# On the coefficients of singularity of a bi-harmonic problem on a truncated non-convex sector near the angle π by Fourier analysis

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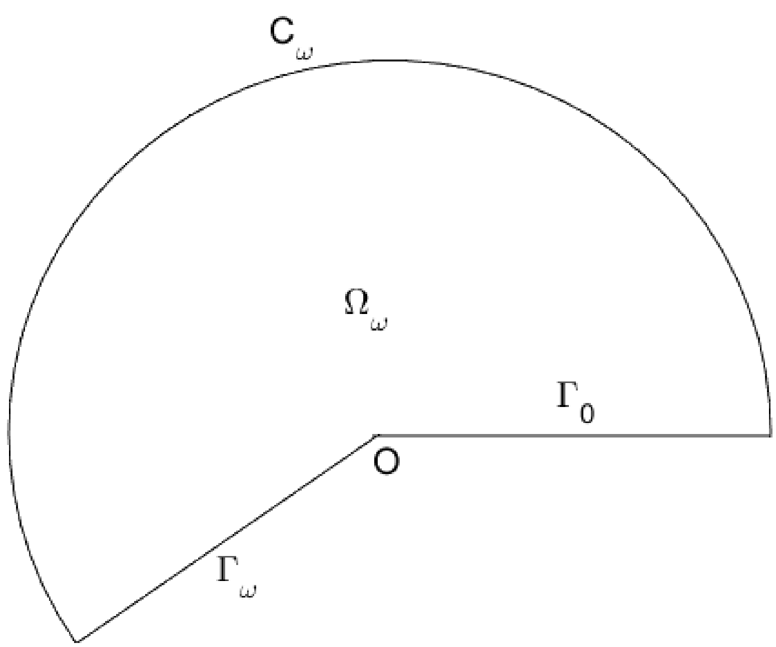
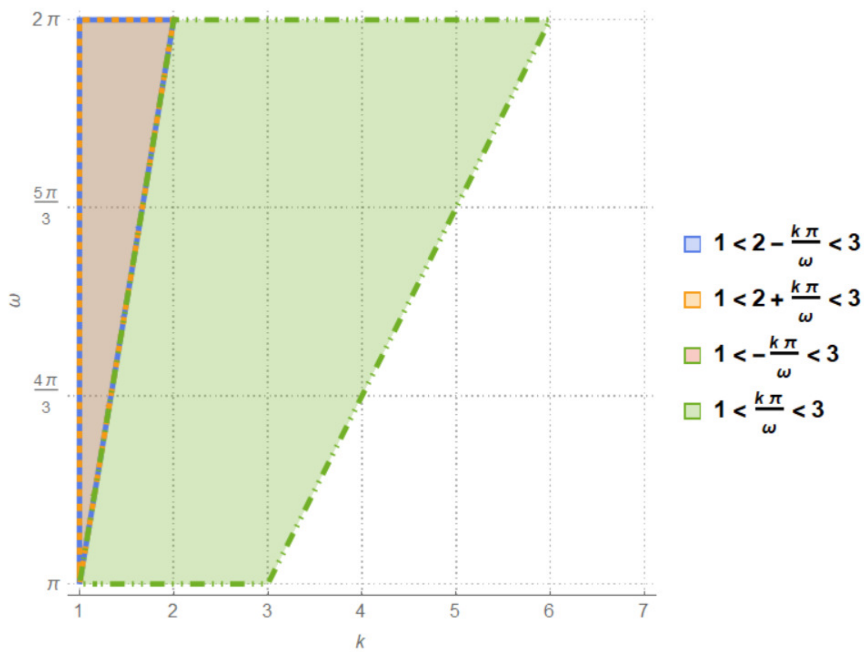
**Abstract**

Based on Fourier series, we adapt an approach discussed in a recent work on the Laplace operator to classical results obtained in the literature, describing the singularities of solutions to a fourth-order elliptic problem on a polygonal domain of the plane that may appear near a concave corner. We demonstrate how the Fourier series method provides explicit decomposition and precise description of the coefficients of singularities of the solution. As a main result, explicit and sharp estimates with respect to the opening angle parameter can be obtained via this method. We recall that such estimates can be useful for the asymptotic analysis of solutions near corners where the opening angle generates a jump in singularity in Sobolev's exponent.





2π
5π/3
4π/3
π
ω
1
2
3
4
5
6
7
k
1 < 2 − kπ/ω < 3
1 < 2 + kπ/ω < 3
1 < −kπ/ω < 3
1 < kπ/ω < 3

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RESEARCH ARTICLE

# On the coefficients of singularity of a bi-harmonic problem on a truncated non-convex sector near the angle $\pi$ by Fourier analysis

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## Abstract

Based on Fourier series, we adapt an approach discussed in a recent work on the Laplace operator to classical results obtained in the literature, describing the singularities of solutions to a fourth-order elliptic problem on a polygonal domain of the plane that may appear near a concave corner. We demonstrate how the Fourier series method provides explicit decomposition and precise description of the coefficients of singularities of the solution. As a main result, explicit and sharp estimates with respect to the opening angle parameter can be obtained via this method. We recall that such estimates can be useful for the asymptotic analysis of solutions near corners where the opening angle generates a jump in singularity in Sobolev's exponent.

**KEYWORDS**

Fourier series, Bi-harmonic operator, Navier's boundary conditions, Concave corner, Singularity.

**MSC CLASSIFICATION**

35J25, 35J40; 35J75; 35B45; 35Q99; 35B40

## 1 | INTRODUCTION

The bi-harmonic equation is a fourth order elliptic partial differential equation which has diverse applications across various fields such as structural engineering, fluid dynamics, biomathematics, geophysics, and electromagnetic modeling. The behavior of solutions to elliptic problems on polygons, particularly near corners, began in the 1960s. In engineering and applied mathematics, there is significant interest in analyzing singular solutions to partial differential equations in non-smooth domains. When approximating solutions to elliptic problems in a regular open set, the order of the approximation error depends on the finite elements, the mesh used, and the regularity of the solutions. This regularity becomes a challenge when the open set has singularities or the boundary data are discontinuous. Specifically, for planar polygons, the solution's regularity is influenced by both the data's regularity and the polygon's geometry. It is well-known that such singularities can significantly reduce convergence in error estimates of standard numerical approximation schemes. For example, some authors discus techniques for computing the singular part or stress intensity factor (SIF) through explicit extraction formulas and numerical methods separately, cf. [1,2,3] and the references therein. Our main aim in this paper was motivated by a slightly similar drawback which is the jump of singularity in Sobolev's exponent in such problems near a critical angle such as $\pi$ (for nearly flat boundaries) or $2\pi$ (for domains with a crack). For example, one could ask the following question: Can we approach a nearly flat boundary by another completely flat one? The answer to such a question will depend on the convexity of the domain near the corner in question. Elements of answers to this question have already been described by [4,5,6] in the case of the bi-harmonic problem on a convex domain and recently [7] for the laplacian operator on a family of non-convex open sectors.

