Decay of solutions of non-homogenous hyperbolic equations

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Abstract

We consider conditions for the decay in time of solutions of non-homogenous hyperbolic equations. It is proven that solutions of the equations go to 0 in L 2 at infinity if and only if an equation's right-hand side uniquely determines the initial conditions in a certain way. We also obtain that a hyperbolic equation has a unique solution that fades when t-[?].

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We consider conditions for the decay in time of solutions of non-homogenous hyperbolic equations. It is proven that solutions of the equations go to 0 in L^2 at infinity if and only if an equation's right-hand side uniquely determines the initial conditions in a certain way. We also obtain that a hyperbolic equation has a unique solution that fades when $t \to \infty$.

Keywords: non-homogenous hyperbolic equation, decay in time of solutions, solutions asymptotics

Mathematics Subject Classification (2020): 35L10, 35B30, 35B40

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. In the paper, we consider the decay in time of solutions to the following problem

$$\begin{cases} u_t + Lu = f & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = g, u_t = h & \text{on } \{0\} \times \Omega, \end{cases} \tag{1}$$

where functions f, g, h are given and we want to find is u. We also require that operator $u_t + Lu$ is a second-order hyperbolic operator. In our considerations, the operator L has the following form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j},$$

where we also assume that a^{ij} depends only on $x \in \Omega$. Moreover, we require that there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) y_i y_j \geq \lambda |y|^2 \text{ for all } x \in \Omega \text{ and } y \in \mathbb{R}^n. \tag{2}$$

In addition, we assume that $a^{ij} \in L^\infty(\Omega)$ and $a^{ij} = a^{ji}$ for $i, j = 1, \dots, n$.

It is well known that u can be interpreted as a displacement of a material. The coefficients a^{ij} come from material principles. In general, equation (1) models wave transmission in the medium. Problems as in (1) were widely studied. There are well-known results about the existence, uniqueness, and regularity of solutions of hyperbolic equations (see [9, 13, 14, 19]).

The following theorem is the main result of the present article.

Theorem 1.1. *Let us assume that* $g \in H_0^1(\Omega), h \in L^2(\Omega), f \in L^2_{\text{loc}}\left([0,\infty); L^2(\Omega)\right) \cap L^1(0,\infty; L^2(\Omega))$ *and* u *is a weak solution of the problem* (1)*. Then,*

$$u(t,\cdot) \to 0 \text{ in } L^2(\Omega) \text{ when } t \to \infty \tag{3}$$

if and only if the following equalities

$$\begin{aligned} (g, \varphi_m)_{L^2(\Omega)} &= \frac{1}{\sqrt{\lambda_m}} \int_0^\infty \sin\left(\sqrt{\lambda_m} s\right) (f(s,\cdot), \varphi_m)_{L^2(\Omega)} ds, \\ (h, \varphi_m)_{L^2(\Omega)} &= -\int_0^\infty \cos\left(\sqrt{\lambda_m} s\right) (f(s,\cdot), \varphi_m)_{L^2(\Omega)} ds \end{aligned} \tag{4}$$

hold for all $m \in \mathbb{N}$.

We see that the conditions in (4) uniquely determine the initial conditions of the problem (1) if we know that (3) is satisfied. The equalities uniquely connect the initial conditions with f in (1).

The decay of wave phenomena is natural and often observed. We see it in the water, string, and many other cases. It also occurs in the hyperbolic-parabolic system of thermoelasticity. It is a problem that describes oscillations and the heat in a medium. It can be written as follows

$$\begin{cases} u_{tt} - \Delta u = \mu \operatorname{div}(\theta I), & \text{in } (0,\infty) \times \Omega, \\ \theta_t - \Delta \theta = \mu \theta \operatorname{div}(u_t), & \text{in } (0,\infty) \times \Omega, \\ u|_{\partial\Omega} = 0, \ \frac{\partial \theta}{\partial n}|_{\partial\Omega} = 0 & \text{on } (0,\infty), \\ u(0,\cdot) = u_0, \ u_t(0,\cdot) = v_0, \ \theta(0,\cdot) = \theta_0 > 0, \end{cases} \tag{5}$$

where μ is a constant, initial data θ_0, u_0, v_0 are given and we want to find $u\colon [0,\infty) \times \Omega \to \mathbb{R}^n$, which is the displacement, and $\theta\colon [0,\infty) \times \Omega \to \mathbb{R}$, which is the temperature. It is usually considered for $n = 1, 2, 3$. We see that the equation for u is the non-homogenous wave equation. System (5) and similar were widely studied in many cases (see [2, 5–8, 10–12, 15–18] and many others).

