Towards Enhancing Nonlinear Observers for Lipschitz System: Exploiting the Matrix Multipliers-based LMIs

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Abstract

This article is focused on the design of an LMI-based observer for the class of disturbance-affected nonlinear systems. Two novel LMI conditions are derived by deploying a more general form of the matrix multiplier compared to the one used in the literature. The first method is based on the use of the H [?] criterion, while the second one utilises an ISS notion. Both LMIs are developed by employing the reformulated Lipschitz property, a well-known LPV approach and the new variant of Young inequality. The key element of the proposed LMI conditions is the incorporation of the novel matrix multipliers which allow us to include some additional decision variables as compared to the methods proposed in the literature. These additional variables add extra degrees of freedom, thus enhancing the LMI feasibility. Furthermore, the effectiveness of the proposed methodologies is showcased through a numerical example.

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Towards Enhancing Nonlinear Observers for Lipschitz System: Exploiting the Matrix Multipliers-based LMIs

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Summary

This article is focused on the design of an LMI-based observer for the class of disturbance-affected nonlinear systems. Two novel LMI conditions are derived by deploying a more general form of the matrix multiplier compared to the one used in the literature. The first method is based on the use of the $\mathcal{H}_\infty$ criterion, while the second one utilises an ISS notion. Both LMIs are developed by employing the reformulated Lipschitz property, a well-known LPV approach and the new variant of Young inequality. The key element of the proposed LMI conditions is the incorporation of the novel matrix multipliers which allow us to include some additional decision variables as compared to the methods proposed in the literature. These additional variables add extra degrees of freedom, thus enhancing the LMI feasibility. Furthermore, the effectiveness of the proposed methodologies is showcased through a numerical example.

KEYWORDS:
LMI-based nonlinear observer design, Lipschitz systems, $\mathcal{H}_\infty$ criterion, Input-to-state stability (ISS), Linear parameter varying (LPV) approach

1 | INTRODUCTION

Over the past few decades, the topic of state estimation for dynamical systems has emerged as a pivotal research interest in control system engineering. This is because grasping real-time information about the state of the system is crucial in various applications. Several operations in the control system domain, for instance, controlling systems, monitoring systems, and decision-making, are executed using such real-time data. One of the techniques used to collect real-time measurements is to deploy sensors on physical systems. However, the quantity and quality of sensors are frequently restricted in practical scenarios due to cost and physical constraints. Hence, observers become an indispensable components in modern-day applications assisting in the collection of current knowledge of systems, for example, autonomous vehicle tracking[1], the state-of-charge estimation of LI-ion battery model[2], cardiovascular application[3], and so on.

Observer design for linear systems has been extensively studied and proven to be quite effective. In the paper[4], the proposed Luenberger observer was the first state estimation method established for linear systems. Compared to linear systems, the development of nonlinear observers is still an arduous problem. As a consequence, an abundant amount of research has been carried out in this domain, and various approaches have been proposed. The authors of the articles[5,6] developed the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) techniques for the state estimation of nonlinear systems. However, the sliding-mode observer and the high-gain observer are deployed for the same task in the papers [7,8]. Recently, linear matrix inequality (LMI)-based methodologies have earned a substantial amount of interest, and several results are outlined in the publications [1,9,10].

Among these approaches, few LMI methods are dependent on the S-Procedure lemma [11], the Riccati equations [12], and the Young inequality [13]. Though each technique provides a conservative LMI condition, there is the possibility for enhancement. The authors of the articles [9,14] utilised the $\mathcal{H}_\infty$ criterion in the observer design for the estimation of the state in the presence of noise. An alternative for the $\mathcal{H}_\infty$ criterion is the use of the input-to-state stability (ISS) property. An ISS notion was introduced in the paper [15]. Further, the authors of the letter [16] had proposed an ISS-Lyapunov function to use the ISS property for the stability of systems. An observer based on the ISS-Lyapunov function was proposed in the publications [17,18,19]. All these cited papers provide efficient state estimation. Along with this, an ISS-Lyapunov function aids in obtaining an LMI condition.

The objective of the proposed article is to design a nonlinear observer for disturbance-affected systems that reconstructs the states of systems with better noise compensation. In order to achieve the aforementioned goal, two novel LMI conditions are formulated to compute the observer gain by utilising the $\mathcal{H}_\infty$ criterion and the ISS condition. These established LMIs are derived by incorporating the well-known linear parameter varying (LPV) approach, a variant of Young inequality, and the reformulated Lipschitz property. The primary component of the novel LMIs is the newly defined matrix multipliers. The integration of this matrix multiplier with the LMI framework is inspired by the work presented in the paper [9,20,21]. The deployment of such a matrix multiplier adds some additional numbers of decision variables inside LMIs and enhances the LMI feasibility.

The remainder of this article is structured in the following manner: Section 2 encompasses the notations and the recapitulation of some preliminaries and background results related to the LMI-based observer design. The articulation of the problem statement is illustrated in Section 3. Further, the development of $\mathcal{H}_\infty$ criterion-based LMI condition is showcased in Section 4. Section 5 includes the LMI synthesis using the notion of ISS. A few comments on the proposed matrix multipliers-based LMIs are discussed in Section 6. Later on, the efficiency of the derived LMI conditions, and the performance of the observer are emphasised in Section 7 through a numerical example. In last, Section 8 comprises some conclusions and future perspectives.

2 | NOMENCLATURE AND SOME BACKGROUND RESULTS

2.1 | Glossary

Throughout the article, the subsequent notations are employed:

- $\|e\|$ and $\|e\|_{\mathcal{L}_2}$ denote the euclidean and the $\mathcal{L}_2$ norms of a vector e, respectively. Its initial value at $t = 0$ is represented by e_0.
- We define a vector of the canonical basis of $\mathbb{R}^s$ in the following manner:

$$e_s(i) = \underbrace{(0, \ldots, 0, \overbrace{\quad 1 \quad}^{i^{\text{th}}}, 0, \ldots, 0)}_{s \text{ components}}^\top \in \mathbb{R}^s, \; s \geq 1.$$

- $\mathbf{I}$ and $\mathbf{O}$ indicate an identity matrix and a null matrix, respectively.
- The transpose of matrix A is symbolised by $A^\top$.
- $A \in \mathbf{S}^n$ implies that a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric. The terms $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ signify the minimum and maximum eigenvalues of the same matrix A, respectively.
- Within a symmetric matrix, repeated blocks are represented by using the symbol $(\star)$.
- For any matrix $A \in \mathbb{R}^{n \times n}$, $A > 0$ ($A < 0$) signifies that A is a positive definite matrix (a negative definite matrix). Similarly, a positive semi-definite matrix (a negative semi-definite matrix) is denoted by $A \geq 0$ ($A \leq 0$).
- $A = \text{block-diag}(A_1, \ldots, A_n)$ is a block-diagonal matrix having elements $A_1, \ldots, A_n$ in the diagonal.

2.2 | Preliminaries

The objective of this segment is to provide an overview of the mathematical tools and background results that will be required in the elaboration of the main outcomes.

Definition 1 (Input-to-State stability [16]). Let us consider a generalised class of nonlinear systems:

$$\dot{\zeta} = f(\zeta, u), \tag{1}$$

where $\zeta \in \mathbb{R}^n$ and $u \in \mathbb{R}^s$ denote the states and input of the systems, respectively. The function $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is piece-wise continuous in t, and it is assumed to be locally Lipschitz in ζ and u. The system (1) is input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state ζ_0, and any bounded input $u(t)$, solution $\zeta(t)$ exists for all $t \geq 0$ and satisfies:

$$\|\zeta(t)\| \leq \beta(\|\zeta_0\|, t) + \gamma(\|u\|_\infty), \forall t \geq 0. \tag{2}$$

Definition 2 (ISS-Lyapunov function [16]). A smooth function $V(\zeta) : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for the system (1) if and only if there exist class $\mathcal{K}_\infty$ functions $\alpha_i \in \ (1 \leq i \leq 4)$, such that it fulfills

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \tag{3}$$

$$\dot{V}(\eta, u) \leq -\alpha_3(\|\eta\|) + \alpha_4(\|u\|), \tag{4}$$

for any $\eta \in \mathbb{R}^n$ and $u \in \mathbb{R}^s$.

