Two Twinning operators when imposing nonlinear modulation in Short-Time Fourier Transform

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Keywords: Short-time Fourier transform, nonlinear modulation, uncertainty principle.

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1 Introduction

Let $\mathcal{F}$ be the conventional Fourier transform

$$\big(\mathcal{F}f\big)(\omega) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-it\omega}\, f(t)\,\mathrm{d}t, \quad \omega\in\mathbb{R} \tag{1.1}$$

for $f\in\mathcal{S}(\mathbb{R})$, the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}$. The Fourier transform can be extended to $L^2(\mathbb{R})$ by density arguments, which maps $L^2(\mathbb{R})$ onto itself. Furthermore, $L^p(\mathbb{R})$ with $0<p\le\infty$, is the standard complex quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} \tag{1.2}$$

with the obvious modification if $p=\infty$(see Refs.[4] and [7]). For the purpose of time-frequency localization, the short-time Fourier transform(STFT) for any function $f\in L^2(\mathbb{R})$ is defined by

$$\mathcal{V}_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}\,e^{-i\omega t}\,\mathrm{d}t, \qquad (x,\omega)\in\mathbb{R}^2, \tag{1.3}$$

where the basic atom g is specially chosen function to guarantee the convergence of the right side integral in (1.3). Essentially, $\mathcal{V}_g f$ is the inner product $\langle f, \mathcal{M}_\omega\mathcal{T}_x g\rangle$ of f and the kernel function $\mathcal{M}_\omega\mathcal{T}_x g$[9], where $\mathcal{M}_\omega$ and $\mathcal{T}_x$ are the modulation and translation operators, given respectively by

$$\mathcal{M}_\omega f(t) = e^{i\omega t} f(t), \quad \mathcal{T}_x f(t) = f(t-x). \tag{1.4}$$

The modulation and translation are important tools in the time-frequency analysis and harmonic analysis, which satisfy the following noncommutative relation $\mathcal{T}_x\mathcal{M}_\omega = e^{-ix\omega}\mathcal{M}_\omega\mathcal{T}_x = \rho_{-x\omega}\mathcal{M}_\omega\mathcal{T}_x$, where ρ_d is the rotation transform defined as $\rho_d: f\to e^{id}f$(see Refs.[11] and [14]). The short-time Fourier transform $\mathcal{V}_g$ is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$ and admits the corresponding inversion formula

$$f(\cdot) = \frac{1}{\langle\tilde g, g\rangle}\iint_{\mathbb{R}^2} \mathcal{V}_g f(x,\omega)\,\mathcal{M}_\omega\mathcal{T}_x\tilde g(\cdot)\,\mathrm{d}\omega\mathrm{d}x, \tag{1.5}$$

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for some synthesis window function $\tilde{g}$(see Ref.[6]).

Remark that the short-time Fourier transform is based on the linear modulation $e^{it\omega}$, and there exist fruitful studies on the short-time Fourier transform(see Refs.[1],[2],[3],[4],[5]). Many scholars investigated the so called nonlinear modulation in different settings(see Ref.[8]). For instance, an integral version of the non-harmonic Fourier series, called the Chirp transform was studied in [8]. It is definded as

$$\big(\mathcal{F}_{\phi,\theta}f\big)(\omega) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\phi(t)\theta(\omega)}\, f(t)\,\mathrm{d}t, \quad \omega\in\mathbb{R}$$

for $f\in L^1(\mathbb{R})$, where $\theta(t)$ as well as $\phi(t)$ is close enough to t in some sense. It is proven that the Chirp transform is a unitary isometric mapping from $L^2(\mathbb{R},\mathrm{d}\mu_\phi)$ onto $L^2(\mathbb{R},\mathrm{d}\theta)$ with the measure $\mathrm{d}\mu_\phi(x)=\frac{1}{\phi'(x)}\mathrm{d}x$. We are motivated to investigate some special STFTs with the nonlinear modulation $e^{it\theta_a(\omega)}$, where the phase function θ_a is originated from the nonlinear Fourier atoms $e^{i\theta_a(t)}$, the boundary value of the Möbius transform $\frac{z-a}{1-az}$ with real parameter a on the unit circle. Nonlinear Fourier atom $e^{i\theta_a(t)}, t\in\mathbb{R}$ with $a\in(-1,1)$ is the boundary values

$$e^{i\theta_a(t)} := \tau_a(e^{it}) \tag{1.6}$$

of the of Blaschke products of order 1 (known also as the Möbius transform)

$$\tau_a(z) := \frac{z-a}{1-\overline{a}z},\ \ z\in\mathbb{C}.$$

for some parameter a in the unit disc. By noting $a=|a|e^{it_a}$, it is well-known that

$$\theta_a(t) = t + 2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}, \qquad \forall t\in\mathbb{R}.$$

One can see that the derivative of θ_a is just the Poisson kernel in periodic setting up to a constant factor, that is,

$$\theta_a^{'}(t) := p_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2},$$

Moreover, we have

$$\theta_a^{-1} = \theta_{-a}. \tag{1.7}$$

In particular, if a is real, then θ_a is odd $\theta_a(-t)=-\theta_a(t)$ and vice versa(see Ref.[8]).

The nonlinear modulation operator $\mathcal{M}_{\theta_a}$ with respect to θ_a is defined as

$$\mathcal{M}_{\theta_a(\omega)} : f(x)\to e^{i\theta_a(\omega)x}f(x),\ \ x\in\mathbb{R} \tag{1.8}$$

and the nonlinear translation $\mathcal{T}_{\theta_a(x)}$ with respect to θ_a is defined as

$$\mathcal{T}_{\theta_a(x)} : f\to f(\cdot-\theta_a(x)). \tag{1.9}$$

In this note, we will introduce a special short-time Fourier transforms in terms of the nonlinear phase function θ_a, named the short-time chirp transform $\mathcal{W}_g^{\theta_a}f$, which is defined by

$$\mathcal{W}_g^{\theta_a}f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}\,e^{-it\theta_a(\omega)}\,\mathrm{d}t, \quad (x,\omega)\in\mathbb{R}^2. \tag{1.10}$$

We remark that, when choosing a suitable windowed function g, the operator $\mathcal{W}_g^{\theta_a}$ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2,dm)$ with $dm=p_a(x)dxd\omega$.

It is crucial to realize that the operator $\mathcal{W}_g^{\theta_a}$ in frequency domain is closely related to the operator

$$\mathcal{V}_g^{\theta_a}f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-\theta_a(x))}\,e^{-it\omega}\,\mathrm{d}t, \quad (x,\omega)\in\mathbb{R}^2, \tag{1.11}$$

which we address it the nonlinearly-sliding window Fourier transform. Obviously, both the operators $\mathcal{W}_g^{\theta_a}$ and $\mathcal{V}_g^{\theta_a}$ are reduced to the conventional STFT $\mathcal{V}g$ in the case $a=0$. What's more, using Fourier transform, (1.3) and (1.11) can be written respectively as

$$\mathcal{V}_gf(x,\omega) = \sqrt{2\pi}\mathcal{F}(f\cdot\mathcal{T}_x\overline{g})(\omega), \tag{1.12}$$

and

$$\mathcal{V}_g^{\theta_a} f(x,\omega) = \mathcal{F}(f\cdot \mathcal{T}_{\theta_a(x)}\overline{g})(\omega). \tag{1.13}$$

In particular, using Chirp transform (restrict $\phi = 1$), (1.10) can be expressed as

$$\mathcal{W}_g^{\theta_a} f(x,\omega) = \mathcal{F}_{\phi,\theta}(f\cdot \mathcal{T}_x\overline{g})(\omega). \tag{1.14}$$

We will represent both of the operators $\mathcal{W}_g^{\theta_a}$ and $\mathcal{V}_g^{\theta_a}$ in different forms, reveal their relationship, establish the formulae of energy-preservation and design the corresponding inversion formulae.

