Stability of stochastic differential delay systems with integral/ fragment-integral term and applications

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Abstract

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Abstract
This article investigates the stochastic stability of stochastic differential delay systems (SDDSs) with path information and their applications in consensus control of multi-agent systems (MASs) based on the path information feedback. Here, the integral path information and fragment-integral path information are considered respectively. The mean square (m.s.) and almost sure (a.s.) exponential stability criteria of the SDDSs with path integral information are established respectively according to the two types of path information. It is shown that the fragment-integral term may work positively for stochastic stability. Moreover, the obtained stochastic stability theorems are applied to design a distributed proportional integral/ fragment-integral control protocol and establish consensus conditions for stochastic MASs under proportional-integral (PI) -type controls. Finally, the effectiveness of the results is verified through two simulation examples.

KEYWORDS
stability, stochastic differential delay systems, path integral, path fragment-integral, consensus

1 | INTRODUCTION

Stochastic differential system is a mathematical model that encompass the combined influence of deterministic and stochastic factors within a given system. It serves as a universal tool for comprehending the intricacies of practical problems influenced by stochastic factors, while also facilitating the modeling of real-world systems that account for environmental noise. Nowadays, stochastic differential systems have emerged as an intriguing field garnering widespread attention from both theoretical and applied domains, assuming a pivotal role in system modeling and control across various disciplines such as physics, epidemiology, economics, network science, and engineering[1–4].

In general, the evolution of a dynamical system is not solely determined by its current state. In fact, its past or history may also exert influence on its future evolution. This justifies the appearance of various types of delay or memory in models, which further indicates that the early states of the system will impact future states [5]. To describe such phenomena, path information is modeled as an important component of the system model, and the concept of path dependence is introduced. Indeed, the path-dependent stochastic models can not only take into account the stochastic modeling of the actual environment, but also effectively capture the dependence of system evolution on path information. Consequently, they have garnered attention from experts in the fields of engineering technology, economics, and social sciences[6, 7]. Additionally, in deterministic situations, incorporating path information into deterministic systems may significantly improve the performance of control systems, such as adaptive control and fully distributed control of multi-agent systems (MASs)[8, 9]. However, there lacks a general theory to support research on stability in stochastic systems with path information. Therefore, investigating stability of path-dependent stochastic systems and their applications in feedback control problems holds great significance.

In stochastic systems, there exist multiple forms of path dependence. Firstly, if a system relies solely on a specific moment or interval along the trajectory/path it follows, it degenerates into a stochastic differential delay systems (SDDS). SDDSs and their applications have been extensively studied over recent decades[10–12]. Secondly, when considering the cumulative impact of the entire or fragmented historical path information, the corresponding system becomes a stochastic system with integral information of the entire or fragmented path[13]. Its typical applications include integral feedback problems in control systems such as proportional-integral-derivative (PID) control, where the corresponding closed-loop system iis inherently path dependent. PID control protocol is widely used for controller design in various engineering systems, with approximately 95% of automatic control systems currently employing PID controllers[14–16]. Despite the extensive exploration of PID control for deterministic systems in many literatures[17–19], research on PID control for stochastic systems remains relatively limited due to the challenges posed by incompletely differentiable states. In their work, reference 20 proposed a PID control term for a second-order stochastic system by utilizing partial differentiable states of the system and provided sufficient conditions for selecting PID parameters to reach mean square (m.s.) asymptotic convergence of the tracking error. Furthermore, reference 21 presented a specific PID controller design method for globally stabilizing nonlinear uncertain stochastic systems with state observers, obtaining explicit formulas for both PID controller and observer gain parameters. Reference 22 demonstrated the capability of PD control to globally stabilize uncertain stochastic control systems in the mean square sense, where the drift and diffusion terms were both nonlinear functions of state and control variables and the upper bound of the partial derivative of the nonlinear functions satisfied certain algebraic inequalities. Additionally, reference 23 designed a PID controller to address the tracking problem in coupled MASs consisting of second-order nonlinear uncertain dynamical agents. Reference 24 solved m.s. consistency issues in directed graphs by designing a PI protocol for stochastic dynamic nonlinear MASs. The above works extend deterministic PID control designs to stochastic systems from a control perspective. Nevertheless, it should be acknowledged that the corresponding closed-loop stochastic systems are path dependent, and their stability criteria have not been well established yet.

Indeed, when considering two types of path information, the corresponding system can be regarded as a SDDS with either global or fragmented path integral information. In deterministic situations, as a typical application of stability problems in delay systems with path integral information, PID control of delay systems have been partially studied[25–28]. However, due to the inherent characteristics of stochastic systems, many methods applicable to deterministic systems cannot be directly applied in stochastic situations. Currently, there is limited research on PID theory for SDDSs due to the absence of corresponding fundamental stability theories for SDDSs with path information.

Drawing upon the aforementioned discussion, this article investigates the stability of SDDSs with path integral information, where both the integral and fragment-integral cases are taken into account, respectively. Specifically, the m.s. and almost sure (a.s.) exponential stability criteria of SDDSs with path integral/ fragment-integral information are established respectively as a foundation for control of such stochastic systems. Based on the stochastic stability theorems, two PI-type controllers are designed and sufficient conditions are proposed to reach m.s. and a.s. consensus for stochastic MASs.

The remainder of this article is structured as follows. Section 2 presents the stability analysis of semi-linear SDDSs with integral and fragment-integral terms. In Section 3, we apply the derived stability criteria to establish consensus conditions for stochastic MASs. In Section 4, two simulation examples are provided to illustrate our theoretical results. Section 5 concludes the article.

2 | STABILITY OF SEMI-LINEAR SDDSS WITH INTEGRAL/ FRAGMENT-INTEGRAL TERM

Throughout this article, we use the following notations. $\mathbb{R}^{a\times b}$ is the set of $a \times b$ real matrices. M^T and $\|M\|$ represent the transpose and Euclidean norm of matrix M. For matrices $M, M_1, M_2, M > 0$ and $M_1 \geq M_2$ indicate that M is positive definite and $M_1 - M_2$ is positive semidefinite respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. For $m_1, m_2 \in \mathbb{R}$, min $\{m_1, m_2\}$(or max $\{m_1, m_2\}$) is denoted by $m_1 \wedge m_2$ (or $m_1 \vee m_2$). For continuous martingales $M_1(t)$ and $M_2(t)$, $\langle M_1, M_2\rangle (t)$ represents their quadratic variation. For $\tau > 0$, $C([-\tau, 0]; \mathbb{R}^n)$ denotes the space of all continuous $\mathbb{R}^n$-valued functions φ defined on $[-\tau, 0]$ with the norm $\|\varphi\|_C = \sup_{t\in[-\tau,0]} \|\varphi(t)\|$. I_N denotes the N-dimensional identity matrix.

2.1 | Stability of semi-linear SDDSs with integral term

Consider the following semi-linear SDDS with integral term

$$dx(t) = [A_0x(t) + A_1x(t-\tau_1) + f(x(t), x(t-\tau_1)) + A_2\int_0^t x(s)ds]dt + g(x(t), x(t-\tau_2))dw(t), \tag{1}$$

where $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}, f_i, g_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau_1, \tau_2 \geq 0$, $w(t)$ is standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $\tau = \tau_1 \vee \tau_2$. The initial data is $x(t) = \varphi(t) \in C([-\tau, 0], \mathbb{R}^n)$, $t \in [-\tau, 0]$. $f_i(x)$ and $g_i(x)$ satisfy the following assumption.

Assumption 1. For matrix $P > 0$, there exist matrices $D_{1p}, D_{2p}, D_{3p}, D_{4p} \geq 0$ such that

$$\begin{aligned} f^T(x_1, x_2)Pf(x_1, x_2) &\leq q^2(x_1^T D_{1p}x_1 + x_2^T D_{2p}x_2), \\ g^T(x_1, x_2)Pg(x_1, x_2) &\leq x_1^T D_{3p}x_1 + x_2^T D_{4p}x_2, \end{aligned} \tag{2}$$

where $q \geq 0$.

Definition 1. The solution to stochastic system (1) is called m.s. exponentially stable (or a.s. exponentially stable) if for any initial data $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, there exist $C_0, \gamma_0 > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq C_0e^{-\gamma_0 t}(or \lim_{t\to\infty} \sup \frac{1}{t}\log\|x(t)\| \leq \frac{-\gamma_0}{2}, a.s.). \tag{3}$$

Lemma 1 (Jensen's Inequality). *For any matrix $\Omega > 0$, the following inequality holds:*

$$(m_1 - m_2)\int_{m_2}^{m_1} \varpi(\xi)^T\Omega varpi(\xi)d\xi \geq \left[\int_{m_2}^{m_1} \varpi(\xi)d\xi\right]^T \Omega \left[\int_{m_2}^{m_1} \varpi(\xi)d\xi\right].$$

Lemma 2 (29). *The LMI $\begin{pmatrix} Q & \Omega \\ \Omega^T & P \end{pmatrix} > 0$ where $Q = Q^T$ and $P = P^T$, is equivalent to either of the following:*

$$\begin{aligned} &1) P > 0, Q - \Omega P^{-1}\Omega^T > 0, \\ &2) Q > 0, P - \Omega^T Q^{-1}\Omega > 0. \end{aligned}$$

Theorem 1. *Suppose the Assumption 1 holds. The semi-linear SDDS with integral term* (1) *is m.s. and a.s. exponentially stable if there exist matrice P, P_1, P_2 where $P > 0$ and $P_2 - P_1P^{-1}P_1^T > 0$ such that*

$$\begin{pmatrix} \Xi_1 & \Xi_3 \\ \Xi_3^T & \Xi_2 \end{pmatrix} < 0, \tag{4}$$

where $\Xi_1 = A_2^T P_1^T + P_1A_2 + \tau_1A_2^T PA_2 + qP_1P^{-1}P_1^T$, $\Xi_2 = (A_0 + A_1)^T P + P(A_0 + A_1) + \tau_1(A_0 + A_1)^T P(A_0 + A_1) + (q + 3)\tau_1A_1^T PA_1 + P_1 + P_1^T + \tau_1P_1P^{-1}P_1^T + qP + q(\tau_1 + 2)(D_{1p} + D_{2p}) + D_{3p} + D_{4p}$ *and* $\Xi_3 = A_2^T P + P_2 + P_1(A_0 + A_1)$.