Definition 3. Let us consider two vectors

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}^\top, \text{ and } B = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}^\top.$$

Then one can define an auxiliary vector $A^{B_i} \in \mathbb{R}^n, \forall i \in \{0, \dots, n\}$ corresponding to A and B in the following manner:

$$A^{B_i} = \begin{cases} \begin{pmatrix} b_1 & b_2 & \dots & b_i & a_{i+1} & \dots & a_n \end{pmatrix}^\top, & \text{for } i = 1, \dots, n \\ A, & \text{for } i = 0. \end{cases} \tag{5}$$

Lemma 1 ([10]). Let $\chi : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear function. Then, the ensuing two statements are equivalent:

i) The function χ satisfies the subsequent inequality:

$$\|\chi(X) - \chi(Y)\| \leq \chi_\chi \|X - Y\|, \quad \forall X, Y \in \mathbb{R}^n, \tag{6}$$

i.e., it is globally Lipschitz.

ii) For all, $i, j = 1, \dots, n$, there exist functions $\chi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and constants $\chi_{ij_{\min}}$ and $\chi_{ij_{\max}}$ such that $\forall\, X,\ Y \in \mathbb{R}^n$,

$$\chi(X) - \chi(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \chi_{ij} \mathcal{H}_{ij}(X - Y), \tag{7}$$

where $\mathcal{H}_{ij} = e_n(i) e_n^\top(j)$, and $\chi_{ij} \triangleq \chi_{ij}(X^{Y_{j-1}}, X^{Y_j})$. The functions $\chi_{ij}(\cdot)$ are globally bounded as follows:

$$\chi_{ij_{\min}} \leq \chi_{ij} \leq \chi_{ij_{\max}}. \tag{8}$$

Lemma 2. For any two vectors $X, Y \in \mathbb{R}^n$ and a matrix $Z > 0 \in \mathbf{S}^n$, the following matrix inequality holds:

$$X^\top Y + Y^\top X \leq X^\top Z^{-1} X + Y^\top Z Y. \tag{9}$$

In the paper [9], the authors had introduced a new variant of (9), which is given by

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY). \tag{10}$$

In this article, both of the aforementioned Young inequalities (9) and (10) are employed to prevent bilinear multiplications between some unknown decision variables, and to tackle the nonlinearities in the Lyapunov analysis, respectively.

In the sequel, the main contributions of this article are presented.

3 | ARTICULATING PROBLEM STATEMENT

Let us consider the subsequent equations which represent a class of disturbance-affected nonlinear systems with nonlinear outputs:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E_1\omega_1, \\ y &= Cx + Fg(x) + D_1\omega_2, \end{aligned} \tag{11}$$

where

i) $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ denote the states and the measurements of the systems, respectively. The input of the system is described by $u \in \mathbb{R}^s$.

ii) $\omega_1 \in \mathbb{R}^{q_1}$ and $\omega_2 \in \mathbb{R}^{q_2}$ are the exogenous signals such as noise or disturbances affecting the system dynamics and outputs, respectively.

iii) A, G, B, C, F, E_1 and D_1 are known constant matrices of appropriate dimensions.

Since there are no specific constraints imposed on the dimension of the disturbances ω_1 and ω_2, or on the structure of the matrices E_1 and D_1, the model (11) can be reformulated as:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + Fg(x) + D\omega, \end{aligned} \tag{12}$$

where $E = \begin{bmatrix} E_1 & \mathbf{O} \end{bmatrix}$, $D = \begin{bmatrix} \mathbf{O} & D_1 \end{bmatrix}$, and $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$.

The system (11) depicts a more generalized form and is often encountered in practical scenarios. Whereas, the form (12) facilitates the simplification of the observer design.

The functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^r$ are assumed to be globally Lipschitz. The detailed form of $f(\cdot)$ and $g(\cdot)$ are illustrated as follows:

$$f(x) = \begin{bmatrix} f_1(H_1x) \\ \vdots \\ f_i(\overbrace{H_ix}^{v_i}) \\ \vdots \\ f_m(H_mx) \end{bmatrix}, \tag{13}$$

and

$$g(x) = \begin{bmatrix} g_1(G_1x) \\ \vdots \\ g_i(\overbrace{G_ix}^{\theta_i}) \\ \vdots \\ g_r(G_rx) \end{bmatrix}, \tag{14}$$

where $H_i \in \mathbb{R}^{\bar{n}\times n}$ $\forall i \in \{1, \dots, m\}$ and $G_i \in \mathbb{R}^{\bar{p}\times n}$ $\forall i \in \{1, \dots, r\}$.

The ensuing observer form is employed for the state estimation purpose:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Gf(\hat{x}) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + Fg(\hat{x}). \end{aligned} \tag{15}$$

where $\hat{x}$ is the estimated state, and $L \in \mathbb{R}^{n\times p}$ is the observer gain matrix.

The estimation error of the observer (15) is defined as $\tilde{x} = x - \hat{x}$. From (12) and (15), the subsequent estimation error dynamic is obtained:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + G\tilde{f}(x, \hat{x}) - LF\tilde{g}(x, \hat{x}) + \mathbb{E}\omega, \tag{16}$$

where

$$\mathbb{A} = A - LC, \tag{17}$$

$$\mathbb{E} = E - LD, \tag{18}$$

$$\tilde{f}(x, \hat{x}) = f(x) - f(\hat{x}) = \begin{bmatrix} f_1(v_1) \\ f_2(v_2) \\ \vdots \\ f_m(v_m) \end{bmatrix} - \begin{bmatrix} f_1(\hat{v}_1) \\ f_2(\hat{v}_2) \\ \vdots \\ f_m(\hat{v}_m) \end{bmatrix}, \tag{19}$$

$$\tilde{g}(x, \hat{x}) = g(x) - g(\hat{x}) = \begin{bmatrix} g_1(\theta_1) \\ g_2(\theta_2) \\ \vdots \\ g_r(\theta_r) \end{bmatrix} - \begin{bmatrix} g_1(\hat{\theta}_1) \\ g_2(\hat{\theta}_2) \\ \vdots \\ g_r(\hat{\theta}_r) \end{bmatrix}. \tag{20}$$

Since $f(\cdot)$ and $g(\cdot)$ are globally Lipschitz, then through utilisation of Lemma 1, there exist functions $f_{ij} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $g_{ij} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \to \mathbb{R}$, and known constants $f_{a_{ij}}, f_{b_{ij}}, g_{a_{ij}}$ and $g_{b_{ij}}$, which fulfill:

$$\tilde{f}(x, \hat{x}) = \sum_{i,j=1}^{m,\bar{n}} f_{ij} \mathcal{H}_{ij} H_i \tilde{x}, \tag{21}$$

$$\tilde{g}(x, \hat{x}) = \sum_{i,j=1}^{r,\bar{p}} g_{ij} \mathcal{G}_{ij} G_i \tilde{x}, \tag{22}$$

where $\mathcal{H}_{ij} = e_n(i) e_n^\top(j)$, $\mathcal{G}_{ij} = e_n(i) e_n^\top(j)$, $f_{ij} \triangleq f_{ij}(v_i^{\hat{v}_{i,j-1}}, v_i^{\hat{v}_{i,j}})$ and $g_{ij} \triangleq g_{ij}(\theta_i^{\hat{\theta}_{i,j-1}}, \theta_i^{\hat{\theta}_{i,j}})$.
The functions f_{ij}, g_{ij} satisfy:

$$f_{a_{ij}} \leq f_{ij} \leq f_{b_{ij}};\ g_{a_{ij}} \leq g_{ij} \leq g_{b_{ij}}.$$

Without loss of generality, we presume that $f_{a_{ij}} = 0$ and $g_{a_{ij}} = 0$, that is,

$$0 \leq f_{ij} \leq f_{b_{ij}}, \tag{23}$$

$$0 \leq g_{ij} \leq g_{b_{ij}}. \tag{24}$$

One can refer to the paper[9] for additional information about this.

By employing (21) and (22), the error dynamic (16) is reformulated as follows:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} g_{ij} LF \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega. \tag{25}$$

Remark 1. In various practical applications, it is possible to have $f_{a_{ij}}, g_{a_{ij}} \neq 0$. In such cases, (25) is rewritten as

$$\dot{\tilde{x}} = \underbrace{\left(\mathbb{A} + \sum_{i,j=1}^{m,\bar{n}} f_{a_{ij}} G \mathcal{H}_{ij} H_i - \sum_{i,j=1}^{r,\bar{p}} g_{a_{ij}} LF \mathcal{G}_{ij} G_i \right)}_{\tilde{\mathcal{A}}} \tilde{x} + \sum_{i,j=1}^{m,\bar{n}} \underbrace{(f_{ij} - f_{a_{ij}})}_{\tilde{f}_{ij}} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \underbrace{(g_{ij} - g_{a_{ij}})}_{\tilde{g}_{ij}} LF \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega.$$

It yields:

$$\dot{\tilde{x}} = \tilde{\mathcal{A}}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} \tilde{f}_{ij} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \tilde{g}_{ij} LF \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega. \tag{26}$$

For the error dynamic (26), the functions $\tilde{f}_{ij}$ and $\tilde{g}_{ij}$ satisfy (23) and (24), respectively. Moreover, both forms, i.e., (25) and (26) are analogous.

The objective of this letter is to develop new LMI conditions that compute the observer gain L, such that

1) When $\omega = 0$, the estimation error dynamic (25) is converging towards zero at $t \to \infty$.

2) When $\omega \neq 0$, the estimation error dynamic (25) converges asymptotically with maximum noise attenuation at $t \to \infty$.

In the literature, numerous LMI-based methodologies have been established for tackling the aforementioned problem. For instance, one can refer to the articles [14,9,10]. Each of these approaches provides an improved LMI condition through the utilisation of different mathematical tools. Despite advances in this area of LMI relaxations, the resulting LMIs remain conservative, so there is a potential for further enhancements. In the subsequent sections, two novel LMIs will be proposed by employing a newly defined matrix multiplier.

