WELL-POSEDNESS OF REYNOLDS AVERAGED EQUATIONS FOR COMPRESSIBLE FLUIDS WITH A VANISHING PRESSURE

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Abstract

We show that the Reynolds averaged equations for compressible fluids (neglecting third order correlations) are well-posed in H s when the pressure vanishes in dimensions d=2 and 3. In order to do this, we show that the system is Friedrichs-symmetrizable. This model belongs to the class of non-conservative hyperbolic systems. Hence the usual symmetrisation method for conservation laws can not be used here.

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1. Introduction and main results

We study the Reynolds averaged equations for compressible fluids, where third-order correlations are neglected. This system can be written in Eulerian coordinates as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & \text{(1.1a)} \\ \partial_t v + (v \cdot \nabla v) + \dfrac{1}{\rho}\left(\nabla p + \operatorname{div}(\rho P)^T\right) = 0, & \text{(1.1b)} \\ \partial_t P + (v \cdot \nabla) P + \dfrac{\partial v}{\partial x} P + P \left(\dfrac{\partial v}{\partial x}\right)^T = 0. & \text{(1.1c)} \end{cases}$$

The variables are the averaged density $\rho > 0$, the averaged velocity $v \in \mathbb{R}^d$ ($d = 2$ or 3), and P is the Reynolds stress tensor, $P \in S_d^{++}(\mathbb{R})$. The function p is the pressure of the fluid and is a function of the density ρ, through an equation of state $p = p(\rho)$. The map $\rho \mapsto p(\rho)$ is supposed to be of class C^1 and non-decreasing. Typical pressure laws are of the form $p(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma > 0$ two constants.

The tensor P is a classical Reynolds tensor appearing in the Reynolds averaging of turbulent flows for barotropic compressible fluids (see [14], [18], [12]). It also appears in the description of free surface shear flows, where the averaging operator is the depth averaging (cf. Annexe C, A7 in [15] for a derivation of the model). In that latter case, the density ρ must be replaced by the water depth, often denoted h. The pressure is then given by $p(h) = gh^2/2$ (cf. [15] for instance).

System (1.1) is hyperbolic whenever $p'(\rho) \geq 0$ and P is definite positive, as it has been proved in [15]. Equation (1.1a) shows that the density ρ is conserved. The conservation of momentum ρv also holds: (1.1b) rewrites as

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + pI_d + \rho P)^T = 0. \tag{1.2}$$

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One can also deduce from (1.1) the conservation of energy:

$$\partial_t e + \operatorname{div}(ev + pv + \rho P v) = 0, \text{ with } e := \frac{1}{2}\rho|v|^2 + \rho\mathcal{E}(\rho) + \frac{1}{2}\operatorname{Tr}(\rho P). \tag{1.3}$$

The map $\rho \mapsto \rho\mathcal{E}(\rho)$ is called the volumic internal energy, and is linked to the pressure via the relation $p(\rho) = \rho^2\mathcal{E}'(\rho)$. The term $\rho|v|^2/2$ is the volumic kinetic energy of the fluid, and the term $\operatorname{Tr}(\rho P)/2$ is the energy associated to the tensor P.

In [6], it was shown that system (1.1) admits a variational formulation, as it is often the case in physics when the energy of a system is conserved. Define the Lagrangian density

$$\mathcal{L}(\rho, v, P) := \frac{1}{2}\rho|v|^2 - \rho\mathcal{E}(\rho) - \frac{1}{2}\operatorname{Tr}(\rho P), \tag{1.4}$$

and the corresponding action

$$\mathcal{A} := \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \mathcal{L}\mathrm{d}x\mathrm{d}t. \tag{1.5}$$

Then one can show that (1.1b) is the Euler-Lagrange equation given by the stationnary action principle applied to the action (1.5), under the two constraints (1.1a) and (1.1c).

The tensor P admits an additional conservation law, sometimes called conservation of enstrophy, that is a consequence of (1.1c) and can be written

$$\partial_t\left(\frac{\det P}{\rho^2}\right) + v\cdot\nabla\left(\frac{\det P}{\rho^2}\right) = 0. \tag{1.6}$$

Note that equation (1.1c) implies that the symmetry of P is conserved by the evolution. Equation (1.6) then implies that, if $P(t) \in S_d^{++}(\mathbb{R})$ for some instant t, then this property is true for all times.