We consider a model case of Fourth order equation from linear elasticity, in planar polygonal domains with concave corner type singularities. This occurs for example in the linear model problem for a hinged plate where mixed or Navier's type boundary conditions are used on straight boundaries and away from the corners. Well-posedeness of such problems and description of their singularities near corners with different boundary conditions were addressed, whether in the harmonic or bi-harmonic case, by many authors in the literature, cf., Grisvard[8,9,10], Kondrat'ev[11], Blum[12], Maz'ya[13,14,15], Nicaise[16,17], Dauge[18,19], Stylianou[20], Gerasimov[21], Tami[4,5,6], Douah[7] and the references therein. To study the behavior of solutions near a polygon's corner, standard localization techniques involving suitable cut-off functions and partition of unity are commonly used, cf.[4] and the references therein. Thus, the problem turns out to consider a family of open planar non-convex sectors with opening angle $\omega \in (\pi, 2\pi)$. We aim at studying by means of partial Fourier analysis in polar coordinates, cf.[22,7] and references therein, w.r.t to the polar angle $\theta$, the asymptotic behavior of solutions, when $\omega \to \pi^+$, by deriving explicit formulas for the coefficients of the singularities (or SIF), along with explicit estimates that show the behavior (in $H^4$ norm) of the family of solutions $u_\omega$ near the critical angle $\omega = \pi$. As in the case of a recent work on the Laplace's operator[7], a main result of our approach is the lack of uniformity in the estimates with respect to the opening angle parameter $\omega$, contrarily to those obtained for convex corners, cf.[4,5]. To the knowledge of the authors, no such study was performed on a family of non-convex open sets, that is, when $\omega \in (\pi, 2\pi)$, for bi-harmonic problems.

Throughout this paper, a same generic constant $C > 0$ independent of $\omega$ in all estimates that follow will be used at different occurrences. In Section 2, we present the problem setting and the main result, along with a partial proof of $H^2$ uniform estimates w.r.t the opening angle $\omega \in (\pi, 2\pi)$ of the family of weak solutions $u_\omega$ for our problem. The proof of $H^4$ estimates will be justified progressively in the sections that follow. In addition, some preliminaries, such as Sobolev's spaces in polar coordinates, Sobolev's norms expressed via Fourier coefficients, and some fundamental tools useful for estimating Fourier coefficients are presented. In Section 3, formal determination of corner singularity via Fourier series is presented. Fourier coefficients of the regular part of the solution and the coefficient of singularity are given explicitly. Explicit and sharp estimates are given w.r.t to $\omega \in (\pi, 4\pi/3)$. These estimates are not uniform in the vicinity of $\pi$, even for the regular part taken separately in the case of the first frequency $k = 1$ in the Fourier series. Section 4 is devoted to the completion of the proof of the main result, in particular the characterization of the coefficient of singularity and the estimates on of the regular part in the norm $H^4$. Concluding remarks and comments are presented in the last section.

## 2 | PROBLEM SETTING AND MAIN RESULTS

Let us denote by $\{\Omega_\omega\}_{\omega\in(\pi,2\pi)}$ a family of open bounded sectors of radius 1 centered all at the origin $O$ (here $O$ represents the reentering corner where the localization has been performed). In polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, one has , cf. Figure 1,

$$\Omega_\omega = \{(x, y),\ 0 < r < 1,\ 0 < \theta < \omega\},$$

with boundary $\partial\Omega_\omega = \overline{\Gamma_0} \cup \overline{C_\omega} \cup \overline{\Gamma_\omega}$ where

$$\begin{aligned} \Gamma_\alpha &= \{(x, y),\ 0 < r < 1,\ \theta = \alpha\}, \\ C_\omega &= \{(x, y),\ r = 1,\ 0 < \theta < \omega\}. \end{aligned}$$

For a right hand side (r.h.s) $f_\omega \in L^2(\Omega_\omega)$ given and assumed continuously depending on the parameter $\omega \in (\pi, 2\pi)$, we look for solutions $u_\omega$ to the following fourth order boundary value problem:

$$\begin{cases} \Delta^2 u_\omega = f_\omega \text{ in } \Omega_\omega, \\ u_\omega = \Delta u_\omega = 0 \text{ on } \partial\Omega_\omega. \end{cases} \tag{1}$$

### 2.1 | Existence of weak solution and a uniform estimates

Let $\Omega_\omega$ a planar sector with concave corner at the origin as described in Figure 1, and $f_\omega \in L^2(\Omega_\omega)$. Let $V_\omega := H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ the Hilbert space equipped with the norm of $H^2(\Omega_\omega)$.

**Definition 1.** A function $u_\omega$ is called a weak solution of (1) if $u_\omega \in V_\omega$, and if

$$\int_{\Omega_\omega} (\Delta u_\omega \Delta v - f_\omega v)dxdy = 0,\ \forall v \in V_\omega. \tag{2}$$

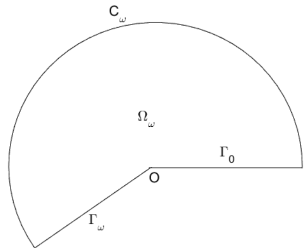


**FIGURE 1** Planar sector $\Omega_\omega$ with concave corner at the origin: opening angle $\omega \in (\pi, 2\pi)$.

**Theorem 1.** *Problem (1) admits a unique weak solution in $V_\omega$ that depends continuously on the r.h.s $f_\omega \in L^2(\Omega_\omega)$ and uniformly on the parameter $\omega \in (\pi, 2\pi)$, i.e. there exists a constant independent of $\omega$, $C > 0$, such that:*

$$\|u_\omega\|_{H^2(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{3}$$

*If, in addition, $\Delta^2 u_\omega \in L^2(\Omega_\omega)$ then $u_\omega$ satisfies the equations:*

$$\Delta^2 u_\omega = f_\omega \text{ in } L^2(\Omega_\omega), \text{and } u_\omega = \Delta u_\omega = 0 \text{ on } L^2(\partial\Omega_\omega)$$

*which gives sense to the trace of the second boundary condition on $\partial\Omega_\omega$.*