4 | THE SYNTHESIS OF A ROBUST CIRCLE-CRITERION BASED LMI CONDITION

This section is dedicated to the development of the new LMI condition. In order to mitigate the impact of external disturbances/noise ω on the estimated states, the $\mathcal{H}_\infty$ criterion is commonly used in the control system domain (Refer to the papers [9,14,22]), and it is illustrated as

$$\|\tilde{x}\|_{\mathcal{L}_2^n} \leq \sqrt{\mu\|\omega\|^2_{\mathcal{L}_2^q} + \nu\|\tilde{x}_0\|^2}, \tag{27}$$

where $\mu > 0$. The term $\sqrt{\mu}$ indicates the disturbance attenuation level, and $\nu > 0$ is to be estimated. The condition stated in the criterion (27) ensures the asymptotic stability of the error dynamic (25) at $t \to \infty$ along with optimal disturbance compensation.

In order to analyse the $\mathcal{H}_\infty$ stability of the error dynamic (25), let us consider the following quadratic Lyapunov function:

$$V(\tilde{x}) = \tilde{x}^\top P\tilde{x}, \text{ where } P > 0 \in \mathbf{S}^n. \tag{28}$$

The error dynamic (25) fulfills the $\mathcal{H}_\infty$ criterion (27) if it admits a Lyapunov function (28) such that

$$\mathcal{W} \triangleq \dot{V}(\tilde{x}) + \|\tilde{x}\|^2 - \mu\|\omega\|^2 \leq 0. \tag{29}$$

Remark 2. If the inequality (29) is true, then one can obtain:

$$\int_0^t \dot{V}(\tilde{x},\tau) + \|\tilde{x}(\tau)\|^2 - \mu\|\omega(\tau)\|^2 \leq 0 \tag{30}$$

Since $\forall t \geq 0 \;\; V(\tilde{x}(t)) \geq 0$, the following inequality is derived:

$$-V(\tilde{x}_0) + \|\tilde{x}\|^2_{\mathcal{L}_2^n} - \mu\|\omega\|^2_{\mathcal{L}_2^q} \leq 0 \text{ when } t \to \infty. \tag{31}$$

It leads to

$$\|\tilde{x}\|^2_{\mathcal{L}_2^n} \leq \mu\|\omega\|^2_{\mathcal{L}_2^q} + V(\tilde{x}_0). \tag{32}$$

Additionally, we have $V(\tilde{x}_0) \leq \lambda_{\max}(P)\|\tilde{x}_0\|^2$, and it yields:

$$\|\tilde{x}\|^2_{\mathcal{L}_2^n} \leq \mu\|\omega\|^2_{\mathcal{L}_2^q} + \lambda_{\max}(P)\|\tilde{x}_0\|^2. \tag{33}$$

Inequality (33) is equivalent to (27) if we consider $\nu = \lambda_{\max}(P) > 0$. Hence, the error dynamic (25) satisfies the $\mathcal{H}_\infty$ criterion (27) if it possesses a Lyapunov function (28) which fulfills (29).

Further, $\dot{V}(\tilde{x})$ is calculated along the trajectories of (25), and illustrated as follows:

$$\begin{aligned}\dot{V}(\tilde{x}) = {} & \tilde{x}^\top(\mathbb{A}^\top P + P\mathbb{A})\tilde{x} + \tilde{x}^\top\left[\left(\sum_{i,j=1}^{m,\bar{n}} f_{ij}PG\mathcal{H}_{ij}H_i\right) + \left(\sum_{i,j=1}^{m,\bar{n}} f_{ij}PG\mathcal{H}_{ij}H_i\right)^\top\right]\tilde{x} \\ & - \tilde{x}^\top\left[\left(\sum_{i,j=1}^{r,\bar{p}} g_{ij}PLF\mathcal{G}_{ij}G_i\right) + \left(\sum_{i,j=1}^{r,\bar{p}} g_{ij}PLF\mathcal{G}_{ij}G_i\right)^\top\right]\tilde{x} + \tilde{x}^\top(P\mathbb{E})\omega + \omega^\top(\mathbb{E}^\top P)\tilde{x}.\end{aligned} \tag{34}$$

From (29) and (34), we get:

$$\begin{aligned}\mathcal{W} = {} & \begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top \begin{bmatrix} A^\top P + PA - PLC - C^\top L^\top P + \mathbf{I}_n & PE - PLD \\ (\star) & -\mu\mathbf{I}_q\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix} \\ & + \begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top\left[\sum_{i,j=1}^{m,\bar{n}}\left(\begin{bmatrix}(PG\mathcal{H}_{ij})\\ \mathbf{O}\end{bmatrix}\begin{bmatrix} f_{ij}H_i & \mathbf{O}\end{bmatrix} + \begin{bmatrix} f_{ij}H_i & \mathbf{O}\end{bmatrix}^\top\begin{bmatrix}(PG\mathcal{H}_{ij})\\ \mathbf{O}\end{bmatrix}^\top\right)\right]\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix} \\ & + \begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top\left[\sum_{i,j=1}^{r,\bar{p}}\left(\begin{bmatrix}(-PLF\mathcal{G}_{ij})\\ \mathbf{O}\end{bmatrix}\begin{bmatrix} g_{ij}G_i & \mathbf{O}\end{bmatrix} + \begin{bmatrix} g_{ij}G_i & \mathbf{O}\end{bmatrix}^\top\begin{bmatrix}(-PLF\mathcal{G}_{ij})\\ \mathbf{O}\end{bmatrix}^\top\right)\right]\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}.\end{aligned} \tag{35}$$

Now, the necessary condition to obtain $\mathcal{W} \leq 0$ is established as follows:

$$\mathcal{A}_L + \mathbb{N}_1 + \mathbb{N}_2 \leq 0, \tag{36}$$

where

$$\mathcal{A}_L = \begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \mathbf{I}_n & PE - PLD \\ (\star) & -\mu \mathbf{I}_q \end{bmatrix}, \tag{37}$$

$$\mathbb{N}_1 = \sum_{i,j=1}^{m,\bar{n}} \Bigg(\underbrace{\begin{bmatrix} (PG\mathcal{H}_{ij}) \\ \mathbf{O} \end{bmatrix}}_{U_{ij}^\top} \underbrace{f_{ij} \overbrace{\begin{bmatrix} H_i & \mathbf{O} \end{bmatrix}}^{\mathbb{H}_i}}_{V_{ij}} + V_{ij}^\top U_{ij} \Bigg), \tag{38}$$

$$\mathbb{N}_2 = \sum_{i,j=1}^{r,\bar{p}} \Bigg(\underbrace{\begin{bmatrix} (-R^\top F\mathcal{G}_{ij}) \\ \mathbf{O} \end{bmatrix}}_{M_{ij}^\top} \underbrace{g_{ij} \overbrace{\begin{bmatrix} G_i & \mathbf{O} \end{bmatrix}}^{\mathbb{G}_i}}_{N_{ij}} + N_{ij}^\top M_{ij} \Bigg), \tag{39}$$

and $R^\top = PL$.

To improve the readability, the subsequent notations are introduced for the further part of this section:

$$U = \begin{bmatrix} U_{11}^\top & \dots & U_{1\bar{n}}^\top & \dots & U_{m1}^\top & \dots & U_{m\bar{n}}^\top \end{bmatrix}^\top, \tag{42}$$

$$V = \begin{bmatrix} V_{11}^\top & \dots & V_{1\bar{n}}^\top & \dots & V_{m1}^\top & \dots & V_{m\bar{n}}^\top \end{bmatrix}^\top, \tag{43}$$

$$M = \begin{bmatrix} M_{11}^\top & \dots & M_{1\bar{p}}^\top & \dots & M_{r1}^\top & \dots & M_{r\bar{p}}^\top \end{bmatrix}^\top, \tag{44}$$

$$N = \begin{bmatrix} N_{11}^\top & \dots & N_{1\bar{p}}^\top & \dots & N_{r1}^\top & \dots & N_{r\bar{p}}^\top \end{bmatrix}^\top, \tag{45}$$

where U_{ij}, V_{ij}, M_{ij} and N_{ij} are defined in (38) and (39).