Proof. Let $\vartheta(t) = \int_0^t x(s)ds$ and $\varrho(t) = (\vartheta^T(t), x^T(t))^T$. Then, (1) can be expressed as:

$$d\varrho(t) =(L_0\varrho(t) + L_1\varrho(t-\tau_1) + F(\varrho(t), \varrho(t-\tau_1))dt + G(\varrho(t), \varrho(t-\tau_2))dw(t), \tag{5}$$

where $L_0 = \begin{pmatrix} 0 & I_n \\ A_2 & A_0 \end{pmatrix}$, $L_1 = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}$, $F(\varrho(t), \varrho(t-\tau_1)) = (0, f(x(t), x(t-\tau_1))^T)^T$ and $G(\varrho(t), \varrho(t-\tau_2)) = (0, g(x(t), x(t-\tau_2))^T)^T$. The initial data is $\{\varrho(t) = (\vartheta(t), x(t))^T|\vartheta(0) = 0, x(t) = \varphi(t), t \in [-\tau, 0]\}$. We choose Lyapunov functional as

$$V(\varrho, t) = V_1(\varrho) + V_2(\varrho),$$

where $V_1(\varrho) = [\varrho(t) + L_1 \int_{t-\tau_1}^t \varrho(s)ds]^T \bar{P}[\varrho(t) + L_1 \int_{t-\tau_1}^t \varrho(s)ds]$, and $\bar{P} \in \mathbb{R}^{2n\times 2n}$, $\bar{P} > 0$. $V_2(\varrho)$ will be given later. According to the Itô formula, it can be deduced that $\mathcal{L}V_1(\varrho)$ has the following form:

$$\begin{aligned}\mathcal{L}V_1(\varrho) =&\varrho^T(t)\left[(L_0+L_1)^T\bar{P} + \bar{P}(L_0+L_1)\right]\varrho(t) + 2\varrho^T(t)\bar{P}F(\varrho(t),\varrho(t-\tau_1))\\ &+2\int_{t-\tau_1}^t \varrho^T(s)dsL_1^T\bar{P}(L_0+L_1)\varrho(t) + 2\int_{t-\tau_1}^t \varrho^T(s)dsL_1^T\bar{P}F(\varrho(t),\varrho(t-\tau_1))\\ &+G^T(\varrho(t),\varrho(t-\tau_2))\bar{P}G(\varrho(t),\varrho(t-\tau_2))\\ =:&\Delta_0+\Delta_1+\Delta_2+\Delta_3+\Delta_4.\end{aligned} \tag{6}$$

Let $\bar{P} = \begin{pmatrix} P_2 & P_1 \\ P_1^T & P \end{pmatrix}$. Note that $\bar{P} > 0$ according to Lemma 2 since $P_2 - P_1P^{-1}P_1^T > 0$ and $P > 0$. Using Lemma 1 and the elementary inequality: $2a^T\Gamma b \le \varepsilon a^T\Gamma a + \frac{1}{\varepsilon}b^T\Gamma b$, for $a, b \in \mathbb{R}^n$, $\Gamma > 0$ and $\varepsilon > 0$, ones get 1) $\Delta_1 = 2\vartheta^T(t)P_1f(x(t), x(t-\tau_1)) + 2x^T(t)Pf(x(t), x(t-\tau_1)) \le q\vartheta^T(t)P_1P^{-1}P_1^T\vartheta(t) + qx^T(t)(P+2D_{1p})x(t) + 2qx^T(t-\tau_1)D_{2p}x(t-\tau_1)$, 2) $\Delta_2 = 2\int_{t-\tau_1}^t x^T(s)dsA_1^TPA_2\vartheta(t) + 2\int_{t-\tau_1}^t x^T(s)dsA_1^TP_1^Tx(t) + 2\int_{t-\tau_1}^t x^T(s)dsA_1^TP(A_0+A_1)x(t) \le 3\int_{t-\tau_1}^t x^T(s)A_1^TPA_1x(s)ds + \tau_1x^T(t)P_1P^{-1}P_1^Tx(t) + \tau_1\vartheta^T(t)A_2^TPA_2\vartheta(t) + \tau_1x^T(t)(A_0+A_1)^TP(A_0+A_1)x(t)$, 3) $\Delta_3 = 2\int_{t-\tau_1}^t x^T(s)dsA_1^TPf(x(t), x(t-\tau_1)) \le q\tau_1x^T(t)D_{1p}x(t) + q\tau_1x^T(t-\tau_1)D_{2p}x(t-\tau_1) + q\int_{t-\tau_1}^t x^T(s)A_1^TPA_1x(s)ds$ and 4) $\Delta_4 = g^T(x(t), x(t-\tau_2))Pg(x(t), x(t-\tau_2)) \le x^T(t)D_{3p}x(t) + x^T(t-\tau_2)D_{4p}x(t-\tau_2)$. Substituting the above inequalities into previous formula (6) yields

$$\begin{aligned}\mathcal{L}V_1(\varrho) \le\ &\varrho^T(t)\left[(L_0+L_1)^T\bar{P} + \bar{P}(L_0+L_1)\right]\varrho(t) + q\vartheta^T(t)P_1P^{-1}P_1^T\vartheta(t) + \tau_1\vartheta^T(t)A_2^TPA_2\vartheta(t)\\ &+\tau_1x^T(t)P_1P^{-1}P_1^Tx(t) + x^T(t)D_{3p}x(t) + qx^T(t)(P+\tau_1D_{1p}+2D_{1p})x(t)\\ &+\tau_1x^T(t)(A_0+A_1)^TP(A_0+A_1)x(t) + qx^T(t-\tau_1)(\tau_1D_{2p}+2D_{2p})x(t-\tau_1)\\ &+(q+3)\int_{t-\tau_1}^t x^T(s)A_1^TPA_1x(s)ds + x^T(t-\tau_2)D_{4p}x(t-\tau_2).\end{aligned}$$

Let $V_2(\varrho) = (q+3)\int_{-\tau_1}^0\int_{t+s}^t \varrho^T(\theta)L_1^T\bar{P}L_1\varrho(\theta)d\theta ds$. According to the Itô formula and the definition of $V_2(\varrho)$, we can get

$$\mathcal{L}V_2(\varrho) = (q+3)\tau_1x^T(t)A_1^TPA_1x(t) - (q+3)\int_{t-\tau_1}^t x^T(s)A_1^TPA_1x(s)ds. \tag{7}$$

Combining with the estimation of $\mathcal{L}V_1(\varrho)$, we obtain

$$\begin{aligned}\mathcal{L}V(\varrho) \le&\varrho^T(t)S_1\varrho(t) + \vartheta^T(t)S_2\vartheta(t) + x^T(t)S_3x(t) + q(\tau_1+2)x^T(t-\tau_1)D_{2p}x(t-\tau_1)\\ &+x^T(t-\tau_2)D_{4p}x(t-\tau_2),\end{aligned}$$

where $S_1 = (L_0+L_1)^T\bar{P} + \bar{P}(L_0+L_1)$, $S_2 = \tau_1A_2^TPA_2 + qP_1P^{-1}P_1^T$ and $S_3 = \tau_1(A_0+A_1)^TP(A_0+A_1) + (q+3)\tau_1A_1^TPA_1 + qP + q(\tau_1+2)D_{1p} + D_{3p} + \tau_1P_1P^{-1}P_1^T$. Then, we get $dV(\varrho) = \mathcal{L}V(\varrho)dt + dM(t)$, where $M(t) = 2\int_0^t \varrho^T(s)\bar{P}G(\varrho(s), \varrho(s-\tau_2), s)dw(s) = \int_0^t 2x^T(s)Pg(x(s), x(s-\tau_2), s)dw(s)$. Futhermore, it can be inferred $dV(\varrho) = \varrho^T(t)S_4\varrho(t)dt + H_1(t-\tau_1)dt + H_2(t-\tau_2)dt + dM(t)$, where $S_4 = \begin{pmatrix} \Xi_1 & \Xi_3 \\ \Xi_3^T & \Xi_2 \end{pmatrix}$, $\Xi_1 = A_2^TP_1^T + P_1A_2 + \tau_1A_2^TPA_2 + qP_1P^{-1}P_1^T$, $\Xi_2 = (A_0+A_1)^TP + P(A_0+A_1) + \tau_1(A_0+A_1)^TP(A_0+A_1) + (q+3)\tau_1A_1^TPA_1 + P_1 + P_1^T + \tau_1P_1P^{-1}P_1^T + qP + q(\tau_1+2)D_{1p} + D_{3p}$, $\Xi_3 = A_2^TP + P_2 + P_1(A_0+A_1)$, $H_1(t) = \varrho^T(t)\bar{D}_{2p}\varrho(t)$, $H_2(t) = \varrho^T(t)\bar{D}_{4p}\varrho(t)$, $\bar{D}_{2p} = \begin{pmatrix} 0 & 0 \\ 0 & q(\tau_1+2)D_{2p} \end{pmatrix}$ and $\bar{D}_{4p} = \begin{pmatrix} 0 & 0 \\ 0 & D_{4p} \end{pmatrix}$. Applying the Itô formula to $e^{\gamma t}V(\varrho)$, for any $\gamma > 0$ yields

$$\begin{aligned}d\left[e^{\gamma t}V(\varrho)\right] =&\gamma e^{\gamma t}V(\varrho)dt + e^{\gamma t}dV(\varrho)\\ \le&\gamma e^{\gamma t}V(\varrho)dt + e^{\gamma t}\varrho^T(t)S_4\varrho(t)dt + e^{\gamma t}dM(t) + e^{\gamma t}H_1(t-\tau_1)dt + e^{\gamma t}H_2(t-\tau_2)dt.\end{aligned} \tag{8}$$

Integrating the above inequality (8) and taking expectation, we can obtain

$$\begin{aligned}e^{\gamma t}\mathbb{E}V(\varrho) \le&\mathbb{E}V(\varrho_0) + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(\varrho_s)ds + \mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)S_4\varrho(s)ds\\ &+\mathbb{E}\int_0^t e^{\gamma s}H_1(s-\tau_1)ds + \mathbb{E}\int_0^t e^{\gamma s}H_2(s-\tau_2)ds.\end{aligned}$$

Note that $\int_0^t e^{\gamma s}H_1(s-\tau_1)ds \leq e^{\gamma\tau_1}\int_{-\tau_1}^0 H_1(s)ds+e^{\gamma\tau_1}\int_0^t e^{\gamma s}H_1(s)ds$ and $\int_0^t e^{\gamma s}H_2(s-\tau_2)ds \leq e^{\gamma\tau_2}\int_{-\tau_2}^0 H_2(s)ds+e^{\gamma\tau_2}\int_0^t e^{\gamma s}H_2(s)ds$. Then, it can be inferred that

$$e^{\gamma t}\mathbb{E}V(\varrho) \leq C_1(\gamma) + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(\varrho_s)ds + \mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)S_5(\gamma)\varrho(s)ds, \tag{9}$$

where $S_5(\gamma) = S_4 + e^{\gamma\tau_1}\bar{D}_{2p} + e^{\gamma\tau_2}\bar{D}_{4p}$ and $C_1(\gamma) = \mathbb{E}V(\varrho_0) + e^{\gamma\tau_1}\int_{-\tau_1}^0 H_1(s)ds + e^{\gamma\tau_2}\int_{-\tau_2}^0 H_2(s)ds$. According to the definition of the Lyapunov functional $V(\varrho)$ and the elementary inequality $(a+b)^T O(a+b) \leq 2a^T Oa + 2b^T Ob$, for $a, b \in \mathbb{R}^n$ and $O > 0$, we can obtain $V(\varrho_s) \leq C_2\int_{s-\tau_1}^s \|\varrho(\theta)\|^2 d\theta + 2\|\bar{P}\|\|\varrho(s)\|^2$, where $C_2 = (q+5)\tau_1\|L_1\|^2\|\bar{P}\|$. Substituting this inequality into previous formula (9) yields

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(\varrho) \leq & 2\|P\|\int_0^t \gamma e^{\gamma s}\mathbb{E}\|\varrho(s)\|^2 ds + C_1(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)S_5(\gamma)\varrho(s)ds \\ & + C_2\int_0^t \gamma e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|\varrho(\theta)\|^2 d\theta ds. \end{aligned}$$

