Asymptotic behavior of Navier-Stokes-Voigt equations in a thin domain with damping term and Tresca friction law

Mokhtar Kirane[1], Mohamed Dilmi[2], Mourad Dilmi[3], and Hamid Benseridi[3]

[1]Khalifa University
[2]Universite Saad Dahlab Blida Faculte des Sciences
[3]Universite Ferhat Abbas Setif 1 Faculte des Sciences

October 3, 2023

Abstract

In this work, we consider a mathematical model of viscoelastic incompressible fluid governed by the Navier-Stokes-Voigt equations in a three dimensional thin domain Ω ε , with damping term and Tresca friction law. First, we give the problem statement and the weak variational formulation of the considered problem. Then we study the asymptotic analysis of the problem when a dimension of the domain tends to zero. The limit problem and the specific equation of Reynolds are obtained.

Received <day> <Month>, <year>; Revised <day> <Month>, <year>; Accepted <day> <Month>, <year>

DOI: xxx/xxxx

ARTICLE TYPE

Asymptotic behavior of Navier-Stokes-Voigt equations in a thin domain with damping term and Tresca friction law

Mohamed Dilmi*[1] | Mokhtar Kirane[2] | Mourad Dilmi[3] | Amid Benseridi[3]

[1]Department of Mathematics, Faculty of Science, University of Saad Dahlab – Blida 1. Algeria.
Email: mohamed77dilmi@gmail.com

[2]Department of Mathematics, Faculty of Arts and Science, Khalifa University, P.O. Box: 127788, Abu Dhabi, UAE
Email: mokhtar.kirane@ku.ac.ae

[3]Applied Mathematics Laboratory, Department of Mathematics, Setif 1 University, 19000, Algeria
Email: mouraddil@yahoo.fr
Email: m_benseridi@yahoo.fr

Abstract

In this work, we consider a mathematical model of viscoelastic incompressible fluid governed by the Navier-Stokes-Voigt equations in a three dimensional thin domain Ω^ϵ, with damping term and Tresca friction law. First, we give the problem statement and the weak variational formulation of the considered problem. Then we study the asymptotic analysis of the problem when a dimension of the domain tends to zero. The limit problem and the specific equation of Reynolds are obtained.

KEYWORDS:

Navier-Stokes-Voigt equations, Reynolds equation, Variational formulation, Tresca friction law.

1 | INTRODUCTION

Fluid flow problems arise in several physical phenomena and play an important role in many industrial applications. Navier-Stokes equations are well known to be effectively useful in modeling turbulence in fluid phenomena, which has been intensively studied.

The Navier-Stokes-Voigt (or Navier-Stokes-Voight) model, is a modification of the Navier-Stokes equation by adding a pseudoparabolic regularization $-\nu\Delta\frac{\partial u}{\partial t}$ for the velocity field u. This model was introduced by Oskolkov[21]. Recently proposed as a regularization of the 3D-Navier-Stokes equation for the purpose of direct numerical simulations in[6]. The presence of the regularizing term $-\nu\Delta\frac{\partial u}{\partial t}$ is of great importance, because it changes the parabolic character of the limit Navier-Stokes equations, into a damped hyperbolic system. This model has worked well in many applications where it has recently been used in image processing (see[12]). In fact, the Navier-Stokes-Voigt system is perhaps the newest model in the the so-called α-models in fluid mechanics[13].

In recent years, this model has attracted the attention of many mathematicians. In[18] Levant et al, studied the statistical properties of the three-dimensional Navier-Stokes-Voigt system. Anh and Thanh[1] proved the existence and uniqueness of solutions for the Navier-Stokes-Voigt model, then they examined the mean square exponential stability and the almost sure exponential stability of the stationary solutions.

Many interesting results on the existence and long-time behavior of solutions in terms of existence of attractors to Navier-Stokes-Voigt equations can be found in ([3],[7],[15],[23]).

Furthermore, the existence and time decay rates of solutions for Navier-Stokes-Voigt model have been studied in ([2],[24]). Layton and Rebholz[16] studied analytically and numerically the relaxation time of flow evolution governed by the Navier-Stokes-Voigt model.

It is known that partial differential equations in thin domains appear in many applications such as (thin elastic bodies, thin rods, plates or shells) for solid mechanics, and (lubrication, meteorology, ocean dynamics) for fluid mechanics. When the study of the three-dimensional Navier-Stokes equations in thin domains, Raugel, and Sell[14], investigated the Navier-Stokes equations in a flat thin domain with a nonflat top boundary. Temam and Ziane[22], looked at the Navier-Stokes equations with free boundary conditions in a thin spherical shell, to give a mathematical justification for the derivation of the primitive equations for the atmosphere and ocean dynamics. In the work[20], the author formally derived limit equations of the Navier-Stokes equations in a thin tubular neighborhood of an evolving closed surface.

The study of asymptotic behavior of Stokes equations in a thin domain with Tresca friction law, is performed in ([4],[8],[17]). The reader is referred to some works related to the study of elastic and viscosity materials in a thin domain with Tresca friction law ([5],[9],[10]).

Motivated by lubrication theory, where the domain of flow is usually very thin, we examine in this paper the asymptotic behavior of the solution of the boundary value problem of the Navier-Stokes-Voigt model, with damping term and Tresca friction law in a thin domain Ω^{ε}. Our ai is to know the behavior of the solution to this problem when one dimension of the domain tends to zero. Besides, we will derive the Reynolds equation which allows to determine the operating characteristics of the system such as load capacity, dissipative power, friction number, etc.

This article is organized as follows. Section 2 is devoted to the mechanical problem governed by the Navier-Stokes-Voigt model. In section 3, we derive the variational formulation of the problem. In section 4, we use the change of variable $z = x_3/\varepsilon$ to transform the initial problem posed in the domain Ω^{ε}, to a new problem posed on a fixed domain Ω independent of parameter ε. Then we establish some a priori estimates for the velocity and pressure fields, independently of ε. Finally, in Section 5, we derive and study the limit problem when ε tends to zero. Moreover we show that the limit pressure is given as the unique solution of a Reynolds equation.

2 | SETTING OF THE PROBLEM

Let Ω^{ε} be a bounded domain of $\mathbb{R}^3$ and $\bar{\Gamma}^{\varepsilon} = \overline{\Gamma}_1^{\varepsilon} \cup \overline{\Gamma}_L^{\varepsilon} \cup \overline{\omega}$, the boundary of Ω^{ε}, such that Γ_1^{ε} is the upper surface of equation $x_3 = \varepsilon h(x_1, x_2)$, Γ_L^{ε} is the lateral surface and ω is a bounded domain of $\mathbb{R}^2$ of equation $x_3 = 0$ which constitutes the bottom of the domain Ω^{ε}. We suppose that $h(.)$ is a function of class C^1 defined on ω such that

$$0 < \underline{h} = h_{min} \leq h(x') \leq h_{max} = \overline{h}, \forall (x', 0) \in \omega.$$

More precisely the domain of the flow given by

$$\Omega^{\varepsilon} = \{(x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \omega, 0 < x_3 < \varepsilon h(x')\}.$$

In this thin domain, we study the asymptotic behavior of the following mechanical problem:

Find the velocity field $u^{\varepsilon} : \Omega^{\varepsilon} \times]0, T[\to \mathbb{R}^3$ and the pressure $\pi^{\varepsilon} : \Omega^{\varepsilon} \times]0, T[\to \mathbb{R}$, such that

$$\frac{\partial u^{\varepsilon}}{\partial t} - \operatorname{div}(\sigma^{\varepsilon}(u^{\varepsilon}, \pi^{\varepsilon})) + \sqrt{\varepsilon}\,(u^{\varepsilon}.\nabla u^{\varepsilon}) + \alpha^{\varepsilon}(1 + |u^{\varepsilon}|)\,u^{\varepsilon} = f^{\varepsilon} \text{ in } \Omega^{\varepsilon} \times]0, T[, \tag{1}$$

$$\sigma^{\varepsilon}(u^{\varepsilon}, \pi^{\varepsilon}) = \mu \nabla u^{\varepsilon} + \nu \nabla\left(\frac{\partial u^{\varepsilon}}{\partial t}\right) - \pi^{\varepsilon} I_3 \text{ in } \Omega^{\varepsilon} \times]0, T[, \tag{2}$$

$$\operatorname{div}(u^{\varepsilon}) = 0 \text{ in } \Omega^{\varepsilon} \times]0, T[, \tag{3}$$

$$u^{\varepsilon} = 0 \text{ on } \Gamma_1^{\varepsilon} \times]0, T[, \tag{4}$$

$$u^{\varepsilon} = 0 \text{ on } \Gamma_L^{\varepsilon} \times]0, T[, \tag{5}$$

$$u^{\varepsilon}.n = 0 \text{ on } \omega \times]0, T[, \tag{6}$$

$$\left.\begin{array}{l} \left|\sigma_{\tau}^{\varepsilon}\right| < k^{\varepsilon} \Rightarrow u_{\tau}^{\varepsilon} = 0, \\ \left|\sigma_{\tau}^{\varepsilon}\right| = k^{\varepsilon} \Rightarrow \exists \beta > 0, \text{ such that } u_{\tau}^{\varepsilon} = -\beta \sigma_{\tau}^{\varepsilon}, \end{array}\right\} \text{ on } \omega \times]0, T[; \tag{7}$$

where (1) represents the equation of the equilibrium of the fluid, f^{ε} is a given external force and α^{ε} the damping coefficient. The relation (2) represents the law of behavior of the fluid for the Navier-Stokes-Voigt model, such that $\mu > 0$ is the kinematic viscosity coefficient and $\nu > 0$ is the length-scale parameter characterizing the elasticity of the fluid. The equation (3) describes the condition of incompressibility of the fluid. The conditions $(4) - (6)$ are a non-slip boundary condition on $\Gamma_1^{\varepsilon} \times]0, T[$ and