Further, V and N can be expressed as

$$V = \begin{bmatrix} V_{11} \\ V_{12} \\ \vdots \\ V_{1\bar{n}} \\ \vdots \\ V_{m1} \\ \vdots \\ V_{m\bar{n}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{H}_1 & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \mathbb{H}_1 & \dots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \ddots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \mathbb{H}_1 & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \star & \ddots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \star & \star & \mathbb{H}_m & \dots & \mathbf{O} \\ \star & \star & \star & \star & \star & \star & \ddots & \mathbf{O} \\ \star & \star & \star & \star & \star & \star & \star & \mathbb{H}_m \end{bmatrix}}_{\mathbb{H}} \underbrace{\begin{bmatrix} f_{11}\mathbf{I} \\ f_{12}\mathbf{I} \\ \vdots \\ f_{1\bar{n}}\mathbf{I} \\ \vdots \\ f_{m1}\mathbf{I} \\ \vdots \\ f_{m\bar{n}}\mathbf{I} \end{bmatrix}}_{\Phi} = \mathbb{H}\Phi, \tag{46}$$

$$N = \begin{bmatrix} N_{11} \\ N_{12} \\ \vdots \\ N_{1\bar{p}} \\ \vdots \\ N_{r1} \\ \vdots \\ N_{r\bar{p}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{G}_1 & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \mathbb{G}_1 & \dots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \ddots & \mathbf{O} & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \mathbb{G}_1 & \dots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \star & \ddots & \mathbf{O} & \dots & \mathbf{O} \\ \star & \star & \star & \star & \star & \mathbb{G}_r & \dots & \mathbf{O} \\ \star & \star & \star & \star & \star & \star & \ddots & \mathbf{O} \\ \star & \star & \star & \star & \star & \star & \star & \mathbb{G}_r \end{bmatrix}}_{\mathbb{G}} \underbrace{\begin{bmatrix} g_{11}\mathbf{I} \\ g_{12}\mathbf{I} \\ \vdots \\ g_{1\bar{p}}\mathbf{I} \\ \vdots \\ g_{r1}\mathbf{I} \\ \vdots \\ g_{r\bar{p}}\mathbf{I} \end{bmatrix}}_{\Psi} = \mathbb{G}\Psi, \tag{47}$$

where $\mathbb{H}_i$ and $\mathbb{G}_i$ are described in (38) and (39), respectively. By incorporating all these notations, (46) and (47), one can rewrite $\mathbb{N}_1$ and $\mathbb{N}_2$ as

$$\mathbb{N}_1 = \mathbb{U}^\top(\mathbb{H}\Phi) + (\mathbb{H}\Phi)^\top\mathbb{U}, \tag{48}$$

$$\mathbb{N}_2 = \mathbb{M}^\top(\mathbb{G}\Psi) + (\mathbb{G}\Psi)^\top\mathbb{M}. \tag{49}$$

$$
\mathbb{Z}=\begin{bmatrix}
Z_{11} & Z_{a^1_{12}} & \dots & Z_{a^1_{1\bar n}} & Z_{b^{11}_{21}} & Z_{b^{11}_{22}} & \dots & Z_{b^{11}_{2\bar n}} & \dots & Z_{b^{11}_{m1}} & Z_{b^{11}_{m2}} & \dots & Z_{b^{11}_{m\bar n}} \\
Z_{a^1_{12}} & Z_{12} & \dots & Z_{a^2_{1\bar n}} & Z_{b^{12}_{21}} & Z_{b^{12}_{22}} & \dots & Z_{b^{12}_{2\bar n}} & \dots & Z_{b^{12}_{m1}} & Z_{b^{12}_{m2}} & \dots & Z_{b^{12}_{m\bar n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{a^1_{1\bar n}} & Z_{a^2_{1\bar n}} & \dots & Z_{1\bar n} & Z_{b^{1\bar n}_{21}} & Z_{b^{1\bar n}_{22}} & \dots & Z_{b^{1\bar n}_{2\bar n}} & \dots & Z_{b^{1\bar n}_{m1}} & Z_{b^{1\bar n}_{m2}} & \dots & Z_{b^{1\bar n}_{m\bar n}} \\
Z_{b^{11}_{21}} & Z_{b^{12}_{21}} & \dots & Z_{b^{1\bar n}_{21}} & Z_{21} & Z_{a^1_{22}} & \dots & Z_{a^1_{2\bar n}} & \dots & Z_{b^{21}_{m1}} & Z_{b^{21}_{m2}} & \dots & Z_{b^{21}_{m\bar n}} \\
Z_{b^{11}_{22}} & Z_{b^{12}_{22}} & \dots & Z_{b^{1\bar n}_{22}} & Z_{a^1_{22}} & Z_{22} & \dots & Z_{a^2_{2\bar n}} & \dots & Z_{b^{22}_{m1}} & Z_{b^{22}_{m2}} & \dots & Z_{b^{22}_{m\bar n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{b^{11}_{2\bar n}} & Z_{b^{12}_{2\bar n}} & \dots & Z_{b^{1\bar n}_{2\bar n}} & Z_{a^1_{2\bar n}} & Z_{a^2_{2\bar n}} & \dots & Z_{2\bar n} & \dots & Z_{b^{2\bar n}_{m1}} & Z_{b^{2\bar n}_{m2}} & \dots & Z_{b^{2\bar n}_{m\bar n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
Z_{b^{11}_{m1}} & Z_{b^{12}_{m1}} & \dots & Z_{b^{1\bar n}_{m1}} & Z_{b^{21}_{m1}} & Z_{b^{22}_{m1}} & \dots & Z_{b^{2\bar n}_{m1}} & \dots & Z_{m1} & Z_{a^1_{m2}} & \dots & Z_{a^1_{m\bar n}} \\
Z_{b^{11}_{m2}} & Z_{b^{12}_{m2}} & \dots & Z_{b^{1\bar n}_{m2}} & Z_{b^{21}_{m2}} & Z_{b^{22}_{m2}} & \dots & Z_{b^{2\bar n}_{m2}} & \dots & Z_{a^1_{m2}} & Z_{m2} & \dots & Z_{a^2_{m\bar n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{b^{11}_{m\bar n}} & Z_{b^{12}_{m\bar n}} & \dots & Z_{b^{1\bar n}_{m\bar n}} & Z_{b^{21}_{m\bar n}} & Z_{b^{22}_{m\bar n}} & \dots & Z_{b^{2\bar n}_{m\bar n}} & \dots & Z_{a^1_{m\bar n}} & Z_{a^2_{m\bar n}} & \dots & Z_{m\bar n}
\end{bmatrix}, \tag{40}
$$

where $Z_{ij} > 0 \in \mathbf{S}^{\bar n}$, $Z_{a^k_{ij}} \geq 0 \in \mathbf{S}^{\bar n}$ $\forall i,k \in \{1,\dots,m\}, \& j \in \{1,\dots,\bar n\}$; $Z_{b^{kj}_{ij}} \geq 0 \in \mathbf{S}^{\bar n}, \forall i \in \{2,\dots,m\}, k \in \{1,\dots,m-1\}, \& j \in \{1,\dots,\bar n\}$ such that $\mathbb{Z} > 0$.

$$
\mathbb{S}=\begin{bmatrix}
S_{11} & S_{a^1_{12}} & \dots & S_{a^1_{1\bar p}} & S_{b^{11}_{21}} & S_{b^{11}_{22}} & \dots & S_{b^{11}_{2\bar p}} & \dots & S_{b^{11}_{r1}} & S_{b^{11}_{r2}} & \dots & S_{b^{11}_{r\bar p}} \\
S_{a^1_{12}} & S_{12} & \dots & S_{a^2_{1\bar p}} & S_{b^{12}_{21}} & S_{b^{12}_{22}} & \dots & S_{b^{12}_{2\bar p}} & \dots & S_{b^{12}_{r1}} & S_{b^{12}_{r2}} & \dots & S_{b^{12}_{r\bar p}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
S_{a^1_{1\bar p}} & S_{a^2_{1\bar p}} & \dots & S_{1\bar p} & S_{b^{1\bar p}_{21}} & S_{b^{1\bar p}_{22}} & \dots & S_{b^{1\bar p}_{2\bar p}} & \dots & S_{b^{1\bar p}_{r1}} & S_{b^{1\bar p}_{r2}} & \dots & S_{b^{1\bar p}_{r\bar p}} \\
S_{b^{11}_{21}} & S_{b^{12}_{21}} & \dots & S_{b^{1\bar p}_{21}} & S_{21} & S_{a^1_{22}} & \dots & S_{a^1_{2\bar p}} & \dots & S_{b^{21}_{r1}} & S_{b^{21}_{r2}} & \dots & S_{b^{21}_{r\bar p}} \\
S_{b^{11}_{22}} & S_{b^{12}_{22}} & \dots & S_{b^{1\bar p}_{22}} & S_{a^1_{22}} & S_{22} & \dots & S_{a^2_{2\bar p}} & \dots & S_{b^{22}_{r1}} & S_{b^{22}_{r2}} & \dots & S_{b^{22}_{r\bar p}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
S_{b^{11}_{2\bar p}} & S_{b^{12}_{2\bar p}} & \dots & S_{b^{1\bar p}_{2\bar p}} & S_{a^1_{2\bar p}} & S_{a^2_{2\bar p}} & \dots & S_{2\bar p} & \dots & S_{b^{2\bar p}_{r1}} & S_{b^{2\bar p}_{r2}} & \dots & S_{b^{2\bar p}_{r\bar p}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
S_{b^{11}_{r1}} & S_{b^{12}_{r1}} & \dots & S_{b^{1\bar p}_{r1}} & S_{b^{21}_{r1}} & S_{b^{22}_{r1}} & \dots & S_{b^{2\bar p}_{r1}} & \dots & S_{r1} & S_{a^1_{r2}} & \dots & S_{a^1_{r\bar p}} \\
S_{b^{11}_{r2}} & S_{b^{12}_{r2}} & \dots & S_{b^{1\bar p}_{r2}} & S_{b^{21}_{r2}} & S_{b^{22}_{r2}} & \dots & S_{b^{2\bar p}_{r2}} & \dots & S_{a^1_{r2}} & S_{r2} & \dots & S_{a^2_{r\bar p}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
S_{b^{11}_{r\bar p}} & S_{b^{12}_{r\bar p}} & \dots & S_{b^{1\bar p}_{r\bar p}} & S_{b^{21}_{r\bar p}} & S_{b^{22}_{r\bar p}} & \dots & S_{b^{2\bar p}_{r\bar p}} & \dots & S_{a^1_{r\bar p}} & S_{a^2_{r\bar p}} & \dots & S_{r\bar p}
\end{bmatrix}, \tag{41}
$$

where $S_{ij} > 0 \in \mathbf{S}^{\bar p}$, $S_{a^k_{ij}} \geq 0 \in \mathbf{S}^{\bar p}$ $\forall i,k \in \{1,\dots,r\}, \& j \in \{1,\dots,\bar p\}$; $S_{b^{kj}_{ij}} \geq 0 \in \mathbf{S}^{\bar p}$, $\forall i \in \{2,\dots,r\}, k \in \{1,\dots,r-1\}, \& j \in \{1,\dots,\bar p\}$ so that $\mathbb{S} > 0$.