Note that $\int_0^t e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|\varrho(\theta)\|^2 d\theta ds \leq \int_{-\tau_1}^0 \mathbb{E}\|\varrho(\theta)\|^2\int_\theta^{\theta+\tau_1} e^{\gamma s}ds d\theta + \int_0^t \mathbb{E}\|\varrho(\theta)\|^2\int_\theta^{\theta+\tau_1} e^{\gamma s}ds d\theta \leq \tau_1 e^{\gamma\tau_1}\int_0^t e^{\gamma\theta}\mathbb{E}\|\varrho(\theta)\|^2 d\theta + \tau_1^2 e^{\gamma\tau_1}\|\varphi\|_C^2$. Hence, we can get

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(\varrho) \leq & C_3(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)S_5(\gamma)\varrho(s)ds + C_4(\gamma)\gamma\int_0^t e^{\gamma s}\mathbb{E}\|\varrho(s)\|^2 ds \\ \leq & C_3(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)S_6(\gamma)\varrho(s)ds, \end{aligned} \tag{10}$$

where $C_3(\gamma) = C_1(\gamma) + C_2\gamma\tau_1^2 e^{\gamma\tau_1}\|\varphi\|_C^2$, $C_4(\gamma) = C_2\tau_1 e^{\gamma\tau_1} + 2\|\bar{P}\|$ and $S_6(\gamma) = S_5(\gamma) + \gamma C_4(\gamma)I_n$. Considering S_6 as a function of γ, we can see that $S_6(0) < 0$ under condition (4). Therefore, if condition (4) holds, then there exists a $\bar{\gamma} > 0$ such that for any $\gamma < \bar{\gamma}$, $S_6(\gamma) = S_5(\gamma) + \gamma C_4(\gamma)I_n = S_4 + e^{\gamma\tau_1}\bar{D}_{2p} + e^{\gamma\tau_2}\bar{D}_{4p} + \gamma C_4(\gamma)I_n < 0$. This, together with inequality (10), can lead to $\mathbb{E}\int_0^t e^{\gamma s}\varrho^T(s)(-S_6(\gamma))\varrho(s)ds < C_3(\gamma)$, which implies $\mathbb{E}\|\varrho(t)\|^2 \leq C_0 e^{-\gamma_0 t}$. Therefore, this proof is completed.

□

Remark 1. A delay-dependent stability criterion for the semi-linear SDDS with integral term (1) is provided inTheorem 1 . In condition (4), the stability condition does not involve time delay τ_2. This implies that the stability of semi-linear SDDSs with integral term is independent of delay in the diffusion term, which is in consistent with our previous work 30.

Note that if the integral term vanishes, i.e. $A_2 = 0$, the stochastic system (1) degenerates to the first-order case:

$$dx(t) = [A_0x(t) + A_1x(t-\tau_1) + f(x(t), x(t-\tau_1), t)]dt + g(x(t), x(t-\tau_2), t)dw(t). \tag{11}$$

Then, P_1, P_2 vanish in condition (4) and the following corollary can be obtained, which is consistent with Theorem 4.4 in 30.

Corollary 1. *If there exist matrix $P > 0$ such that*

$$\begin{aligned} &(A_0 + A_1)^T P + P(A_0 + A_1) + \tau_1(A_0 + A_1)^T P(A_0 + A_1) + (q+3)\tau_1 A_1^T P A_1 \\ &+ qP + q(\tau_1 + 2)(D_{1p} + D_{1p}) + D_{3p} + D_{4p} < 0, \end{aligned} \tag{12}$$

then the semi-linear SDDS (11) *is m.s. and a.s. exponentially stable.*

In addition, for the case $f = 0$, that is, consider the following SDDS with integral term, Theorem 1 can directly lead to the following corollary.

$$dx(t) = \left[A_0x(t) + A_1x(t-\tau_1) + A_2\int_0^t x(s)ds\right]dt + g(x(t), x(t-\tau_2))dw(t). \tag{13}$$

Corollary 2. *Suppose Assumption 1 holds. If there exist matrice P, P_1, P_2 where $P > 0$ and $P_2 - P_1P^{-1}P_1^T > 0$ such that*

$$\begin{pmatrix} \Xi_1 & \Xi_3 \\ \Xi_3^T & \Xi_2 \end{pmatrix} < 0, \tag{14}$$

where $\Xi_1 = A_2^T P_1^T + P_1 A_2 + \tau_1 A_2^T P A_2$, $\Xi_2 = (A_0+A_1)^T P + P(A_0+A_1) + \tau_1 (A_0+A_1)^T P(A_0+A_1) + 3\tau_1 A_1^T P A_1 + P_1 + P_1^T + \tau_1 P_1 P^{-1} P_1^T + D_{3p} + D_{4p}$ *and* $\Xi_3 = A_2^T P + P_2 + P_1(A_0 + A_1)$*, then the SDDS with integral term* (13) *is m.s. and a.s. exponentially stable.*

The previous theorem gives a stability criterion in the form of matrix inequality. For the scalar case, we can further obtain an explicit condition of the stability criterion. Let $A_0 = a_0, A_1 = a_1, A_2 = a_2$ and $g = \sigma x(t-\tau_2)$. Consider the following linear scalar stochastic system

$$dx(t) = (a_0 x(t) + a_1 x(t-\tau_1) + a_2 \int_0^t x(s)ds)dt + \sigma x(t-\tau_2)dw(t). \tag{15}$$

Corollary 3. *The linear scalar SDDS with integral term* (15) *is m.s. and a.s. exponentially stable if there exist constants* μ, θ *which satisfy* $\mu - \theta^2 > 0$ *such that* $a_2\theta + 2a_2^2\tau_1 < 0$ *and* $\xi_1\xi_2 - \xi_3^2 > 0$ *hold, where* $\xi_1 = 2a_2\theta + a_2^2\tau_1$, $\xi_2 = 2(a_0 + a_1) + \tau_1(a_0 + a_1)^2 + 3\tau_1 a_1^2 + 2\theta + \tau_1\theta^2 + \sigma^2$*and* $\xi_3 = \mu + (a_0 + a_1)\theta + a_2$.

Proof. Let $P_2 = \mu, P_1 = \theta, P = 1$ in Theorem 1. Then, we can obtain the desired result. □

Remark 2. Corollary 3 provides stability criteria for a scalar linear stochastic system with integral term. If $\tau_1, \tau_2 = 0$ in (15), then the linear scalar delay-free stochastic system is m.s. and a.s. exponentially stable if there exist constants μ, θ satisfying $\mu - \theta^2 > 0$ such that $\eta_1 < 0$ and $\eta_1\eta_2 - \eta_3^2 > 0$, where $\eta_1 = 2\theta a_2$, $\eta_2 = 2(a_0 + a_1) + 2\theta + \sigma^2$, $\eta_3 = a_2 + \mu + \theta(a_0 + a_1)$. Let $\mu = -a_2 - \theta(a_0 + a_1)$, then the above inequalities can be reduced to $a_2 + \theta(a_0 + a_1) + \theta^2 < 0$, $\theta a_2 < 0$, and $2(a_0 + a_1) + 2\theta + \sigma^2 < 0$. Let $\theta \in (0, \bar{\theta})$, where $\bar{\theta} = \frac{-a_0 - a_1 + \sqrt{(a_0+a_1)^2 - 4a_2}}{2} \wedge -a_0 - a_1 - \frac{1}{2}\sigma^2$. If $a_2 < 0$ and $a_0 + a_1 + \frac{1}{2}\sigma^2 < 0$, then the above three inequalities hold, that is, the linear scalar delay-free stochastic system is m.s. and a.s. exponentially stable, which is consistent with Theorem 1 in 20.

In what follows, a more concise stability criterion than Theorem 1 is proposed to facilitate the application of stability theorem to feedback control design.

Theorem 2. *Suppose the Assumptions 1 holds. The semi-linear SDDS with integral term* (1) *is m.s. and a.s. exponentially stable if* $A_2 < 0$ *and there is* $P > 0$ *such that* $\mho_P < 0$ *holds and* $A_2 P = P A_2$*, where* $\mho_P = (A_0 + A_1)^T P + P(A_0 + A_1) + \tau_1(A_0 + A_1)^T P(A_0 + A_1) + (q+3)\tau_1 A_1^T P A_1 + qP + q(\tau_1 + 2)(D_{1p} + D_{2p}) + D_{3p} + D_{4p} < 0$.

Proof. Note that condition $\mho_P < 0$ implies that the system matrix $A_0 + A_1$ must be Hurwitz. In condition (4), we can choose $P_1 = -(A_2 P + P_2)(A_0 + A_1)^{-1}$, and then $\Xi_3 = 0$. Moreover, $-A_2 P$ is positive definite since $PA_2 = A_2 P$ and $A_2 < 0$. Now, let $P_2 = -(1-\alpha)A_2 P$ with $\alpha \in (0, 1)$, and then $P_1 = -\alpha A_2 P(A_0 + A_1)^{-1}$. One can see that for any $\alpha \in (0, \alpha_1)$,

$$\begin{aligned} P_2 - P_1 P^{-1} P_1^T &= -(1-\alpha)A_2 P - \alpha^2 A_2 P \bar{A} P A_2 \\ &\geq -(1-\alpha)A_2 P - \alpha^2 (A_2 P)^2 \lambda_{\max}\bar{A} \\ &\geq \lambda_{\min}(-A_2 P)(1-\alpha)I_n - \lambda_{\max}^2(-A_2 P)\lambda_{\max}\bar{A}\alpha^2 I_n > 0, \end{aligned} \tag{16}$$

where $\bar{A} = (A_0 + A_1)^{-1} P^{-1}((A_0 + A_1)^{-1})^T$, $\alpha_1 < \frac{-\lambda_{\min}(-A_2P) + \sqrt{\lambda_{\min}^2(-A_2P) + 4\lambda_{\min}(-A_2P)\lambda_{\max}(-A_2P)^2\lambda_{\max}\bar{A}}}{2\lambda_{\max}^2(-A_2P)\lambda_{\max}\bar{A}}$.Note that $\mho_P = (A_0 + A_1)^T P + P(A_0 + A_1) + \tau_1(A_0 + A_1)^T P(A_0 + A_1) + (q+3)\tau_1 A_1^T P A_1 + qP + q(\tau_1 + 2)(D_{1p} + D_{2p}) + D_{3p} + D_{4p} < 0$ and

$$P_1^T + P_1 = -\alpha[A_2 P(A_0 + A_1)^{-1} + ((A_0 + A_1)^{-1})^T P A_2]. \tag{17}$$

Then, there is a small $\alpha_2 > 0$ such that for $\alpha < \alpha_2$

$$\begin{aligned} \Xi_2 &= \mho_P + P_1^T + P_1 + \tau_1 P_1 P^{-1} P_1^T \\ &= \mho_P - \alpha[A_2 P(A_0 + A_1)^{-1} + ((A_0 + A_1)^{-1})^T P A_2] + \tau_1\alpha^2 A_2 P\bar{A} P A_2 < 0. \end{aligned} \tag{18}$$

Moreover, since $(A_0 + A_1)^T P + P(A_0 + A_1) < 0$,

$$\begin{aligned} \Xi_1 &= A_2^T P_1^T + P_1 A_2 + \tau_1 A_2^T P A_2 + qP_1 P^{-1} P_1^T \\ &= \alpha A_2((A_0 + A_1)^{-1})^T((A_0 + A_1)^T P + P(A_0 + A_1))(A_0 + A_1)^{-1} A_2 + q\alpha^2 A_2 P\bar{A} P A_2 + \tau_1 A_2^T P A_2 \\ &\leq -\alpha a + \alpha^2 b + c, \end{aligned} \tag{19}$$

where $a = \lambda_{\min}(A_2((A_0 + A_1)^{-1})^T(-(A_0 + A_1)^T P - P(A_0 + A_1))(A_0 + A_1)^{-1} A_2)$, $b = q\lambda_{\max}(A_2 P\bar{A} P A_2)$ and $c = \tau_1\lambda_{\max}(A_2^T P A_2)$. Then, we can obtain $\Xi_2 < 0$, for a small τ_1 and $\alpha < \alpha_3 := \frac{a}{b}$. That is, choosing $P_1 = -(A_2 P + P_2)(A_0 + A_1)^{-1}$ and $P_2 = -(1-\alpha)A_2 P$ with $\alpha < \alpha_1 \wedge \alpha_2 \wedge \alpha_3$ yields (4). Therefore, the desired result follows from Theorem 1.