$\Gamma_L^\varepsilon \times]0,T[$, the condition (6) is the non penetration on $\omega \times]0,T[$, and (7) is the Tresca boundary conditions on $\omega \times]0,T[$, with k^ε the coefficient of friction, $|.|$ is the $\mathbb{R}^2$ Euclidean norm, $n = (n_1, n_2, n_3)$ is the unit outward normal to Γ^ε, and

$$u_n^\varepsilon = u^\varepsilon.n,\;\; u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon.n, \qquad \sigma_n^\varepsilon = \left(\sigma_n^\varepsilon\right).n,\;\; \sigma_\tau^\varepsilon = \sigma^\varepsilon.n - \left(\sigma_n^\varepsilon\right).n,$$

are the normal and the tangential velocity u^ε, and the components of the normal and the tangential stress tensor σ^ε, respectively.

The problem consists in finding u^ε which fulfills $(1)-(7)$, with the following initial condition

$$u^\varepsilon(x,0) = u_0^\varepsilon, \quad \forall x \in \Omega^\varepsilon. \tag{8}$$

Remark 1. On $\omega \times]0,T[$ the third component of the velocity is zero.

Indeed, according to condition (6) we have

$$u^\varepsilon.n = u_1^\varepsilon.n_1 + u_2^\varepsilon.n_2 + u_3^\varepsilon.n_3 = 0 \text{ on } \omega \times]0,T[\,,$$

where $n = (n_1, n_2, n_3) = (0,0,-1)$ is the unit normal to ω. So $u_3^\varepsilon = 0$, on $\omega \times]0,T[$.

3 | NOTATION AND VARIATIONAL FORMULATION

• $L^p(\Omega^\varepsilon)$ is the Lebesgue space with the norm

$$\|u\|_{L^p(\Omega^\varepsilon)} = \left[\int\limits_{\Omega^\varepsilon} |u|^p\,dx\right]^{\frac{1}{p}}, \;\; 1 \le p < \infty.$$

• $H^1(\Omega)^3$ is the Sobolev space

$$H^1(\Omega^\varepsilon)^3 = \left\{u \in L^2(\Omega^\varepsilon)^3 \,:\, \nabla u \in L^2(\Omega^\varepsilon)^{3\times 3}\right\},$$

with inner product and norm

$$\langle u, v\rangle_{H^1(\Omega^\varepsilon)} = \int\limits_{\Omega^\varepsilon} u.v dx + \int\limits_{\Omega^\varepsilon} \nabla u.\nabla v dx, \quad \|u\|_{H^1(\Omega^\varepsilon)} = \left[\|u\|_{L^2(\Omega^\varepsilon)^3} + \|\nabla u\|_{L^2(\Omega^\varepsilon)^{3\times 3}}\right]^{\frac{1}{2}}.$$

• $H_0^1(\Omega^\varepsilon)^3$ is the closure of $D(\Omega^\varepsilon)$ in $H^1(\Omega^\varepsilon)^3$, and $H^{-1}(\Omega^\varepsilon)^3$ is the dual space of $H_0^1(\Omega^\varepsilon)^3$.

Let X be a real Banach space with the norm $\|.\|_X$. We denote by $L^p(0,T;X)$ the standard Banach space of all functions from $(0,T)$ to X , endowed with the norm

$$\|u\|_{L^p(0,T;X)} = \left[\int\limits_0^T \|u(s)\|_X^p\,ds\right]^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

Lemma 1 ([11]). The condition of Tresca (7) is equivalent to

$$\sigma_\tau^\varepsilon.u_\tau^\varepsilon + k^\varepsilon\left|u_\tau^\varepsilon\right| = 0 \;\text{ on }\; \omega \times]0,T[\,.$$

Let u^ε be the solution of $(1)-(8)$; Multiplying the equation (1) by $(\varphi - u^\varepsilon)$ and integrating the resulting equation over Ω^ε, then using the Green's formula and the lemma above, we obtain the following variational problem:

$$\begin{gathered}\text{Find } (u^\varepsilon, \pi^\varepsilon) \in L^2\left(0,T,V_{\text{div}}^\varepsilon\right) \times L^2\left(0,T,L_0^2(\Omega^\varepsilon)\right), \text{ such that}\\ \left(\frac{\partial u^\varepsilon}{\partial t}, \varphi - u^\varepsilon\right) + a_\mu(u^\varepsilon, \varphi - u^\varepsilon) + a_\nu\left(\frac{\partial u^\varepsilon}{\partial t}, \varphi - u^\varepsilon\right) + \sqrt{\varepsilon} b(u^\varepsilon, u^\varepsilon, \varphi - u^\varepsilon)\\ +\alpha^\varepsilon((1+|u^\varepsilon|)u^\varepsilon, \varphi - u^\varepsilon) - (\pi^\varepsilon, \text{div}(\varphi)) + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon)\\ \ge (f^\varepsilon, \varphi - u^\varepsilon), \forall \varphi \in V^\varepsilon,\\ u^\varepsilon(x,0) = u_0^\varepsilon, \quad \forall x \in \Omega^\varepsilon;\end{gathered} \tag{9}$$

with

$$V^\varepsilon = \left\{v \in H^1(\Omega^\varepsilon)^3 \,:\, v = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon,\, v.n = 0 \text{ on } \omega\right\},$$

$$V_{\text{div}}^\varepsilon = \{v \in V^\varepsilon \,:\, \text{div}(v) = 0\},$$

$$L_0^2(\Omega^\varepsilon) = \left\{ \varphi \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} \varphi dx = 0 \right\},$$

$$a_\kappa(u,v) = \kappa \int_{\Omega^\varepsilon} \nabla u.\nabla v dx, \; j^\varepsilon(v) = \int_\omega k^\varepsilon |v| \, dx', \; (f,v) = \int_{\Omega^\varepsilon} f.v dx,$$

$$b(u,v,w) = \int_{\Omega^\varepsilon} u.\nabla v.w dx = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} u_i \frac{\partial v_i}{\partial x_j} w_j dx.$$

The bilinear form $a_\kappa(.,.)$ is continuous and coercive on $V_{\text{div}}^\varepsilon \times V_{\text{div}}^\varepsilon$ and $j^\varepsilon(.)$ is a convex and continuous function on $V_{\text{div}}^\varepsilon$. The trilinear form $b(.,.,.)$ satisfies

$$b(u,v,v) = 0, \forall u, v \in V_{\text{div}}^\varepsilon,$$
$$|b(u,v,w)| \le \|u\|_{L^3(\Omega^\varepsilon)^3} \|\nabla v\|_{L^2(\Omega^\varepsilon)^3} \|w\|_{L^6(\Omega^\varepsilon)^3}, \forall u, v, w \in V_{\text{div}}^\varepsilon.$$

Theorem 1. Under the assumptions

$$\begin{aligned} &f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t} \in L^2\left(0,T,L^2(\Omega^\varepsilon)^3\right), \\ &u_0 \in H^1(\Omega^\varepsilon)^3, \; \left(u_0\right)_\tau = 0, \\ &k^\varepsilon \in L^\infty(\omega), k^\varepsilon > 0, \end{aligned} \tag{10}$$

the problem (9) admits a unique solution $u^\varepsilon \in L^2\left(0,T,V_{\text{div}}^\varepsilon\right)$, such that

$$\frac{\partial u^\varepsilon}{\partial t} \in L^2\left(0,T,V_{\text{div}}^\varepsilon\right).$$