The deployment of the new variant of Young inequality (10) on (48) and (49) yield:

$$
\mathbb{N}_1 \leq \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi), \tag{50}
$$

$$
\mathbb{N}_2 \leq \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi), \tag{51}
$$

where the matrices $\mathbb{Z}$ and $\mathbb{S}$ are defined in (40) and (41), respectively.
Hence, the condition (36) is reformulated as

$$
\mathcal{A}_L + \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi) + \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi) \leq 0. \tag{52}
$$

Inequalities (23) and (24) imply that each element inside Φ and Ψ is bounded and belong to convex sets $\mathcal{F}_m$ and $\mathcal{G}_r$, respectively. The sets $\mathcal{F}_m$ and $\mathcal{G}_r$ are defined as follows:

$$
\mathcal{F}_m \triangleq \left\{\Phi : 0 \leq f_{ij} \leq f_{b_{ij}}, \forall i \in \{1,\dots,m\} \,\&\, j \in \{1,\dots,\bar n\}\right\},
$$

$$
\mathcal{G}_r \triangleq \left\{\Psi : 0 \leq g_{ij} \leq g_{b_{ij}}, \forall i \in \{1,\dots,r\} \,\&\, j \in \{1,\dots,\bar p\}\right\}.
$$

The set of vertices of $\mathcal{F}_m$ and $\mathcal{G}_r$ are given by

$$\mathcal{H}_\phi = \left\{ \{\mathcal{F}_{11}, \dots, \mathcal{F}_{1\bar{n}}, \dots, \mathcal{F}_{m1}, \dots, \mathcal{F}_{m\bar{n}}\} : \mathcal{F}_{ij} \in [0, f_{b_{ij}}] \right\}, \tag{53}$$

$$\mathcal{H}_\psi = \left\{ \{\mathcal{G}_{11}, \dots, \mathcal{G}_{1\bar{p}}, \dots, \mathcal{G}_{r1}, \dots, \mathcal{G}_{r\bar{p}}\} : \mathcal{G}_{ij} \in [0, g_{b_{ij}}] \right\}. \tag{54}$$

Therefore, the inequality (52) is rewritten as

$$\mathcal{A}_L + \left[\frac{1}{2}(U + \mathbb{ZH}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{ZH}\Phi)\right]_{\forall\ \Phi\in\mathcal{H}_\phi} + \left[\frac{1}{2}(M + \mathbb{SG}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{SG}\Psi)\right]_{\forall\ \Psi\in\mathcal{H}_\psi} \leq 0. \tag{55}$$

Now, we are ready to state the following theorem:

Theorem 1. Let us consider the matrices $\mathbb{Z}$, $\mathbb{S}$, which are expressed in the form of (40), (41), respectively. The estimation error dynamic (25) is $\mathcal{H}_\infty$ asymptotically stable if there exist two matrices $P > 0 \in \mathbf{S}^n$ and $R \in \mathbb{R}^{p\times n}$ such that the ensuing optimization problem is solvable:

$$\min\ \mu \text{ subject to}$$
$$\begin{bmatrix} \mathcal{A}_L & (U + \mathbb{ZH}\Phi)^\top & (M + \mathbb{SG}\Psi)^\top \\ (\star) & -2\mathbb{Z} & \mathbf{O} \\ (\star) & (\star) & -2\mathbb{S} \end{bmatrix} < 0, \ \ \forall \Phi \in \mathcal{H}_\phi, \ \forall \Psi \in \mathcal{H}_\psi, \tag{56}$$

where $\mathcal{A}_L$, U, $\mathbb{H}$, M and $\mathbb{G}$ are specified in (37), (42), (46), (44) and (47), respectively. The gain matrix L is calculated by utilising $L = P^{-1}R^\top$.

Proof. The implementation of the Schur Lemma on the expression (55) yields the LMI (56). From convexity principle which is proposed in the paper[11], the error dynamics (25) satisfies $\mathcal{H}_\infty$ criterion (27) with $\nu = \lambda_{\max}(P) > 0$ and minimum μ obtained from the solution of LMI (56) if LMI (56) is solved for all $\Phi \in \mathcal{F}_{F_m}$ and $\Psi \in \mathcal{G}_{G_r}$. Hence, proved. ☐

In order to ensure the stability of the estimation error dynamic (25) in the absence of the exogenous disturbances (i.e., $\omega = 0$), one can follow the succeeding remark:

Remark 3. At $\omega = 0$, an inequality (29) becomes:

$$\dot{V}(\tilde{x}) + \|\tilde{x}\|^2 \leq 0. \tag{57}$$

An inequality (57) leads to the exponential stability condition $\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x})$, along with $\sigma = \frac{1}{\lambda_{\max}(P)} > 0$. Since the error dynamics (25) admits the exponential stability criterion, it ensures that the error dynamics (25) is exponentially stable when $\omega = 0$.

In the next part, we propose a second LMI approach.

5 | NEW LMI DESIGN BY EXPLORING ISS CRITERION

In this section, the novel LMI condition is derived by incorporating the ISS notion with a popular LPV approach, which ensures the stability of the error dynamic (25).

In order to facilitate the lucidity of the presentation and to enhance the comprehensibility of the contributions, this section is divided as follows:

1. In the first part, we will derive certain conditions which guarantee that the error dynamic (25) is ISS w.r.t. ω.
2. Later on, a necessary criterion in the form of an LMI is deduced by deploying these conditions.

5.1 | Establishing the essential criterion for ISS

The following theorem provides the required conditions which ensure the ISS behaviour of the system (25) w.r.t. ω:

Theorem 2. I) The error dynamic (25) is ISS with respect to ω if it possesses an ISS-Lyapunov function (28).

II) The trajectories of the system (25) satisfy the following constraints:

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t} \|\tilde{x}_0\| + \sqrt{\frac{\delta}{\sigma\lambda_{\min}(P)}} \|\omega\|^2_{\mathcal{L}_2}, \tag{58}$$

for any $\mathcal{L}_2$ bounded $\omega \in \mathbb{R}^q$.

III) In addition to this, $\tilde{x}(t)$ is bounded at $t \to \infty$, such that:

$$\|\tilde{x}(\infty)\| \leq \sqrt{\frac{\delta}{\sigma\lambda_{\min}(P)}} \|\omega\|^2_{\mathcal{L}_2}. \tag{59}$$

Proof. For simplicity, the proof of the theorem is provided in the subsequent parts:

- **ISS-Lyapunov function:**
  Let us consider the Lyapunov function (28). One can easily notice that the function $V(\tilde{x})$ satisfies:

$$\lambda_{\min}(P)\|\tilde{x}\|^2 \leq V(\tilde{x}) \leq \lambda_{\max}(P)\|\tilde{x}\|^2. \tag{60}$$

Further, the derivative of the function $V(\tilde{x})$ along the trajectories of (25) is calculated and illustrated in (34). By implementing the inequality (9) on the term $\tilde{x}^\top P\mathbb{E}\omega + \omega^\top \mathbb{E}^\top P\tilde{x}$ of (34), we obtain:

$$\tilde{x}^\top P\mathbb{E}\omega + \omega^\top \mathbb{E}^\top P\tilde{x} \leq \delta^{-1}\tilde{x}^\top (P\mathbb{E})^\top (P\mathbb{E})\tilde{x} + \delta\omega^\top\omega,$$

where $\delta > 0$.
Thus, the inequality (34) is reformulated as follows:

$$\begin{aligned}\dot{V}(\tilde{x}) \leq \tilde{x}^\top \left[\mathbb{A}^\top P + P\mathbb{A} + \mathbb{E}^\top P(\delta^{-1}\mathbf{I})P\mathbb{E}\right]\tilde{x} + \tilde{x}^\top \left[\left(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\right) + \left(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\right)^\top\right]\tilde{x} \\ - \tilde{x}^\top \left[\left(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\right) + \left(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\right)^\top\right]\tilde{x} + \delta\omega^\top\omega.\end{aligned} \tag{61}$$

Let us consider a positive scalar σ such that the following inequality holds:

$$\begin{aligned}\left[\mathbb{A}^\top P + P\mathbb{A} + \mathbb{E}^\top P(\delta^{-1}\mathbf{I})P\mathbb{E}\right] + \left[\left(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\right) + \left(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\right)^\top\right] \\ + \left[\left(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\right) + \left(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\right)^\top\right] + \sigma P \leq O.\end{aligned} \tag{62}$$

It leads to:

$$\dot{V}(\tilde{x}) \leq -\tilde{x}^\top(\sigma P)\tilde{x} + \delta\omega^\top\omega. \tag{63}$$

which gives:

$$\dot{V}(\tilde{x}) \leq -\sigma\lambda_{\max}(P)\|\tilde{x}\|^2 + \delta\|\omega\|^2. \tag{64}$$

From (60) and (64), one can notice that the Lyapunov function (28) fulfills (3) and (4) along with

$$\alpha_1(\tilde{x}) = \lambda_{\min}(P)\|\tilde{x}\|^2,\ \alpha_2(\tilde{x}) = \lambda_{\max}(P)\|\tilde{x}\|^2,\ \alpha_3(\tilde{x}) = -\sigma\lambda_{\max}(P)\|\tilde{x}\|^2 \text{ and } \alpha_4(\omega) = \delta\|\omega\|^2.$$

Since $V(\tilde{x})$ meets the criterion illustrated in (3) and (4), it is an ISS-Lyapunov function. Therefore, the system (25) is ISS with respect to ω as it admits an ISS-Lyapunov function (28). Hence, statement I of Theorem 2 is proved.

