control constraints in common for all agents.

Variable	min	max
$\mathbf{x}$	$\begin{bmatrix}-2\\-1\\-2\\-1\end{bmatrix}$	$\begin{bmatrix}2\\1\\2\\1\end{bmatrix}$
$\mathbf{u}$	$\begin{bmatrix}-0.1\\-0.1\end{bmatrix}$	$\begin{bmatrix}0.1\\0.1\end{bmatrix}$

each scenario, and the initial configuration is fully connected. The computation time for the solutions in seconds is denoted by t_{comp}.

Exploration feasibility and cost efficiency examples are reported in the following subsections. We define k-critical instants as time steps in which all available k hops are necessary to maintain connectivity, i.e., the network is not $(k-1)$-hop connected. The examples were implemented in Matlab 2022 with YALMIP 2021[24] and GUROBI 9.5[25], and run on an Intel Core i7 @ 4.0GHz processor.

4.1 | Results

In the first scenario, we varied T until we obtained the minimum necessary for a feasible solution. The 2-hop and 3-hop cases required 7 time steps to attain feasibility, while with 4-hops a feasible solution exists with 6 time steps. One can conclude that there exists a trade-off between the duration of the maneuver and the number of hops. A maneuver may be made feasible within a shorter time T by allowing a larger number of hops.

To evaluate the effect of varying k in the performance considering the cost, in the second scenario, we adopted $T = 8$ for varying $k \in \{2, 3, 4\}$, ensuring feasibility. Figure 1 showcases the obtained solutions with $t_{\text{comp}} = 10$ s and $t_{\text{comp}} = 100$ s of optimization time. All cases used the total available time steps $T = 8$. We obtained progressively better total and control costs as k increased because the formation could be better spatially distributed, requiring a less energetically aggressive maneuver to visit the targets as a group. Regarding the cost, from Fig. 1, we can see that in $t_{\text{comp}} = 10$ s, the more hops are allowed, the smaller the total cost. The trend was kept with $t_{\text{comp}} = 100$ s, with a reduction in the costs for every $k \in \{2, 3, 4\}$. It is worth noting that the trajectories obtained with $t_{\text{comp}} = 10$ and 100 s are qualitatively similar and, in terms of the cost reduction, with $t_{\text{comp}} = 10$ s high-quality solutions are already obtained. Therefore the refinement as t_{comp} is increased may be exchanged for a reduction in computation time, as a means to cope with limited time budgets in applications. In particular, for $k = 4$, the trajectories obtained are visually and quantitatively improved compared to $k = 2, 3$, pointing that, even though larger k increases the number of constraints and binary variables, it is not necessarily detrimental considering the best solution achieved within limited computation time.

5 | CONCLUSION

In this note, we presented a graph-theoretic MAS topological modeling approach, from which we derived connectivity constraints that ensure k-hop connectivity of the network at all time instants. The selected k value asserts the maximum allowed diameter that the time-varying communication graph may assume.

The k-hop connectivity property is encoded by binarizing the powers of the adjacency matrix and converting the necessary and sufficient conditions into equivalent linear constraints. By additionally enforcing a linear edge generation restriction based on the distance between pairs of agents, the network is guaranteed to be k-hop connected.

A MILP trajectory and control planner framework is constructed from the developed constraint set, including target exploration constraints, and a cost function encompassing fuel and time minimization. The simulation examples showcase the potential advantages of increasing the k value. There are benefits in exploration feasibility and fuel efficiency. However, in application scenarios, larger k may render the network connectivity frail and prone to loss of communication. Therefore, the proposal in this note enables the selection of appropriate k values following the mission requirements.

Possible directions for further research include (i) encompassing topological qualities in the cost function J, allowing the robustness of the communication network to be better represented within the cost, and (ii) studying the correlation between graph diameter, delay, and packet loss in communication.