Proof. The proof is based on the regularization method, which is based on an approximation of nondifferentiable term $j(.)$ by a family of differentiable once $j_\zeta(.)$, where

$$j_\zeta(v) = \int_\omega k^\varepsilon\left(x'\right)\phi_\zeta\left(|v|^2\right)dx', \text{ with } \phi_\zeta(v) = \frac{1}{1+\zeta}|v|^{1+\zeta}, \zeta > 0,$$

and we build the approximate problem

$$\begin{aligned} &\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \varphi\right) + a_\mu\left(u_\zeta^\varepsilon, \varphi\right) + a_\nu\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \varphi\right) + \sqrt{\varepsilon} b\left(u_\zeta^\varepsilon, u_\zeta^\varepsilon, \varphi\right) + \alpha^\varepsilon\left(\left(1 + \left|u_\zeta^\varepsilon\right|\right)u_\zeta^\varepsilon, \varphi\right) + \left(\left(j_\zeta^\varepsilon\right)'(u^\varepsilon), \varphi\right) \\ &= (f^\varepsilon, \varphi), \quad \forall \varphi \in V_{\text{div}}^\varepsilon. \end{aligned} \tag{11}$$

Using Galerkin's method, we show that there exists a unique solution u_ζ^ε of (11) (see the works of Duvaut and Lions ([11],[19]). Then, the limit of u_ζ^ε when ζ tends to zero is a solution of (9). □

4 | A PRIORI ESTIMATES

For the asymptotic analysis of the studied problem, we introduce the change of variable $z = x_3/\varepsilon$ to obtain a fixed domain which is independent of ε,

$$\Omega = \{(x', z) \in \mathbb{R}^3 : \left(x', 0\right) \in \omega, 0 < z < h(x')\},$$

and $\bar\Gamma = \bar\Gamma_1 \cup \bar\Gamma_L \cup \bar\omega$ its boundary.

Now, we define on Ω new unknowns

$$\left.\begin{aligned} &\hat u_i^\varepsilon\left(x', z, t\right) = u_i^\varepsilon\left(x', x_3, t\right), i = 1, 2, \\ &\hat u_3^\varepsilon\left(x', z, t\right) = \varepsilon^{-1} u_3^\varepsilon\left(x', x_3, t\right), \\ &\hat\pi^\varepsilon\left(x', z, t\right) = \varepsilon^2 \pi^\varepsilon\left(x', x_3, t\right). \end{aligned}\right\} \tag{12}$$

For the data of the problem (9), it is assumed that they depend on ε as follows

$$\left.\begin{aligned} &\hat f\left(x', z, t\right) = \varepsilon^2 f^\varepsilon\left(x', x_3, t\right), \\ &\hat k = \varepsilon k^\varepsilon, \hat\alpha = \varepsilon^2 \alpha^\varepsilon, \end{aligned}\right\} \tag{13}$$

with $\hat{f}$, $\hat{k}$ and $\hat{\alpha}$ not depending on ε. We now introduce the functional framework on Ω as follows

$$V = \left\{ \varphi \in H^1(\Omega)^3 : \varphi = 0 \text{ on } \Gamma_L \cup \Gamma_1 \text{ and } \varphi.n = 0 \text{ on } \omega \right\},$$

$$V_{\text{div}} = \{\varphi \in V : \operatorname{div}(\varphi) = 0\},$$

$$\Pi(V) = \left\{ \varphi \in H^1(\Omega)^2 : \varphi = \left(\varphi_1, \varphi_2\right), \varphi_i = 0 \text{ on } \Gamma_L \cup \Gamma_1, i = 1, 2 \right\},$$

$$L_0^2(\Omega) = \left\{ \varphi \in L^2(\Omega) : \int_\Omega \varphi dx' dz = 0 \right\},$$

$$V_z = \left\{ v = \left(v_1, v_2\right) \in L^2(\Omega)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2 \text{ and } v = 0 \text{ on } \Gamma_1 \right\},$$

V_z is a Banach space with the norm

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \left(\|v_i\|_{L^2(\Omega)}^2 + \left\| \frac{\partial v_i}{\partial z} \right\|_{L^2(\Omega)}^2 \right) \right)^{\frac{1}{2}}.$$

From (12) and (13), we deduce that the problem (9) is equivalent to the following variational inequality:

$$\text{Find } (\hat{u}^\varepsilon, \hat{\pi}^\varepsilon) \in L^2\left(0, T, V_{\text{div}}\right) \times L^2\left(0, T, L_0^2(\Omega)\right), \text{ such that}$$

$$\begin{gathered}
\varepsilon^2 \sum_{i=1}^{2} \left(\tfrac{\partial \hat{u}_i^\varepsilon}{\partial t}, \hat{\varphi}_i - \hat{u}_i^\varepsilon \right) + \varepsilon^4 \left(\tfrac{\partial \hat{u}_3^\varepsilon}{\partial t}, \hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) + \hat{a}_\mu (\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) + \hat{a}_\nu \left(\tfrac{\partial \hat{u}^\varepsilon}{\partial t}, \hat{\varphi} - \hat{u}^\varepsilon \right) \\
+ \sqrt{\varepsilon} \hat{b} (\hat{u}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) + \hat{\alpha} \sum_{i=1}^{2} \left(\left(1 + \left|\hat{u}_i^\varepsilon\right|\right) \hat{u}_i^\varepsilon, \hat{\varphi}_i - \hat{u}_i^\varepsilon \right) + \hat{\alpha} \left(\left(\varepsilon^2 + \varepsilon^3 \left|\hat{u}_3^\varepsilon\right| \right) \hat{u}_3^\varepsilon, \hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) \\
- (\hat{\pi}^\varepsilon, \operatorname{div}(\hat{\varphi})) + \hat{J}(\hat{\varphi}) - \hat{J}(\hat{u}^\varepsilon) \\
\geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \hat{u}_i^\varepsilon \right) + \varepsilon \left(\hat{f}_3, \hat{\varphi}_3 - \hat{u}_3^\varepsilon \right), \ \forall \hat{\varphi} \in V, \\
\hat{u}^\varepsilon \left(x', z, 0\right) = \hat{u}_0, \quad \forall \left(x', z\right) \in \Omega,
\end{gathered} \tag{14}$$

where

$$\hat{J}(\hat{\varphi}) = \int_\omega \hat{k} \, |\hat{\varphi}| \, dx',$$

$$\begin{aligned}
\hat{a}_\kappa (\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) = \ & \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \kappa \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \frac{\partial \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right)}{\partial x_j} dx' dz \\
& + \sum_{i=1}^{2} \int_\Omega \kappa \left[\frac{\partial \hat{u}_i^\varepsilon}{\partial z} \frac{\partial \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right)}{\partial z} + \varepsilon^4 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \frac{\partial \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right)}{\partial x_i} \right] dx' dz \\
& + \varepsilon^2 \int_\Omega \kappa \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right)}{\partial z} dx' dz,
\end{aligned}$$

$$\begin{aligned}
\hat{b} (\hat{u}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) = \ & \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \hat{u}_i^\varepsilon \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \left(\hat{\varphi}_j - \hat{u}_j^\varepsilon\right) dx' dz + \varepsilon^4 \sum_{i=1}^{2} \int_\Omega \hat{u}_i^\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) dx' dz \\
& + \varepsilon^2 \sum_{i=1}^{2} \int_\Omega \hat{u}_3^\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right) dx' dz + \varepsilon^4 \sum_{i=1}^{2} \int_\Omega \hat{u}_3^\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) dx' dz,
\end{aligned}$$

and

$$(\hat{\pi}^\varepsilon, \operatorname{div}(\hat{\varphi})) = \sum_{j=1}^{2} \int_\Omega \hat{\pi}^\varepsilon \frac{\partial \hat{\varphi}_j}{\partial x_j} dx' dz + \int_\Omega \hat{\pi}^\varepsilon \frac{\partial \hat{\varphi}_3}{\partial z} dx' dz.$$

In what follows we will use the following inequalities

- Poincaré inequality

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq \varepsilon \bar{h} \, \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}}. \tag{15}$$

- Cauchy-Schwarz inequality $\forall u^\varepsilon, v^\varepsilon \in L^2(\Omega^\varepsilon)^3$

$$\int_{\Omega^\varepsilon} |u^\varepsilon . v^\varepsilon|\, dx \leq \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} . \|v^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} . \tag{16}$$

- Sobolev's inequality

$$\begin{aligned} \|u^\varepsilon\|_{L^3(\Omega^\varepsilon)^3} &\leq c_s \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} , \\ \|u^\varepsilon\|_{L^6(\Omega^\varepsilon)^3} &\leq c_s \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} . \end{aligned} \tag{17}$$

First, we will obtain estimates on the velocity $\hat{u}^\varepsilon$ and the acceleration $\frac{\partial \hat{u}^\varepsilon}{\partial t}$.