The writing plan is as follows. Section 2 is devoted to the twin in time-frequency domain of $\mathcal{V}_g^{\theta_a}$ and $\mathcal{W}_g^{\theta_a}$. Section 3 contributes to the exploration of the orthogonality relations of $\mathcal{V}_g^{\theta_a}$ and $\mathcal{W}_g^{\theta_a}$. Section 4 focuses on establishing inversion formulae for $\mathcal{V}_g^{\theta_a}$ and $\mathcal{W}_g^{\theta_a}$. Section 5 is concentrated to the Lieb type inequality and uncertainty principles of $\mathcal{V}_g^{\theta_a}$.

2 Twin of $\mathcal{W}_g^{\theta_a}$ and $\mathcal{V}_g^{\theta_a}$ in time-frequency domain

From now on, we restrict the parameter a to be a real number. As mentioned above, both of the operators $\mathcal{W}_g^{\theta_a}$ and $\mathcal{V}_g^{\theta_a}$ are reduced to the conventional STFT $\mathcal{V}g$ in the case $a = 0$, the hint of which suggests some close relationships between the operators are waiting for figuring out.

Noting that the nonlinearly-sliding window Fourier transform can be regarded as the inner product form $\mathcal{V}_g^{\theta_a} f(x,\omega) = \langle f, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)} g\rangle$ and applying the Plancherel formula, it arrives at

$$\begin{aligned}\mathcal{V}_g^{\theta_a} f(x,\omega) &= \langle \widehat{f}, \mathcal{T}_\omega \mathcal{M}_{-\theta_a(x)}\widehat{g}\rangle \\ &= \int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi-\omega)\, e^{-i(\xi-\omega)\theta_a(x)}}\,\mathrm{d}\xi \\ &= e^{-i\theta_a(x)\omega}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi-\omega)}\, e^{i\xi\theta_a(x)}\,\mathrm{d}\xi.\end{aligned}$$

Similarly, the definition of the short-time Chirp transform gives rise to

$$\begin{aligned}\mathcal{W}_g^{\theta_a}(x,\omega) &= \int_{\mathbb{R}} f(t)\overline{g(t-x)}\, e^{-it\theta_a(\omega)}\,\mathrm{d}t \\ &= \langle f, \mathcal{M}_{\theta_a(\omega)}\mathcal{T}_x g\rangle \\ &= \langle \widehat{f}, \mathcal{T}_{\theta_a(\omega)}\mathcal{M}_{-x}\widehat{g}\rangle \\ &= \int_{\mathbb{R}} \widehat{f}(\xi)\,\overline{e^{-ix(\xi-\theta_a(\omega))}\,\widehat{g}(\xi-\theta_a(\omega))}\,\mathrm{d}\xi \\ &= e^{-i\theta_a(\omega)x}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi-\theta_a(\omega))}\, e^{ix\xi}\,\mathrm{d}\xi.\end{aligned}$$

Then we obtain the following theorem.

Theorem 2.1. *If $f, g \in L^2(R)$, then*

$$\mathcal{V}_g^{\theta_a} f(x,\omega) = e^{-i\theta_a(x)\omega}\mathcal{W}_{\widehat{g}}^{\theta_a}\widehat{f}(\omega,-x), \quad (x,\omega)\in\mathbb{R}^2. \tag{2.15}$$

and

$$\mathcal{W}_g^{\theta_a} f(x,\omega) = e^{-i\theta_a(\omega)x}\mathcal{V}_{\widehat{g}}^{\theta_a}\widehat{f}(\omega,-x), \quad (x,\omega)\in\mathbb{R}^2 \tag{2.16}$$

Proof: From the above calculations, we know that

$$\mathcal{W}_g^{\theta_a}(x,\omega) = e^{-i\theta_a(\omega)x}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi-\theta_a(\omega))}\, e^{ix\xi}\,\mathrm{d}\xi.$$

According to the definition of $\mathcal{V}_g^{\theta_a} f$, it is easy to conclude the identity (2.16).

Equation (2.15) is a direct consequence of the identity

$$\mathcal{V}_g^{\theta_a} f(x,\omega) = e^{-i\theta_a(x)\omega}\int_{\mathbb{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi-\omega)}\, e^{i\xi\theta_a(x)}\,\mathrm{d}\xi$$

by noting that $\theta_a(\cdot)$ is an odd functions. The proof of this theorem is completed. □

Remark: We see from above formulae that two affine transforms are useful in time-frequency plane

$$\mathcal{I}_1 : (x,\omega) \longleftarrow (x,\omega)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\omega, x)$$

and

$$\mathcal{I}_2 : (x,\omega) \longleftarrow (x,\omega)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (\omega, -x).$$

Define $E(x,\omega) = e^{-ix\theta_a(\omega)}, (x,\omega) \in \mathbb{R}^2$. Then $E(\mathcal{I}_1(x,\omega)) = e^{-i\omega\theta_a(x)}$. The formulae (2.16) and (2.15) can be rewritten as, respectively,

$$\mathcal{V}_g^{\theta_a} f(x,\omega) = E(\mathcal{I}_1(x,\omega))\mathcal{W}_{\widehat{g}}^{\theta_a}\widehat{f}(\mathcal{I}_2(x,\omega)), \quad (x,\omega) \in \mathbb{R}^2. \tag{2.17}$$

and

$$\mathcal{W}_g^{\theta_a} f(x,\omega) = E(x,\omega)\mathcal{V}_{\widehat{g}}^{\theta_a}\widehat{f}(\mathcal{I}_2(x,\omega)), \quad (x,\omega) \in \mathbb{R}^2. \tag{2.18}$$

We also remark that, when $a = 0$, the twin formulae (2.17) and (2.18) are reduced to the identity

$$V_g f(x,\omega) = e^{-ix\omega} V_{\hat{g}}\hat{f}(\mathcal{I}_2(x,\omega)),$$

which is just the alternative form of the conventional short-time Fourier transform in terms of the window g and the signal f simultaneously in frequency domain.

3 Orthogonality Relations

We are ready to reconstruct any signal $f \in L^2(\mathbb{R})$ from the image spaces $V_g^{\theta_a}(L^2(\mathbb{R}^2))$ and $W_g^{\theta_a}(L^2(\mathbb{R}^2))$, both the subspaces of $L^2(\mathbb{R}^2, dm)$ with the Lebesgue measure $\mathrm{d}m = p_a(x)\,\mathrm{d}x\mathrm{d}\omega$ and the periodic Poisson kernel $p_a(x) = \theta_a^{'}(x) = |\frac{1-a^2}{e^{ix}-a}|$ with real parameter a.