It has been proved in [7] that system (1.1) does not admit any further conservation law. Thus, in dimension $d = 2$ or 3, system (1.1) is not conservative. Hence the usual symmetrisation method of Godunov (cf. [8]) and Lax and Friedrichs (cf. [5]) for hyperbolic systems of conservation laws (see for instance [17], pages 83-84) can not be used here.

However, one can show that system (1.1) is Friedrichs-symmetrizable when the pressure vanishes. More precisely, we state the following proposition:

Proposition 1. *Let $d = 2$ or 3. Suppose that ρ takes values in $\mathbb{R}_+^*$ and P takes values in $S_d^{++}(\mathbb{R})$. Then the two following properties are equivalent:*

1. *System (1.1) is Friedrichs-symmetrizable*
2. *The pressure p is constant: $p'(\rho) = 0$, or the tensor P is a scalar matrix, i.e. there exists $\lambda = \lambda(t, x) \in \mathbb{R}$ such that $P = \lambda I_d$.*

Let us make few comments about this proposition:

- The tensor $P = \lambda I_d$ is a solution of system (1.1) only for trivial velocities; hence, for applications of this result, the case $p' = 0$ seems more interesting.
- This property holds in the variables (ρ, v, P). It could be possible that system (1.1), written in different variables, appears to be symmetrizable even when $p' \neq 0$.

- When $d = 1$, system (1.1) is symmetrizable, even when $p' \neq 0$. In fact, one can prove that one-dimensional hyperbolic systems are always Friedrichs-symmetrizable (cf. [13]). In higher dimension $d \geq 2$, this does not hold anymore.

As a consequence of Proposition 1, we have the following result regarding the well-posedness of system (1.1):

Theorem 1. *Let $d = 2$ or 3, $s > 1 + d/2$ and $\mathcal{U} := \mathbb{R}_+^* \times \mathbb{R}^d \times S_d^{++}(\mathbb{R})$. Let $\bar{Y} := (\bar{\rho}, \bar{u}, \bar{P}) \in \mathcal{U}$ and Y_0 taking values in $\mathcal{U}$ such that $Y_0 - \bar{Y} \in H^s(\mathbb{R}^d)$. We consider the Cauchy problem associated to (1.1) with initial data Y_0. There exists $T > 0$ such that (1.1) with $p' = 0$ has a unique classical solution $Y(t)$ in $\mathcal{C}^1([0, T] \times \mathbb{R}^d)$ with values in $\mathcal{U}$ achieving the initial data $Y(0) = Y_0$. Furthermore, $Y - \bar{Y}$ belongs to $\mathcal{C}([0, T], H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^d))$.*

Theorem 1 is a consequence of Proposition 1 and of the theory of well-posedness of Kato for quasilinear evolution equations (cf. [11]). For a detailed proof of this result, see for instance Sect. 10 in [1].

System (1.1) has been used over the last years in the modeling of turbulent flows, including for numerical simulations (see [16], [10], [7], [9]). However, the well-posedness of (1.1) in dimension $d \geq 2$ is still uncertain today. Theorem 1 states an answer to this question in the case of a vanishing pressure. One could also obtain system (1.1) with $p' = 0$ when modeling a fluid for which the pressure gradient ∇p is negligible compared to the turbulence of the fluid $\mathrm{div}(\rho P)$ in (1.1b).

The hypothesis $p' = 0$ can also be found in the litterature, in a model called "pressureless gas dynamics". The pressureless Euler equations are obtained from the Euler equations with a pressure assumed to be 0. The pressureless gas dynamics are used to model astrophysical systems (see [19] for instance) and is only weakly hyperbolic. As a consequence, phenomena like creation of vacuum or high concentrations (delta shocks) can occur (see for instance [3], [4], [2]). System (1.1) with $p' = 0$ can thus be seen as a hyperbolic version of the pressureless gas model. In this sense, the Reynolds tensor P brings more regularity to the model.