Theorem 2. Let $(u^\varepsilon, \pi^\varepsilon) \in L^2\left(0, T, V^\varepsilon_{\text{div}}\right) \times L^2\left(0, T, L^2_0(\Omega^\varepsilon)\right)$ be a solution of the problem (9), where $k^\varepsilon \in L^\infty_+(\omega)$. Then, there exists a constant c independent of ε such that

$$\begin{aligned} &\sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)}\right) + \sum_{i,j=1}^{2}\left\|\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} \\ &+ \left\|\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\|\hat{u}_i^\varepsilon\|^3_{L^3(0,T,L^3(\Omega))} + \left\|\varepsilon \hat{u}_3^\varepsilon\right\|^3_{L^3(0,T,L^3(\Omega))} \\ &\leq c, \end{aligned} \tag{18}$$

$$\begin{aligned} &\sum_{i=1}^{2}\left(\left\|\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)}\right) + \sum_{i,j=1}^{2}\left\|\varepsilon \frac{\partial}{\partial x_j}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)} \\ &+ \left\|\varepsilon \frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left\|\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} \\ &\leq c. \end{aligned} \tag{19}$$

Proof. Let u^ε be the solution of the problem (9); choosing $\varphi = (0,0,0)$, we obtain

$$\left(\frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon\right) + a_\mu(u^\varepsilon, u^\varepsilon) + a_\nu\left(\frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon\right) + \sqrt{\varepsilon}\, b(u^\varepsilon, u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon\left((1 + |u^\varepsilon|)\, u^\varepsilon, u^\varepsilon\right) + j^\varepsilon(u^\varepsilon) \leq (f^\varepsilon, u^\varepsilon).$$

Using the identity $b(u^\varepsilon, u^\varepsilon, u^\varepsilon) = 0$, and $j^\varepsilon(u^\varepsilon) \geq 0$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\left(\|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{d}{dt} a_\nu(u^\varepsilon, u^\varepsilon)\right) + a_\mu(u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon\left((1 + |u^\varepsilon|)\, u^\varepsilon, u^\varepsilon\right) \leq (f^\varepsilon, u^\varepsilon),$$

which integrated from 0 to t, leads to

$$\begin{aligned} &\frac{1}{2}\|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \int_0^t \mu\, \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\, ds + \frac{\nu}{2}\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \alpha^\varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\, ds \\ &+ \alpha^\varepsilon \int_0^t \|u^\varepsilon(s)\|^3_{L^3(\Omega^\varepsilon)^3}\, ds \\ &\leq \frac{1}{2}\left\|u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{1}{2}\nu\left\|\nabla u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \int_0^t \int_{\Omega^\varepsilon} f^\varepsilon(s) . u^\varepsilon(s)\, dx ds. \end{aligned}$$

On the other hand, applying Cauchy-Schwarz (16), Poincaré's (15) and Young's inequalities, we find the following inequality

$$\begin{aligned}\int_0^t\int_{\Omega^\varepsilon} f^\varepsilon(s).u^\varepsilon(s)\,dxds &\le \int_0^t\int_{\Omega^\varepsilon}|f^\varepsilon(s)|\,|u^\varepsilon(s)|\,dx'dx_3ds\\ &\le \int_0^t\left(\frac{1}{\sqrt{\mu}}\varepsilon\bar h\,\|f^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^3}\right)\left(\sqrt{\mu}\,\|\nabla u^\varepsilon(s)\|_{L^2(\Omega^\varepsilon)^{3\times3}}\right)\\ &\le \frac{1}{\mu}\left(\varepsilon\bar h\right)^2\int_0^t\|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3}\,ds+\mu\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\,ds.\end{aligned}$$

Whereupon,

$$\begin{aligned}\frac{\nu}{2}\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\alpha^\varepsilon\int_0^t\|u^\varepsilon(s)\|^3_{L^3(\Omega^\varepsilon)^3}\,ds &\le \frac{1}{\mu}\left(\varepsilon\bar h\right)^2\|f^\varepsilon\|^2_{L^2\left(0,T,L^2(\Omega^\varepsilon)^3\right)}\\ &\quad+\frac12\left\|u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^3}+\frac12\nu\left\|\nabla u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}.\end{aligned}\tag{20}$$

Now, as

$$\begin{aligned}\varepsilon^2\|f^\varepsilon\|^2_{L^2\left(0,T,L^2(\Omega^\varepsilon)^3\right)} &= \varepsilon^{-1}\left\|\hat f\right\|^2_{L^2\left(0,T,L^2(\Omega)^3\right)},\\ \left\|\frac{\partial u_i^\varepsilon}{\partial x_3}\right\|^2_{L^2(\Omega^\varepsilon)} &= \varepsilon^{-1}\left\|\frac{\partial \hat u_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)},\ i=1,2,\end{aligned}$$

multiplying the inequality (20) by ε, we deduce that

$$\frac{\nu}{2}\left[\varepsilon\,\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]+\hat\alpha\left[\frac{1}{\varepsilon}\|u^\varepsilon\|^3_{L^3\left(0,T,L^3(\Omega^\varepsilon)^3\right)}\right]\le A,\tag{21}$$

where $A=\frac{1}{\mu}\bar h^2\left\|\hat f\right\|^2_{L^2\left(0,T,L^2(\Omega)^3\right)}+\frac12\|\hat u_0\|^2_{L^2(\Omega)^3}+\frac12\nu\|\nabla\hat u_0\|^2_{L^2(\Omega)^{3\times3}}$ is a constant independent of ε.

From (21), we find (18).

Differentiating (11) with respect to t and taking $\varphi=\frac{\partial u_\zeta^\varepsilon}{\partial t}$, we get

$$\begin{aligned}&\left(\frac{\partial^2u_\zeta^\varepsilon}{\partial t^2},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)+a_\mu\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)+a_\nu\left(\frac{\partial^2u_\zeta^\varepsilon}{\partial t^2},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)+\sqrt{\varepsilon}b\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},u_\zeta^\varepsilon,\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\\ &+\sqrt{\varepsilon}b\left(u_\zeta^\varepsilon,\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)+\alpha^\varepsilon\left(1+\left|u_\zeta^\varepsilon\right|\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)+\left(\frac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon\right),\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\\ &=\left(\frac{\partial f^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right).\end{aligned}$$

Noting that $b\left(u_\zeta^\varepsilon,\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)=0$, and $a_\mu\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)$, $\alpha^\varepsilon\left(\left(1+\left|u_\zeta^\varepsilon\right|\right)\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)$, $\left(\frac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon\right),\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)$ are positive terms, so we have

$$\frac12\frac{d}{dt}\left(\left\|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+a_\nu\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\right)+\sqrt{\varepsilon}b\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},u_\zeta^\varepsilon,\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\le\left(\frac{\partial f^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right).$$

Integrating this inequality over $(0,t)$, we find

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2+\nu\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2 \\ &\leq\frac{\left(\varepsilon\bar{h}\right)^2}{\nu}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2ds+\nu\int_0^t\left\|\nabla\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2ds \\ &+\frac{1}{2}\left\|\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2+\nu\left\|\nabla\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2 \\ &+\int_0^t\sqrt{\varepsilon}\left|b\left(\frac{\partial u_\zeta^\varepsilon(s)}{\partial t},u_\zeta^\varepsilon(s),\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right)\right|ds.\end{aligned}\tag{22}$$

Using Hölder and Sobolev's inequalities, we obtain

$$\begin{aligned}\sqrt{\varepsilon}\int_0^t\left|b\left(\frac{\partial u_\zeta^\varepsilon(s)}{\partial t},u_\zeta^\varepsilon(s),\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right)\right|ds&\leq\sqrt{\varepsilon}\int_0^t\left\|\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^3(\Omega^\varepsilon)^3}\left\|\nabla u_\zeta^\varepsilon(s)\right\|_{L^2(\Omega^\varepsilon)^3}\left\|\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^6(\Omega^\varepsilon)^3}ds \\ &\leq c_s\int_0^t\sqrt{\varepsilon}\left\|\nabla u_\zeta^\varepsilon(s)\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\left\|\nabla\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2ds.\end{aligned}$$

Using the estimates above, the inequality (22) leads to

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2+\nu\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2 \\ &\leq\frac{\left(\varepsilon\bar{h}\right)^2}{\nu}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2ds+\int_0^t\left(\nu+c_s\sqrt{\varepsilon}\left\|\nabla u_\zeta^\varepsilon(s)\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\right)\left\|\nabla\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2ds \\ &+\frac{1}{2}\left\|\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^3}^2+\nu\left\|\nabla\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2.\end{aligned}\tag{23}$$