□

Remark 3. Theorem 2 provides a relatively conservative stability condition since $A_2 < 0$ and $A_2P = PA_2$ are required in Theorem 2, while A_2 does not need to be symmetric and negatively definite in Theorem 1. However, the dimension and complexity of the matrix inequality in Theorem 2 is lower than the matrix inequality (4), indicating that the stability condition in Theorem 2 is more concise and easier to calculate than Theorem 1. Therefore, Theorem 2 is more applicable to controller design in feedback control problems, which will be shown in Section 3.

2.2 | Stability of semi-linear SDDSs with fragment-integral term

In this subsection, we will investigate the stability of SDDSs with fragment-integral term. In fact, fragment-integral term, also namely the distributed delay term, has been addressed in many previous studies. This subsection aims to investigate the specific role of this term in the stability of SDDSs and provide a more concise stability theorem for the feedback control and applications. The specific content is as follows.

Consider the semi-linear SDDS with fragment-integral term

$$dx(t) = [A_0x(t) + A_1x(t-\tau_1) + f(x(t), x(t-\tau_1)) + A_2\int_{t-h}^{t} x(s)ds]dt + g(x(t), x(t-\tau_2))dw(t), \tag{20}$$

where $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$, $\tau_1, \tau_2, h \geq 0, f, g : \mathbb{R}^n \to \mathbb{R}^n$, $w(t)$ is the independent Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $\tau = \tau_1 \vee \tau_2 \vee h$. The initial data is $x(t) = \varphi(t) \in C([-\tau, 0], \mathbb{R}^n)$, $t \in [-\tau, 0]$. $f(x)$ and $g(x)$ satisfy Assumption 1.

Theorem 3. *Suppose Assumptions 1 holds. If there exists a matrix $P > 0$ such that*

$$\begin{aligned}&(A_0 + A_1 + hA_2)^TP + P(A_0 + A_1 + hA_2) + \tau_1(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2)\\&+ h(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2) + (q + 1)\tau_1(A_1 + hA_2)^TP(A_1 + hA_2)\\&+ \frac{1}{3}(q + 1)h^3A_2^TPA_2 + qP + (q + \tau_1q + hq)(D_{1p} + D_{2p}) + D_{3p} + D_{4p} < 0,\end{aligned} \tag{21}$$

then the semi-linear SDDS with fragment-integral term (20) *is m.s. and a.s. exponentially stable.*

Proof. We choose the Lyapunov functional

$$V(x) = V_1(x) + V_2(x)$$

with $V_1(x) = \varpi^T(x)P\varpi(x)$ and $V_2(x) = (q+1)\int_{-h}^{0}\int_{t+s}^{t}(s+h)^2x^T(\theta-\tau_1)A_2^TPA_2x(\theta-\tau_1)d\theta ds + (q+1)\int_{-\tau_1}^{0}\int_{t+s}^{t}x^T(\theta)(A_1+hA_2)^TP(A_1+hA_2)x(\theta)d\theta ds$, where $\varpi(x) = x(t) + (A_1 + hA_2)\int_{t-\tau_1}^{t}x(s)ds + A_2\int_{t-h}^{t}(s-t+h)x(s-\tau_1)ds$. Using the Itô formula, we get

$$\begin{aligned}dV_1(x) &= x^T(t)\left[(A_0 + A_1 + hA_2)^TP + P(A_0 + A_1 + hA_2)\right]x(t)dt + 2x^T(t)Pf(x(t), x(t-\tau_1))dt\\&+ 2\int_{t-\tau_1}^{t}x^T(s)ds(A_1 + hA_2)^TP(A_0 + A_1 + hA_2)x(t)dt + 2\int_{t-\tau_1}^{t}x^T(s)ds(A_1 + hA_2)^TPf(x(t), x(t-\tau_1))dt\\&+ 2\int_{t-h}^{t}(s-t+h)x^T(s-\tau_1)dsA_2^TP(A_0 + A_1 + hA_2)x(t)dt + 2\int_{t-h}^{t}(s-t+h)x^T(s-\tau_1)dsA_2^TPf(x(t), x(t-\tau_1))dt\\&+ dM_1(t) + d\langle M, PM\rangle(t)\\&=: \tilde{\Delta}_0 + \tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 + \tilde{\Delta}_4 + \tilde{\Delta}_5 + dM_1(t) + d\langle M, PM\rangle(t),\end{aligned} \tag{22}$$

where $\langle M, PM\rangle(t) = \int_0^t g^T(x(s), x(s-\tau_2))Pg(x(s), x(s-\tau_2))ds$, $M_1(t) = 2\int_0^t[x(s) + (A_1 + hA_2)\int_{s-\tau_1}^{s}x(\theta)d\theta + A_2\int_{s-h}^{s}(\theta-s+h)x(\theta-\tau_1)d\theta]^TPdM(s)$ and $M(t) = \int_0^t g(x(s), x(s-\tau_2))dw(s)$. By Lemma 1 and the elementary inequality: $2a^T\Gamma b \leq \varepsilon a^T\Gamma a + \frac{1}{\varepsilon}b^T\Gamma b$, for $a, b \in \mathbb{R}^n$, $\Gamma > 0$ and $\varepsilon > 0$, it can be deduced that 1) $\tilde{\Delta}_1 \leq qx^T(t)Px(t) + qx^T(t)D_{1p}x(t) + qx^T(t-\tau_1)D_{2p}x(t-\tau_1)$, 2) $\tilde{\Delta}_2 \leq \tau_1x^T(t)(A_0+A_1+hA_2)^TP(A_0+A_1+hA_2)x(t) + \int_{t-\tau_1}^{t}x^T(s)(A_1+hA_2)^TP(A_1+hA_2)x(s)ds$, 3) $\tilde{\Delta}_3 \leq q\int_{t-\tau_1}^{t}x^T(s)(A_1+hA_2)^TP(A_1+hA_2)x(s)ds + \tau_1q(x^T(t)D_{1p}x(t) + x^T(t-\tau_1)D_{2p}x(t-\tau_1))$, 4) $\tilde{\Delta}_4 \leq hx^T(t)(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2)x(t) + \int_{t-h}^{t}(s-t+h)^2x^T(s-\tau_1)A_2^TPA_2x(s-\tau_1)ds$ and 5)$\tilde{\Delta}_5 \leq hq(x^T(t)D_{1p}x(t) + x^T(t-\tau_1)D_{2p}x(t-\tau_1)) + q\int_{t-h}^{t}(s-t+h)^2x^T(s-\tau_1)A_2^TPA_2x(s-\tau_1)ds$. Substituting the above inequalities into (22) yields

$$dV_1(x) \le x^T(t)U_1x(t)dt + x^T(t-\tau_1)U_2x(t-\tau_1)dt + d\langle M, PM\rangle(t) + dM_1(t) + (q+1)\int_{t-h}^{t}(s-t+h)^2x^T(s-\tau_1)A_2^TPA_2x(s-\tau_1)dsdt + (q+1)\int_{t-\tau_1}^{t}x^T(s)(A_1+hA_2)^TP(A_1+hA_2)x(s)dsdt,$$

where $U_1 = (A_0 + A_1 + hA_2)^TP + P(A_0 + A_1 + hA_2) + \tau_1(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2) + h(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2) + qP + (q + \tau_1q + hq)D_{1p}$ and $U_2 = (q + \tau_1q + hq)D_{2p}$. According to the definition of $V_2(x)$, we have

$$\begin{aligned} dV_2(x) = &\frac{1}{3}(q+1)h^3x^T(t-\tau_1)A_2^TPA_2x(t-\tau_1)dt + (q+1)\tau_1x^T(t)(A_1+hA_2)^TP(A_1+hA_2)x(t)dt \\ &-(q+1)\int_{t-h}^{t}(s-t+h)^2x^T(s-\tau_1)A_2^TPA_2x(s-\tau_1)dsdt - (q+1)\int_{t-\tau_1}^{t}x^T(s)(A_1+hA_2)^TP(A_1+hA_2)x(s)dsdt. \end{aligned} \tag{23}$$

Combining with the estimation of $dV_1(x)$ and Assumption 1 yields

$$dV(x) \le x^T(t)U_3x(t)dt + J_1(t-\tau_1)dt + J_2(t-\tau_2)dt + dM_1(t),$$

where $U_3 = (A_0 + A_1 + hA_2)^TP + P(A_0 + A_1 + hA_2) + \tau_1(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2) + h(A_0 + A_1 + hA_2)^TP(A_0 + A_1 + hA_2) + (q + 1)\tau_1(A_1 + hA_2)^TP(A_1 + hA_2) + qP + (q + \tau_1q + hq)D_{1p} + D_{3p}$, $J_1(t) = x^T(t)(\frac{1}{3}(q + 1)h^3A_2^TPA_2 + (q + \tau_1q + hq)D_{2p})x(t)$ and $J_2(t) = x^T(t)D_{4p}x(t)$. Applying the Itô formula to $e^{\gamma t}V(x)$, for any $\gamma > 0$, ones obtain

$$\begin{aligned} d\left[e^{\gamma t}V(x)\right] =&\gamma e^{\gamma t}V(x)dt + e^{\gamma t}dV(x) \\ \le&\ \gamma e^{\gamma t}V(x)dt + e^{\gamma t}x^T(t)U_3x(t)dt + e^{\gamma t}dM_1(t) + e^{\gamma t}J_1(t-\tau_1)dt + e^{\gamma t}J_2(t-\tau_2)dt. \end{aligned} \tag{24}$$

Integrating the above inequality (24) and taking expectation, we can get

$$e^{\gamma t}\mathbb{E}V(x) \le \mathbb{E}V(x_0) + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(x_s)ds + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_3x(s)ds + \mathbb{E}\int_0^t e^{\gamma s}J_1(s-\tau_1)ds + \mathbb{E}\int_0^t e^{\gamma s}J_2(s-\tau_2)ds.$$