*Proof.* The proof is standard unless the uniformity of the constant $C$ w.r.t $\omega$ that one can sketch as follows: Let $a(u, v) = \int_{\Omega_\omega} \Delta u \Delta v dxdy$ and $l(v) = \int_{\Omega_\omega} f v dxdy$ the corresponding bi-linear an linear forms associated to (2) on the Hilbert space $V_\omega$ equipped with the norm $\|.\|_{H^2(\Omega_\omega)}$. Then, the continuity of $a(.,.)$ and $l(.)$ and the uniformity of constants w.r.t $\omega$ are straightforward. However, the uniformity in the coercivity constant of the bi-linear form $a(.,.)$ needs more precision. In fact, it comes from Poincare's inequality on the one hand, cf.[6], for all $u \in V_\omega$,

$$\|u\|_{H^2(\Omega_\omega)} \leq c\||\nabla^2 u\||_{L^2(\Omega_\omega)}, \tag{4}$$

which yields equivalence, with uniform constant, between the norm and semi-norm $H^2$ in the space $V_\omega$, and, on the other hand, the "second fundamental inequality" , cf. ([20], Corollary 2.3.6 p.31), which can be reformulated in the case of a planar sector $\Omega_\omega$ as follows: Using the Green's formula in $H^3(\Omega_\omega) \cap H^1_0(\Omega_\omega)$, as in ([20], p.29),

$$\int_{\Omega_\omega} (\Delta v)^2 \, dxdy = \int_{\partial\Omega_\omega} \kappa(m) \, (\partial_n v)^2 \, dm + \||\nabla^2 u\||^2_{L^2(\Omega_\omega)} \geq \||\nabla^2 u\||^2_{L^2(\Omega_\omega)} \tag{5}$$

where $\partial_n$ represents the normal derivative operator outward to $\partial\Omega_\omega$, and

$$\kappa(m) = \begin{cases} 0 & \text{if } m \in \Gamma_0 \cup \Gamma_\omega \\ 1 & \text{if } m \in C_\omega \end{cases}$$

designates the curvature of $\partial\Omega_\omega$ which is essentially positive in the case of the planar sector $\Omega_\omega$, no matter if it is concave or convex.

Hence, and by same arguments as in[20], based on density of $H^3(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ in $V_\omega$, one arrives at the inequality (5) for all $v \in V_\omega$. We conclude that, for all $v \in V_\omega$:

$$a(v, v) = \int_{\Omega_\omega} \Delta v \Delta v dxdy \geq c\|u\|^2_{H^2(\Omega_\omega)}$$

Therefore, the Lax-Milgram theorem gives the existence and uniqueness of $u_\omega \in V_\omega$ and we have:

$$c\|u_\omega\|^2_{H^2(\Omega_\omega)} \leq a(u_\omega, u_\omega) = \int_{\Omega_\omega} f_\omega \Delta u_\omega dxdy \leq \|f_\omega\|_{L^2(\Omega_\omega)}\|u_\omega\|_{H^2(\Omega_\omega)}$$

which yields the desired estimate independent of $\omega$, $\|u_\omega\|_{H^2(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}$. On the other hand, if $\Delta^2 u_\omega \in L^2(\Omega_\omega)$ then one applies the Green's formula to the variational equation in (2) to obtain

$$\int_{\Omega_\omega} (\Delta^2 u_\omega v - f_\omega v) dxdy = 0,\ \forall v \in C_c^\infty(\Omega_\omega)$$

which gives, by density argument, $\Delta^2 u_\omega = f_\omega$ in $L^2(\Omega_\omega)$. By a second integration by part, we obtain

$$\int_{\partial\Omega_\omega} \Delta u_\omega \partial_n v dm = 0,\ \forall v \in H_0^1(\Omega_\omega) \cap C^\infty(\overline{\Omega_\omega}),$$

and by the fact that the range of the trace function $v \mapsto \partial_n v$ defined on $H_0^1(\Omega_\omega) \cap C^\infty(\overline{\Omega_\omega})$ is dense in $L^2(\partial\Omega_\omega)$ , one concludes that $\Delta u_\omega = 0$ on $L^2(\partial\Omega_\omega)$. The first condition $u_\omega = 0$ on $L^2(\partial\Omega_\omega)$ is contained in the definition of $H_0^1(\Omega_\omega)$ and the proof is ended. □

## 2.2 | Sobolev's norms in polar coordinates by Fourier modes

In polar coordinates, $\phi : (r, \theta) \mapsto (x = r\cos\theta, y = r\sin\theta)$, if a distribution $v$ lies in $D'(\Omega_\omega)$, let us denote by $\tilde{v}(r, \theta) := (v \circ \phi)(r, \theta)$, $\tilde{v}$ is a distribution w.r.t $r, \theta$. Then, we denote by $\tilde{H}^m(\Omega_\omega)$ the Sobolev's space

$$\tilde{H}^m(\Omega_\omega) := \left\{\tilde{v};\ v = \tilde{v} \circ \phi^{-1} \in H^m(\Omega_\omega)\right\},$$

which is the image of $H^m(\Omega_\omega)$ by the mapping $u \mapsto \tilde{u} = u \circ \phi$. Equipped with the norm $\|\tilde{v}\|_{\tilde{H}^m(\Omega_\omega)} := \|v\|_{H^m(\Omega_\omega)}$ , $\tilde{H}^m(\Omega_\omega)$ is a Hilbert space for the natural scalar product $(\tilde{v}, \tilde{w})_{\tilde{H}^m(\Omega_\omega)} := (v, w)_{H^m(\Omega_\omega)}$ . In what follows, we denote by

$$V^{(k_1,k_2)} := \frac{\partial^{k_1+k_2}}{\partial r^{k_1}\partial\theta^{k_2}}\tilde{V},\ \forall k_1, k_2 : 0 \leq k_1 + k_2 \leq m,$$

where $V^{(0,0)}$ denotes simply $\tilde{V}$. For $m \leq 4$, we can thus characterize the $H^m$ semi-norms of a distribution $V$ in polar coordinates as follows:

**Lemma 1.** *Let $V \in H^m(\Omega_\omega)$, $m \leq 4$.*

$$H^m(\Omega_\omega) = \left\{V \in L^2(\Omega_\omega);\ \nabla^l V \in L^2(\Omega_\omega);\ \forall l : 0 \leq l \leq m\right\},$$