Now, we are going to estimate $\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}$. From (10) and (11), we deduce

$$\begin{aligned}&\left(\frac{\partial u_\zeta^\varepsilon(0)}{\partial t},\varphi\right)+a_\nu\left(\frac{\partial u_\zeta^\varepsilon(0)}{\partial t},\varphi\right)+a_\mu\left(u_\zeta^\varepsilon(0),\varphi\right)+\sqrt{\varepsilon}b\left(u_\zeta^\varepsilon(0),u_\zeta^\varepsilon(0),\varphi\right) \\ &+\alpha^\varepsilon\left(\left(1+\left|u_\zeta^\varepsilon(0)\right|\right)u_\zeta^\varepsilon(0),\varphi\right) \\ &=\left(f^\varepsilon(0),\varphi\right),\end{aligned}$$

using Cauchy-Schwarz's and Poincarė's inequalities, we obtain

$$\begin{aligned}&\left|\left(\frac{\partial u_\zeta^\varepsilon(0)}{\partial t},\varphi\right)+a_\nu\left(\frac{\partial u_\zeta^\varepsilon(0)}{\partial t},\varphi\right)\right| \\ &\leq\mu\left\|\nabla u_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}}+\hat{\alpha}\bar{h}^2\left\|\nabla u_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}} \\ &+c_s\sqrt{\varepsilon}\left\|\nabla u_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{\hat{\alpha}\bar{h}}{\sqrt{\varepsilon}}\left(\int_\Omega\left|\hat{u}_0\right|^4dx'dz\right)^{\frac{1}{2}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}} \\ &+\varepsilon\bar{h}\left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)^3}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^{3\times3}}.\end{aligned}$$

Therefore

$$\left|\left\langle \frac{\partial u_\zeta^\varepsilon(0)}{\partial t},\varphi\right\rangle_{H^1(\Omega^\varepsilon)^3}\right| \le \left(\mu\left\|\nabla u_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}+\hat{\alpha}\bar{h}^2\left\|\nabla u_0^\varepsilon\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}+c_s\sqrt{\varepsilon}\left\|\nabla u_0^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right)\|\varphi\|_{H^1(\Omega^\varepsilon)^3}$$

$$+\left(\frac{\hat{\alpha}\bar{h}}{\sqrt{\varepsilon}}\left(\int_\Omega\left|\hat{u}_0\right|^4dx'dz\right)^{\frac{1}{2}}+\varepsilon\bar{h}\left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)^3}\right)\|\varphi\|_{H^1(\Omega^\varepsilon)^3}.$$

Multiplying this inequality by $\sqrt{\varepsilon}$, we get

$$\sqrt{\varepsilon}\left\|\frac{\partial u_\zeta^\varepsilon(0)}{\partial t}\right\|_{H^1(\Omega^\varepsilon)^3}\le C_0,$$

where $C_0=\left(\mu+\hat{\alpha}\bar{h}^2\right)\|\hat{u}_0\|_{H^1(\Omega)^3}+c_s\left(\hat{\alpha}\bar{h}+1\right)\|\hat{u}_0\|^2_{H^1(\Omega)^3}+\bar{h}\left\|\hat{f}(0)\right\|_{L^2(\Omega)^3}$ is a constant independent of ε.

Now, passing to the limit in (23) when ζ tends to zero, we deduce that

$$\frac{1}{2}\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\nu\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}$$

$$\le\int_0^t\left[\frac{1}{2}\left\|\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\left(\nu+c_s\sqrt{\varepsilon}\left\|\nabla u^\varepsilon(s)\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\right)\left\|\nabla\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]ds$$

$$+\frac{\left(\varepsilon\bar{h}\right)^2}{\nu}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}ds+\left(\frac{1}{2}+\nu\right)\frac{\left(C_0\right)^2}{\varepsilon}.$$

Multiplying this inequality by ε and use the fact that $\sqrt{\varepsilon}\left\|\nabla u^\varepsilon(s)\right\|_{L^2(\Omega^\varepsilon)^{3\times3}}\le\sqrt{\frac{2.A}{\nu}}$, we obtain

$$\varepsilon\left[\frac{1}{2}\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\nu\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right] \tag{24}$$

$$\le\int_0^t\varepsilon\left[\frac{1}{2}\left\|\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\left(\nu+c_s.\sqrt{\frac{2.A}{\nu}}\right)\left\|\nabla\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]ds+B,$$

where $B=\frac{\bar{h}^2}{\nu}\left\|\frac{\partial\hat{f}}{\partial t}\right\|^2_{L^2\left(0,T,L^2(\Omega)^3\right)}+\left(C_0\right)^2$.

Applying the Gronwall inequality to (24), we have

$$\varepsilon\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]\le c,$$

where c is a constant independent of ε. The proof is complete. □

Now, we are looking for a priori estimates on the pressure $\hat{\pi}^\varepsilon$. For this we need to establish the following result.

Theorem 3. Under the hypotheses of Theorem 2, there exists a constant c independent of ε such that

$$\left\|\frac{\partial\hat{\pi}^\varepsilon}{\partial x_1}\right\|_{L^2\left(0,T,H^{-1}(\Omega)\right)}\le c, \tag{25}$$

$$\left\|\frac{\partial\hat{\pi}^\varepsilon}{\partial x_2}\right\|_{L^2\left(0,T,H^{-1}(\Omega)\right)}\le c, \tag{26}$$

$$\left\|\frac{\partial\hat{\pi}^\varepsilon}{\partial z}\right\|_{L^2\left(0,T,H^{-1}(\Omega)\right)}\le\varepsilon.c. \tag{27}$$

Proof. According to (9), we have

$$\left(\pi^\varepsilon,\operatorname{div}(\psi)\right)=\left(\frac{\partial u^\varepsilon}{\partial t},\psi\right)+a_\mu(u^\varepsilon,\psi)+a_\nu\left(\frac{\partial u^\varepsilon}{\partial t},\psi\right)+\sqrt{\varepsilon}b\left(\hat{u}^\varepsilon,\hat{u}^\varepsilon,\psi\right)$$

$$+\alpha^\varepsilon\left(\left(1+|u^\varepsilon|\right)u^\varepsilon,\psi\right)+\left(f^\varepsilon,\psi\right),\forall\psi\in L^2\left(0,T,H_0^1\left(\Omega^\varepsilon\right)^3\right).$$

Using Cauchy-Schwarz (16), Poincaré (15), Sobolev (17) and Young inequalities, we find

$$\left|\int_0^t \left(\pi^\varepsilon, \operatorname{div}(\psi)\right) ds\right| \tag{28}$$

$$\leq \left(\mu + \hat{\alpha}\bar{h}^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds + \left(\bar{h}^2 + \nu\right)\int_0^t \left\|\nabla \frac{\partial u^\varepsilon}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds$$

$$+ \frac{\left(\varepsilon\bar{h}\right)^2}{\mu+\nu}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3} ds + 2\left(1+\mu+\nu\right)\int_0^t \|\nabla\psi(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds$$

$$+ c_s\int_0^t \sqrt{\varepsilon}\,\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\|\nabla\psi(s)\|_{L^2(\Omega^\varepsilon)^{3\times 3}} ds + \alpha^\varepsilon\int_0^t\int_{\Omega^\varepsilon} |u^\varepsilon(s)|\,u^\varepsilon(s).\psi(s)\,dxds.$$

The last term of (28) can be estimated as follows

$$\left|\alpha^\varepsilon\int_0^t\int_{\Omega^\varepsilon} |u^\varepsilon(s)|\,u^\varepsilon(s)\,\psi\,dx'dx_3ds\right| \leq \left|\frac{\hat{\alpha}}{\varepsilon^2}\int_0^t\left(\int_{\Omega^\varepsilon}|u^\varepsilon(s)|^3\,dx'dx_3\right)^{\frac{2}{3}}\left(\int_{\Omega^\varepsilon}|\psi|^3\,dx'dx_3\right)^{\frac{1}{3}}ds\right|$$

$$\leq \frac{\hat{\alpha}}{\varepsilon}\int_0^t \frac{1}{\varepsilon}\|u^\varepsilon(s)\|^3_{L^3(\Omega^\varepsilon)^3} ds + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^3_{L^3(\Omega^\varepsilon)^3} ds$$

$$\leq \frac{A}{\varepsilon} + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^3_{L^3(\Omega^\varepsilon)^3} ds.$$

So, we get

$$\left|\int_0^t \left(\pi^\varepsilon, \operatorname{div}(\psi)\right) ds\right| \leq \left(\mu + \hat{\alpha}\bar{h}^2\right)\int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \left(\bar{h}^2 + \nu\right)\int_0^t \left\|\nabla \frac{\partial u^\varepsilon}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds \tag{29}$$

$$+ \frac{\left(\varepsilon\bar{h}\right)^2}{\mu+\nu}\int_0^t \|f^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^3} ds + 2\left(1+\mu+\nu+c_s\right)\int_0^t \|\nabla\psi(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds$$

$$+ c_s\int_0^t \varepsilon\,\|\nabla u^\varepsilon(s)\|^4_{L^2(\Omega^\varepsilon)^{3\times 3}} ds + \frac{A}{\varepsilon} + \frac{\hat{\alpha}}{\varepsilon^2}\int_0^t \|\psi(s)\|^3_{L^3(\Omega^\varepsilon)^3} ds.$$

Now, we choose $\psi = \left(\psi_1, 0, 0\right)$, $\psi = \left(0, \psi_2, 0\right)$, $\psi = \left(0, 0, \psi_3\right)$ respectively in (29), and change the variables, to get (25) − (27). □

5 | STUDY OF THE LIMIT PROBLEM

The aim of this section is to study the limit behavior of the sequences $(\hat{u}^\varepsilon)$ and $(\hat{\pi}^\varepsilon)$ as $\varepsilon \to 0$. We will demonstrate the weak convergence of these sequences and give the limit problem that characterizes these limits.