- **The proof of statements II and III:**
  From (63),

$$\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x}) + \delta\|\omega\|^2. \tag{65}$$

It implies that the trajectories of $V(\tilde{x})$ hold:

$$V(\tilde{x}(t)) \leq V(\tilde{x}_0)e^{-\sigma t} + \delta e^{-\sigma t}\int_0^t \mathrm{e}^{\sigma s}\|\omega(s)\|_2^2 \mathrm{d}s \leq V(\tilde{x}_0)\mathrm{e}^{-\sigma t} + \frac{\delta}{\sigma}\left(1 - \mathrm{e}^{-\sigma t}\right)\sup_{s\in[0,t]}\|\omega(s)\|_2^2. \tag{66}$$

Since $0 \leq 1 - \mathrm{e}^{-\sigma t} \leq 1$ and $\sup_{s\in[0,t]} \|\omega(s)\|_2^2 \leq \|\omega\|_{\mathcal{L}_2}^2$, the inequality (66) is altered as:

$$V(\tilde{x}(t)) \leq V(\tilde{x}_0)\mathrm{e}^{-\sigma t} + \frac{\delta}{\sigma}\|\omega\|_{\mathcal{L}_2}^2. \tag{67}$$

As we have $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq V(\tilde{x},t) \leq \lambda_{\max}(P)\|\tilde{x}(t)\|^2$, it is easy to derive:

$$\|\tilde{x}(t)\|^2 \leq \frac{V(\tilde{x},t)}{\lambda_{\min}(P)} \leq \frac{e^{-\sigma t}V(\tilde{x},0) + \delta\|\omega(.)\|_2^2}{\lambda_{\min}(P)} \leq \frac{\lambda_{\max}(P)\|\tilde{x}_0\|^2 e^{-\sigma t} + \delta\sigma^{-1}\|\omega\|_{\mathcal{L}_2}^2}{\lambda_{\min}(P)}. \tag{68}$$

Thus, for any $\mathcal{L}_2$ bounded ω,

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t}\|\tilde{x}_0\| + \sqrt{\frac{\delta}{\sigma\lambda_{\min}(P)}}\|\omega\|_{\mathcal{L}_2}^2.$$

Hence, statement II is proved.
At $t \to \infty$, the inequality (58) becomes:

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\delta}{\sigma\lambda_{\min}(P)}}\|\omega\|_{\mathcal{L}_2}^2.$$

Therefore, $\tilde{x}(t)$ is bounded at $t \to \infty$.

This ends the proof of the theorem. □

Remark 4. In the case of $\omega = 0$, the condition specified in (58) is reformulated as

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t}\|\tilde{x}_0\|.$$

Hence, the error dynamic (25) is exponentially stable in the absence of disturbances (i.e., at $\omega = 0$).

5.2 | Formulating a matrix-multipliers based LMI

This segment of the section encompasses the development of an LMI condition which is based on the aforementioned conditions described for stability.

Theorem 3. Let us assume that there exist two symmetric positive definite matrices $\mathbb{Z}$ and $\mathbb{S}$, which are defined in (40), (41), respectively. The estimation error dynamic (25) is ISS w.r.t. ω if the following optimization problem is solvable:

$$\min\ \delta \text{ subject to}$$
$$\begin{bmatrix} \mathbb{L}_1 & \left(\mathbb{U} + \mathbb{Z}\mathbb{F}\Phi\right)^\top & \left(\mathbb{M} + \mathbb{S}\mathbb{H}\Psi\right)^\top \\ \star & -2\mathbb{Z} & \mathbf{O} \\ \star & \star & -2\mathbb{S} \end{bmatrix} < 0\ ,\ \forall\Phi \in \mathcal{F}_m,\ \forall\Psi \in \mathcal{H}_r, \tag{69}$$

where $P = P^\top > 0 \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{p\times n}$, $\delta, \sigma > 0$ and

$$\mathbb{L}_1 = \begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \sigma P & PE - R^\top D \\ (\star) & -\delta\mathbf{I} \end{bmatrix}. \tag{70}$$

Other variables remain consistent with those specified in Theorem 1. In addition to this, $L = P^{-1}R^\top$ aids in the determination of the gain matrix L.

Proof. One of the essential conditions described in the proof of Theorem 1 is showcased in (62). Further, one can rewrite the inequality (62) in the subsequent manner:

$$\mathbb{L}_1 + \mathbb{N}_1 + \mathbb{N}_2 \leq 0, \tag{71}$$

where $\mathbb{L}_1$, $\mathbb{N}_1$ and $\mathbb{N}_2$ are defined in (70), (38) and (39), respectively. Additionally, $R^\top = PL$.
From (50) and (51), we obtain:

$$\mathbb{L}_1 + \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top\mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi + \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top\mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi) \leq 0. \tag{72}$$

Analogous to (55), the inequality (72) is rewritten as

$$\mathbb{L}_1 + \left[\frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi)\right]_{\forall\, \Phi\in\mathcal{H}_\phi} + \left[\frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi)\right]_{\forall\, \Psi\in\mathcal{H}_\psi} \leq 0. \tag{73}$$

The Schur compliment of (73) yeilds LMI (69). According to the convexity principal[11], if LMI (69) is evaluated for every elements of $\Phi \in \mathcal{F}_{F_m}$ and $\Psi \in \mathcal{G}_{G_r}$, then the error dynamic (25) satisfies (62). It ensures that all conditions specified in Theorem 2 are fulfilled by the error dynamics (25). Hence, from Theorem 2, system (25) is ISS w.r.t. ω. Hence, the proof is completed. □

In the following segment, the effectiveness of the proposed LMIs is discussed.

6 | COMMENT ON THE PROPOSED LMIs: EXPLOITING THE NUMBER OF DECISION VARIABLES

The introduction of the newly defined matrix multipliers aids in the improvement of the LMI conditions as compared to the existing approaches. These advancements are mainly because of the additional number of decision variables in the proposed LMIs. Hence, these matrix multipliers play a vital role in LMI enhancement. In addition to this, one must know how these matrix multipliers add extra numbers of decision variables. The objective of this section is to tackle such questions and to prove the uniqueness of the proposed LMIs. First, the computation of the number of decision variables inside the derived LMIs is presented. Further, the comparison with existing LMI approaches is provided to validate the novelty of the proposed method.

As stated earlier, the use of the matrices $\mathbb{Z}$ and $\mathbb{S}$ in LMIs (56) and (69) allows the inclusion of additional numbers of decision variables. Both LMIs contain the ensuing number of decision variables:

$$\mathrm{N}_{dv_1} = \underbrace{np + \frac{n(n+1)}{2}}_{\mathcal{N}_P} + q + \underbrace{\left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right)}_{\mathcal{N}_{\text{add}_1}} + \underbrace{\left(\frac{r\bar{p}(r\bar{p}+1)}{2}\right)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right)}_{\mathcal{N}_{\text{add}_2}}, \tag{74}$$

where $\mathcal{N}_P$, $\mathcal{N}_{\text{add}_1}$ and $\mathcal{N}_{\text{add}_2}$ are the number of variables obtained from matrices $\mathbb{L}_1$, $\mathbb{Z}$ and $\mathbb{S}$, respectively. Moreover, the terms $\mathcal{N}_{\text{add}_1}$ and $\mathcal{N}_{\text{add}_2}$ represent the additional number of decision variables in the proposed LMIs. The total number of additional variables is given by,

$$\mathcal{N}_{\text{add}} = \mathcal{N}_{\text{add}_1} + \mathcal{N}_{\text{add}_2} = \left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + \left(\frac{r\bar{p}(r\bar{p}+1)}{2}\right)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right). \tag{75}$$

Now, let us determine the number of decision variables in the subsequent cases:

1. Case 1: Block-diagonal matrix multipliers (similar to the paper[9])
   If one deploys the block-diagonal matrices in the proposed LMIs, then the following number of decision variables is obtained:

   $$\mathrm{N}_{dv_2} = np + \frac{n(n+1)}{2} + q + \mathcal{N}^1_{\text{add}},$$

   where

   $$\mathcal{N}^1_{\text{add}} = m\bar{n}\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + r\bar{p}\left(\frac{\bar{p}(\bar{p}+1)}{2}\right). \tag{76}$$

   $\mathcal{N}^1_{\text{add}}$ denotes the number of variables obtained from block-diagonal matrices.