2. Proof of the proposition

Notations. We denote $\partial_i := \partial/\partial x_i$ the partial derivative of the variable x_i, for $1 \leq i \leq d$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a scalar function, we denote by $\nabla f \in \mathbb{R}^d$ the gradient of f, i.e. the vector field of components $\partial_i f$, $1 \leq i \leq d$.

If $Z = (Z_1, \dots, Z_n) : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, the divergence of Z is the scalar function defined by

$$\mathrm{div}(Z) := \partial_1 Z_1 + \cdots + \partial_n Z_n.$$

We also denote by $\partial Z/\partial x$ the Jacobian matrix of Z, i.e. the matrix of coefficients $(\partial Z/\partial x)_{i,j} = \partial Z_i/\partial x_j$, for $1 \leq i, j \leq d$.

If $Z = (Z_i)_{1 \leq i \leq d}$ and $Z' = (Z'_i)_{1 \leq i \leq d}$ are two vector fields, we denote $Z \otimes Z'$ the second order tensor defined by $Z \otimes Z' := Z(Z')^T$, i.e. the matrix of coefficients $(Z \otimes Z')_{i,j} = Z_i Z'_j$, for $1 \leq i, j \leq d$.

If $A : \mathbb{R}^d \to M_d(\mathbb{R})$ is a second order tensor, we defined the divergence of A as the line vector of $\mathbb{R}^d$ whose i-th component is given by the divergence of the i-th column of A.

For any positive integer d, we denote $I_d \in M_d(\mathbb{R})$ the identity matrix of size d. We denote $S_d^{++}(\mathbb{R})$ the set of symmetric definite positive matrices, i.e. the symmetric matrices of size d with a positive spectrum.

We now give the proof of Proposition 1.

Proof. We write system (1.1) in matricial form:

$$\partial_t Y + A(Y,\nabla)Y = 0, \text{ with } Y := \begin{pmatrix} \rho \\ v \\ \tilde{P} \end{pmatrix} \in \mathbb{R}^{1+d+d(d+1)/2}$$

and, if $\xi = (\xi_i)_{1\le i\le d} \in \mathbb{R}^d$,

$$A(Y,\xi) := \begin{pmatrix} v\cdot\xi & \rho\xi^T & 0 \\ \frac{1}{\rho}(p'(\rho)I_d + P)\xi & (v\cdot\xi)I_d & C(\xi) \\ 0 & D(\xi) & (v\cdot\xi)I_{d(d+1)/2} \end{pmatrix}. \tag{2.1}$$

When $d=2$, the symmetric matrix $P = (P_{ij})_{1\le i,j\le 2}$ can be identified as a vector $\tilde{P} \in \mathbb{R}^3$ given by

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \text{ and } D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When $d=3$, the symmetric matrix P can be identified as a vector $\tilde{P} \in \mathbb{R}^6$:

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}$$

and

$$D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where $(P\xi)_i$ denotes the i-th component of the vector $P\xi$.

We first show the implication (2) $\Rightarrow$ (1)**.** Namely, if $p' = 0$ or $P = \lambda I_d$, then system (1.1) is Friedrichs-symmetrizable.

We thus suppose that $p' = 0$ or $P = \lambda I_d$.

Consider $S = S(Y) \in S_n^{++}(\mathbb{R})$ (with $n = (d+1)(d+2)/2$) defined as a block matrix, compatible with A:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}, \text{ with } S_2 \in M_d(\mathbb{R}) \text{ and } S_3 \in M_{d(d+1)/2}(\mathbb{R}). \tag{2.2}$$

Note that S is symmetric definite positive if and only if the matrices S_2 and S_3 are also symmetric definite positive.