Lemma 2. Under the hypotheses of Theorem 2 and 3, there exists $u_i^* \in L^2\left(0, T, V_z\right)$, $i = 1, 2$ and $\pi^* \in L^2\left(0, T, L_0^2(\Omega)\right)$ *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^*,\ i = 1, 2, \text{ weakly } in\ L^2\left(0, T, V_z\right) \tag{30}$$

$$\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_i^*}{\partial t},\ i = 1, 2, \text{ weakly } in\ L^2\left(0, T, V_z\right),$$

$$\left|\hat{u}_i^\varepsilon\right|\hat{u}_i^\varepsilon \rightharpoonup \left|u_i^*\right|u_i^*,\ i=1,2,\ \text{weakly } in\ L^{\frac{3}{2}}\left(0,T,L^{\frac{3}{2}}(\Omega)\right), \tag{31}$$

$$\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0,\ \varepsilon\frac{\partial}{\partial x_j}\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial t}\right) \rightharpoonup 0,\ i,j=1,2,\ \text{weakly } in\ L^2\left(0,T,L^2(\Omega)\right), \tag{32}$$

$$\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0,\ \varepsilon\frac{\partial}{\partial x_i}\left(\frac{\partial\hat{u}_3^\varepsilon}{\partial t}\right) \rightharpoonup 0\ i=1,2,\ \text{weakly } in\ L^2\left(0,T,L^2(\Omega)\right), \tag{33}$$

$$\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0,\ \varepsilon\frac{\partial}{\partial z}\left(\frac{\partial\hat{u}_3^\varepsilon}{\partial t}\right) \rightharpoonup 0\ \text{weakly } in\ L^2\left(0,T,L^2(\Omega)\right), \tag{34}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0,\ \varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0\ \text{weakly } in\ L^2\left(0,T,L^2(\Omega)\right), \tag{35}$$

$$\hat{\pi}^\varepsilon \rightharpoonup \pi^*,\ \text{weakly } in\ L^2\left(0,T,L_0^2(\Omega)\right). \tag{36}$$

Proof. From (18) and (19) there exists a fixed constant C_T independent on ε such that

$$\left\|\frac{\partial\hat{u}_i^\varepsilon(s)}{\partial z}\right\|^2_{L^2\left(0,T,L^2(\Omega)\right)} \le C_T,\quad \left\|\frac{\partial}{\partial z}\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial t}\right)\right\|^2_{L^2\left(0,T,L^2(\Omega)\right)} \le C_T,\ i=1,2.$$

Using these estimates with Poincaré's inequality in the domain Ω

$$\left\|\hat{u}_i^\varepsilon(s)\right\|_{L^2\left(0,T,L^2(\Omega)\right)} \le \bar{h}\left\|\frac{\partial\hat{u}_i^\varepsilon(s)}{\partial z}\right\|_{L^2\left(0,T,L^2(\Omega)\right)},$$

we obtain (30). The results (32) – (33) are deduced directly from (18), (19) and (30). For (31), we have

$$\int_0^T\int_\Omega\left(\left|\left|\hat{u}_i^\varepsilon\right|\hat{u}_i^\varepsilon\right|\right)^{\frac{3}{2}}dx'dzds \le \int_0^T\left\|\hat{u}_i^\varepsilon(s)\right\|^3_{L^3(\Omega)}ds \le C_T.$$

So $\left|\hat{u}_i^\varepsilon\right|\hat{u}_i^\varepsilon$, $i=1,2$ is bounded in $L^{\frac{3}{2}}\left(0,T,L^{\frac{3}{2}}(\Omega)\right)$, then we conclude that $\left|\hat{u}_i^\varepsilon\right|\hat{u}_i^\varepsilon$, $i=1,2$, weakly converges to $\left|u_i^*\right|u_i^*$ *in* $L^{\frac{3}{2}}\left(0,T,L^{\frac{3}{2}}(\Omega)\right)$. Because $\operatorname{div}(\hat{u}^\varepsilon)=0$, by (18) and with a particular choice of the test function, we obtain (34) and (35) see[4]. To find (36) we use (25) – (27) and the following inequality

$$\left\|\hat{\pi}^\varepsilon\right\|_{L^2\left(0,T,L^2(\Omega)\right)} \le c'\left\|\nabla\hat{\pi}^\varepsilon\right\|_{L^2\left(0,T,H^{-1}(\Omega)^3\right)}.$$

□

Theorem 4. The weak limit (u^*,π^*) satisfies the following properties

$$\pi^*\left(x',z,t\right)=\pi^*\left(x',t\right),\ \text{a,e in }\Omega, \tag{37}$$

$$\int_\Omega\pi^*\left(\frac{\partial u_1^*}{\partial x_1}+\frac{\partial u_2^*}{\partial x_2}\right)dx'dz=0. \tag{38}$$

Also (u^*,π^*) is the solution of the variational inequality and the limit problem

$$\begin{aligned}&\sum_{i=1}^2\int_\Omega\mu\frac{\partial u_i^*}{\partial z}.\frac{\partial\left(\hat{\varphi}_i-u_i^*\right)}{\partial z}dx'dz+\sum_{i=1}^2\int_\Omega\nu\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right).\frac{\partial\left(\hat{\varphi}_i-u_i^*\right)}{\partial z}dx'dz\\&+\sum_{i=1}^2\int_\Omega\hat{\alpha}\left(1+\left|u_i^*\right|\right)u_i^*\left(\hat{\varphi}_i-u_i^*\right)dx'dz-\int_\Omega\pi^*\left(\frac{\partial\hat{\varphi}_1}{\partial x_1}+\frac{\partial\hat{\varphi}_2}{\partial x_2}\right)dx'dz+\int_\omega\hat{k}\left(|\hat{\varphi}|-|u^*|\right)dx'\\&\ge\sum_{i=1}^2\int_\Omega\hat{f}_i.\left(\hat{\varphi}_i-u_i^*\right)dx'dz,\ \forall\hat{\varphi}\in\Pi(V),\end{aligned} \tag{39}$$

$$-\frac{\partial^2}{\partial z^2}\left[\mu.u_i^*+\nu.\frac{\partial u_i^*}{\partial t}\right]+\hat{\alpha}\left(1+\left|u_i^*\right|\right)u_i^*+\frac{\partial\pi^*}{\partial x_i}=\hat{f}_i,\ \text{ in } L^2(\Omega)\ \text{ for } i=1,2. \tag{40}$$

Proof. From the formula (29), we have

$$\int_0^t \int_\Omega \hat{\pi}^\varepsilon(s) \cdot \frac{\partial \psi(s)}{\partial z} dx' dz ds \leq \varepsilon c, \forall \psi \in L^2\left(0, T, H_0^1(\Omega)\right),$$

therefore

$$\lim_{\varepsilon \to 0} \int_0^t \int_\Omega \hat{\pi}^\varepsilon(s) \cdot \frac{\partial \psi}{\partial z}(s) dx' dz ds = 0, \forall \psi \in L^2\left(0, T, H_0^1(\Omega)\right).$$

By using (36), we deduce (37).