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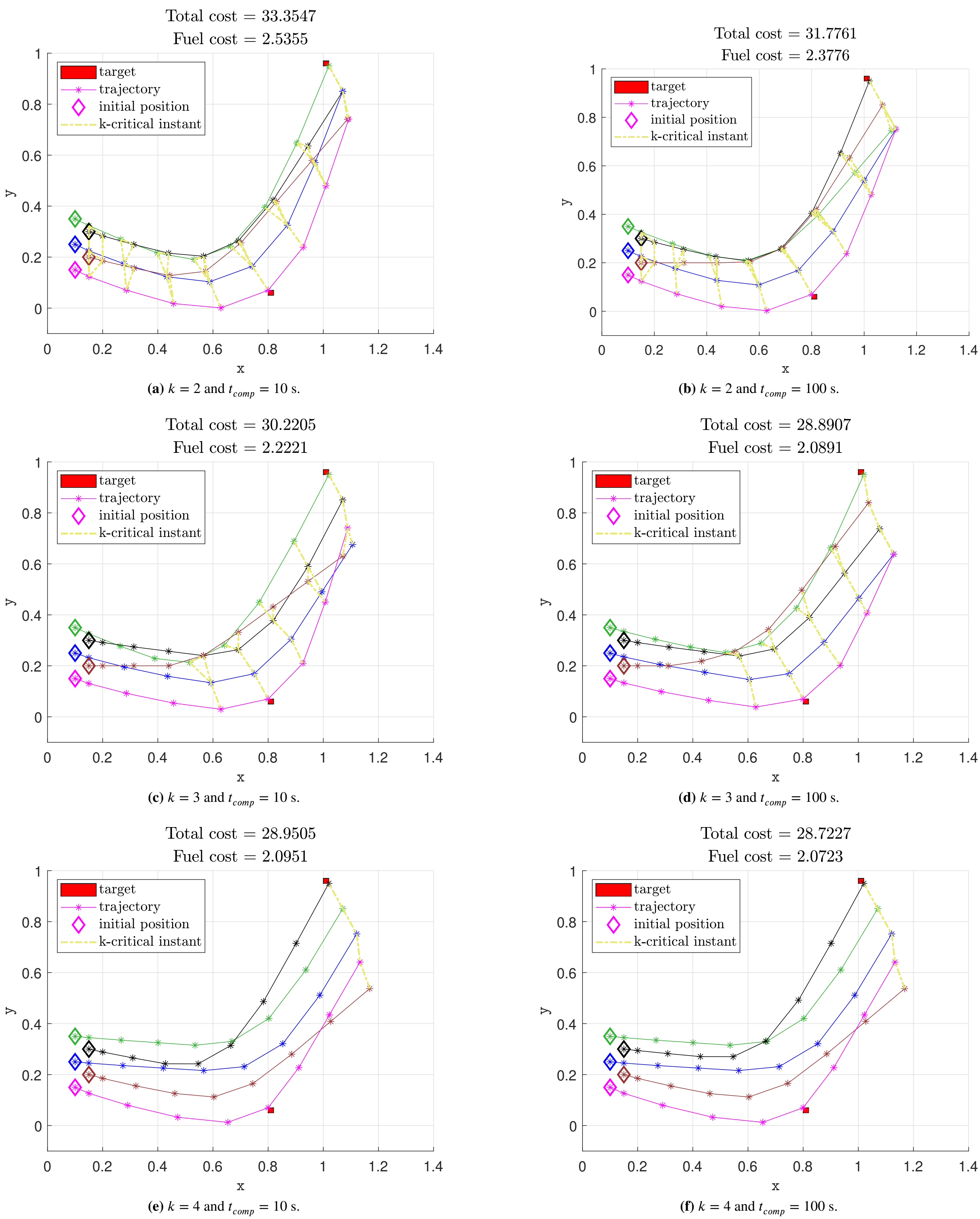

(a) $k = 2$ and $t_{comp} = 10$ s.

(b) $k = 2$ and $t_{comp} = 100$ s.

(c) $k = 3$ and $t_{comp} = 10$ s.

(d) $k = 3$ and $t_{comp} = 100$ s.

(e) $k = 4$ and $t_{comp} = 10$ s.

(f) $k = 4$ and $t_{comp} = 100$ s.

FIGURE 1 Trajectories and costs with $T = 8$ for $k \in \{2, 3, 4\}$ and $t_{comp} \in \{10, 100\}$ s. The paths of each agent are represented by different colors.