Theorem 3.1. *Suppose that f_1, f_2, g_1, g_2 are functions in $L^2(\mathbb{R})$. Then $V_{g_j}^{\theta_a} f_j \in L^2(\mathbb{R}^2, \mathrm{d}m)$ for $j = 1, 2$, and*

$$\langle \mathcal{V}_{g_1}^{\theta_a} f_1, \mathcal{V}_{g_2}^{\theta_a} f_2\rangle_{L^2(\mathbb{R}^2,\mathrm{d}m)} = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{3.19}$$

Proof: We first restrict the windows g_j in $L^1 \cap L^\infty \subseteq L^2(\mathbb{R})$. Then we know that $f_j \cdot \mathcal{T}_{\theta_a(x)}\overline{g_j} \in L^2(\mathbb{R})$ for all $x \in \mathbb{R}$. Combining the definition of $\mathcal{V}_g^{\theta_a}$ and applying the Parseval's identity, it yields

$$\begin{aligned}
&\langle \mathcal{V}_{g_1}^{\theta_a} f_1, \mathcal{V}_{g_2}^{\theta_a} f_2\rangle_{L^2(\mathbb{R}^2,\mathrm{d}m)} \\
&= \iint_{\mathbb{R}^2} \mathcal{V}_{g_1}^{\theta_a} f_1(x,\omega)\,\overline{\mathcal{V}_{g_2}^{\theta_a} f_2(x,\omega)}\,\mathrm{d}m(x,\omega) \\
&= \iint_{\mathbb{R}^2} \mathcal{V}_{g_1}^{\theta_a} f_1(x,\omega)\,\overline{\mathcal{V}_{g_2}^{\theta_a} f_2(x,\omega)}\,\mathrm{d}\omega\, p_a(x)\mathrm{d}x \\
&= \iint_{\mathbb{R}^2} \mathcal{F}(f_1 \cdot \mathcal{T}_{\theta_a(x)}\overline{g_1})(\omega)\,\overline{\mathcal{F}(f_2 \cdot \mathcal{T}_{\theta_a(x)}\overline{g_2})(\omega)}\,\mathrm{d}\omega\, p_a(x)\mathrm{d}x \\
&= \iint_{\mathbb{R}^2} f_1(t)\overline{f_2(t)}\,\overline{g_1(t-\theta_a(x))}g_2(t-\theta_a(x))\,\mathrm{d}t\, p_a(x)\mathrm{d}x.
\end{aligned}$$

Applying the integral transformation $\begin{cases} t = \tilde{t} \\ \theta_a(x) = \tilde{t} - \tilde{x} \end{cases}$ to above integral and noting that the Jacobbi factor $\frac{\partial(t,x)}{\partial(\tilde{t},\tilde{x})} = \begin{vmatrix} 1 & 0 \\ (\theta_a^{-1})'(x) & -(\theta_a^{-1})'(x) \end{vmatrix} = -\frac{1}{\theta_a'(x)} = -\frac{1}{\theta_a'(\theta_a^{-1}(\tilde{t}-\tilde{x}))}$, noting that $f_1\overline{f_2}$, $\overline{g_1}g_2 \in L^1(\mathbb{R})$ and using Fubini's theorem, it gives rise to

$$\begin{aligned}
\langle \mathcal{V}_{g_1}^{\theta_a} f_1, \mathcal{V}_{g_2}^{\theta_a} f_2\rangle_{L^2(\mathbb{R}^2,\mathrm{d}m)} &= \iint_{\mathbb{R}^2} f_1(\tilde{t})\overline{f_2(\tilde{t})}\,\overline{g_1(\tilde{x})}g_2(\tilde{x})\,|-\frac{1}{\theta_a'(\theta_a^{-1}(\tilde{t}-\tilde{x}))}|\, p_a(\theta_a^{-1}(\tilde{t}-\tilde{x}))\,\mathrm{d}\tilde{t}\mathrm{d}\tilde{x} \\
&= \int_{\mathbb{R}} f_1(\tilde{t})\,\overline{f_2(\tilde{t})}\big[\int_{\mathbb{R}} \overline{g_1(\tilde{x})}\,g_2(\tilde{x})\,\mathrm{d}\tilde{x}\big]\mathrm{d}\tilde{t} \\
&= \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}.
\end{aligned}$$

By a standard density argument, we can extend above orthogonal relationship to $L^2(\mathbb{R})$. For fixed $g_1 \in L^1 \cap L^\infty$, the mapping $g_2 \longmapsto \langle \mathcal{V}_{g_1}^{\theta_a} f_1, \mathcal{V}_{g_2}^{\theta_a} f_2 \rangle_{L^2(\mathbb{R}^2, \mathrm{d}m)}$ is a linear functional that coincides with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on the dense subspace $L^1 \cap L^\infty$. It is therefore bounded and extends to all $g_2 \in L^2(\mathbb{R})$. In the same way, for arbitrary f_1, f_2 and $g_2 \in L^2(\mathbb{R})$, the conjugate linear functional $g_1 \longmapsto \langle \mathcal{V}_{g_1}^{\theta_a} f_1, \mathcal{V}_{g_2}^{\theta_a} f_2 \rangle_{L^2(\mathbb{R}^2, \mathrm{d}m)}$ equals to $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on $L^1 \cap L^\infty$ and extends to all of L^2. We therefore conclude the orthogonality relations for all $f_j, g_j \in L^2(\mathbb{R})$ and $\mathcal{V}_g^{\theta_a} \in L^2(\mathbb{R}^2, \mathrm{d}m)$. □

It fails to apply the same techniques in Theorem 3.1 to the short-time chirp transform $\mathcal{W}_g^{\theta_a} f$ when dealing with the issue of orthogonality of $\mathcal{W}_g^{\theta_a} f$ since $\mathcal{W}_g^{\theta_a} f$ can only be rewritten in terms of chirp Fourier transform (see (1.14)) rather than conventional Fourier transform. Fortunately, by using Theorem 2.1, we are able to establish the following orthogonality relationship for the short-time chirp transform.

Theorem 3.2. *For arbitrary f_1, f_2, g_1, $g_2 \in L^2(\mathbb{R})$, it holds $\mathcal{W}_{g_j}^{\theta_a} f_j \in L^2(\mathbb{R}^2, \mathrm{d}\tilde{m})$ with $\mathrm{d}\tilde{m} = p_a(\omega)\mathrm{d}x\mathrm{d}\omega$ for $j = 1, 2$, and*

$$\langle \mathcal{W}_{g_1}^{\theta_a} f_1, \mathcal{W}_{g_2}^{\theta_a} f_2 \rangle_{L^2(\mathbb{R}^2, \mathrm{d}\tilde{m})} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \tag{3.20}$$