Note that $\int_0^t e^{\gamma s}J_1(s-\tau_1)ds \le e^{\gamma\tau_1}\int_{-\tau_1}^0 J_1(s)ds + e^{\gamma\tau_1}\int_0^t e^{\gamma s}J_1(s)ds$ and $\int_0^t e^{\gamma s}J_2(s-\tau_2)ds \le e^{\gamma\tau_2}\int_{-\tau_2}^0 J_2(s)ds + e^{\gamma\tau_2}\int_0^t e^{\gamma s}J_2(s)ds$. Then, we can obtain

$$e^{\gamma t}\mathbb{E}V(x) \le C_1(\gamma) + \int_0^t \gamma e^{\gamma s}\mathbb{E}V(x_s)ds + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_4(\gamma)x(s)ds, \tag{25}$$

where $U_4(\gamma) = U_3 + e^{\gamma\tau_1}(\frac{1}{3}(q+1)h^3A_2^TPA_2 + (q+\tau_1q+hq)D_{2p}) + e^{\gamma\tau_2}D_{4p}$ and $C_1(\gamma) = \mathbb{E}V(x_0) + e^{\gamma\tau_1}\int_{-\tau_1}^0 J_1(s)ds + e^{\gamma\tau_2}\int_{-\tau_2}^0 J_2(s)ds$. According to the definition of the Lyapunov functional $V(x)$ and the elementary inequality $(a + b)^TO(a + b) \le 2a^TOa + 2b^TOb$, for $a, b \in \mathbb{R}^n$ and $O > 0$, we can obtain $V(x_s) \le C_2\int_{s-\tau_1}^s \|x(\theta)\|^2d\theta + C_3\int_{s-h}^s \|x(\theta-\tau_1)\|^2d\theta + 3\|P\|\|x(s)\|^2$, where $C_2 = (q + 4)\tau_1\|A_1 + hA_2\|^2\|P\|$ and $C_3 = (q + 4)h^2\|A_2\|^2\|P\|$. Substituting this inequality into previous formula (25), we can get

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(x) \le&3\|P\|\int_0^t \gamma e^{\gamma s}\mathbb{E}\|x(s)\|^2ds + C_1(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_4(\gamma)x(s)ds \\ &+ C_2\int_0^t \gamma e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|x(\theta)\|^2d\theta ds + C_3\int_0^t \gamma e^{\gamma s}\int_{s-h}^s \mathbb{E}\|x(\theta-\tau_1)\|^2d\theta ds. \end{aligned}$$

Note that $\int_0^t e^{\gamma s}\int_{s-\tau_1}^s \mathbb{E}\|x(\theta)\|^2d\theta ds \le \int_{-\tau_1}^0 \mathbb{E}\|x(\theta)\|^2\int_\theta^{\theta+\tau_1} e^{\gamma s}dsd\theta + \int_0^t \mathbb{E}\|x(\theta)\|^2\int_\theta^{\theta+\tau_1} e^{\gamma s}dsd\theta \le \tau_1e^{\gamma\tau_1}\int_0^t e^{\gamma\theta}\mathbb{E}\|x(\theta)\|^2d\theta + \tau_1^2e^{\gamma\tau_1}\|\varphi\|_C^2$. Hence, ones obtain

$$\begin{aligned} e^{\gamma t}\mathbb{E}V(x) \le&C_4(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_4(\gamma)x(s)ds + C_5(\gamma)\gamma\int_0^t e^{\gamma s}\mathbb{E}\|x(s)\|^2ds + C_6(\gamma)\int_0^t e^{\gamma s}\mathbb{E}\|x(s-\tau_1)\|^2ds \\ \le&C_4(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_5(\gamma)x(s)ds + C_6(\gamma)\int_0^t e^{\gamma s}\mathbb{E}\|x(s-\tau_1)\|^2ds, \end{aligned} \tag{26}$$

where $C_4(\gamma) = C_1(\gamma) + (C_2 + C_3)\gamma\tau_1^2 e^{\gamma\tau_1}\|\varphi\|_C^2$, $C_5(\gamma) = C_2\tau_1 e^{\gamma\tau_1} + 3\|P\|$, $C_6(\gamma) = C_3 h e^{\gamma h}\gamma$ and $U_5(\gamma) = U_4(\gamma) + \gamma C_5(\gamma) I_n$. Note that $\int_0^t e^{\gamma s} x^2(s-\tau_1)ds \le e^{\gamma\tau_1}\int_{-\tau_1}^0 x^2(s)ds + e^{\gamma\tau_1}\int_0^t e^{\gamma s}x^2(s)ds$. Then, it can be deduced that

$$e^{\gamma t}\mathbb{E}V(x) \le C_5(\gamma) + \mathbb{E}\int_0^t e^{\gamma s}x^T(s)U_6(\gamma)x(s)ds, \tag{27}$$

where $C_5(\gamma) = C_4(\gamma) + C_6(\gamma)e^{\gamma\tau_1}\int_{-\tau_1}^0 \|x(s)\|^2 ds$ and $U_6(\gamma) = U_5(\gamma) + \gamma C_5(\gamma)I_n + e^{\gamma\tau_1}C_6(\gamma)I_n$. Considering U_6 as a function of γ, we can see that $U_6(0) < 0$ under condition (21). Therefore, if condition (21) holds, then there exists a $\gamma^* > 0$ such that for any $\gamma < \gamma^*$, $U_6(\gamma) < 0$. This, together with inequality (27), can lead to $\mathbb{E}\int_0^t e^{\gamma s}x^T(s)(-U_6(\gamma))x(s)ds < C_5(\gamma)$, which implies $\mathbb{E}\|x(t)\|^2 \le C_0 e^{-\gamma_0 t}$. This proof is completed.

□

Remark 4. Theorem 3 develops the sufficient conditions for the semi-linear SDDS with fragment-integral term to be m.s. and a.s. exponentially stable. In the previous works [31, 32] the discrete delay term like $x(t-\tau)$ was proved to be positive for stochastic stability. It is revealed in Theorem 3 that both the discrete and distributed delay terms can have a positive impact on stochastic stability since condition (21) does not require A_0 to be Hurwitz. Additionally, the condition (21) in Theorem 3 is a simple matrix inequality containing one decision variable, which has lower computational complexity and can more intuitively reflect the role of proportional and fragment-integral terms. A concise stability criterion is also more conducive to applications.

In addition, for the case $f = 0$, Theorem 3 can directly lead to the following corollary for the following SDDS with fragment-integral term

$$dx(t) = \left[A_0x(t) + A_1x(t-\tau_1) + A_2\int_{t-h}^t x(s)ds\right]dt + g(x(t), x(t-\tau_2))dw(t). \tag{28}$$

Corollary 4. *Suppose Assumptions 1 holds. If there exists a matrix $P > 0$ such that*

$$\begin{aligned}&(A_0 + A_1 + hA_2)^T P + P(A_0 + A_1 + hA_2) + \tau_1(A_0 + A_1 + hA_2)^T P(A_0 + A_1 + hA_2)\\&+ h(A_0 + A_1 + hA_2)^T P(A_0 + A_1 + hA_2) + \tau_1(A_1 + hA_2)^T P(A_1 + hA_2)\\&+ \frac{1}{3}h^3 A_2^T P A_2 + D_{3p} + D_{4p} < 0,\end{aligned} \tag{29}$$

then the SDDS with fragment-integral term (28) *is m.s. and a.s. exponentially stable.*

In the next section, we apply the stochastic stability criteria obtained in this section to establish consensus conditions for stochastic MASs under distributed proportional integral (PI) and proportional fragment-integral (PFI) control protocols.

3 | CONSENSUS OF STOCHASTIC MASS UNDER PI AND PFI PROTOCOLS

The interaction topology among MASs is modeled as an connected undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, ..., N\}$ represents the node set with i being the ith agent, $\mathcal{E}$ represents the edge set, and $\mathcal{A} = [a_{ij}]_{N\times N}$ represents the adjacency matrix with $a_{ij} = 1$ representing there exists an information flow between j and i, otherwise $a_{ij} = 0$. The set of agent i's neighbors is represented as N_i, that is, for $j \in N_i$, $a_{ij} = 1$. The degree of i is denoted by $\deg(i) = \sum_{j=1}^N a_{ij}$. The Laplacian matrix of $\mathcal{G}$ is denoted as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\{\deg(i), i = 1, ..., N\}$. Since the graph $\mathcal{G}$ is connected and undirected, the eigenvalues of $\mathcal{L}$ are denoted by $\lambda_1 = 0$ and $0 < \lambda_2 \le \ldots \le \lambda_N$. Denote $\Lambda = diag(\lambda_2, \lambda_3, ..., \lambda_N)$.

Consider a system with $N(N \ge 2)$ agents, where the dynamic model of each agent is described as follows

$$dx_i(t) = Ax_i(t) + Bu_i(t) + dM_i(t), \quad i = 1, 2, ..., N, \tag{30}$$

where $x_i(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u_i(t) \in \mathbb{R}^m$ is the control input of the ith agent, $M_i(t) = \sum_{l=1}^d \int_0^t g_l(x_i(s))dw_l(s)$, $g_l : \mathbb{R}^n \to \mathbb{R}^n$, $d > 0$, $\{w_l(t)\}_{l=1}^d$ are independent Brownian motions. $g_l(x)$ satisfies Assumption 1. We consider the following PI and PFI control protocols for the ith agent:

$$u_i(t) = \sum_{j\in N_i}\left(K_p\epsilon_{ji}(t) + K_I\int_0^t \epsilon_{ji}(s)ds\right), \tag{31}$$

and

$$u_i(t) = \sum_{j \in N_i} \left(K_p \epsilon_{ji}(t) + K_I \int_{t-\tau}^{t} \epsilon_{ji}(s)ds \right), \tag{32}$$

where $\epsilon_{ji}(t) = x_j(t) - x_i(t)$ and $K_p, K_I \in \mathbb{R}^{m \times n}$ are the proportional and integral feedback gain matrices to be designed. τ represents the duration of the integral action, $\tau > 0$. Denote $x(t) = [x_1^T(t), ..., x_N^T(t)]^T$. We also give the initial data $x(t) = \varphi(t)$ for $t \in [-\tau, 0]$, $\varphi \in C([-\tau, 0], \mathbb{R}^{nN})$.

Definition 2. The MAS (30) reachs m.s. (or a.s.) consensus if for any initial data $\varphi \in C([-\tau, 0]; \mathbb{R}^{nN})$ and all distinct $i, j \in \mathcal{V}$, $\lim_{t\to\infty} \mathbb{E}\|x_i(t) - x_j(t)\|^2 = 0$ (or $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, a.s.).

3.1 | Consensus of stochastic linear MASs under PI protocol

Theorem 4. *Suppose that $g_l(x) = \sigma_l x$, $\sigma_l \geq 0$. The stochastic linear MASs* (30) *under PI control protocol* (31) *can reach m.s. and a.s. consensus if there exist three matrice $\bar{P}_1 = I_{N-1} \otimes P_1$, $\bar{P}_2 = I_{N-1} \otimes P_2$, $\bar{P} = I_{N-1} \otimes P$, $P_1, P_2, P \in \mathbb{R}^{n\times n}$, $P > 0$ and $P_2 - P_1 P^{-1} P_1^T > 0$ such that*

$$\begin{bmatrix} \Phi_1 & \Phi_3 \\ \Phi_3^T & \Phi_2 \end{bmatrix} < 0, \tag{33}$$

where $\Phi_1 = Q_2^T \bar{P}_1^T + \bar{P}_1 Q_2$, $\Phi_2 = (Q_0 + Q_1)^T \bar{P} + \bar{P}(Q_0 + Q_1) + \bar{P}_1 + \bar{P}_1^T + D_{\bar{P}}$, $\Phi_3 = Q_2^T \bar{P} + \bar{P}_2 + \bar{P}_1(Q_0 + Q_1)$, $Q_0 = I_{N-1} \otimes A$, $Q_1 = -\Lambda \otimes BK_p$, $Q_2 = -\Lambda \otimes BK_I$, $D_{\bar{P}} = \bar{\sigma}^2 \bar{P}$ and $\bar{\sigma} = \max(\sigma_l)$.