2. Case 2: Diagonal matrix multipliers (same as the one proposed in[14])
   Here, if we use $\mathbb{Z}$ and $\mathbb{S}$ as the diagonal matrices, then the number of additional variables is achieved as follows:

   $$\mathrm{N}_{dv_2} = np + \frac{n(n+1)}{2} + q + \mathcal{N}^2_{\text{add}},$$

   along with

   $$\mathcal{N}^2_{\text{add}} = m\bar{n}^2 + r\bar{p}^2. \tag{77}$$

   $\mathcal{N}^2_{\text{add}}$ signifies the number of variables obtained from diagonal matrices.

3 Case 3: Matrix-multiplier showcased in the paper[21]
Here, if we employ the matrices $\mathbb{Z}$ and $\mathbb{S}$ in the form of Equation (28) of the article[21], then we get:

$$\mathrm{N}_{dv_3} = np + \frac{n(n+1)}{2} + q + \mathcal{N}^3_{\text{add}},$$

along with

$$\mathcal{N}^3_{\text{add}} = (2m-1)(2\bar{n}-1)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + (2r-1)(2\bar{p}-1)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right). \tag{78}$$

$\mathcal{N}^3_{\text{add}}$ represents the number of variables obtained in Case 3.

Since m and $\bar{n}$ are positive integers, it is easy to interpret

$$\mathcal{N}^2_{\text{add}} \leq \mathcal{N}^1_{\text{add}} \leq \mathcal{N}^3_{\text{add}} \leq \mathcal{N}_{\text{add}}. \tag{79}$$

Hence, the number of additional decision variables obtained from the proposed matrix multipliers is greater than the one employed in the existing methods. These additional variables add extra degrees of freedom and improve the feasibility of LMI.

In the sequel, the effectiveness of the derived LMI approaches is highlighted through a numerical example.

7 | EVALUATING THE PERFORMANCE OF THE PROPOSED LMIS AND THE OBSERVERS

The primary aim of this section is to emphasise the significance of the derived LMI conditions. In order to achieve this objective, a numerical example of the Lipschitz nonlinear system is utilised.

Let us consider a nonlinear system under the form of (12) with the subsequent parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix};\ G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix};\ E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};\ F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The nonlinearities of the dynamics and outputs are illustrated as

$$f(x) = \begin{bmatrix} f_1(H_1x) \\ f_2(H_2x) \end{bmatrix} = \begin{bmatrix} \sin(0.3x_2) \\ \cos(0.3x_2x_3) \end{bmatrix}, \text{ and } g(x) = \begin{bmatrix} g_1(F_1x) \\ g_2(F_2x) \end{bmatrix} = \begin{bmatrix} \sin(0.5x_2) \\ \sin(0.5x_3) \end{bmatrix},$$

where
$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ H_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ F_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Thus, $m = 2,\ r = 2,\ \bar{n} = 4$ and $\bar{p} = 3$.

It is easy to infer that both functions $f(x)$ and $g(x)$ fulfill (23) and (24), respectively. Therefore, LMIs (56) and (69) can be implemented to design the observer (15).

Let us consider the ensuing cases for the analysis of the proposed LMI performance:

I) **Case 1:** LMI (56) with the following matrices:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{b_{21}} & Z_{b_{22}} \\ Z_{b_{21}} & Z_{21} & Z_{a_{22}} \\ Z_{b_{22}} & Z_{a_{22}} & Z_{22} \end{bmatrix} \text{ and } \mathbb{S} = \begin{bmatrix} S_{11} & S_{b_{21}} \\ S_{b_{21}} & S_{21} \end{bmatrix}, \tag{80}$$

where $Z_{ij}, Z_{a_{ij}}, Z_{b_{ij}} \in \mathbb{R}^{\bar{n}\times\bar{n}}\ \forall\, i,j \in \{1,2\}$ and $S_{11}, S_{21}, S_{b_{21}} \in \mathbb{R}^{\bar{p}\times\bar{p}}$ are symmetric matrices such that $\mathbb{Z} > 0$ and $\mathbb{S} > 0$. The form of the matrices illustrated in (80) is equivalent to the one described in (40) and (41).

II) **Case 2:** LMI (56) along with

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & \alpha Z_{21} & \alpha Z_{22} \\ \alpha Z_{21} & Z_{21} & \alpha Z_{22} \\ \alpha Z_{22} & \alpha Z_{22} & Z_{22} \end{bmatrix} \text{ and } \mathbb{S} = \begin{bmatrix} S_{11} & \beta S_{21} \\ \beta S_{21} & S_{21} \end{bmatrix}, \tag{81}$$

where $Z_{ij} = Z_{ij}^\top \in \mathbb{R}^{\bar{n}\times\bar{n}}, S_{ij} = S_{ij}^\top \in \mathbb{R}^{\bar{p}\times\bar{p}}\ \forall\, i,j \in \{1,2\}; \alpha = \beta = 0.2$ so that $\mathbb{Z} > 0$ and $\mathbb{S} > 0$. The structure of the matrices specified in (81) is equivalent to the one proposed in the paper[20].

III) **Case 3:** LMI (56) by using

$$\mathbb{Z} = \text{block-diag}(Z_{11}, Z_{21}, Z_{22}) \text{ and } \mathbb{S} = \text{block-diag}(S_{11}, S_{21}), \tag{82}$$

where all the matrices are same as the one defined in (80) such that $\mathbb{Z} > 0$ and $\mathbb{S} > 0$. The structure of the matrices illustrated in (82) is similar to the one utilised in the article[9].

IV) **Case 4:** LMI (69) by utilising the matrices specified in (80).

V) **Case 5:** LMI (69) by deploying the matrices illustrated in (81).

VI) **Case 6:** LMI (69) by employing the matrices described in (82).

VII) **Case 7:** LMI approach proposed in the article[9]. (We have considered $K_i = 0$)

Table 1 A synopsis of LMI solutions obtained in several cases

No.	Cases	Parmeters obtained from LMI solution		
		$\sqrt{\mu}$	δ	$\gamma = \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}}$
1	**Case 1**	1.7226	N.A. (Not Applicable)	
2	**Case 2**	3.4457	N.A.	
3	**Case 3**	2.8374	N.A.	
4	**Case 4**	N.A.	0.2652	1.6284
5	**Case 5**	N.A.	1.0273	32.0516
6	**Case 6**	N.A.	0.7351	2.7113
7	**Case 7**	2.0153	N.A.	

The feasibility of LMIs (56) and (69) is tested in all the aforementioned cases using MATLAB LMI toolbox by considering $\sigma = 0.01$. The optimal values obtained from LMI solutions are outlined in Table 1. It showcases that the value of $\sqrt{\mu}$ obtained in **Case 1** is better as compared to the one obtained in **Case 2**, **Case 3** and **Case 7**. It interprets that the proposed LMI (56) provides a more optimal solution with the newly defined matrix multipliers compared to other matrices used in literature and the existing methods. Additionally, Table 1 conveys that LMI (69) provides the optimal values of δ and γ with the proposed matrix multipliers (i.e., (80)) than with the existing matrix multipliers (that is, (81) and (82)). Thus, Table 1 aids in highlighting the superiority of the proposed matrix-multiplier-based LMIs over the existing methods.

Further, the performance of the observer (15) is analysed for the above-mentioned cases. The initial conditions of the systems are as follows: $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$. The input of the system is considered as $u = 2\sin t \; \forall t \in [0, 20]$. Let us presume that the dynamics and outputs of the systems are corrupted with the Gaussian noise ($\omega \sim N(0, 1)$). Through the utilisation of the observer gain matrices obtained from LMI solutions, the observer (15) is implemented in a MATLAB environment. Figure 1 represents the plot of the estimation error ($\tilde{x}$) obtained in **Case 1**. Whereas, the estimation error achieved in **Case 4** and **Case 7** are shown in Figure 2 and 3, respectively. All these figures highlight the asymptotic convergence of the estimation error. In addition to this, these figures infer that the observer (34) with the proposed LMIs (i.e., (56) and (69)) provides a better noise compensation as compared to the method proposed in the article[9]. To validate this, the RMSE values of the estimation errors are computed over