*where:*

$$\begin{aligned}
\|V\|^2_{L^2(\Omega_\omega)} &= \int_{\Omega_\omega} \left|\tilde{V}\right|^2 rdrd\theta \\
\|\nabla V\|^2_{L^2(\Omega_\omega)} &= \int_{\Omega_\omega} \left(\left|V^{(1,0)}\right|^2 + \left|\frac{V^{(0,1)}}{r}\right|^2\right) rdrd\theta \\
\left\|\nabla^2 V\right\|^2_{L^2(\Omega_\omega)} &= \int_{\Omega_\omega} \left(\left|V^{(2,0)}\right|^2 + 2\left|\frac{rV^{(1,1)} - V^{(0,1)}}{r^2}\right|^2 + \left|\frac{V^{(0,2)} + rV^{(1,0)}}{r^2}\right|^2\right) rdrd\theta \\
\left\|\nabla^3 V\right\|^2_{L^2(\Omega_\omega)} &= \int_{\Omega_\omega} \left(\left|V^{(3,0)}\right|^2 + 3\left|\frac{V^{(2,1)}r^2 - 2V^{(1,1)}r + 2V^{(0,1)}}{r^3}\right|^2\right. \\
&\quad + 3\left|\frac{r\left(-V^{(1,0)} + V^{(1,2)} + rV^{(2,0)}\right) - 2V^{(0,2)}}{r^3}\right|^2 \\
&\quad \left. + \left|\frac{-2V^{(0,1)} + V^{(0,3)} + 3rV^{(1,1)}}{r^3}\right|^2\right) rdrd\theta
\end{aligned}$$

$$\begin{aligned}\left\|\nabla^4 V\right\|^2_{L^2(\Omega_\omega)} = \int_{\Omega_\omega} \Bigg( & \left|V^{(4,0)}\right|^2 + 4\left|\frac{r\left(6V^{(1,1)} + r\left(rV^{(3,1)} - 3V^{(2,1)}\right)\right) - 6V^{(0,1)}}{r^4}\right|^2 \\ & + 6\left|\frac{6V^{(0,2)} + r\left(2V^{(1,0)} - 4V^{(1,2)} + r\left(-2V^{(2,0)} + V^{(2,2)} + rV^{(3,0)}\right)\right)}{r^4}\right|^2 \\ & + 4\left|\frac{6V^{(0,1)} - 3V^{(0,3)} + r\left(-8V^{(1,1)} + V^{(1,3)} + 3rV^{(2,1)}\right)}{r^4}\right|^2 \\ & + \left|\frac{-8V^{(0,2)} + V^{(0,4)} + 3r\left(-V^{(1,0)} + 2V^{(1,2)} + rV^{(2,0)}\right)}{r^4}\right|^2 \Bigg) rdrd\theta\end{aligned}$$

*Proof.* Results from calculus and transformation of iterated gradient form Cartesian to polar coordinates. ☐

In what follows, we will use the same notation $V$ for both $V$ and its image $\tilde{V}$.

**Lemma 2.** *Let $V \in H^m(\Omega_\omega)$ a periodic distribution such that $V(.,0) = V(.,\omega) = 0$, and $m \leq 4$. Let us denote by $V(r,\theta) = \sum_{k\geq 1} V_k(r)\sin\frac{k\pi}{\omega}\theta$ a.e in $\Omega_\omega$ the partial Fourier series of $V$ in $\theta$, where*

$$V_k(r) = \frac{2}{\omega}\int_0^\omega V(r,\theta)\sin\frac{k\pi}{\omega}\theta d\theta$$

*is the $k^{th}$ Fourier mode seen as a function of $r \in (0,1)$. Then,*

$$\|V\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left|V_k\right|^2 rdr,$$

$$\|\nabla V\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left(\left|V_k'\right|^2 + \left(\frac{k\pi}{\omega}\right)^2\left|\frac{V_k}{r}\right|^2\right) rdr,$$

$$\|\nabla^2 V\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left(\left|V_k''\right|^2 + 2\left(\frac{k\pi}{\omega}\right)^2\left|\frac{V_k'}{r} - \frac{V_k}{r^2}\right|^2 + \left|\frac{V_k'}{r} - \left(\frac{k\pi}{\omega}\right)^2\frac{V_k}{r^2}\right|^2\right) rdr.$$

$$\begin{aligned}\left\|\nabla^3 V\right\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \Bigg( & \left|V_k'''\right|^2 + 3\left(\frac{k\pi}{\omega}\right)^2\left|\frac{V_k''}{r} - \frac{2V_k'}{r^2} + \frac{2V_k}{r^3}\right|^2 \\ & + 3\left|\frac{V_k''}{r} - \left(1 + \left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k'}{r^2} + 2\left(\frac{k\pi}{\omega}\right)^2\frac{V_k}{r^3}\right|^2 \\ & + \left(\frac{k\pi}{\omega}\right)^2\left|\frac{3V_k'}{r^2} - \left(2 + \left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k}{r^3}\right|^2\Bigg) rdr\end{aligned}$$

$$\left\|\nabla^4 V\right\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left( \left|V_k^{(4)}\right|^2 + 4\left(\frac{k\pi}{\omega}\right)^2 \left|\frac{V_k'''}{r} - \frac{3V_k''}{r^2} + \frac{6V_k'}{r^3} - \frac{6V_k}{r^4}\right|^2 \right.$$
$$+ 6\left|\frac{V_k'''}{r} - \left(2+\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k''}{r^2} + \left(2+4\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k'}{r^3} - 6\left(\frac{k\pi}{\omega}\right)^2\frac{V_k}{r^4}\right|^2$$
$$+ 4\left(\frac{k\pi}{\omega}\right)^2\left|3\frac{V_k''}{r^2} - \left(8+\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k'}{r^3} + \left(6+3\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k}{r^4}\right|^2$$
$$\left. + \left|3\frac{V_k''}{r^2} - \left(3+6\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k'}{r^3} + \left(\frac{k\pi}{\omega}\right)^2\left(8+\left(\frac{k\pi}{\omega}\right)^2\right)\frac{V_k}{r^4}\right|^2 \right) rdr$$

*Proof.* Follows, with the help of Lemma 1, from orthogonality of Fourier basis and Parseval's type identities. □

## 2.3 | Fundamental Lemmas for Fourier analysis

Let us denote by $\int_0^1 \left|\phi(r)\right|^2 rdr := \|\phi\|^2_{L^2(rdr)}$. The two following lemmas, which result from a variant of Hardy's type weighted inequalities, (G. Hardy 1927)[23], are essential for the estimation of the norms in $H^4(\Omega_\omega)$ of the Fourier modes of the regular and singular parts of the solution $u_\omega$ of problem (1). In fact, Fourier series method is efficient to obtain the decomposition of $u_\omega$ into its regular/singular part $u_\omega = u_{\omega,r} + u_{\omega,s}$ just by handling some critical powers of $r$ and balancing integral limits between those from 0 to $r$ and others from $r$ to 1 in order to isolate what is called the roots of a transcendent equation as obtained in the literature. In the following section, these roots are extracted directly by imposing the condition on the singular part $u_{\omega,s}$ to belong to the space $H^\sigma(\Omega_\omega)$ with $2 \leq \sigma < 4$. The two following lemmas are fundamental in the uniform estimates for the Fourier coefficients that will be given later.