Proof. With the distributed PI control protocol (31), the stochastic linear MASs (30) can be expressed as:

$$dx(t) = (I_N \otimes A)x(t)dt - (\mathcal{L} \otimes BK_p)x(t)dt - \int_0^t (\mathcal{L} \otimes BK_I)x(s)dsdt + d\Theta_1(t),$$

where $\Theta_1(t) = \sum_{l=1}^{d} \int_0^t [I_N \otimes (\sigma_l I_n)]\, x(s)dw_l(s)$. Denote $\Upsilon_N = \frac{1}{\sqrt{N}} \mathbf{1}_N \mathbf{1}_N^T$ and $\mho(t) = [(I_N - \Upsilon_N) \otimes I_n]\, x(t)$, where column vector $\mathbf{1}_N = [1, 1, ..., 1]^T$.

$$d\mho(t) = (I_N \otimes A)\mho(t)dt - (\mathcal{L} \otimes BK_p)\mho(t)dt - \int_0^t (\mathcal{L} \otimes BK_I)\mho(s)dsdt + d\Theta_2(t),$$

where $\Theta_2(t) = \sum_{l=1}^{d} \int_0^t [I_N \otimes (\sigma_l I_n)]\, \mho(s)dw_l(s)$. Denote the unitary matrix $S_{\mathcal{L}} = [\frac{1_N}{\sqrt{N}}, \kappa_2, ..., \kappa_N]$, where $\kappa_i^T \mathcal{L} = \lambda_i \kappa_i^T$, $\left\|\kappa_i^T\right\| = 1$, $i = 2, ..., N$. Define $\mho(t) = (S_{\mathcal{L}} \otimes I_n)\tilde{\mho}(t)$ and $\tilde{\mho}(t) = [\tilde{\mho}_1^T(t), ..., \tilde{\mho}_N^T(t)]^T$, where $\tilde{\mho}_1(t) \equiv 0$. Denote $\bar{\mho}(t) = [\tilde{\mho}_2^T(t), ..., \tilde{\mho}_N^T(t)]^T$.

$$\begin{aligned} d\bar{\mho}(t) &= (I_{N-1} \otimes A)\bar{\mho}(t)dt - (\Lambda \otimes BK_p)\bar{\mho}(t)dt - \int_0^t (\Lambda \otimes BK_I)\bar{\mho}(s)dsdt + d\Theta_3(t) \\ &= (Q_0 \bar{\mho}(t) + Q_1 \bar{\mho}(t))dt + Q_2 \int_0^t \bar{\mho}(s)dsdt + d\Theta_3(t), \end{aligned} \tag{34}$$

where $Q_0 = I_{N-1} \otimes A$, $Q_1 = -\Lambda \otimes BK_p$, $Q_2 = -\Lambda \otimes BK_I$ and $\Theta_3(t) = \sum_{l=1}^{d} \int_0^t [I_{N-1} \otimes (\sigma_l I_n)]\, \bar{\mho}(s)dw_l(s)$. Therefore, the consensus problem of stochastic MAS (30) is transformed into the stability problem of the stochastic differential equation (34).

Let $\bar{P} = \begin{bmatrix} \bar{P}_2 & \bar{P}_1 \\ \bar{P}_1^T & \bar{P} \end{bmatrix}$, where $\bar{P}_1 = I_{N-1} \otimes P_1$, $\bar{P}_2 = I_{N-1} \otimes P_2$, $\bar{P} = I_{N-1} \otimes P$, $P_1, P_2, P \in \mathbb{R}^{n\times n}$, $P > 0$ and $P_2 - P_1 P^{-1} P_1^T > 0$. Denote $R_l = I_{N-1} \otimes (\sigma_l I_n)$. Then, we can get $\bar{\mho}^T(t) R_l^T \bar{P}_2 R_l \bar{\mho}(t) = \sigma_l^2 \bar{\mho}^T(t) \bar{P} \bar{\mho}(t) = \bar{\mho}^T(t) D_{\bar{P}} \bar{\mho}(t)$, where $D_{\bar{P}} = \bar{\sigma}^2 \bar{P}$ and $\bar{\sigma} = \max(\sigma_l)$. Applying Theorem 1 to equation (34), we can obtain the following stability criteria of (34)

$$\begin{bmatrix} \Phi_1 & \Phi_3 \\ \Phi_3^T & \Phi_2 \end{bmatrix} < 0,$$

where $\Phi_1 = Q_2^T \bar{P}_1^T + \bar{P}_1 Q_2$, $\Phi_2 = (Q_0 + Q_1)^T \bar{P} + \bar{P}(Q_0 + Q_1) + \bar{P}_1 + \bar{P}_1^T + D_{\bar{P}}$ and $\Phi_3 = Q_2^T \bar{P} + \bar{P}_2 + \bar{P}_1(Q_0 + Q_1)$. □

The above theorem provides sufficient condition for the consensus of stochastic MASs in the form of linear matrix inequalities. On this basis, the stochastic algebraic Riccati equation (SAREs) can be used to design control gains, thereby obtaining more intuitive explicit consensus conditions. Firstly, consider the following SARE

$$A^TP + PA + C^TPC - PBR^{-1}B^TP + Q = 0, R > 0. \tag{35}$$

Remark 5. The solvability of this SARE has been well investigated in 33. It indicates that the existence and uniqueness of the positive definite solution $P > 0$ to (35) is equivalent to the corresponding stochastic system $dx(t) = [Ax(t) + Bu(t)]dt + Cx(t)dw(t)$ is m.s. stabilizable, that is, the m.s. stabilization of the stochastic system can guarantee the existence of the solution to SARE. In this case, the stabilizing and optimal controller is $u(t) = -R^{-1}B^TPx(t)$. In particular, if B is invertible, then the SARE (35) exists a positive definite solution.

Lemma 3. *If $Q_2 < 0$ and there exists $\bar{P} = I_{N-1} \otimes P > 0$ such that $\Pi_P = (Q_0 + Q_1)^T\bar{P} + \bar{P}(Q_0 + Q_1) + D_{\bar{P}} < 0$ holds and $Q_2\bar{P} = \bar{P}Q_2$, where $Q_0 = I_{N-1} \otimes A$, $Q_1 = -\Lambda \otimes BK_p$, $Q_2 = -\Lambda \otimes BK_I$, $D_{\bar{P}} = \bar{\sigma}^2\bar{P}$ and $\bar{\sigma} = \max(\sigma_l)$, then the stochastic linear MASs* (30) *under PI control protocol* (31) *can reach m.s. and a.s. consensus.*

Proof. Let $\tau_1 = 0$ in Theorem 2, then we can get that if $Q_2 < 0$ and there is $\bar{P} > 0$ such that $\Pi_P = (Q_0+Q_1)^T\bar{P}+\bar{P}(Q_0+Q_1)+D_{\bar{P}} < 0$ holds and $Q_2\bar{P} = \bar{P}Q_2$, then (33) holds. Then, by Theorem 4, we can obtain the desired result. □

Corollary 5. *Assume that the SARE* (35) *has a positive definite solution. Suppose that $g_l(x) = \sigma_l x$, $\sigma_l \geq 0$ and B is invertible. The stochastic linear MASs* (30) *under PI control protocol* (31) *with $K_p = k_pB^TP$, $K_I = k_IB^{-1}$ can reach m.s. and a.s. consensus if $k_p > \frac{1}{2\lambda_i}$ and $k_I > 0$, where $P > 0$ is the solution to the SARE* (35) *with $C = \sigma_l$.*

Proof. Note that $\Pi_2 < 0$ can be guaranteed by

$$(A - \lambda_iBK_p)^TP + P(A - \lambda_iBK_p) + D_P < 0,$$

where $D_P = \bar{\sigma}^2P$ and $i = 2, ..., N$. By the elementary inequality $(m_1 + m_2)^TO(m_1 + m_2) \leq 2m_1^TOm_1 + 2m_2^TOm_2$, for $m_1, m_2 \in \mathbb{R}^n$ and $O > 0$, we can get

$$A^TP + PA - \lambda_iK_p^TB^TP - \lambda_iPBK_p + \bar{\sigma}^2P < 0.$$

Note that B is invertible and $K_I = k_IB^{-1}, k_I > 0$, which satisfy $Q_2 = -\Lambda \otimes BK_I < 0$ and $Q_2P = PQ_2$. Then, according to Lemma 3, K_p can be selected based on the SARE (35). Let $K_p = k_pB^TP$, the above inequality can be transformed into

$$A^TP + PA - 2\lambda_ik_pPBB^TP + \bar{\sigma}^2P < 0.$$

In this case, $P > 0$ is the solution P to equation (35) with $C = \sigma_l$. Let $R = I_m$ in the equation (35), then we can get

$$A^TP + PA - 2\lambda_ik_pPBB^TP + \bar{\sigma}^2P = (1 - 2\lambda_ik_p)PBB^TP - Q < 0. \tag{36}$$

Note that $1 - 2\lambda_ik_p < 0$. Then, the above inequality (36) holds, that is, $\Phi_2 < 0$ holds. By Lemma 3, if $K_p = k_pB^TP$, $K_I = k_IB^{-1}$, $P > 0$, $1 - 2\lambda_ik_p < 0$ and $k_I < 0$, then the stochastic differential equation (34) is m.s. and a.s. exponentially stable. This, combined with the definition of $\bar{\mho}(t)$, implies the m.s. and a.s. consensus. □

Remark 6. The above corollary utilizes Lemma 3 and SARE to design controller gains. On the basis that the integral term will not have a negative effect on consensus, a SARE is used to design and solve the gain of the proportional control term. However, due to the integral term, the controller designed in this corollary is no longer an optimal controller like typical proportional feedback, and the integral control gain also needs to be further designed, which is the next step in the future.

Specifically, for the scalar case, we can obtain more compact explicit conditions for the consensus of stochastic MASs with respect to system parameters. Consider a scalar MAS, where the dynamic model of ith agent is described as

$$dx_i(t) = ax_i(t) + bu_i(t) + dm_i(t), \quad i = 1, 2, ..., N, \tag{37}$$

where a, b are constants, $m_i(t) = \int_0^t g(x_i(s))dw(s)$. We consider the following distributed PI control protocol for the ith agent:

$$u_i(t) = \sum_{j \in N_i} \left(k_p \epsilon_{ji}(t) + k_I \int_0^t \epsilon_{ji}(s)ds \right), \tag{38}$$

where $\epsilon_{ji}(t) = x_j(t) - x_i(t)$ and $k_p, k_I \in \mathbb{R}$ are the proportional and integral feedback gain matrices to be designed.