Proof: Recalling the identity (2.16), we know that

$$\begin{aligned}
&\langle \mathcal{W}_{g_1}^{\theta_a} f_1, \mathcal{W}_{g_2}^{\theta_a} f_2 \rangle_{L^2(\mathbb{R}^2, \mathrm{d}\tilde{m})} \\
&= \iint_{\mathbb{R}^2} \mathcal{W}_{g_1}^{\theta_a} f_1(x,\omega)\, \overline{\mathcal{W}_{g_2}^{\theta_a} f_2(x,\omega)}\, \mathrm{d}\tilde{m}(x,\omega) \\
&= \iint_{\mathbb{R}^2} e^{-i\theta_a(\omega)x}\, \mathcal{V}_{\widehat{g_1}}^{\theta_a} \widehat{f_1}(\omega, -x)\, \overline{e^{-i\theta_a(\omega)x}\, \mathcal{V}_{\widehat{g_2}}^{\theta_a} \widehat{f_2}(\omega, -x)}\, p_a(\omega)\mathrm{d}x\mathrm{d}\omega \\
&= \iint_{\mathbb{R}^2} \mathcal{V}_{\widehat{g_1}}^{\theta_a} \widehat{f_1}(\omega, -x)\, \mathcal{V}_{\widehat{g_2}}^{\theta_a} \widehat{f_2}(\omega, -x)\ p_a(\omega)\mathrm{d}\omega\mathrm{d}x.
\end{aligned}$$

Applying the integral transform $\begin{cases} x = -\tilde{\omega}, \\ \omega = \tilde{x} \end{cases}$, we get

$$\begin{aligned}
&\langle \mathcal{W}_{g_1}^{\theta_a} f_1, \mathcal{W}_{g_2}^{\theta_a} f_2 \rangle_{L^2(\mathbb{R}^2, \mathrm{d}\tilde{m})} \\
=\ & \iint_{\mathbb{R}^2} \mathcal{V}_{\widehat{g_1}}^{\theta_a} \widehat{f_1}(x,\omega)\, \mathcal{V}_{\widehat{g_2}}^{\theta_a} \widehat{f_2}(x,\omega)\ p_a(x)\mathrm{d}x\mathrm{d}\omega \\
=\ & \langle \mathcal{V}_{\widehat{g_1}}^{\theta_a} \widehat{f_1}, \mathcal{V}_{\widehat{g_2}}^{\theta_a} \widehat{f_2} \rangle_{L^2(\mathbb{R}^2, \mathrm{d}m)}.
\end{aligned}$$

Finally, the orthogonality (3.19) of $\mathcal{V}_g^{\theta_a}$ concludes that of $\mathcal{W}_g^{\theta_a}$. □

As a consequence of Theorems 3.1 and 3.2, it follows the next corollary, which shows that when choosing the windowed function g normalized by $\|g\|_2 = 1$, both $\mathcal{V}_g^{\theta_a}$ and $\mathcal{W}_g^{\theta_a}$ are isometries from $L^2(\mathbb{R})$ onto their image spaces $\mathcal{V}_g^{\theta_a}(L^2(\mathbb{R}))$ and $\mathcal{W}_g^{\theta_a}(L^2(\mathbb{R}))$, respectively.

Corollary 3.3. *For $f, g \in L^2(\mathbb{R})$ and $\|g\|_2 = 1$, it holds*

$$\|\mathcal{W}_g^{\theta_a} f\|_2 = \|f\|_2, \qquad \|\mathcal{V}_g^{\theta_a} f\|_2 = \|f\|_2.$$

4 Inversion Formula

We now investigate the inversions of $\mathcal{V}_g^{\theta_a}$ and $\mathcal{W}_g^{\theta_a}$. We deal with the nonlinearly-sliding window Fourier transform first.

Theorem 4.1. *Suppose that $g, \gamma \in L^2(\mathbb{R})$ and $\langle g, \gamma \rangle \neq 0$. Then for all $f \in L^2(\mathbb{R})$, we have*

$$f = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^2} \mathcal{V}_g^{\theta_a} f(x,\omega)\, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)} \gamma\, \mathrm{d}m \tag{4.21}$$

where the measure $\mathrm{d}m = p_a(x)\mathrm{d}x\mathrm{d}\omega$ and the convergence is in the weak sense.

Proof: For any fixed $f \in L^2(\mathbb{R})$, Theorem 3.1 shows that the following integral

$$\ell_f := \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{V}_g^{\theta_a} f(x,\omega)\, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)}\gamma \,\mathrm{d}m \tag{4.22}$$

defines a boundary linear functional on $L^2(\mathbb{R})$ and thus is a well-defined function in $L^2(\mathbb{R})$. Moreover, for arbitrary $h \in L^2(\mathbb{R})$, the orthogonality relationship gives rise to

$$\begin{aligned} \ell_f(h) &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{V}_g^{\theta_a} f(x,\omega) \langle \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)}\gamma, h\rangle \,\mathrm{d}m \\ &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{V}_g^{\theta_a} f(x,\omega) \overline{\langle h, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)}\gamma\rangle} \,\mathrm{d}m \\ &= \frac{1}{\langle \gamma, g\rangle} \langle \mathcal{V}_g^{\theta_a} f, \mathcal{V}_\gamma^{\theta_a} h\rangle \;=\; \langle f, h\rangle. \end{aligned}$$

Thus $\ell_f = f$ and the inversion formula concludes from the arbitrariness of f. □

The next theorem shows the inversion formula of the short-time chirp transform, which is essentially a consequence of Theorem 2.1 and 3.1.

Theorem 4.2. *Suppose that $g, \gamma \in L^2(\mathbb{R})$ and $\langle g, \gamma\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R})$*

$$f = \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{W}_g^{\theta_a} f(x,\omega)\, \mathcal{M}_{\theta_a(\omega)} \mathcal{T}_x \gamma \,\mathrm{d}\tilde{m}, \tag{4.23}$$

where $\mathrm{d}\tilde{m} = p_a(\omega)\mathrm{d}\omega\mathrm{d}x$ and the convergence is in the weak sense.

Proof: For any fixed $f \in L^2(\mathbb{R})$, Theorem 3.2 shows that the following integral

$$\ell_f := \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{W}_g^{\theta_a} f(x,\omega)\, \mathcal{M}_{\theta_a(\omega)} \mathcal{T}_x \gamma \,\mathrm{d}\tilde{m} \tag{4.24}$$

defines a boundary linear functional on $L^2(\mathbb{R})$ and thus is a well-defined function in $L^2(\mathbb{R})$. Moreover, for arbitrary $h \in L^2(\mathbb{R})$, Theorem 2.1 and the orthogonality relationships of $\mathcal{V}_g^{\theta_a}$ gives rise to

$$\begin{aligned} \ell_f(h) &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{W}_g^{\theta_a} f(x,\omega) \langle \mathcal{M}_{\theta_a(\omega)} \mathcal{T}_x \gamma, h\rangle \mathrm{d}\tilde{m} \\ &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{W}_g^{\theta_a} f(x,\omega) \overline{\langle h, \mathcal{M}_{\theta_a(\omega)} \mathcal{T}_x \gamma\rangle} \mathrm{d}\tilde{m} \\ &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{W}_g^{\theta_a} f(x,\omega) \overline{\mathcal{W}_\gamma^{\theta_a} h(x,\omega)} \mathrm{d}\tilde{m} \\ &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} e^{-i\theta_a(\omega)x}\, \mathcal{V}_{\widehat{g}}^{\theta_a} \widehat{f}(\omega, -x)\, \overline{e^{-i\theta_a(\omega)x}\, \mathcal{V}_{\widehat{\gamma}}^{\theta_a} \widehat{h}(\omega, -x)} \mathrm{d}\tilde{m} \\ &= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^2} \mathcal{V}_{\widehat{g}}^{\theta_a} \widehat{f}(\omega, -x)\, \overline{\mathcal{V}_{\widehat{\gamma}}^{\theta_a} \widehat{h}(\omega, -x)} \mathrm{d}\tilde{m} \\ &= \frac{1}{\langle \gamma, g\rangle} \langle \mathcal{V}_{\widehat{g}}^{\theta_a} \widehat{f}, \mathcal{V}_{\widehat{\gamma}}^{\theta_a} \widehat{h}\rangle \;=\; \langle f, h\rangle. \end{aligned}$$