**Lemma 3.** *For any $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(rdr)$, let $I_f^{\alpha,\beta}(r) := r^\alpha \int_0^r f(s)s^\beta ds$ defined for $r \in (0,1)$. If $\beta > 0$ and $\alpha + \beta \geq -1$ then $I_f^{\alpha,\beta} \in L^2(rdr)$ and we have:*

$$\|I_f^{\alpha,\beta}\|_{L^2(rdr)} \leq \frac{1}{2\sqrt{\beta\,(\alpha+\beta+1)}}\|f\|_{L^2(rdr)},\ \text{if}\ \alpha+\beta > -1, \tag{6}$$

$$\|I_f^{\alpha,\beta}\|_{L^2(rdr)} \leq \frac{1}{\beta}\|f\|_{L^2(rdr)},\ \text{if}\ \alpha+\beta = -1. \tag{7}$$

*Proof.* See[7]. □

**Lemma 4.** *For any $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(rdr)$, let $f \mapsto J_f^{\alpha,\beta}$ where $J_f^{\alpha,\beta}(r) := r^\alpha \int_r^1 f(s)s^\beta ds$ defined for $r \in (0,1)$. If $\beta < 0$ and $\alpha + \beta \geq -1$ then $J_f^{\alpha,\beta} \in L^2(rdr)$ and we have*

$$\|J_f^{\alpha,\beta}\|_{L^2(rdr)} \leq \frac{1}{2\sqrt{|\beta|\,(\alpha+\beta+1)}}\|f\|_{L^2(rdr)},\ \text{if}\ \alpha+\beta > -1, \tag{8}$$

$$\|J_f^{\alpha,\beta}\|_{L^2(rdr)} \leq \frac{1}{|\beta|}\|f\|_{L^2(rdr)},\ \text{if}\ \alpha+\beta = -1. \tag{9}$$

*Proof.* See[7]. □

*Remark 1.* As a consequence of optimality results of Hardy's inequalities, the estimates given by Lemmas 3 and 4 are optimal in the sens that one can not, for example, expect better that $1/|\beta|$ in the inequalities (7) and (9), in particular in the critical case when $\beta \to 0$.

**Lemma 5.** *Let $F \in L^2(\Omega_\omega)$ such that $F(r,\theta) := f(r)\sin\frac{\pi}{\omega}\theta$ in polar coordinates. Let us define:*

$$V(r,\theta) := \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r f(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r f(s)s^{1+\frac{\pi}{\omega}}ds\right)\sin\frac{\pi}{\omega}\theta = v(r)\sin\frac{\pi}{\omega}\theta,$$

*Then, $V \in H^1(\Omega_\omega)$, and there exists a constant $C > 0$ independent of $F$ and $\omega \in (\pi, 2\pi)$, such that:*

$$\sqrt{1-\frac{\pi}{\omega}}\|V\|_{H^1(\Omega_\omega)} \leq C\|F\|_{L^2(\Omega_\omega)}. \tag{10}$$

*Proof.* We have

$$\|\nabla V\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\left(\|v'\|^2_{L^2(rdr)} + \left(\frac{\pi}{\omega}\right)^2\left\|\frac{v}{r}\right\|^2_{L^2(rdr)}\right),$$

where

$$\frac{v(r)}{r} = \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}-1}\int_0^r f(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}-1}\int_0^r f(s)s^{1+\frac{\pi}{\omega}}ds\right)$$

$$v'(r) = \frac{1}{2}\left(r^{\frac{\pi}{\omega}-1}\int_0^r f(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}-1}\int_0^r f(s)s^{1+\frac{\pi}{\omega}}ds\right)$$

Thus, applying Lemma 3, Inequality (10) follows immediately. □

**Lemma 6.** *Let $c \in L^2(rdr)$ and $b$ defined by the following expression (assuming the four integrals exist):*

$$b(r) = r^{2-\frac{\pi}{\omega}}\int_0^r \frac{c(s)s^{1+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2-\frac{\pi}{\omega}\right)}ds + r^{2+\frac{\pi}{\omega}}\int_0^r \frac{c(s)s^{1-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2+\frac{\pi}{\omega}\right)}ds$$
$$- r^{\frac{\pi}{\omega}}\int_0^r \frac{c(s)s^{3-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2-\frac{\pi}{\omega}\right)}ds - r^{-\frac{\pi}{\omega}}\int_0^r \frac{c(s)s^{3+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2+\frac{\pi}{\omega}\right)}ds. \tag{11}$$

*Then, b can be written in terms of two integrals recursively as follows:*

$$b(r) = \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r v(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r v(s)s^{1+\frac{\pi}{\omega}}ds\right), \tag{12}$$

*where*

$$v(r) = \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r c(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r c(s)s^{1+\frac{\pi}{\omega}}ds\right). \tag{13}$$

*Proof.* Lemma 5 ensures that $v(r)\sin\frac{\pi}{\omega}\theta \in H^1(\Omega_\omega)$ ,i.e., $v'$ is $L^2(rdr)$. Starting from (13) and (12), replace the expression of $v$ given by(13) inside each integral of the r.h.s of (12) and use integration by part to obtain the equation given by (11). □

## 2.4 | Main result

After having introduced the framework of Sobolev's spaces and Fourier series in polar coordinates, we are now in position to state the main result of this paper. It should be pointed that, as in the previous work on the Laplace operator[7], only the first frequency in the Fourier series of the data $f_\omega$ is responsible of the lack of uniformity of estimates in the vicinity of $\pi$.

**Theorem 2** (Main Theorem)**.** *Let $\omega \in (\pi, 4\pi/3)$ and $f_\omega \in L^2(\Omega_\omega)$ with Fourier coefficients*

$$c_{k,\omega}(r) = \frac{2}{\omega}\int_0^\omega f_\omega(r,\theta)\sin\frac{k\pi}{\omega}\theta d\theta,\ k \geq 1.$$

*The family of solutions $(u_\omega)_{\omega\in(\pi,4\pi/3)}$ of Problem 1 satisfies the following:*

*i) $u_\omega \in H^{2+\sigma}(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ , for all $\sigma < 1 - \frac{\pi}{\omega}$, and $u_\omega$ admits in $\Omega_\omega$ the decomposition into regular and singular parts: $u_\omega = u_{\omega,r} + u_{\omega,s}$, where*

$$u_{\omega,r} = u^{(1)}_{\omega,r} + u^{(2)}_{\omega,r} + u^{(3)}_{\omega,r} + u^{[4]}_{\omega,r}, \tag{14}$$

$$u_{\omega,s} = \left(\lambda_{1,1}r^{2-\frac{\pi}{\omega}} + \lambda_{2,1}r^{2+\frac{\pi}{\omega}}\right)\sin\frac{\pi}{\omega}\theta + +\lambda_{3,2}r^{\frac{2\pi}{\omega}}\sin\frac{2\pi}{\omega}\theta + \lambda_{3,3}r^{\frac{3\pi}{\omega}}\sin\frac{3\pi}{\omega}\theta, \tag{15}$$