We can now compute the product SA by block matrix mutiplication. We obtain that

$$SA = \begin{pmatrix} v\cdot\xi & \rho\xi^T & 0 \\ \frac{1}{\rho}S_2(p'(\rho)Id + P)\xi & (v\cdot\xi)S_2 & S_2C(\xi) \\ 0 & S_3D(\xi) & (v\cdot\xi)S_3 \end{pmatrix} \tag{2.3}$$

Case $d = 2$. Let us choose

$$S_2 := \mu P^{-1} \text{ and } S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{12}q_{11} & \frac{1}{2}q_{12}^2 \\ q_{12}q_{11} & q_{11}q_{22} + q_{12}^2 & q_{12}q_{22} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & \frac{1}{2}q_{22}^2 \end{pmatrix},$$

where we denoted

$$P^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in S_2^{++}(\mathbb{R}), \tag{2.4}$$

and

$$\mu := \begin{cases} \rho^2 & \text{when } p'(\rho) = 0, \\ \rho^2 \dfrac{\lambda}{p'(\rho)+\lambda} & \text{when } P = \lambda Id. \end{cases} \tag{2.5}$$

The matrix S_2 is symmetric definite positive.

We see that the principal minors of $\mu^{-1}S_3$ are given by $M_1 = \frac{1}{2}q_{11}^2 > 0$, $M_2 = \frac{1}{2}q_{11}^2(q_{11}q_{22} - q_{12}^2) > 0$, and $M_3 = \frac{1}{4}(q_{11}q_{22} - q_{12}^2)^3 > 0$ (recall that P is positive definite). Hence by Sylvester's criterion, S_3 is definite positive, and S, defined by (2.2), is a positive definite matrix.

We compute that

$$\frac{1}{\rho}S_2\left(p'(\rho)Id + P\right)\xi = \rho\xi, \tag{2.6}$$

$$S_2C(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{11}\xi_2 + q_{12}\xi_1 & q_{12}\xi_2 \\ q_{12}\xi_1 & q_{12}\xi_2 + q_{22}\xi_1 & q_{22}\xi_2 \end{pmatrix},$$

and, since $q_{i1}P_{1j} + q_{i2}P_{2j} = \delta_{ij}$ by (2.4),

$$S_3D(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{12}\xi_1 \\ q_{12}\xi_1 + q_{11}\xi_2 & q_{22}\xi_1 + q_{12}\xi_2 \\ q_{12}\xi_2 & q_{22}\xi_2 \end{pmatrix} = [S_2C(\xi)]^T.$$

Hence for any $Y \in \mathcal{U}$, and for any $\xi \in \mathbb{R}^2$, the matrix $S(Y)$ is symmetric definite positive and (2.3) shows that the matrix $S(Y)A(Y,\xi)$ is symmetric. As a consequence, (1.1) is Friedrichs-symmetrizable when $d = 2$.

Case $d = 3$. Define again S by equation (2.2). Choose $S_2 = \mu P^{-1}$ with μ as in (2.5), such that S_2 is symmetric definite positive and (2.6) holds again. Define S_3 by

$$S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{11}q_{12} & q_{11}q_{13} & \frac{1}{2}q_{12}^2 & q_{12}q_{13} & \frac{1}{2}q_{13}^2 \\ q_{11}q_{12} & q_{11}q_{22}+q_{12}^2 & q_{11}q_{23}+q_{12}q_{13} & q_{12}q_{22} & q_{12}q_{23}+q_{22}q_{13} & q_{13}q_{23} \\ q_{11}q_{13} & q_{11}q_{23}+q_{13}q_{12} & q_{11}q_{33}+q_{13}^2 & q_{12}q_{23} & q_{12}q_{33}+q_{23}q_{13} & q_{13}q_{33} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & q_{12}q_{23} & \frac{1}{2}q_{22}^2 & q_{22}q_{23} & \frac{1}{2}q_{23}^2 \\ q_{12}q_{13} & q_{12}q_{23}+q_{13}q_{22} & q_{12}q_{33}+q_{13}q_{23} & q_{22}q_{23} & q_{22}q_{33}+q_{23}^2 & q_{23}q_{33} \\ \frac{1}{2}q_{13}^2 & q_{13}q_{23} & q_{13}q_{33} & \frac{1}{2}q_{23}^2 & q_{23}q_{33} & \frac{1}{2}q_{33}^2 \end{pmatrix},$$

where we denoted again

$$P^{-1} := \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}.$$

We see that S_3 is symmetric. Furthermore, the principal minors of $\mu^{-1}S_3$ are given by $M_1 = q_{11}^2/2 > 0$, $M_2 = q_{11}^2(q_{11}q_{22} - q_{12}^2)/2 > 0$, $M_3 = q_{11}^3 \det(P^{-1})/2 > 0$, $M_4 = q_{11}(q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})/4 > 0$, $M_5 = (q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})^2/4 > 0$ and $M_6 = \det(P^{-1})^4/8 > 0$. Hence, by Sylvester's criterion, S_3 is definite positive and S is symmetric definite positive.