Corollary 6. *Suppose that $g(x) = \sigma x$, $\sigma \geq 0$. The stochastic linear MASs* (37) *under PI control protocol* (38) *can reach m.s. and a.s. consensus if $\exists\, \theta \in (\theta_1, \theta_2)$ where $\theta_1 = \frac{bk_p\lambda_2 - \sqrt{b^2k_p^2\lambda_2^2 + 4bk_I\lambda_2}}{2}$, $\theta_2 = \frac{bk_p\lambda_2 + \sqrt{b^2k_p^2\lambda_2^2 + 4bk_I\lambda_2}}{2}$ and $b^2k_p^2\lambda_2 + 4bk_I > 0$, such that $-2bk_I\theta\lambda_i + \theta a < 0$, $2(a - bk_p\lambda_i) + (a+2)\theta + \sigma^2 < 0$.*

Proof. Let $\bar{P} = \begin{bmatrix} \bar{P}_2 & \bar{P}_1 \\ \bar{P}_1^T & \bar{P} \end{bmatrix}$, where $\bar{P}_2 = \mu\Lambda$, $\bar{P} = I_{N-1}$, $\bar{P}_1 = \theta I_{N-1}$ with μ, θ to be determined. Indeed, $\mu\lambda_2 > \theta^2$ is necessary to ensure the positive definiteness of P. By Theorem 4, we can obtain the following stability criteria of (34)

$$U = \begin{bmatrix} \Phi_1 & \Phi_3 \\ \Phi_3^T & \Phi_2 \end{bmatrix} < 0,$$

where $\Phi_1 = -2bk_I\theta\Lambda$, $\Phi_2 = 2(aI_{N-1} - bk_p\Lambda) + 2\theta I_{N-1} + \sigma^2 I_{N-1}$ and $\Phi_3 = \mu\Lambda - bk_I\Lambda + \theta(aI_{N-1} - bk_p\Lambda)$. Let $\mu = bk_I + \theta bk_p$, then $\Phi_3 = \theta a I_{N-1}$. Note that

$$\begin{bmatrix} 0 & \theta a I_{N-1} \\ \theta a I_{N-1} & 0 \end{bmatrix} \leq \begin{bmatrix} \theta a I_{N-1} & 0 \\ 0 & \theta a I_{N-1} \end{bmatrix}.$$

Then, we can get

$$U \leq \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix},$$

where $\eta_1 = -2bk_I\theta\Lambda + \theta a I_{N-1}$ and $\eta_2 = 2(aI_{N-1} - bk_p\Lambda) + 2\theta I_{N-1} + \sigma^2 I_{N-1} + \theta a I_{N-1}$. Then, we need $\eta_1 < 0$ and $\eta_2 < 0$. It is easy to verify that $-2bk_I\theta\lambda_i + \theta a < 0$ implies $\eta_1 < 0$, $2(a - bk_p\lambda_i) + (a+2)\theta + \sigma^2 < 0$ implies $\eta_2 < 0$ and $\theta \in (\theta_1, \theta_2)$ where $\theta_1 = \frac{bk_p\lambda_2 - \sqrt{b^2k_p^2\lambda_2^2 + 4bk_I\lambda_2}}{2}$, $\theta_2 = \frac{bk_p\lambda_2 + \sqrt{b^2k_p^2\lambda_2^2 + 4bk_I\lambda_2}}{2}$, $b^2k_p^2\lambda_2 + 4bk_I > 0$ implies $\mu\lambda_2 > \theta^2$. Therefore, $U < 0$. That is, the stochastic linear MASs (37) under PI control protocol (38) can reach m.s. and a.s. consensus.

□

3.2 | Consensus of stochastic linear MASs under PFI protocol

Theorem 5. *Suppose that $g_l(x) = \sigma_l x$, $\sigma_l \geq 0$. The stochastic linear MASs* (30) *under PFI control protocol* (32) *with $K_p = k_p B^T P$, $K_I = k_I B^T P$ and $P > 0$ can reach m.s. and a.s. consensus if*

$$A^T P + PA + 3\tau A^T PA - \rho_i PBI_m^{-1} B^T P + \bar{\sigma}^2 P < 0, \tag{39}$$

where $\rho_i = 2k_p\lambda_i + 2k_I\tau\lambda_i - 3\tau k_p^2\lambda_i^2 - \frac{10}{3}\tau^3 k_I^2\lambda_i^2$ and $\bar{\sigma} = \max(\sigma_l)$.

Proof. With the distributed PFI control protocol (32), the stochastic linear MASs (30) can be expressed as:

$$dx(t) = (I_N \otimes A - \mathcal{L} \otimes BK_p)x(t)dt - \int_{t-\tau}^{t} (\mathcal{L} \otimes BK_I)x(s)dsdt + dM_1(t),$$

where $M_1(t) = \sum_{l=1}^{d} \int_0^t [I_N \otimes (\sigma_l I_n)]\, x(s)dw_l(s)$. Denote $\Upsilon_N = \frac{1}{\sqrt{N}} \mathbf{1}_N \mathbf{1}_N^T$ and $\mho(t) = [(I_N - \Upsilon_N) \otimes I_n]\, x(t)$, where column vector $\mathbf{1}_N = [1, 1, ..., 1]^T$.

$$d\mho(t) = (I_N \otimes A - \mathcal{L} \otimes BK_p)\mho(t)dt - \int_{t-\tau}^{t} (\mathcal{L} \otimes BK_I)\mho(s)dsdt + dM_2(t),$$

where $M_2(t) = \sum_{l=1}^{d} \int_0^t [I_N \otimes (\sigma_l I_n)] \mho(s) dw_l(s)$. Denote $S_{\mathcal{L}} = [\frac{1_N}{\sqrt{N}}, \kappa_2, ..., \kappa_N]$, where $\kappa_i^T \mathcal{L} = \lambda_i \kappa_i^T$, $\left\|\kappa_i^T\right\| = 1, i = 2, ..., N$. Define $\mho(t) = (S_{\mathcal{L}} \otimes I_n)\tilde{\mho}(t)$ and $\tilde{\mho}(t) = [\tilde{\mho}_1^T(t), ..., \tilde{\mho}_N^T(t)]^T$, where $\tilde{\mho}_1(t) \equiv 0$. Denote $\bar{\mho}(t) = [\tilde{\mho}_2^T(t), ..., \tilde{\mho}_N^T(t)]^T$.

$$
\begin{aligned}
d\bar{\mho}(t) &= (I_{N-1} \otimes A - \Lambda \otimes BK_p)\bar{\mho}(t)dt - \int_{t-\tau}^{t} (\Lambda \otimes BK_I)\bar{\mho}(s)dsdt + dM_3(t) \\
&= L_0 \bar{\mho}(t)dt + L_1 \int_{t-\tau}^{t} \bar{\mho}(s)dsdt + dM_3(t),
\end{aligned}
\tag{40}
$$

where $M_3(t) = \sum_{l=1}^{d} \int_0^t [I_{N-1} \otimes (\sigma_l I_n)] \bar{\mho}(s) dw_l(s)$, $L_0 = I_{N-1} \otimes A - \Lambda \otimes BK_p$ and $L_1 = -\Lambda \otimes BK_I$. Therefore, the consensus problem of (30) is transformed into the stability problem of (40). Denote $Q_l = I_{N-1} \otimes (\sigma_l I_n)$ and $\bar{P} = I_{N-1} \otimes P (P \in \mathbb{R}^{n \times n})$. Then, we can get $\bar{\mho}^T(t) Q_l^T \bar{P} Q_l \bar{\mho}(t) = \sigma_l^2 \bar{\mho}^T(t) \bar{P} \bar{\mho}(t) \leq \bar{\mho}^T(t) D_{\bar{P}} \bar{\mho}(t)$, where $D_{\bar{P}} = \bar{\sigma}^2 I_{N-1} \otimes P$ and $\bar{\sigma} = \max(\sigma_l)$. Applying Theorem 3 to equation (40) and substituting $D_{\bar{P}}$ into it, we can obtain the following stability criteria of (40)

$$
(L_0 + \tau L_1)^T \bar{P} + \bar{P}(L_0 + \tau L_1) + \tau (L_0 + \tau L_1)^T \bar{P} (L_0 + \tau L_1) + \frac{1}{3} \tau^3 L_1^T \bar{P} L_1 + D_{\bar{P}} < 0.
$$

Let $H_i = A - \lambda_i BK_p - \tau \lambda_i BK_I$, $i = 2, ..., N$. According to the definition of L_0, L_1 and $\bar{P}$, the above inequality can be written as

$$
H_i^T P + PH_i + \tau H_i^T PH_i + \frac{1}{3} \tau^3 \lambda_i^2 K_I^T B^T PBK_I + \bar{\sigma}^2 P < 0. \tag{41}
$$

By elementary inequality $(m_1 + m_2 + m_3)^T O (m_1 + m_2 + m_3) \leq 3m_1 T O m_1 + 3m_2^T O m_2 + 3m_3^T O m_3, m_1, m_2, m_3 \in \mathbb{R}^n, O > 0$, then $H_i^T P H_i \leq 3A^T PA + 3\lambda_i^2 K_p^T B^T PBK_p + 3\tau^2 \lambda_i^2 K_I^T B^T PBK_I$. Substituting this inequality into (41) and letting $K_p = k_p B^T P$, $K_I = k_I B^T P$, we can obtain

$$
A^T P + PA + 3\tau A^T PA - \rho_i PBI_m^{-1} B^T P + \bar{\sigma}^2 P < 0, \tag{42}
$$

where $\rho_i = 2k_p \lambda_i + 2k_I \tau \lambda_i - 3\tau k_p^2 \lambda_i^2 - \frac{10}{3} \tau^3 k_I^2 \lambda_i^2$. Thus, by Theorem 3, if (42) holds, then (40) is m.s. and a.s. exponentially stable, which implies m.s. and a.s. consensus of stochastic MAS (30).

□

Remark 7. Theorem 5 establishes a sufficient condition for the m.s. and a.s. consensus of a stochastic MAS with proportional and fragment-integral control, which indicates that fragment-integral term can have a positive impact on consensus for a stochastic MASs since condition (39) does not require A to be Hurwitz. In addition, when A does not meet the Hurwitz condition, the stochastic MAS can also achieve consensus by selecting appropriate control parameters of proportional and fragment-integral terms.

Especially, for the scalar case, i.e. $A = a$ and $B = b$, we obtain Corollary 7.

Corollary 7. *Suppose that $g(x) = \sigma x$, $\sigma \geq 0$. The stochastic linear MASs* (30) *under PI control protocol* (32) *with $K_p = k_p b$, $K_I = k_I b$ can reach m.s. and a.s. consensus if*

$$
2a + 3\tau a^2 - \rho_i b^2 + \sigma^2 < 0, \tag{43}
$$

where $\rho_i = 2k_p \lambda_i + 2k_I \tau \lambda_i - 3\tau k_p^2 \lambda_i^2 - \frac{10}{3} \tau^3 k_I^2 \lambda_i^2$.

4 | SIMULATIONS

In this section, two simulations are provided to validate the effectiveness of the theoretical results. Consider the linear MAS (30) composed of four scalar agents, where $A = -0.01$, $B = 1$ and $g_l(x) = 0.0001x$. The interaction topology is modeled as an connected undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where the node set $\mathcal{V} = \{1, 2, 3, 4\}$, the edge set $\mathcal{E} = \{(3, 4), (4, 3), (2, 3), (3, 2), (1, 2), (2, 1)\}$, and the adjacency matrix $\mathcal{A} = [a_{ij}]_{4\times4}$ with $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = 1$ and other being 0. We can obtain the eigenvalues of the Laplacian matrix $\mathcal{L}$ are $\lambda_1 = 0$, $\lambda_2 = 0.5858$, $\lambda_3 = 2$, $\lambda_4 = 3.4142$. The initial value is $x(0) = [7, 2, -4, -8]^T$.

The control input $u_i(t)$ adopts distributed proportional integral(PI) (31) and proportional fragment-integral(PFI) control protocols(32). We will select appropriate proportional and integral control gains K_p and K_I to ensure consensus.