*$u^{(j)}_{\omega,r} \in H^4(\Omega_\omega)$, for $j = 1, 2, 3, 4$, are given by*

$$u^{(j)}_{\omega,r}(r,\theta) = b_{j,\omega}(r)\sin\frac{j\pi}{\omega}\theta, \text{ for } j = 1, 2, 3 \tag{16}$$

$$u^{[4]}_{\omega,r}(r,\theta) = \sum_{k\geq 4}\left(b_{k,\omega}(r) + A_{k,\omega}r^{2+\frac{k\pi}{\omega}} + B_{k,\omega}r^{\frac{k\pi}{\omega}}\right)\sin\frac{k\pi}{\omega}\theta \tag{17}$$

*where $b_{j,\omega}(r)$ ($j \geq 1$) , $\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,2}, \lambda_{3,3}$, and $A_{k,\omega}, B_{k,\omega}$ ($k \geq 4$) are (respectively) given by their explicit extraction formulas (27), (52), (71), (75), (29), (30), (54), (73), and (77).*

*ii) Moreover, there exists $C > 0$ independent of $\omega \in (\pi, 4\pi/3)$ and $f_\omega \in L^2(\Omega_\omega)$ such that the following estimate holds and is sharp:*

$$\sqrt{1-\frac{\pi}{\omega}}\left(\left|\lambda_{1,1}\right| + \left|\lambda_{2,1}\right| + \left\|u^{(1)}_{\omega,r}\right\|_{H^4(\Omega_\omega)}\right) \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{18}$$

$$\left|\lambda_{3,2}\right| + \left|\lambda_{3,3}\right| + \left\|u^{(2)}_{\omega,r}\right\|_{H^4(\Omega_\omega)} + \left\|u^{(3)}_{\omega,r}\right\|_{H^4(\Omega_\omega)} + \left\|u^{[4]}_{\omega,r}\right\|_{H^4(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{19}$$

## 3 | COEFFICIENTS OF SINGULARITIES REVISITED BY FOURIER COEFFICIENTS

Following the results in [12], we can summarize that a solution $u_\omega$ of (1) admits near the origin the following decomposition:

$$u_\omega = u_{\omega,r} + u_{\omega,s},$$

such that

$$u_{\omega,r} \in H^4_{loc}(\Omega_\omega), \text{ and } u_{\omega,s}(r,\theta) = \sum_{-2<\Im m\zeta_k<0} r^{1+i\zeta_k}\psi_k(\theta),$$

where the $\zeta_k$ are roots of the transcendent equation $\sinh^2(\zeta\omega) + \sin^2\omega = 0$ with imaginary part in $]-2, 0[$ and the $\psi_k$ are $C^\infty$ functions of $\theta$. In this section, we will give some results on the Fourier series method applied to this model case of bi-harmonic problem with Navier's boundary conditions on a family of planar non convex sectors. This method allows us to retrieve such a decomposition and to extract the roots systematically. Moreover, both regular and singular parts are given explicitly and explicit and sharp estimates w.r.t the opening angle $\omega$ are obtained in the vicinity of $\pi$.

Since singularities are caused by the geometry of the domain, it follows that they are found in the kernel of the bi-harmonic operator, i.e., they are solutions to the homogeneous equation

$$\Delta^2 u_{\omega,s} = 0 \text{ in } L^2(\Omega_\omega). \tag{20}$$

Let

$$u_{\omega,s}(r,\theta) = \sum_{k\geq 1} a_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta,$$

be the Fourier series of $u_{\omega,s}(r,\theta)$ in polar coordinates. Thus, one obtains, at least formally, by putting the Fourier coefficients of $\Delta^2 u_{\omega,s}$ all equal zero, that $a_{k,\omega}(r)$ is solution to the following differential equation, for all $k \geq 1$,

$$a^{(4)}_{k,\omega}(r) + \frac{2}{r}a^{(3)}_{k,\omega}(r) - \frac{1}{r^2}\left(1 + 2\left(\frac{k\pi}{\omega}\right)^2\right)a''_{k,\omega}(r) + \frac{1}{r^3}\left(1 + 2\left(\frac{k\pi}{\omega}\right)^2\right)a'_{k,\omega}(r) + \frac{1}{r^4}\left(\left(\frac{k\pi}{\omega}\right)^4 - 4\left(\frac{k\pi}{\omega}\right)^2\right)a_{k,\omega}(r) = 0, \tag{21}$$

whose general solution is given by

$$a_{k,\omega}(r) = \lambda_{1,k}r^{2-\frac{k\pi}{\omega}} + \lambda_{2,k}r^{2+\frac{k\pi}{\omega}} + \lambda_{3,k}r^{\frac{k\pi}{\omega}} + \lambda_{4,k}r^{-\frac{k\pi}{\omega}}, \tag{22}$$

where $\lambda_{j,k}$ are constants, $j = 1, 2, 3, 4$, that can be determined by imposing the condition on the singular part $u_{\omega,s}$ to belong to the space $H^\sigma(\Omega_\omega)$ with $2 \leq \sigma < 4$. As far as we know, the power function in $r$, $(r,\theta) \mapsto r^{\alpha_k}\sin\frac{k\pi}{\omega}\theta$, $\alpha_k$ not integer, belongs to $H^\sigma(\Omega_\omega)$ as long as $\sigma < \alpha_k + 1$. Henceforth, we look for the Fourier coefficient of $u_{\omega,s}$ that satisfy

$$a_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta \in H^\sigma(\Omega_\omega),\ 2 \leq \sigma < 4,$$

or, in other words, the problem turns out to find non-integer powers $\alpha_k$ in the r.h.s of (22) such that $1 < \alpha_k < 3$. Figure 2 illustrates regions plot for different powers $\alpha_k = 2 - \frac{k\pi}{\omega}, 2 + \frac{k\pi}{\omega}, -\frac{k\pi}{\omega}, \frac{k\pi}{\omega}$, respectively for $\omega \in (\pi, 2\pi)$. If we denote by $A_{\alpha_k}$ the set of integers $k$ such that $1 < \alpha_k < 3$, then we observe the following:

$$A_{-\frac{k\pi}{\omega}} = \emptyset, \text{ and } A_{2\pm\frac{k\pi}{\omega}} = \{1\}, \forall \omega \in (\pi, 2\pi),$$

$$A_{\frac{k\pi}{\omega}} = \begin{cases} \{2,3\} & \text{if } \omega \in (\pi, \frac{4\pi}{3}], \\ \{2,3,4\} & \text{if } \omega \in (\frac{4\pi}{3}, \frac{5\pi}{3}], \\ \{2,3,4,5\} & \text{if } \omega \in (\frac{5\pi}{3}, 2\pi). \end{cases}$$