As $\mathrm{div}(\hat{u}^\varepsilon) = 0$ in Ω, we obtain for $\theta \in D(]0, T[\times \omega)$,

$$\begin{aligned}
&\int_0^t \int_\Omega \theta\left(x', s\right)\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1} + \frac{\partial \hat{u}_2^\varepsilon}{\partial x_2} + \frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right) dx' dz ds \\
&= \int_0^t \int_\Omega \theta\left(x', s\right)\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1} + \frac{\partial \hat{u}_2^\varepsilon}{\partial x_2}\right) dx' dz ds \\
&= 0.
\end{aligned}$$

By using $\hat{u}_i^\varepsilon \rightharpoonup u_i^*$, $i = 1, 2$ in $L^2\left(0, T, V_z\right)$, we obtain

$$\int_0^t \int_\Omega \theta\left(x', s\right)\left(\frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\right) dx' dz ds = 0. \tag{41}$$

Using (37), π^* is now in $L^2\left(0, T, L^2(w)\right)$, then there exists $\left(\theta_m\right)$ in $D(]0, T[\times \omega)$ such that $\theta_m \to \pi^*$ in $L^2\left(0, T, L^2(w)\right)$. So from (41), we obtain (38) when $m \to \infty$.

By passing to the limit in (14), using the result of the Lemma 2, and the fact that $\hat{J}(.)$ is convex and lower semi-continuous, we obtain

$$\begin{aligned}
&\sum_{i=1}^2 \int_\Omega \mu \frac{\partial u_i^*}{\partial z} \cdot \frac{\partial\left(\hat{\varphi}_i - u_i^*\right)}{\partial z} dx' dz + \sum_{i=1}^2 \int_\Omega \nu \frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right) \cdot \frac{\partial\left(\hat{\varphi}_i - u_i^*\right)}{\partial z} dx' dz \\
&+ \sum_{i=1}^2 \int_\Omega \hat{\alpha}\left(1 + \left|u_i^*\right|\right) u_i^*\left(\hat{\varphi}_i - u_i^*\right) dx' dz - \int_\Omega \pi^*\left(\frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2}\right) dx' dz + \int_\omega \hat{k}\left(|\hat{\varphi}| - |u^*|\right) dx' \\
&\geq \sum_{i=1}^2 \int_\Omega \hat{f}_i \cdot \left(\hat{\varphi}_i - u_i^*\right) dx' dz, \quad \forall \hat{\varphi} \in \Pi(V).
\end{aligned} \tag{42}$$

Now, we choose in the variational inequality (42)

$$\hat{\varphi}_i = u_i^* \pm w_i, \text{ such that } w_i \in H_0^1(\Omega), i = 1, 2,$$

we find

$$\begin{aligned}
&\sum_{i=1}^2 \int_\Omega \left[\mu \frac{\partial u_i^*}{\partial z} + \nu \frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right)\right] \frac{\partial w_i}{\partial z} dx' dz + \sum_{i=1}^2 \int_\Omega \hat{\alpha}\left(1 + \left|u_i^*\right|\right) u_i^* w_i dx' dz - \sum_{i=1}^2 \int_\Omega \pi^* \frac{\partial w_i}{\partial x_i} dx' dz \\
&= \sum_{i=1}^2 \int_\Omega \hat{f}_i w_i dx' dz.
\end{aligned}$$

Using Green's formula and choosing $w_1 = 0$ and $w_2 \in H_0^1(\Omega)$, then $w_2 = 0$ and $w_1 \in H_0^1(\Omega)$, we obtain

$$-\int_\Omega \frac{\partial}{\partial z}\left[\mu\frac{\partial u_i^*}{\partial z} + \nu\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right)\right] w_i dx' dz + \int_\Omega \hat{\alpha}\left(1 + |u_i^*|\right) u_i^* w_i dx' dz + \int_\Omega \frac{\partial \pi^*}{\partial x_i} w_i dx' dz$$
$$= \int_\Omega \hat{f}_i w_i dx' dz, \forall w_i \in H_0^1(\Omega).$$

Thus

$$-\frac{\partial^2}{\partial z^2}\left[\mu.u_i^* + \nu.\frac{\partial u_i^*}{\partial t}\right] + \hat{\alpha}\left(1 + |u_i^*|\right) u_i^* + \frac{\partial \pi^*}{\partial x_i} = \hat{f}_i, \text{ in } H^{-1}(\Omega) \text{ for } i = 1,2. \tag{43}$$

Since $\hat{f}_i \in L^2(\Omega)$, then (43) is valid in $L^2(\Omega)$. □

Theorem 5. Let

$$\tau^*\left(x', t\right) = \frac{\partial u^*}{\partial z}\left(x', 0, t\right) \text{ and } s^*\left(x', t\right) = u^*\left(x', 0, t\right),$$

be the traces of the velocity u^* on ω. The traces τ^* and s^* satisfy the following inequality

$$\int_\omega \hat{k}\left(|\psi + s^*| - |s^*|\right) dx' - \int_\omega \left[\mu\tau^* + \nu\frac{\partial \tau^*}{\partial t}\right] \psi dx' \geq 0, \forall \psi \in L^2(\omega)^2, \tag{44}$$

and the following limit form of the Tresca boundary conditions

$$\left.\begin{array}{l} \left|\mu\tau^* + \nu\frac{\partial \tau^*}{\partial t}\right| < \hat{k} \Rightarrow s^* = 0, \\ \left|\mu\tau^* + \nu\frac{\partial \tau^*}{\partial t}\right| = \hat{k} \Rightarrow \exists \lambda > 0 : s^* = \lambda\left(\mu\tau^* + \nu\frac{\partial \tau^*}{\partial t}\right), \end{array}\right\} \text{ a.e on } \omega \times]0, T[. \tag{45}$$

Moreover, the pair (u^*, π^*) satisfies the weak generalized equation of Reynolds

$$\int_\omega \left(\frac{h^3\left(x'\right)}{12}\nabla\pi^* - \frac{h\left(x'\right)}{2}\left(\mu s^* + \nu\frac{\partial s^*}{\partial t}\right) + \int_0^h \left(\mu u^* + \nu\frac{\partial u^*}{\partial t}\right)\left(x', \zeta, t\right) d\zeta + \tilde{F} + \tilde{U}\right)\nabla\psi dx' \tag{46}$$
$$= 0, \ \forall \psi \in H^1(\omega)^2,$$

with

$$\tilde{F}\left(x', h, t\right) = \int_0^h F\left(x', z, t\right) dz - \frac{h}{2}F\left(x', h, t\right), \tilde{U}(x, h, t) = -\hat{\alpha}\int_0^h U(x, z, t) dz + \frac{\hat{\alpha}h}{2}U(x, h, t),$$
$$F\left(x', z, t\right) = \int_0^z \int_0^\zeta \hat{f}\left(x', \eta, t\right) d\eta d\zeta, U(x, z, t) = \int_0^z \int_0^\zeta \left(1 + |u^*|\right) u^*(x, \eta, t) d\eta d\zeta.$$

Proof. For (44), (45), it is enough to follow the same techniques of[4]. To prove (46) we integrate (40) over $(0, z)$, we note that

$$-\mu\frac{\partial u_i^*}{\partial z} - \nu\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right) + \mu\frac{\partial u_i^*}{\partial z}\left(x', 0, t\right) + \nu\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}\right)\left(x', 0, t\right) + z\frac{\partial \pi^*}{\partial x_i}$$
$$= -\hat{\alpha}\int_0^z \left(1 + |u^*|\right) u^*\left(x', \eta, t\right) d\eta + \int_0^z \hat{f}_i\left(x', \eta, t\right) d\eta.$$

By integrating between 0 and z, we obtain

$$\left(\mu.u^* + \nu.\frac{\partial u_i^*}{\partial t}\right)\left(x', z, t\right) = \mu z\tau_i^* + \nu z\frac{\partial \tau_i^*}{\partial t} + \mu s_i^* + \nu\frac{\partial s_i^*}{\partial t} + \frac{z^2}{2}\frac{\partial \pi^*}{\partial x_i} \tag{47}$$
$$+ \hat{\alpha}\int_0^z \int_0^\zeta \left(1 + |u^*|\right) u^*\left(x', \eta, t\right) d\eta d\zeta - \int_0^z \int_0^\zeta \hat{f}_i\left(x', \eta, t\right) d\eta d\zeta.$$

We replace z by $h\left(x'\right)$, hence

$$h\left(x'\right)\left[\mu\tau_i^* + \nu\frac{\partial\tau_i^*}{\partial t}\right] + \mu s_i^* + \nu\frac{\partial s_i^*}{\partial t} + \frac{h\left(x'\right)^2}{2}\frac{\partial\pi^*}{\partial x_i} + \hat{\alpha}\int_0^z\int_0^\zeta \left(1+\left|u^*\right|\right)u^*\left(x',\eta,t\right)d\eta d\zeta \tag{48}$$
$$= \int_0^h\int_0^\zeta f_i\left(x',\eta,t\right)d\eta d\zeta.$$

Integrating (47) from 0 to $h\left(x'\right)$, we obtain

$$\mu\int_0^h u^*\left(x',\zeta,t\right)d\zeta + \nu\int_0^h \frac{\partial u^*\left(x',\zeta,t\right)}{\partial t}d\zeta \tag{49}$$
$$= h\left(x'\right)\left[\mu s_i^* + \nu\frac{\partial s_i^*}{\partial t}\right] + \frac{h\left(x'\right)^2}{2}\left[\mu\tau_i^* + \nu\frac{\partial\tau_i^*}{\partial t}\right] + \frac{h\left(x'\right)^3}{6}\frac{\partial\pi^*}{\partial x_i}$$
$$+ \hat{\alpha}\int_0^h\int_0^z\int_0^\zeta \left(1+\left|u^*\right|\right)u^*\left(x',\eta,t\right)d\eta d\zeta dz - \int_0^h\int_0^z\int_0^\zeta \hat{f}_i\left(x',\eta,t\right)d\eta d\zeta dz.$$

From (48) and (49), we deduce (46). □

Theorem 6. The solution (u^*, π^*) of the limit problem $(39)-(40)$ is unique in $L^2\left(0,T,V_z\right)\times L^2\left(0,T,L_0^2(\omega)\right)$.