In our considerations, it is important that, by the second law of thermodynamics, we predict that

$$\lim_{t\to\infty} u(t,\cdot) = 0 \quad \text{and} \quad \lim_{t\to\infty} \theta(t,\cdot) = \text{constant}.$$

It has been shown lately in paper [3] for $n = 1$. It means that Theorem 1.1 perhaps can be applied to the upper equation in (5).

The paper is divided into three sections. We consider the non-homogenous harmonic oscillator equation in Section 2. It is a well-known ordinary differential equation of the second order. It can be said that this is an ordinary version of a hyperbolic equation. It is shown that the solution of the harmonic oscillator equation disappears when $t \to \infty$ if and only if the initial values and the right-hand side satisfy certain conditions. They uniquely connect them. As a corollary, we obtain that the non-homogenous oscillator equation with a fading solution at infinity has a unique solution. In the last section, we consider a non-homogenous hyperbolic equation. There, we prove the main theorem. Again, it occurs that the equation has a unique solution in the class of functions that disappear when time goes to infinity.

2 Harmonic oscillator

In this section, we consider an ordinary differential equation

$$y''(t) + \mu^2 y(t) = f(t) \quad \text{for } t \geq 0, \tag{6}$$

where the constant $\mu > 0$ and the function $f\colon [0, \infty) \to \mathbb{R}$ are given.

Theorem 2.1. *Let $f\colon [0, \infty) \to \mathbb{R}$ be such that the integrals*

$$\int_0^\infty \sin(\mu s) f(s) ds, \quad \int_0^\infty \cos(\mu s) f(s) ds \tag{7}$$

are convergent in the Riemann sense and let y satisfy equation (6)*. Then,*

$$\lim_{t \to \infty} y(t) = 0$$

if and only if

$$y(0) = \frac{1}{\mu} \int_0^\infty \sin(\mu s) f(s) ds, \quad y'(0) = -\int_0^\infty \cos(\mu s) f(s) ds. \tag{8}$$

Proof. We know that the general solution of equation (6) is given by the formula

$$y(t) = \cos(\mu t) \left(C_1 - \frac{1}{\mu} \int_0^t \sin(\mu s) f(s) ds \right) + \sin(\mu t) \left(C_2 + \frac{1}{\mu} \int_0^t \cos(\mu s) f(s) ds \right)$$

for $t \geq 0$ and for arbitrary $C_1, C_2 \in \mathbb{R}$.

Thus, we see that the solution of (6) with initial values (8) is as follows

$$\begin{aligned} y(t) = \cos(\mu t) &\left(\frac{1}{\mu} \int_0^\infty \sin(\mu s) f(s) ds - \frac{1}{\mu} \int_0^t \sin(\mu s) f(s) ds \right) \\ &+ \sin(\mu t) \left(-\frac{1}{\mu} \int_0^\infty \cos(\mu s) f(s) ds + \frac{1}{\mu} \int_0^t \cos(\mu s) f(s) ds \right). \end{aligned} \tag{9}$$

Hence, we have that $y(t) \to 0$, when $t \to \infty$. Therefore, if equalities (8) are satisfied, then $y(t) \to 0$, when $t \to \infty$.

Let us assume that one of the equalities from (8) is not satisfied. Then, we have that

$$\left(C_1 - \frac{1}{\mu} \int_0^t \sin \mu s f(s) ds \right) \not\to 0 \text{ or } \left(C_2 + \frac{1}{\mu} \int_0^t \cos \mu s f(s) ds \right) \not\to 0, \text{ when } t \to \infty.$$

Therefore, we see that the solution cannot $y(t) \to 0$ if $t \to \infty$. □

We finish the section with the following corollary.

Corollary 2.2. *Let us assume that f is such that the integrals in* (7) *are convergent in the Riemann sense, then the problem*

$$\begin{cases} y''(t) + \mu^2 y(t) = f(t) \quad \text{for } t \geq 0, \\ \lim_{t \to \infty} y(t) = 0 \end{cases}$$

has a unique solution.