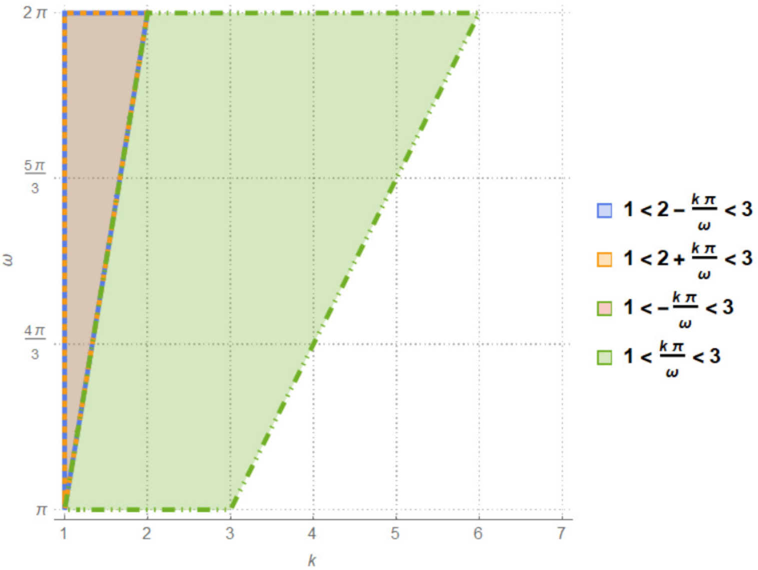


**FIGURE 2** Region Plot of $\alpha_k = 2 - \frac{k\pi}{\omega}, 2 + \frac{k\pi}{\omega}, -\frac{k\pi}{\omega}, +\frac{k\pi}{\omega}$ respectively for $\omega \in (\pi, 2\pi)$.

It follows immediately $\lambda_{4,k} = 0$ for all $k$ and that $a_{k,\omega}(r) = 0$ for all $k \geq 6$. More precisely, we conclude that the singular part of $u_\omega$ takes the following expression, in the three regions of $\omega$, denoted by $u^I_{\omega,s}, u^{II}_{\omega,s}, u^{III}_{\omega,s}$:

$$u_{\omega,s} = \begin{cases} \left(\lambda_{1,1} r^{2-\frac{\pi}{\omega}} + \lambda_{2,1} r^{2+\frac{\pi}{\omega}}\right) \sin \dfrac{\pi}{\omega}\theta & \\ +\lambda_{3,2} r^{\frac{2\pi}{\omega}} \sin \dfrac{2\pi}{\omega}\theta + \lambda_{3,3} r^{\frac{3\pi}{\omega}} \sin \dfrac{3\pi}{\omega}\theta = u^I_{\omega,s} & \text{if } \omega \in (\pi, \frac{4\pi}{3}], \\ u^I_{\omega,s} + \lambda_{3,4} r^{\frac{4\pi}{\omega}} \sin \dfrac{4\pi}{\omega}\theta = u^{II}_{\omega,s} & \text{if } \omega \in (\frac{4\pi}{3}, \frac{5\pi}{3}], \\ u^{II}_{\omega,s} + \lambda_{3,5} r^{\frac{5\pi}{\omega}} \sin \dfrac{5\pi}{\omega}\theta = u^{III}_{\omega,s} & \text{if } \omega \in (\frac{5\pi}{3}, 2\pi). \end{cases}$$

In the case $\omega \in (\pi, 2\pi)$, the constants $\lambda_{j,k}$, for $j = 1, 2, 3$ and $k = 1, ..., 5$, are called the coefficients of singularity (or stress intensity factors in the literature of mechanics). They can be determined by boundary conditions on the solution $u_\omega = u_{\omega,r} + u_{\omega,s}$ after having given the expression of the regular part $u_{\omega,r}$. As we will see later, they are linear forms of the r.h.s $f_\omega$ in the original problem (1). As far as we are interested in the asymptotic behavior of the solution in the vicinity of $\pi$, then let us consider only the first case $\omega \in (\pi, 4\pi/3)$ without considering the case $\omega = 4\pi/3$. In this case, we will state later explicit and sharp estimates of $\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,2}$ and $\lambda_{3,3}$ w.r.t the angle parameter $\omega$ in the vicinity of $\pi$. We will look for the regular part $u_{\omega,r}$ of $u_\omega$ as the particular solution of the problem (1) with the regularity property of being in $H^4(\Omega_\omega)$ for all $\omega \in (\pi, 4\pi/3)$. Actually, $u_{\omega,r}$ is solution to the non homogeneous equation $\Delta^2 u_{\omega,r} = f_\omega$ in $L^2(\Omega_\omega)$ such that the global solution $u_\omega \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$. Observe that one does not need homogeneous Dirichlet/Navier's boundary condition on the curved boundary $C_\omega$ as in the case of a convex corner[4]. We start by Fourier series decomposition of the r.h.s $f_\omega$ as follows:

$$f_\omega(r,\theta) = \sum_{k\geq 1} f_{k,\omega}(r,\theta) = \sum_{k\geq 1} c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta,\ c_{k,\omega}(r) = \frac{2}{\omega}\int_0^\omega f_\omega(r,\theta)\sin\frac{k\pi}{\omega}\theta d\theta \tag{23}$$

$$\sum_{k\geq 1}\int_0^1 c_{k,\omega}(r)^2 r dr = \|f_\omega\|^2_{L^2(\Omega_\omega)}(\text{Parseval's identity})$$

$$u_{\omega,r}(r,\theta) = \sum_{k\geq 1}\left(b_{k,\omega}(r) + A_{k,\omega}r^{2+\frac{k\pi}{\omega}} + B_{k,\omega}r^{\frac{k\pi}{\omega}}\right)\sin\frac{k\pi}{\omega}\theta, \tag{24}$$

where the coefficients $A_{k,\omega}, B_{k,\omega}$ are equal zero for $k = 1,2,3$, and they are not in the case $k \geq 4$. we can extract them with the help of the Navier's boundary condition on the global solution $u_\omega = \Delta u_\omega = 0$ on $\partial\Omega_\omega$. We have $u_{\omega,r} \in H^2(\Omega_\omega)$ since $u_{\omega,s}$ is also $H^2$ by construction. Plugging the Fourier series in the non homogeneous equation $\Delta^2 u_{\omega,r} = f_\omega$ in $L^2(\Omega_\omega)$, and since $A_{k,\omega}r^{2+\frac{k\pi}{\omega}} + B_{k,\omega}r^{\frac{k\pi}{\omega}}$ lies in the kernel of bi-harmonic operator ($\Delta^2$), we look for $b_{k,\omega}$ by identifying all the Fourier coefficients in the equation in polar coordinates. We obtain that $b_{k,\omega}(r)$ is solution to the following differential equation, for all $k \geq 1$,

$$\begin{aligned} b^{(4)}_{k,\omega}(r) + \frac{2}{r}b^{(3)}_{k,\omega}(r) - \frac{1}{r^2}\left(1 + 2\left(\frac{k\pi}{\omega}\right)^2\right)b''_{k,\omega}(r) \\ + \frac{1}{r^3}\left(1 + 2\left(\frac{k\pi}{\omega}\right)^2\right)b'_{k,\omega}(r) + \frac{1}{r^4}\left(\left(\frac{k\pi}{\omega}\right)^4 - 4\left(\frac{k\pi}{\omega}\right)^2\right)b_{k,\omega}(r) = c_{k,\omega}(r). \end{aligned} \tag{25}$$