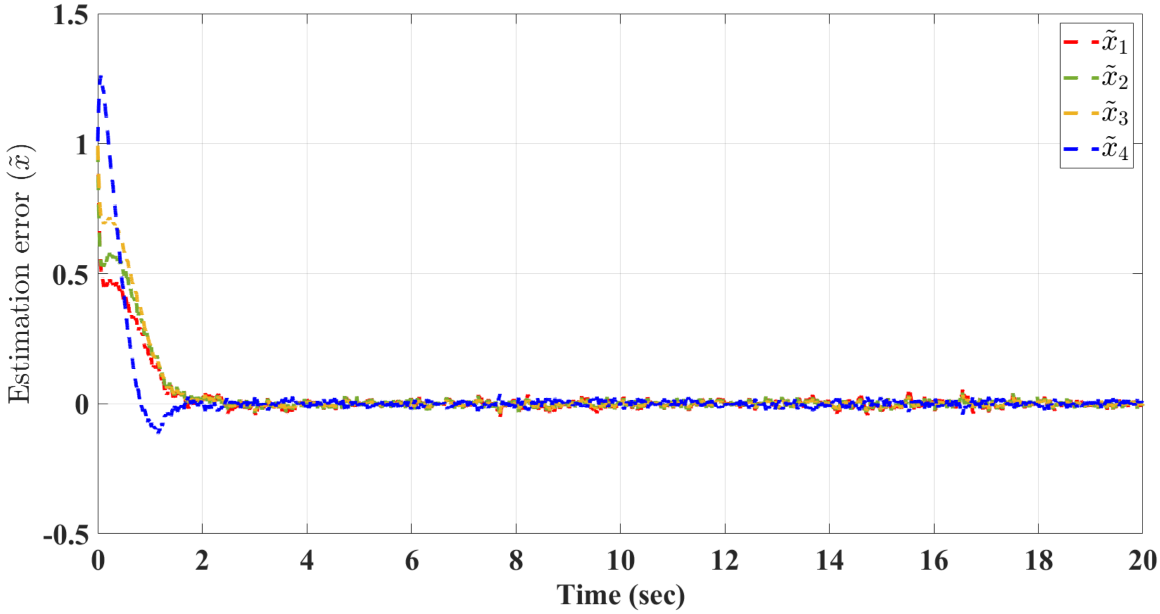

Figure 1 Estimation error ($\tilde{x}$) for Case 1

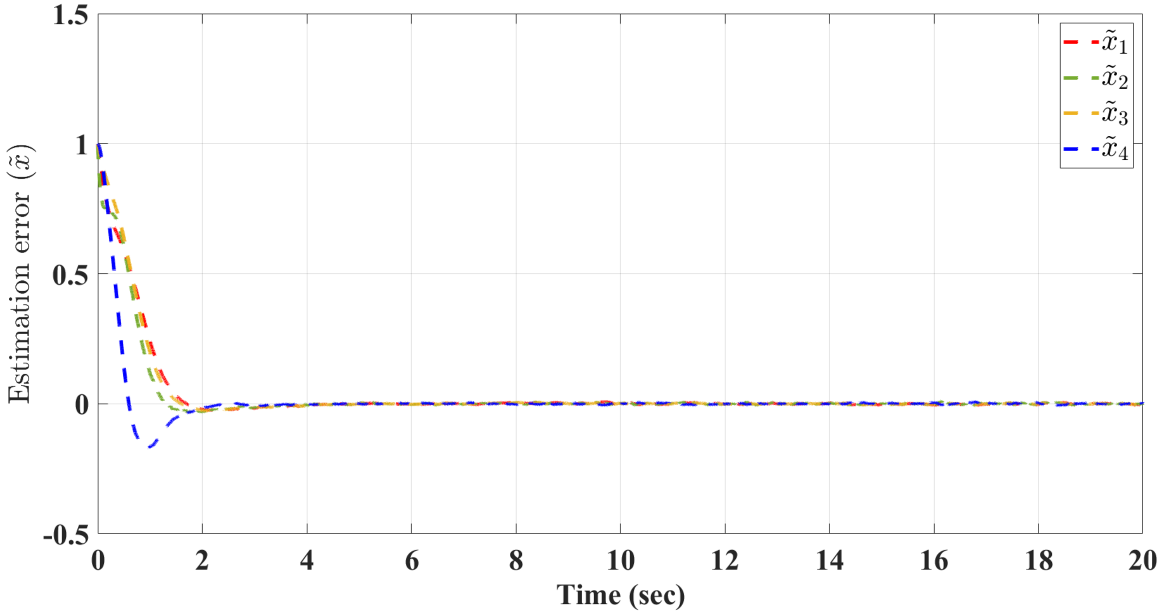

Figure 2 Estimation error ($\tilde{x}$) for Case 4

the duration (5 to 20 sec) and summarised in Table 2. The RMSE values of the estimation error ($\tilde{x}$) are smaller in **Case 1** and **Case 4** than in other cases. It infers that gain obtained in **Case 1** and **Case 4** provides better noise compensation as compared to other cases. Hence, the significance of the proposed LMI-based observer is emphasised.

In the sequel, some concluding remarks regarding the proposed methods are provided.

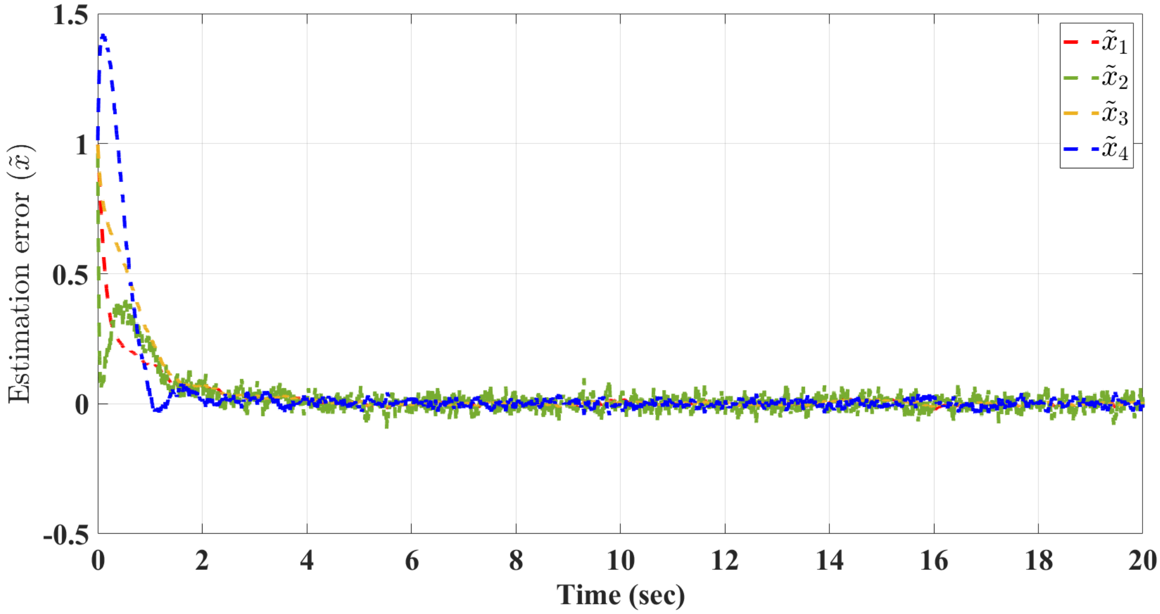

Figure 3 Estimation error ($\tilde{x}$) for Case 7

Table 2 Comparison of RMSE values of $\tilde{x}$ in different cases

Different cases	Estimation error ($\tilde{x}$)			
	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
Case 1	0.0129	0.0106	0.0081	0.0099
Case 2	0.0235	0.0359	0.0169	0.0240
Case 3	0.0180	0.0263	0.0113	0.0221
Case 4	0.0025	0.0029	0.0018	0.0022
Case 5	0.0037	0.0116	0.0027	0.0055
Case 6	0.0038	0.0102	0.0028	0.0049
Case 7	0.0131	0.0254	0.0084	0.0148

8 | CONCLUSIONS

This letter delved into the establishment of an LMI-based observer for nonlinear systems whose system dynamics and outputs are affected by noise/disturbance. To determine the parameter of the proposed observer, two novel LMI conditions are developed in this article by employing the $\mathcal{H}_\infty$ criterion and the ISS notion. Both LMI conditions are formulated through the utilisation of the reformulated Lipschitz property, the well-known LPV approach and a newly defined matrix multiplier. The primary component of this new design approach is the use of a generalized matrix multiplier, which allows us to add some extra numbers of decision variables inside LMIs. The incorporation of such additional decision variables yields the introduction of some extra numbers of degrees of freedom, which enriches the LMI feasibility. Further, the superiority of the newly defined matrix multiplier and the performance of the proposed LMI-based observers are highlighted through a numerical example in the MATLAB environment.

Author contributions

This paper presents a novel matrix-multiplier-based Linear Matrix Inequality (LMI) approach for the observer design of a class of nonlinear systems. The proposed methodologies provide less conservative LMI conditions, compared to the existing literature. This is due to the incorporation of

- the Young inequality in a convenient form;
- a reformulation of the Lipschitz inequality;
- and a novel multiplier matrix including more decision variables compared to the results in the literature.

Due to all the aforementioned mathematical tools, the feasibility of the LMI conditions is enhanced, which then leads to better performances and robustness. The methods are illustrated through a numerical example. To obtain such a new and more general multiplier matrix, compared to those existing in literature, the introduction of a novel mathematical tool, under the form of a new Lemma, was necessary. Hence the importance of the paper, as such a tool may be leveraged in other control design issues, such as stabilization; trajectory tracking; and fault diagnosis.

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Conflict of interest

The authors declare no potential conflict of interest.

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