3 Wave equation

In this section, we consider a second-order non-homogenous hyperbolic equation. We prove the main theorem of the paper here. Let $\{\varphi_m\}$ be a sequence of the operator L eigenfunctions in $H_0^1(\Omega)$. We assume that $\|\varphi_m\|_{L^2(\Omega)} = 1$. We know (see for instance [9]) that span$\{\varphi_m\}$ is a dense set in $H_0^1(\Omega)$. Let $\{\lambda_m\}$ be a sequence of the operator L eigenvalues. We suppose also that $g \in H_0^1(\Omega), h \in L^2(\Omega)$ and $f \in L^2_{\text{loc}}\left([0,\infty); L^2(\Omega)\right)$. Let us remind here that with L is associated the bilinear form

$$B(u,v) = \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} dx.$$

By the separation of variables technique, we know that the solution of the problem (1) is given by the formula

$$u(x,t) = \sum_{m=1}^\infty d_m(t)\varphi_m(x) \text{ in } L^\infty_{\text{loc}}([0,\infty), L^2(\Omega)) \cap C([0,\infty), L^2(\Omega)), \tag{10}$$

where functions $\{d_m\}$ satisfy

$$\begin{cases} d_m'' + \lambda_m d_m = (f, \varphi_m)_{L^2(\Omega)} & \text{on } [0,\infty), \\ d_m(0) = (g,\varphi_m)_{L^2(\Omega)},\ d_m'(0) = (h,\varphi_m)_{L^2(\Omega)} \end{cases} \tag{11}$$

for all $m \in \mathbb{N}$. We shall prove that the function in (10) is well defined.

Proposition 3.1. *Let us assume that $g \in H_0^1(\Omega), h \in L^2(\Omega)$ and $f \in L^2_{\text{loc}}\left([0,\infty); L^2(\Omega)\right)$. Then, the function in formula (10) is well defined.*

Proof. Let us take $T > 0$. We see that

$$u_m := \sum_{k=1}^m d_k \varphi_k$$

is a sequence from the Galerkin method applied to the problem (1). Thus, by [9], we know that $\{u_m\}$ is bounded in $L^\infty\left(0,T; H_0^1(\Omega)\right) \cap W^{1,\infty}\left(0,T; L^2(\Omega)\right) \cap H^2(0,T; H^{-1}(\Omega))$. Hence, we can take a subsequence of $\{u_m\}$ (still denoted as $\{u_m\}$) and $u \in L^\infty\left(0,T; H_0^1(\Omega)\right) \cap W^{1,\infty}\left(0,T; L^2(\Omega))\right) \cap H^2(0,T; H^{-1}(\Omega))$ such that

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u \text{ in } L^\infty\left(0,T; H_0^1(\Omega)\right), \\ u_m' &\overset{*}{\rightharpoonup} u' \text{ in } L^\infty\left(0,T; L^2(\Omega)\right), \\ u_m'' &\rightharpoonup u'' \text{ in } L^2\left(0,T; H^{-1}(\Omega)\right). \end{aligned} \tag{12}$$

We know that u is a solution of (1), which is unique. Therefore, in (12), we can take a whole sequence, not only the subsequence. By Aubin-Lions lemma (see [4]), we get that

$$u_m \to u \text{ in } C\left([0,T], L^2(\Omega)\right).$$

□

Because d_m satisfies (11), so we have an analogous formula as in (9)

$$\begin{aligned} d_m(t) = \cos\left(\sqrt{\lambda_m}t\right)&\left((g,\varphi_m)_{L^2(\Omega)} - \frac{1}{\sqrt{\lambda_m}}\int_0^t \sin\left(\sqrt{\lambda_m}s\right)(f(s,\cdot),\varphi_m)_{L^2(\Omega)}ds\right) \\ &+ \sin\left(\sqrt{\lambda_m}t\right)\left(\frac{1}{\sqrt{\lambda_m}}(h,\varphi_m)_{L^2(\Omega)} + \frac{1}{\sqrt{\lambda_m}}\int_0^t \cos\left(\sqrt{\lambda_m}s\right)(f(s,\cdot),\varphi_m)_{L^2(\Omega)}ds\right) \end{aligned} \tag{13}$$

for all $m \in \mathbb{N}$.

Now, we can prove the paper's main result.

Proof of Theorem 1.1. Let us assume that (3) is satisfied, and let us take $m \in \mathbb{N}$. Then, $(u(t,\cdot),\varphi_m)_{L^2(\Omega)} = d_m(t) \to 0$, when $t \to \infty$. By Theorem 2.1, it implies (4).

Next, let us assume that (4) is satisfied for all $m \in \mathbb{N}$. We need a certain inequality before we will prove the thesis. First, we will justify it for

$$u_m := \sum_{k=1}^{m} d_k \varphi_k,$$

where d_k are defined above.