Thus $\ell_f = f$ and the inversion formula concludes from the arbitrariness of f. □

Next we prove a strong version of the inversion formulas. For arbitrary nested sequence of compact sets $K_n \subseteq \mathbb{R}^2$ exhausting $\mathbb{R}^2$, it means that $K_n \subset K_{n+1}$ and $\bigcup_{n\geq 1} K_n = \mathbb{R}^2$. The common choices are the cubes $[-n, n]^2$ or the balls $\overline{B}(0, n) = \{x \in \mathbb{R}^2 \,:\, |x| \leq n\}$.

Theorem 4.3. *Suppose that $g, \gamma \in L^2(\mathbb{R})$ and $K_n \subseteq \mathbb{R}^2$ for $n \geq 1$ be a nested exhausting sequence of compact sets. Define f_n to be*

$$f_n = \frac{1}{\langle \gamma, g\rangle} \iint_{K_n} \mathcal{V}_g^{\theta_a} f(x,\omega)\, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)}\gamma \,\mathrm{d}m. \tag{4.25}$$

Then $\lim_{n\to\infty} \|f - f_n\|_2 = 0$.

Proof: By Cauchy-Schwartz inequality and Corollary 3.3, we estimate for $h \in L^2(\mathbb{R})$ that

$$\begin{aligned}
|\langle f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|}\Big|\iint_{K_n} \mathcal{V}_g^{\theta_a} f(x,\omega)\,\overline{\mathcal{V}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}m\Big| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|}\|\mathcal{V}_g^{\theta_a} f\|_2\|\mathcal{V}_\gamma^{\theta_a} h\|_2 \\
&= \frac{1}{|\langle \gamma, g\rangle|}\|g\|_2\|f\|_2\|\gamma\|_2\|h\|_2.
\end{aligned}$$

We know that f_n is well-defined for each n in $L^2(\mathbb{R})$ and we get $\|f_n\|_2 \leq |\langle \gamma, g\rangle|^{-1}\|g\|_2\|f\|_2\|\gamma\|_2$. Then

$$\begin{aligned}
|\langle f - f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|}\Big|\Big(\iint_{\mathbb{R}^2} - \iint_{K_n}\Big)\mathcal{V}_g^{\theta_a} f(x,\omega)\overline{\mathcal{V}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}m\Big| \\
&= \frac{1}{|\langle \gamma, g\rangle|}\Big|\iint_{K_n^c} \mathcal{V}_g^{\theta_a} f(x,\omega)\overline{\mathcal{V}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}m\Big| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|}\|\mathcal{V}_\gamma^{\theta_a} h(x,\omega)\|_2\Big(\iint_{K_n^c} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^2\,\mathrm{d}m\Big)^{1/2} \\
&= \frac{1}{|\langle \gamma, g\rangle|}\|\gamma\|_2\|h\|_2\Big(\iint_{K_n^c} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^2\,\mathrm{d}m\Big)^{1/2}.
\end{aligned}$$

The arbitrariness of $h \in L^2(\mathbb{R})$ implies that

$$\begin{aligned}
\|f - f_n\|_2 &= \sup_{\|h\|_2 \leq 1} |\langle f - f_n, h\rangle| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|}\|\gamma\|_2\Big(\iint_{K_n^c} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^2\,\mathrm{d}m\Big)^{1/2}.
\end{aligned}$$

Since $\mathcal{V}_g^{\theta_a} f \in L^2(\mathbb{R}^2, \mathrm{d}m)$, and K_n is exhausting, the right-hand side becomes arbitrarily small as n increases. □

Theorem 4.4. *Fix $g, \gamma \in L^2(\mathbb{R})$ and let $K_n \subseteq \mathbb{R}^2$ for $n \geq 1$ be a nested exhausting sequence of compact sets. Define f_n to be*

$$f_n = \frac{1}{\langle \gamma, g\rangle}\iint_{K_n} \mathcal{W}_g^{\theta_a} f(x,\omega)\,\mathcal{M}_{\theta_a(\omega)}\mathcal{T}_x\gamma\,\mathrm{d}\tilde{m}. \tag{4.26}$$

Then $\lim_{n\to\infty} \|f - f_n\|_2 = 0$.

Proof: By Cauchy-Schwartz inequality and Corollary 3.3, we estimate for $h \in L^2(\mathbb{R})$ that

$$\begin{aligned}
|\langle f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|}\Big|\iint_{K_n} \mathcal{W}_g^{\theta_a} f(x,\omega)\,\overline{\mathcal{W}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}\tilde{m}\Big| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|}\|\mathcal{W}_g^{\theta_a} f\|_2\|\mathcal{W}_\gamma^{\theta_a} h\|_2 \\
&= \frac{1}{|\langle \gamma, g\rangle|}\|g\|_2\|f\|_2\|\gamma\|_2\|h\|_2.
\end{aligned}$$

Therefore, f_n is well-defined for each n in $L^2(\mathbb{R})$ and we get $\|f_n\|_2 \leq |\langle \gamma, g\rangle|^{-1}\|g\|_2\|f\|_2\|\gamma\|_2$. Similarly, We continue that

$$\begin{aligned}
|\langle f - f_n, h\rangle| &= \frac{1}{|\langle \gamma, g\rangle|}\Big|\Big(\iint_{\mathbb{R}^2} - \iint_{K_n}\Big)\mathcal{W}_g^{\theta_a} f(x,\omega)\overline{\mathcal{W}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}\tilde{m}\Big| \\
&= \frac{1}{|\langle \gamma, g\rangle|}\Big|\iint_{K_n^c} \mathcal{W}_g^{\theta_a} f(x,\omega)\overline{\mathcal{W}_\gamma^{\theta_a} h(x,\omega)}\,\mathrm{d}\tilde{m}\Big| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|}\|\mathcal{W}_\gamma^{\theta_a} h(x,\omega)\|_2\Big(\iint_{K_n^c} |\mathcal{W}_g^{\theta_a} f(x,\omega)|^2\,\mathrm{d}\tilde{m}\Big)^{1/2} \\
&= \frac{1}{|\langle \gamma, g\rangle|}\|\gamma\|_2\|h\|_2\Big(\iint_{K_n^c} |\mathcal{W}_g^{\theta_a} f(x,\omega)|^2\,\mathrm{d}\tilde{m}\Big)^{1/2}.
\end{aligned}$$