Proof. Let us suppose that there exist two solutions $\left(u^{*1},\pi^{*1}\right)$ and $\left(u^{*2},\pi^{*2}\right)$ of the limit problem $(39)-(40)$. We take $\hat{\varphi}=u^{*2}$ then $\hat{\varphi}=u^{*1}$ respectively in (39), and by summing the two inequality, we obtain

$$\sum_{i=1}^2 \mu\int_\Omega \left(\frac{\partial u_i^{*1}}{\partial z}-\frac{\partial u_i^{*2}}{\partial z}\right).\left(\frac{\partial u_i^{*1}}{\partial z}-\frac{\partial u_i^{*2}}{\partial z}\right)dx'dz$$
$$+\nu\int_\Omega \left(\frac{\partial}{\partial z}\left(\frac{\partial u_i^{*1}}{\partial t}\right)-\frac{\partial}{\partial z}\left(\frac{\partial u_i^{*2}}{\partial t}\right)\right).\left(\frac{\partial u_i^{*1}}{\partial z}-\frac{\partial u_i^{*2}}{\partial z}\right)dx'dz$$
$$+\sum_{i=1}^2\int_\Omega \hat{\alpha}\left(\left(1+\left|u_i^{*1}\right|\right)u_i^{*1}-\left(1+\left|u_i^{*2}\right|\right)u_i^{*2}\right)\left(u_i^{*1}-u_i^{*2}\right)dx'dz$$
$$\le 0.$$

Thus, we find
$$\left\|\frac{\partial}{\partial z}\left(u^{*1}-u^{*2}\right)\right\|_{L^2(\Omega)^2}^2 + \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial}{\partial z}\left(u^{*1}-u^{*2}\right)\right\|_{L^2(\Omega)^2}^2 \le 0.$$
By integrating this inequality over $(0,t)$, we obtain
$$\left\|\frac{\partial}{\partial z}\left(u^{*1}-u^{*2}\right)\right\|_{L^2\left(0,T,L^2(\Omega)\right)^2}^2 \le 0.$$
Using the Poincaré's inequality, we find
$$\left\|u^{*1}-u^{*2}\right\|_{L^2\left(0,T,V_z\right)} = 0.$$
To prove the uniqueness of the pressure π^* in $L^2\left(0,T,L_0^2(\omega)\right)$, we use the Reynolds equation (46). We get first

$$\int_0^t\int_\omega \frac{h^3}{12}\nabla\left(\pi^{*1}\left(x',s\right)-\pi^{*2}\left(x',s\right)\right)\nabla\psi dx'ds = 0,$$

then choosing $\psi=\pi^{*1}-\pi^{*2}$, and Poincaré's inequality, we get

$$\pi^{*1}=\pi^{*2}, \text{ a.e in } \omega\times\,]0,T[\,.$$

□

CONFLICT OF INTEREST STATEMENT: There are no conflicts of interest declared by the authors.

References

1. C.T. Anh and N.V. Thanh, On the existence and long-time behavior of solutions to stochastic three-dimensional Navier-Stokes-Voigt equations ,Stochastics. 91(2019). DOI: 10.1080/17442508.2018.1551396.

2. C.T. Anh and P.T. Trang, Decay rate of solutions to the 3D Navier-Stokes-Voigt equations in H^m space, Appl. Math. Lett. 61 (2016), 1-7.

3. C.T. Anh and P.T. Trang, Pull-back attractors for three-dimensional Navier-Stokes-Voigt equations in some unbounded domains, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), 223-251.

4. G. Bayada, M. Boukrouche, On a free boundary problem for Reynolds equation derived from the Stokes system with Tresca boundary conditions, J. Math. Anal. Appl. 382 (2003), 212-231.

5. H. Benseridi and M. Dilmi, Some inequalities and asymptotic behavior of a dynamic problem of linear elasticity, Georgian Math. J. 20 (2013), no. 1, pp. 25-41.

6. Y. P. Cao, E. M. Lunasin, and E. S. Titi, Global well-p osedness of the three dim ensional viscous and inviscid simplified Bardina turbulence mo dels, Commun. Math. Sci. 4 (2006), 823–848.

7. M. Conti Zelati and C.G. Gal, Singular limits of Voigt models in fluid dynamics, J. Math. Fluid Mech. 17 (2015), 233-259.

8. M. Dilmi, M. Dilmi and H. Benseridi, Study of generalized Stokes operator in a thin domain with friction law (case p $\prec$ 2). Math Methods Appl Sci. 2018;41(18):1-10. https://doi.org/10.1002/mma.4876.

9. M. Dilmi, M. Dilmi and H. Benseridi, Asymptotic behavior for the elasticity system with a nonlinear dissipative term, Rend. Istit. Mat. Univ. Trieste, vol. 51 (2019) pp. 41-60.

10. M. Dilmi, M. Dilmi and H. Benseridi, A 3D-2D asymptotic analysis of viscoelastic problem with nonlinear dissipative and source terms, Math Meth Appl Sci, 2019. https://doi.org/10.1002/mma.5755.

11. G. Duvaut, J.L. Lions, Les Inéquations en Mécanique et en Physique. Dunod, Paris (1972).

12. M. A. Ebrahimi, Michael Holst, and Evelyn Lunasin. The Navier-Stokes-Voight model for image inpainting. IMA J. Appl. Math., 78(5):869-894, 2013.

13. M. Holst, E. Lunasin and G. Tsogtgerel, Analysis of a general family of regularized NavierStokes and MHD models, J. Nonlinear Sci. 20 (2010), 523-567.

14. D. Iftimie, G. Raugel, and G. R. Sell, Navier-Stokes equations in thin 3D domains with Navier boundary conditions, Indiana Univ. Math. J. 56 (2007), no. 3, 1083-1156.

15. V.K. Kalantarov and E.S. Titi, Global attractor and determining modes for the 3D NavierStokes-Voight equations, Chin. Ann. Math. Ser. B 30 (2009), 697-714.

16. W. J. Layton and L. G. Rebholz, On relaxation times in the Navier-Stokes-Voigt model, Inter. J. Comp. Fluid Dyn., 27 (2013), 184-187.

17. Y. Letoufa, H. Benseridi and M. Dilmi, Study of Stokes dynamical system in a thin domain with Fourier and Tresca boundary conditions, Asian-European Journal of Mathematics. 2019. https://doi.org/10.1142/S1793557121500078.

18. B. Levant, F. Ramos, and E. S. Titi, On the statistical properties of the 3D incompressible Navier-Stokes-Voigt model, Commun. Math. Sci. 8 (2010), 277-293.

19. J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Paris, Dunod, (1969).

20. T.H. Miura, On singular limit equations for incompressible fluids in moving thin domains, Quart. Appl. Math. 76 (2018), 215–251.

21. A. P. Oskolkov, The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 38 (1973), 98-136.

22. R. Temam and M. Ziane, Navier-Stokes equations in thin spherical domains, Optimization methods in partial differential equations (South Hadley, MA, 1996), Contemp. Math., vol. 209, Amer. Math. Soc., Providence, RI, 1997, pp. 281-314.

23. X. Wang and Y. Qin, Three-dimensional Navier-Stokes-Voight equation with a memory and the Brinkman-Forchheimer damping term, Math Meth Appl Sci. 2019;1-17.

24. C. Zhao and H. Zhu, Upper bound of decay rate for solutions to the Navier-Stokes-Voigt equations in R3, Appl. Math. Comput. 256 (2015), 183-191.

$\square$