Let us take u_m' as a test function in the weak formulation of the problem for u_m. We get

$$\int_\Omega u_m'' u_m' dx + B(Du_m, Du_m') = \int_\Omega f u_m' dx.$$

It simply yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u_{m,t}^2 dx + B(Du_m, Du_m)\right) = \int_\Omega f u_m' dx.$$

Now, Let us fix $T > 0$ and let us take $t \in [0,T]$. We integrate the above equality over $[0,t]$. It gives us

$$\begin{aligned}\int_\Omega u_{m,t}^2 dx + B(u_m,u_m) &= 2\int_0^t \int_\Omega f u_t \, dx\, ds + \|h\|_{L^2(\Omega)}^2 + B(g,g) \\ &\le 2\|u_m'\|_{L^\infty(0,T;L^2(\Omega))}\|f\|_{L^1(0,T;L^2(\Omega))} + \|h\|_{L^2(\Omega)}^2 + C\|g\|_{H_0^1(\Omega)}^2 \\ &\le \frac{1}{4}\|u_m'\|_{L^\infty(0,T;L^2(\Omega))}^2 + 4\|f\|_{L^1(0,\infty;L^2(\Omega))}^2 + \|h\|_{L^2(\Omega)}^2 + C\|g\|_{H_0^1(\Omega)}^2,\end{aligned}$$

where $C = (\sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(\Omega)})^{\frac{1}{2}}$. It implies that

$$\begin{aligned}\frac{1}{2}\left(\|u_m'\|_{L^\infty(0,T;L^2(\Omega))}^2 + \lambda\|u_m\|_{L^\infty(0,T;H_0^1(\Omega))}^2\right) &\le \sup_{t\in[0,T]}\left(\int_\Omega u_{m,t}^2 dx + \lambda\int_\Omega |Du_m|^2 dx\right) \\ \le \frac{1}{4}\|u_m'\|_{L^\infty(0,T;L^2(\Omega))}^2 + 4\|f\|_{L^1(0,\infty;L^2(\Omega))}^2 &+ \|h\|_{L^2(\Omega)}^2 + C\|g\|_{H_0^1(\Omega)}^2,\end{aligned}$$

where λ is defined in (2). Hence, we arrive with

$$\begin{aligned}\frac{\lambda}{2}\|u_m\|_{L^\infty(0,T;H_0^1(\Omega))}^2 &\le \frac{1}{4}\|u_m'\|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{\lambda}{2}\|u_m\|_{L^\infty(0,T;H_0^1(\Omega))}^2 \\ &\le 4\|f\|_{L^1(0,\infty;L^2(\Omega))}^2 + \|h\|_{L^2(\Omega)}^2 + \|g\|_{H_0^1(\Omega)}^2.\end{aligned}$$

Because $T > 0$ was arbitrary, so we see that the sequence is bounded $\{u_m\}$ in $L^\infty(0,\infty;H_0^1(\Omega))$. Thus, because u is a *-weak limit of $\{u_m\}$, it derives that $u \in L^\infty(0,\infty;H_0^1(\Omega))$.

Now, we think about u as representative such that $u \in C\left([0,\infty), L^2(\Omega)\right)$. Moreover, let us take a measurable set $X \subset [0,\infty)$ such that $|[0,\infty)\setminus X| = 0$ and the function $u|_X$ is bounded as a function with values in $H_0^1(\Omega)$. Next, let us take a sequence $\{t_k\} \subset X$ such that $t_k \to \infty$. A sequence $\{u(t_k,\cdot)\}$ is bounded in $H_0^1(\Omega)$. Moreover for an arbitrary $m \in \mathbb{N}$, we have

$$(u(t_k,\cdot),\varphi_m)_{H_0^1(\Omega)} = d_m(t_k)\|\varphi_m\|_{H_0^1(\Omega)}^2 = d_m(t_k)\lambda_m \to 0 \text{ when } k \to \infty.$$

Because $\operatorname{span}\{\varphi_m\}$ is dense in $H_0^1(\Omega)$, so we obtain

$$u(t_k,\cdot) \rightharpoonup 0 \text{ in } H_0^1(\Omega).$$

By the compact embedding, we have that

$$u(t_k,\cdot) \to 0 \text{ in } L^2(\Omega).$$

Therefore, we obtain

$$u(t,\cdot) \to 0 \text{ in } L^2(\Omega) \text{ when } t \in X \text{ and } t \to \infty.$$

Because $u \in C\left([0,\infty), L^2(\Omega)\right)$, so one can easy prove that

$$u(t,\cdot) \to 0 \text{ in } L^2(\Omega) \text{ when } t \to \infty.$$

Details are left to the reader. □

Ultimately, we formulate and prove an analogous result as Corollary 2.2 for the wave equation.