Since this holds for all $h \in L^2(\mathbb{R})$, we get

$$\|f - f_n\|_2 = \sup_{\|h\|_2 \leq 1} |\langle f - f_n, h\rangle| \\ \leq \frac{1}{|\langle \gamma, g\rangle|} \|\gamma\|_2 \Big(\iint_{K_n^c} |\mathcal{W}_g^{\theta_a} f(x,\omega)|^2 \, \mathrm{d}\tilde{m} \Big)^{1/2}.$$

Since $\mathcal{W}_g^{\theta_a} f \in L^2(\mathbb{R}^2, \mathrm{d}\tilde{m})$, and K_n is exhausting, the right-hand side becomes arbitrarily small as n increases. □

5 The Uncertainty Principle

It is well-known that the time-frequency resolution of the STFT crucially depends on the choice of the window function g, and it is limited by the size of the essential supports of g and $\hat{g}$. Based on the similarity of the short-time Chirp transform as well as the nonlinearly-sliding window Fourier transform to STFT, the choice of the window function is also of great importance to them. Now we present uncertainty principles that apply directly to them, which explain the following generic principle:

A function cannot be concentrated on small sets in the time-frequency plane, no matter which time-frequency representation is used.

We first give a weak type of such an uncertainty principle, which is analogous to the uncertainty principle of Donoho and Stark for the pair $(f, \hat{f})$.

Theorem 5.1. *Suppose that $\|f\|_2 = \|g\|_2 = 1$ and that $U \subset \mathbb{R}^2$ and $\epsilon \geq 0$ are such that*

$$\iint_U \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}x\mathrm{d}\omega \geq 1 - \epsilon. \tag{5.27}$$

Then $|U| \geq 1 - \epsilon$.

Proof: Using the Cauchy-Schwartz inequality, we get

$$|\mathcal{V}_g^{\theta_a} f(x,\omega)| = |\langle f, \mathcal{M}_\omega \mathcal{T}_{\theta_a(x)} g\rangle| \leq \|f\|_2 \|g\|_2 = 1 \tag{5.28}$$

for all $(x,\omega) \in \mathbb{R}^2$. Therefore,

$$1 - \epsilon \leq \iint_U |\mathcal{V}_g^{\theta_a} f(x,\omega)|^2 \, \mathrm{d}x\mathrm{d}\omega \leq \|\mathcal{V}_g^{\theta_a} f\|_\infty^2 |U| \leq |U|. \tag{5.29}$$

□

Theorem 5.2. *Suppose that $\|f\|_2 = \|g\|_2 = 1$ and that $U \subset \mathbb{R}^2$ and $\epsilon \geq 0$ are such that*

$$\iint_U \left|\mathcal{W}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}x\mathrm{d}\omega \geq 1 - \epsilon. \tag{5.30}$$

Then $|U| \geq 1 - \epsilon$.

Proof: Using the Cauchy-Schwartz inequality, we get

$$|\mathcal{W}_g^{\theta_a} f(x,\omega)| = |\langle f, \mathcal{M}_{\theta_a(\omega)} \mathcal{T}_x g\rangle| \leq \|f\|_2 \|g\|_2 = 1 \tag{5.31}$$

for all $(x,\omega) \in \mathbb{R}^2$. Therefore,

$$1 - \epsilon \leq \iint_U |\mathcal{W}_g^{\theta_a} f(x,\omega)|^2 \, \mathrm{d}x\mathrm{d}\omega \leq \|\mathcal{W}_g^{\theta_a} f\|_\infty^2 |U| \leq |U|. \tag{5.32}$$

□

Next we will establish a strong type inequalities analogous to the Lieb's inequality for $\mathcal{V}_g^{\theta_a}$. To this end, we recall the Babenko-Bechner constant $c_\lambda = \sqrt{\frac{\lambda^{\frac{1}{\lambda}}}{\lambda'^{\frac{1}{\lambda'}}}}$ for positive number λ, which has the property $c_\lambda c_{\lambda'} = 1$

for the conjugate pair (λ, λ'). Moreover, when $\lambda > 2$, setting $s = \frac{2}{\lambda'} = \frac{2(\lambda-1)}{\lambda} \geq 1$ and $t = \frac{\lambda}{\lambda'} = \lambda - 1$, then the conjugate indexes of s and t are as following

$$s' = \frac{2(\lambda-1)}{\lambda-2}, \quad t' = \frac{\lambda-1}{\lambda-2}.$$

Then

$$\begin{aligned} c_s &= \left(\frac{s^{\frac{1}{s}}}{s'^{\frac{1}{s'}}}\right)^{\frac{1}{2}} = \left((\frac{2(\lambda-1)}{\lambda})^{\frac{\lambda}{2(\lambda-1)}}\right)^{\frac{1}{2}} / \left((\frac{2(\lambda-1)}{\lambda-2})^{\frac{\lambda-2}{2(\lambda-1)}}\right)^{\frac{1}{2}} \\ &= \left(\frac{4(\lambda-1)^2(\lambda-2)^{\lambda-2}}{\lambda^\lambda}\right)^{\frac{1}{4(\lambda-1)}} \end{aligned}$$

and

$$\begin{aligned} c_{t'} &= \left(\frac{t'^{\frac{1}{t'}}}{t^{\frac{1}{t}}}\right)^{\frac{1}{2}} = \left((\frac{\lambda-1}{\lambda-2})^{\frac{\lambda-2}{\lambda-1}}\right)^{\frac{1}{2}} / \left((\lambda-1)^{\frac{1}{\lambda-1}}\right)^{\frac{1}{2}}. \\ &= \left(\frac{(\lambda-1)^{\lambda-3}}{(\lambda-2)^{\lambda-2}}\right)^{\frac{1}{2(\lambda-1)}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & c_{\lambda'} c_s^{2/\lambda'} c_{t'}^{1/\lambda'} \\ = & \left(\frac{\lambda^{\lambda-2}}{(\lambda-1)^{\lambda-1}}\right)^{\frac{1}{2\lambda}} \left(\left(\frac{4(\lambda-1)^2(\lambda-2)^{\lambda-2}}{\lambda^\lambda}\right)^{\frac{1}{4(\lambda-1)}}\right)^{\frac{2}{\lambda'}} \left(\left(\frac{(\lambda-1)^{\lambda-3}}{(\lambda-2)^{\lambda-2}}\right)^{\frac{1}{2(\lambda-1)}}\right)^{\frac{1}{\lambda'}} \\ = & \left(\frac{\lambda^{\lambda-2}}{(\lambda-1)^{\lambda-1}}\right)^{\frac{1}{2\lambda}} \left(\left(\frac{4(\lambda-1)^2(\lambda-2)^{\lambda-2}}{\lambda^\lambda}\right)^{\frac{1}{4(\lambda-1)}}\right)^{\frac{2(\lambda-1)}{\lambda}} \left(\left(\frac{(\lambda-1)^{\lambda-3}}{(\lambda-2)^{\lambda-2}}\right)^{\frac{1}{2(\lambda-1)}}\right)^{\frac{\lambda-1}{\lambda}} \\ = & \left(\frac{\lambda^{\lambda-2}}{(\lambda-1)^{\lambda-1}}\right)^{\frac{1}{2\lambda}} \left(\frac{4(\lambda-1)^2(\lambda-2)^{\lambda-2}}{\lambda^\lambda}\right)^{\frac{1}{2\lambda}} \left(\frac{(\lambda-1)^{\lambda-3}}{(\lambda-2)^{\lambda-2}}\right)^{\frac{1}{2\lambda}} \\ = & \left(\frac{\lambda^{\lambda-2}}{(\lambda-1)^{\lambda-1}} \frac{4(\lambda-1)^2(\lambda-2)^{\lambda-2}}{\lambda^\lambda} \frac{(\lambda-1)^{\lambda-3}}{(\lambda-2)^{\lambda-2}}\right)^{\frac{1}{2\lambda}} \\ = & (\frac{2}{\lambda})^{\frac{1}{\lambda}}. \end{aligned}$$

We also need to revisit the Fubini's Theorem, the Young's convolution inequality and the Hausdorff-Young inequality.