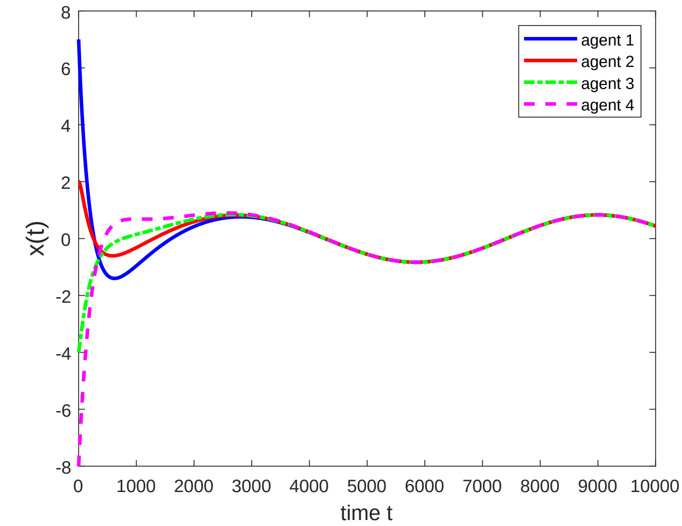

FIGURE 1 The states of a stochastic linear MAS with four agents by PI control.

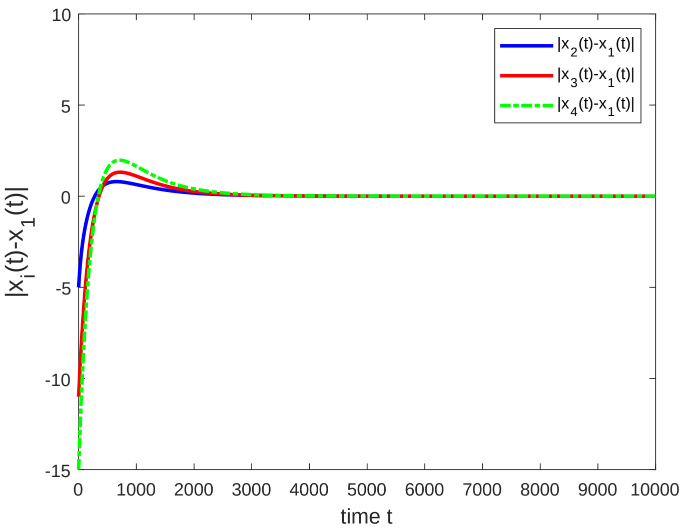

FIGURE 2 The relative state errors of a stochastic linear MAS with four agents by PI control.

4.1 | PI control of linear MAS

According to Corollary 6, we choose $K_p = k_p = 1$ and $K_I = k_I = 0.1$. In Corollary 6, $\theta \in (-0.1741, 1.3457)$ and $b^2k_p^2\lambda_2 + 4bk_I = 0.5775 > 0$. Then, we can choose $\theta = 0.1$ so that $-2bk_I\theta\lambda_i+\theta a \leq -2bk_I\theta\lambda_2+\theta a = -0.012716 < 0$ and $2(a-bk_p\lambda_i)+(a+2)\theta+\sigma^2 \leq 2(a - bk_p\lambda_2) + (a + 2)\theta + \sigma^2 = -0.9926 < 0$ hold. Therefore, the designed control gains satisfy the consensus condition. Here, considering the behavior and sample path of each agent, we have Figure 1, which indicates that states of the four agents tend to be consensus over time. To simulate m.s. and a.s. consensus more accurately, we consider the relative states $\|x_i(t) - x_1(t)\|, i = 2, 3, 4$. Thus, we obtain Figure 2, indicating that the relative states tend to zero, that is, the four agents reach a.s. consensus. For m.s. consensus analysis, we generate 10^4 sample paths. Then, considering the behaviors of the m.s. relative states $E\|x_i(t) - x_1(t)\|^2, i = 2, 3, 4$, we obtain Figure 3, which demonstrates that the four agents reach m.s. consensus. Figures 1-3 reveal that the stochastic MAS (30) can reach m.s. and a.s. consensus by PI control protocol (31).

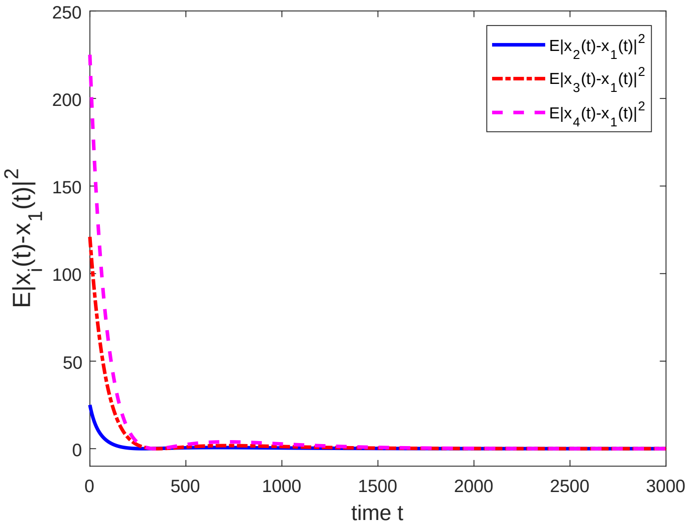

FIGURE 3 The m.s. relative state errors of a stochastic linear MAS with four agents by PI control.

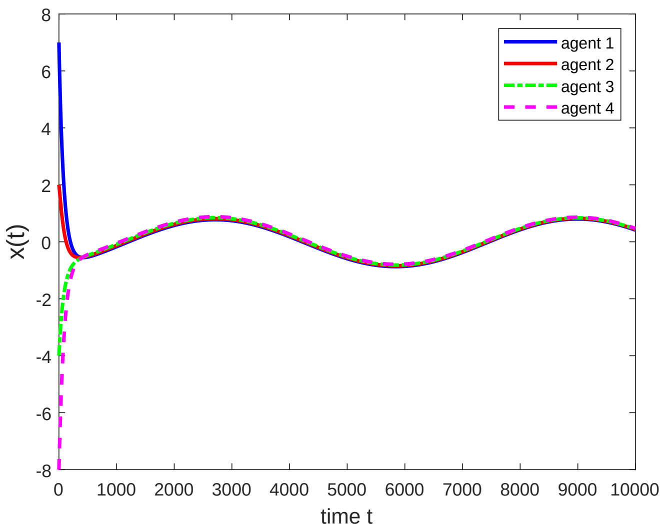

FIGURE 4 The states of a stochastic linear MAS with four agents by PFI control.

4.2 | PFI control of linear MAS

Let $\tau = 0.01$. Owing to Corollary 7, we choose $K_p = k_p = 2$ and $K_I = k_I = 0.01$. In Corollary 7, $\rho_i = 2k_p\lambda_i + 2k_I\tau\lambda_i - 3\tau k_p^2\lambda_i^2 - \frac{10}{3}\tau^3 k_I^2\lambda_i^2 \geq 2k_p\lambda_2 + 2k_I\tau\lambda_2 - 3\tau k_p^2\lambda_4^2 - \frac{10}{3}\tau^3 k_I^2\lambda_4^2 = 0.9445$. Then, $2a + 3\tau a^2 - \rho_i b^2 + \sigma^2 \leq -0.9645 < 0$ holds. Therefore, the designed control gains satisfy the consensus condition. The revolutions of the states for the four agents $x_i(t), i = 1, 2, 3, 4$ of the linear stochastic MAS are displayed in Figure 4, which reveals that the four agents tend to be consensus. Futhermore, consider the relative states $\|x_i(t) - x_1(t)\|, i = 2, 3, 4$. Choosing one sample path, we get Figure 5. From Figure 5, it could be observed that all the relative state trajectories tend to be zero over time, indicating that the four agents reach a.s. consensus. To validate the m.s. consensus, we generate 10^4 sample paths and analyze the behaviors of their corresponding m.s. relative states $E\|x_i(t) - x_1(t)\|^2, i = 2, 3, 4$. Figure 6 illustrates that the four agents reach m.s. consensus. Thus, it can be concluded that the stochastic MAS (30) can reach m.s. and a.s. consensus by PFI protocol (32).

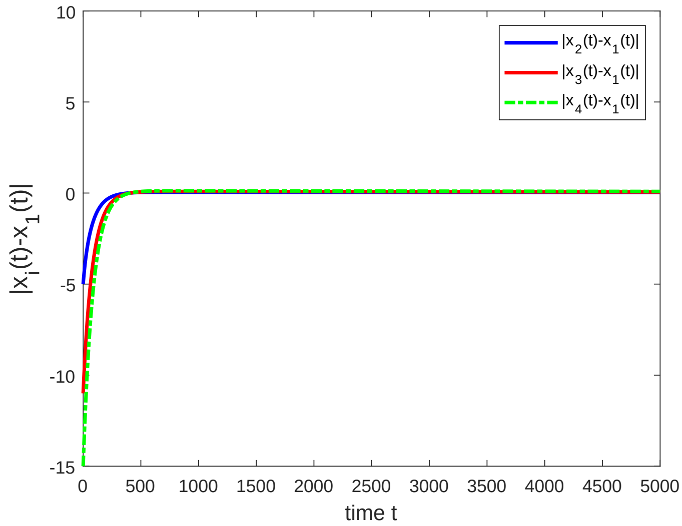

FIGURE 5 The relative state errors of a stochastic linear MAS with four agents by PFI control.

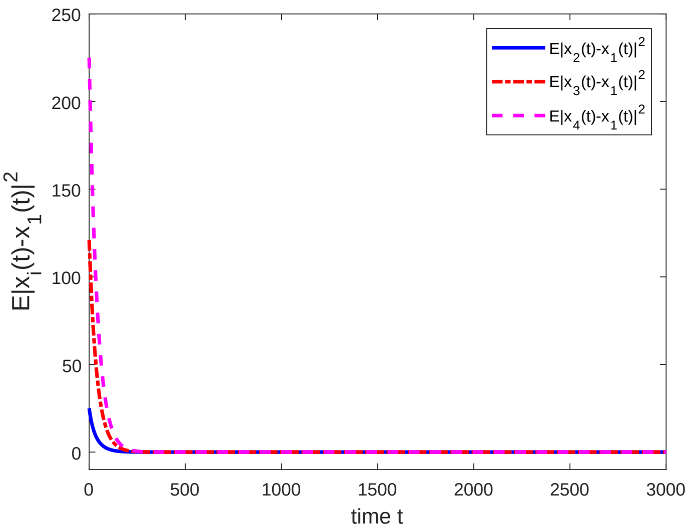

FIGURE 6 The m.s. relative state errors of a stochastic linear MAS with four agents by PFI control.

5 | CONCLUSION

This article investigated the stability of SDDSs with path integral information. For the two cases of integral and fragment-integral, the corresponding m.s. and a.s. exponential stability criteria for two types of SDDSs were obtained. It indicates that the fragment-integral term can play a positive role in stochastic stability. In addition, we applied the obtained stochastic stability theorem to the PI-type control problem of MASs. On this basis, sufficient conditions for the consensus of a stochastic linear MAS under two PI-type control protocols were established, and corresponding controller design methods were provided. Specifically, for the case of PI control, we employed SAREs to design control gains, thereby obtaining more intuitive explicit consensus conditions. The stability results of SDDSs with path information will stimulate future research on more complex nonlinear systems, such as stochastic systems with Markov switching. Moreover, their applications in PI-type control of MASs can also be further expanded for research, such as leader-following and containment control.

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