Proposition 3.2. *Let us assume that the hypotheses of Theorem 1.1 are satisfied. Then, the problem*

$$\begin{cases} u_{tt} + Lu = f & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ \lim\limits_{t\to\infty} \|u(t,\cdot)\|_{L^2(\Omega)} = 0, \end{cases} \tag{14}$$

has a unique solution in $L^\infty_{\text{loc}}\left(0,\infty; H^1_0(\Omega)\right) \cap W^{1,\infty}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right) \cap H^2_{\text{loc}}(0,\infty; H^{-1}(\Omega))$.

Proof. We must show that quantities in (4) define unique initial values for u. Let us remind that $\{\varphi_m\}$ are the eigenfunctions of L and $\{\lambda_m\}$ are eigenvalues of it. The functions $\{\varphi_m\}$ are also normalized in $L^2(\Omega)$. Let us denote

$$f_m(t) := (f(t,\cdot), \varphi)_{L^2(\Omega)}$$

for all $t \in [0,\infty)$ and $m \in \mathbb{N}$. Then, we have

$$f(t,x) := \sum_{m=1}^{\infty} f_m(t)\varphi_m(x)$$

for almost all $(t,x) \in [0,\infty)\times\Omega$ and series is convergent in $L^2(\Omega)$ and pointwise for almost all $t \in [0,\infty)$. It also derives

$$\int_0^\infty \|f(t,\cdot)\|_{L^2(\Omega)} dt = \int_0^\infty \left(\sum_{m=1}^{\infty} f_m^2(t)\right)^{\frac{1}{2}} dt$$

and the integrals are finite because $f \in L^1(0,\infty; L^2(\Omega))$.

First, we will show that the formula in (4) can well define the velocity. Let us denote

$$h_m = -\int_0^\infty \cos\left(\sqrt{\lambda_m}s\right)(f(s,\cdot),\varphi_m)_{L^2(\Omega)} ds$$

for all $m \in \mathbb{N}$. We want to show that $\{h_m\} \in \ell^2$. The Minkowski generalized inequality (see [1])implies

$$\sum_{m=1}^{\infty} h_m^2 = \sum_{m=1}^{\infty}\left(\int_0^\infty \cos\left(\sqrt{\lambda_m}s\right)(f(s,\cdot),\varphi_m)_{L^2(\Omega)} ds\right)^2 \le \left(\int_0^\infty \left(\sum_{m=1}^{\infty} f_m^2(t)\right)^{\frac{1}{2}} dt\right)^2.$$

The integral on the right-hand side is finite, so $\{h_m\} \in \ell^2$. Therefore, the formula

$$h := \sum_{m=1}^{\infty} h_m\varphi_m$$

defines the function h properly.

Next, we are going to the initial value for u. Henceforth, we will write

$$\|w\|_{H_0^1(\Omega)} = (B(w,w))^{\frac{1}{2}}$$

for $w \in H_0^1(\Omega)$. It is equivalent norm to the standard norm in $H_0^1(\Omega)$. Let us denote

$$g_m := \frac{1}{\sqrt{\lambda_m}} \int_0^\infty \sin\left(\sqrt{\lambda_m} s\right) (f(s,\cdot), \varphi_m)_{L^2(\Omega)} ds.$$

One can easily see that

$$\|\varphi_m\|_{H_0^1(\Omega)} = \sqrt{\lambda_m}.$$

On the other hand, we show that $\{h_m \sqrt{\lambda_m}\} \in \ell^2$ similarly as above. Therefore, the formula

$$g := \sum_{m=1}^{\infty} g_m \varphi_m$$

defines $g \in H_0^1(\Omega)$. We see that f, g, h satisfy condition (4).

Let us consider $u \in L^\infty_{\text{loc}}\left(0,\infty; H_0^1(\Omega)\right) \cap W^{1,\infty}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right) \cap H^2_{\text{loc}}(0,\infty; H^{-1}(\Omega))$ the weak solution of the problem

$$\begin{cases} u_t + Lu = f & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = g, u_t = h & \text{on } \{0\} \times \Omega. \end{cases} \tag{15}$$

Then, Theorem 1.1 gives us that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^2(\Omega)} = 0.$$

On the other, let w be another weak solution of (14). Then, by Theorem 1.1, it has to be a weak solution of (15). Because it is unique, so $w = u$. □

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Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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