Lemma 5.3. *(Fubini's Theorem) If $f \in L^1(\mathbb{R}^2, \mu \times \nu)$, then*

$$\begin{aligned} \iint_{\mathbb{R}^2} f(x,\omega) \mathrm{d}(\mu \times \nu) &= \int_{\mathbb{R}} [\int_{\mathbb{R}} f(x,\omega) \mathrm{d}\mu(x)] \mathrm{d}\nu(\omega) \\ &= \int_{\mathbb{R}} [\int_{\mathbb{R}} f(x,\omega) \mathrm{d}\nu(\omega)] \mathrm{d}\mu(x). \end{aligned}$$

Furthermore, for almost all $\omega \in \mathbb{R}$ the section $x \longmapsto f(x,\omega)$ is in $L^1(\mathbb{R}, \mu)$ and for almost all $x \in \mathbb{R}$ the section $\omega \longmapsto f(x,\omega)$ is in $L^1(\mathbb{R}, \nu)$. If φ and ψ are defined by $\varphi(x) = \int_{\mathbb{R}} f(x,\omega)\mathrm{d}\nu(\omega)$ and $\psi(\omega) = \int_{\mathbb{R}} f(x,\omega)\mathrm{d}\mu(x)$, then $\varphi \in L^1(\mathbb{R}, \mu)$ and $\psi \in L^1(\mathbb{R}, \nu)$.

Lemma 5.4. *(Young's convolution inequality)[13] Suppose that p, q, r are positive numbers in $[1, +\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and assume that p', q', r' are the conjugate numbers of p, q, r, respectively. Then for any $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, it holds*

$$\|f * g\|_r \leq c_p c_q c_{r'} \|f\|_p \|g\|_q, \tag{5.33}$$

where c_p is the Babenko-Bechner constant. Moreover, when $p,q>1$, the Young's convolution inequality has the following simplier form

$$\|f*g\|_r \le \|f\|_p\|g\|_q. \tag{5.34}$$

and the optimal constant can be achieved if and only if both of f and g are Gaussian.

Lemma 5.5. *(Hausdorff-Young inequality)[12] Suppose that $p\in[1,2]$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then for $f\in L^p(\mathbb{R})$, it holds that $\hat{f}\in L^{p'}(\mathbb{R})$ and $\|\hat{f}\|_{p'}\le (\frac{p^{1/p}}{(p')^{1/p'}})^{1/2}\|f\|_p$.*

Next lemma is crucial for the proof of Lieb type inequality.

Lemma 5.6. *Suppose that $f,g\in L^2(\mathbb{R})$, $2\le\lambda<\infty$, λ' is the conjugate index of λ and $s=\frac{2}{\lambda'}\ge 1, t=\frac{\lambda}{\lambda'}$. Then both $f^{\lambda'}$ and $(g^*)^{\lambda'}$ are in $L^{2/\lambda'}(\mathbb{R})$ and satisfy that*

$$\|f^{\lambda'}*(g^*)^{\lambda'}\|_t \le c_s^2 c_{t'}\|f\|_2^{\lambda'}\|g\|_2^{\lambda'} \tag{5.35}$$

where $g^*=\overline{g(-\cdot)}$, c_s and $c_{t'}$ are the Babenko-Bechner constants of s and t', respectively.

Proof: Firstly, $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$ and $\lambda\in[2,+\infty)$ imply that $\lambda'\in(1,2]$. Then $f,g\in L^2(\mathbb{R})$ leads to $f^{\lambda'},(g^*)^{\lambda'}\in L^{2/\lambda'}(\mathbb{R})$. Secondly, noting that $\frac{1}{s}+\frac{1}{s}=\frac{1}{t}+1$ and applying the Young's convolution inequality to the functions $f^{\lambda'}$ and $(g^*)^{\lambda'}$ via replacing the triple (p,q,r) with (s,s,t), we obtain

$$\|f^{\lambda'}*(g^*)^{\lambda'}\|_t \le c_s^2 c_{t'}\|f^{\lambda'}\|_s\|(g^*)^{\lambda'}\|_s,$$

Finally, using

$$\|f^{\lambda'}\|_s = \Big(\int_{\mathbb{R}}|f(x)|^{\lambda'\cdot\frac{2}{\lambda'}}\,\mathrm{d}x\Big)^{\lambda'/2} = \|f\|_2^{\lambda'}$$

and similarly $\|(g^*)^{\lambda'}\|_s=\|g\|_2^{\lambda'}$, it arrives at (5.35). □

Theorem 5.7. *Suppose that $f,g\in L^2(\mathbb{R})$ and $2\le\lambda<\infty$. Then*

$$\iint_{\mathbb{R}^2}|\mathcal{V}_g^{\theta_a}f(x,\omega)|^\lambda\,\mathrm{d}m \le \frac{2}{\lambda}\big(\|f\|_2\|g\|_2\big)^\lambda. \tag{5.36}$$

Proof: Let λ' be the conjugate index of λ. Then $1<\lambda'\le 2$ since $2\le\lambda<\infty$ and $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$. By Cauchy-Schwartz inequality, we deduce $f\cdot\mathcal{T}_{\theta_a(x)}\overline{g}\in L^1(\mathbb{R})$ from $f,g\in L^2(\mathbb{R})$.

Recalling $\mathcal{V}_g^{\theta_a}f(x,\omega)=\mathcal{F}(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})(\omega)\in L^2(\mathbb{R}^2,\mathrm{d}m)$ in Theorem 3.1 with the Lebesgue measure $\mathrm{d}m=p_a(x)\,\mathrm{d}x\mathrm{d}\omega$, we know that $|\mathcal{F}(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})(\omega)|^2\in L^1(\mathbb{R}^2,dm)$. By Fubini's theorem, it implies that, for almost all $x\in\mathbb{R}$, $|\mathcal{F}(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})(\omega)|^2\in L^1(\mathbb{R},d\omega)$ and then $\mathcal{F}(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})\in L^2(\mathbb{R})$. The unitary of the Fourier operator concludes that $f\cdot\mathcal{T}_{\theta_a(x)}\overline{g}\in L^2(\mathbb{R})$ for almost all $x\in\mathbb{R}$.