We also check that

$$S_3 D(\xi) = [S_2 C(\xi)]^T.$$

Hence equation (2.3) shows that SA is symmetric, and, consequently, (1.1) is Friedrichs-symmetrizable when $d = 3$.

We now show that (1) $\Rightarrow$ (2)**.** Suppose that system (1.1) is Friedrichs-symmetrizable, i.e. there is a matrix S such that the product SA is symmetric.

We write S as a block matrix, compatible with A:

$$S := \begin{pmatrix} S_1 & \alpha & \beta \\ \alpha^T & S_2 & \gamma \\ \beta^T & \gamma^T & S_3 \end{pmatrix}.$$

We compute the product SA by block multiplication. Since SA is symmetric, for any block of SA, denoted $(SA)_{i,j}$ $(1 \le i, j \le 3)$, we must have $(SA)_{i,j} = (SA)_{j,i}^T$. For $i = j = 3$, we obtain the constraint

$$\gamma^T C(\xi) + (v \cdot \xi) S_3 = \left[\gamma^T C(\xi) + (v \cdot \xi) S_3\right]^T.$$

Since S_3 is symmetric, we deduce that the product $\gamma^T C(\xi)$ is symmetric, for any $\xi \in \mathbb{R}^d$. By computing explicitely the product, we obtain that the only possibility is that $\gamma = 0$.

For $i = 1$ and $j = 3$, we obtain the constraint

$$[\alpha C(\xi) + (v \cdot \xi)\beta]^T = (v \cdot \xi)\beta^T + \frac{1}{\rho}\gamma^T (p'(\rho) Id + P)\xi.$$

Since $\gamma = 0$, we obtain that $\alpha C(\xi) = 0$, for any $\xi \in \mathbb{R}^d$. By computing explicitely the product, we also obtain that $\alpha = 0$. Hence S has to be of the form

$$S = \begin{pmatrix} S_1 & 0 & \beta \\ 0 & S_2 & 0 \\ \beta^T & 0 & S_3 \end{pmatrix}.$$

For $i = 2$ and $j = 3$, we obtain the constraint

$$\rho\beta^T\xi^T + S_3 D(\xi) = [S_2 C(\xi)]^T \tag{2.7}$$

Solving the linear system (2.7) for S_2, S_3 and β gives after some computations that there exists two constants λ_1, λ_2 such that

$$S_2 = \lambda_1 P^{-1} \text{ and } \beta = \begin{cases} \lambda_2(q_{11}, 2q_{12}, q_{22}) & \text{when } d = 2, \\ \lambda_2(q_{11}, 2q_{12}, 2q_{13}, q_{22}, 2q_{23}, q_{33}) & \text{when } d = 3. \end{cases} \tag{2.8}$$

Note that it follows from these computations that $\beta D(\xi) = 2\lambda_2\xi^T$.

For $i = 1$ and $j = 2$ we obtain the constraint

$$\frac{1}{\rho} S_2(p'(\rho)Id + P)\xi = \left[S_1\rho\xi^T + \beta D(\xi)\right]^T = (S_1\rho + 2\lambda_2)\xi. \tag{2.9}$$

Equation (2.9) implies that S_2 is proportional to $(p'(\rho)Id + P)^{-1}$. Since S_2 is also proportional to P^{-1} by (2.8), we obtain that the two matrices P and $P + p'(\rho)Id$ are proportional (recall that S_2 is invertible). Hence $p' = 0$ or $P = \lambda Id$.

Declarations

- Conflict Of Interest statement: there is no competing interest.
- Data availability statement: there is no associated data.

$\square$

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