Above discussion indicates that, for almost all $x\in\mathbb{R}$, $f\cdot\mathcal{T}_{\theta_a(x)}\overline{g}\in L^1\cap L^2(\mathbb{R})$. Therefore, by Marcinkiewicz interpolation theorem, one has $f\cdot\mathcal{T}_{\theta_a(x)}\overline{g}\in L^{\lambda'}(\mathbb{R})$ for almost all $x\in\mathbb{R}$. Moreover, by Hausdorff-Young inequality, it gives rise to

$$\begin{aligned}
\Big(\int_{\mathbb{R}}|\mathcal{V}_g^{\theta_a}f(x,\omega)|^\lambda\,\mathrm{d}\omega\Big)^{1/\lambda} &= \left(\int_{\mathbb{R}}|\mathcal{F}(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})(\omega)|^\lambda\,\mathrm{d}\omega\right)^{1/\lambda}\\
&\le c_{\lambda'}\left(\int_{\mathbb{R}}|(f\cdot\mathcal{T}_{\theta_a(x)}\overline{g})(y)|^{\lambda'}\,\mathrm{d}y\right)^{1/\lambda'}\\
&= c_{\lambda'}\left(\int_{\mathbb{R}}|f(y)|^{\lambda'}|\overline{g}(y-\theta_a(x))|^{\lambda'}\,\mathrm{d}y\right)^{1/\lambda'}\\
&= c_{\lambda'}\left(\big(|f^{\lambda'}*g^*|^{\lambda'}\big)|_{\theta_a(x)}\right)^{1/\lambda'},
\end{aligned}$$

where $c_{\lambda'}$ is the Babenko-Bechner constants of λ'. Hence

$$\begin{aligned}
\|\mathcal{V}_g^{\theta_a} f\|_\lambda &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^\lambda p_a(x)\mathrm{d}\omega\mathrm{d}x\right)^{1/\lambda} \\
&\le c_{\lambda'}\left(\int_{\mathbb{R}}\left((|f^{\lambda'} * g^*|^{\lambda'})|_{\theta_a(x)}\right)^{\lambda/\lambda'} p_a(x)\mathrm{d}x\right)^{1/\lambda} \\
&\le c_{\lambda'}\left(\int_{\mathbb{R}}\left((|f^{\lambda'} * g^*|^{\lambda'})|_{y}\right)^{\lambda/\lambda'} \mathrm{d}y\right)^{1/\lambda} \\
&= c_{\lambda'}\left\||f^{\lambda'} * g^*|^{\lambda'}\right\|_{\lambda/\lambda'}^{1/\lambda'}.
\end{aligned}$$

By noting (5.35), it implies that

$$\begin{aligned}
\|\mathcal{V}_g^{\theta_a} f\|_\lambda &\le c_{\lambda'}\left(c_s^2 c_{t'}\|f\|_2^{\lambda'}\|g\|_2^{\lambda'}\right)^{1/\lambda'} \\
&= c_{\lambda'} c_s^{\frac{2}{\lambda'}} c_{t'}^{\frac{1}{\lambda'}}\|f\|_2\|g\|_2
\end{aligned}$$

with $s = \frac{2}{\lambda'} \ge 1, t = \frac{\lambda}{\lambda'}$. Calculation shows that $c_{\lambda'} c_s^{2/\lambda'} c_{t'}^{1/\lambda'} = (\frac{2}{\lambda})^{\frac{1}{\lambda}}$. Therefore, it concludes (5.36). □

Next theorem is a strong version of Lieb type uncertainty principle for $\mathcal{V}_g^{\theta_a}$.

Theorem 5.8. *Suppose that $\|f\|_2 = \|g\|_2 = 1$ and that $U \subset \mathbb{R}^2$ and $\epsilon \ge 0$ are such that*

$$\iint_U \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}m \ge 1-\epsilon. \tag{5.37}$$

Then $|U| \ge (1-\epsilon)^{\frac{p}{p-2}}(\frac{p}{2})^{\frac{2}{p-2}}$ for all $p>2$. In particular, $|U| \ge \sup_{p>2}(1-\epsilon)^{\frac{p}{p-2}}(\frac{p}{2})^{\frac{2}{p-2}} \ge 2(1-\epsilon)^2$.

Proof: Let $q = \frac{p}{2}$ and then its conjugate index is $q' = \frac{p}{p-2}$. By Theorem 5.7, we know that $\mathcal{V}_g^{\theta_a} f(x,\omega), (x,\omega)$ belongs to $L^p(\mathbb{R}^2, dm)$ for $p>2$. Applying Hölder's inequality with exponents q and q', it gives that

$$\begin{aligned}
&\iint_U \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}m = \iint_{\mathbb{R}^2} \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \chi_U(x,\omega)\,\mathrm{d}m \\
\le\ &(\iint_{\mathbb{R}^2} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^{2\cdot q}\,\mathrm{d}m)^{1/q}(\iint_{\mathbb{R}^2} \chi_U(x,\omega)^{q'}\,\mathrm{d}m)^{\frac{1}{q'}} \\
=\ &(\iint_{\mathbb{R}^2} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^{2\cdot\frac{p}{2}}\,\mathrm{d}m)^{2/p}(\iint_{\mathbb{R}^2} \chi_U(x,\omega)^{q'}\,\mathrm{d}m)^{\frac{p-2}{p}} \\
=\ &(\iint_{\mathbb{R}^2} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^{p}\,\mathrm{d}m)^{2/p}(\iint_{\mathbb{R}^2} \chi_U(x,\omega)\,\mathrm{d}m)^{\frac{p-2}{p}}.
\end{aligned}$$

Using (5.36) and noting that the periodic Poisson kernel p_a has upper bound 1, it follows

$$\begin{aligned}
&\iint_U \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}m \\
\le\ &(\iint_{\mathbb{R}^2} |\mathcal{V}_g^{\theta_a} f(x,\omega)|^{p}\,\mathrm{d}m)^{2/p}(\iint_{\mathbb{R}^2} \chi_U(x,\omega)\,\mathrm{d}x\mathrm{d}\omega)^{\frac{p-2}{p}} \\
\le\ &\left(\frac{2}{p}(\|f\|_2\|g\|_2)^p\right)^{\frac{2}{p}}|U|^{\frac{p-2}{p}} \\
=\ &\left(\frac{2}{p}\right)^{\frac{2}{p}}\|f\|_2^2\|g\|_2^2|U|^{\frac{p-2}{p}}.
\end{aligned}$$

Therefore, the conditions $\|f\|_2 = \|g\|_2 = 1$ and $1-\epsilon \le \iint_U \left|\mathcal{V}_g^{\theta_a} f(x,\omega)\right|^2 \mathrm{d}m$ imply that

$$\left(\frac{2}{p}\right)^{\frac{2}{p}}|U|^{\frac{p-2}{p}} \ge 1-\epsilon.$$

Finally, we conclude

$$|U| \geq (1-\epsilon)^{\frac{p}{p-2}} \Big(\frac{p}{2}\Big)^{\frac{2}{p-2}}$$

for all $p > 2$. In particular, when $p = 4$, we have $|U| \geq (1-\epsilon)^2 2^d$ and then $|U| \geq \sup_{p>2} (1-\epsilon)^{\frac{p}{p-2}} (\frac{p}{2})^{\frac{2}{p-2}} \geq 2(1-\epsilon)^2$. □

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