The general form of a particular solution of this equation is:

$$\begin{aligned} b_{k,\omega}(r) = r^{2-\frac{k\pi}{\omega}}\int_a^r \frac{c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)}ds + r^{2+\frac{k\pi}{\omega}}\int_b^r \frac{c_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 + \frac{k\pi}{\omega}\right)}ds \\ - r^{\frac{k\pi}{\omega}}\int_c^r \frac{c_{k,\omega}(s)s^{3-\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)}ds - r^{-\frac{k\pi}{\omega}}\int_d^r \frac{c_{k,\omega}(s)s^{3+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 + \frac{k\pi}{\omega}\right)}ds, \end{aligned} \tag{26}$$

where $a, b, c, d$ are any constants in $(0,1)$. We will see later that they can be determined together with the coefficients of singularity $\lambda_{j,k}$ and the coefficients $A_{k,\omega}, B_{k,\omega}$ either with the help of boundary conditions on $\partial\Omega_\omega$ and/or the expected regularity of $u_{\omega,r}$ in $H^4$.

*Remark 2.* Integrals in the r.h.s of (26) have the same forms as those in fundamental lemmas 3 and 4 where the right powers $\alpha$ and left ones $\beta$ all satisfy the condition $\alpha + \beta \geq -1$ for all the derivatives w.r.t $r$ until order 4. Hence, the set of parameters $a, b, c, d$ can already be refined according the signs of $\beta$, i.e., the missing integral's limit will be 0 if $\beta > 0$ and 1 if $\beta < 0$.

## 3.1 Lowest frequency term, $k = 1$ and determination of $\lambda_{1,1}$ and $\lambda_{2,1}$

According to Remark 2, and the fact that $\frac{3}{4} < \frac{\pi}{\omega} < 1$, all the powers $\beta$ in the four integrals in expression of $b_{1,\omega}(r)$ given by (26) have positive sign, hence we write:

$$\begin{aligned} b_{1,\omega}(r) = r^{2-\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds + r^{2+\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds \\ - r^{\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds - r^{-\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds. \end{aligned} \tag{27}$$

It follows that the first term $U_{1,\omega}(r,\theta)$ in the Fourier series of the global solution

$$u_\omega = u_{\omega,r} + u_{\omega,s} = \sum_{k\geq 1} u_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta = \sum_{k\geq 1} U_{k,\omega}(r,\theta)$$

is obtained by (recall that $a_{1,\omega}(r) = \lambda_{1,1} r^{2-\frac{\pi}{\omega}} + \lambda_{2,1} r^{2+\frac{\pi}{\omega}}$ is the first Fourier coefficient of the singular part $u_{\omega,s}$):

$$U_{1,\omega}(r,\theta) = \left(b_{1,\omega}(r) + a_{1,\omega}(r)\right)\sin\frac{\pi}{\omega}\theta = \left( r^{2-\frac{\pi}{\omega}}\left(\lambda_{1,1} + \int_0^r \frac{c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds\right) + r^{2+\frac{\pi}{\omega}}\left(\lambda_{2,1} + \int_0^r \frac{c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds\right)\right.$$
$$\left. - r^{\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds - r^{-\frac{\pi}{\omega}}\int_0^r \frac{c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds\right)\sin\frac{\pi}{\omega}\theta \quad (28)$$

and applying the boundary conditions $u_\omega = \Delta u_\omega = 0$ on $C_\omega$ (i.e. at $r = 1$) term by term in the Fourier series expansion of $u_\omega$,one obtains for the lowest frequency $k = 1$, $U_{1,\omega} = \Delta U_{1,\omega} = 0$ at $r = 1$, which implies (we have two equations and two unknowns):

$$\lambda_{1,1} = \frac{\pi+\omega}{2\pi}\left(\int_0^1 \frac{c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds + \int_0^1 \frac{c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds\right) - \int_0^1 \frac{c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds \quad (29)$$

$$\lambda_{2,1} = \frac{\pi-\omega}{2\pi}\left(\int_0^1 \frac{c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 - \frac{\pi}{\omega}\right)}ds + \int_0^1 \frac{c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds\right) - \int_0^1 \frac{c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}}{8\left(\left(\frac{\pi}{\omega}\right)^2 + \frac{\pi}{\omega}\right)}ds \quad (30)$$

**Theorem 3.** *$U_{1,\omega}(r,\theta)$ given by (28) is solution of (1) with r.h.s $f_{1,\omega}(r,\theta) = c_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta$. Moreover, there exists $C > 0$ uniform in $\omega \in (\pi, 4\pi/3)$, such that:*

$$\sqrt{1-\frac{\pi}{\omega}}\left(\left|\lambda_{1,1}\right| + \left|\lambda_{2,1}\right| + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^3(\Omega_\omega)}\right) + \left\|\nabla^4\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)} \leq C\left\|f_{1,\omega}\right\|_{L^2(\Omega_\omega)}. \quad (31)$$

*Moreover, this estimate is sharp, i.e, there exists $f_{1,\omega}$ such that $\|f_{1,\omega}\|_{L^2(\Omega_\omega)} = 1$ and*

$$\sqrt{1-\frac{\pi}{\omega}}\left(\left|\lambda_{1,1}\right| + \left|\lambda_{2,1}\right| + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^3(\Omega_\omega)}\right) = O(1), \ as\ \omega \to \pi^+. \quad (32)$$

*Proof.* We will denote by $C > 0$ a generic constant uniform in $\omega$ which is not necessarily the same for all the inequalities which follow.

First of all, Lemma 6 allows one to rewrite $b_{1,\omega}(r)$ in terms of two integrals without the factor $1-\frac{\pi}{\omega}$ in the denominators, and Lemma 5 leads us directly to the estimate

$$\sqrt{1-\frac{\pi}{\omega}}\left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^1(\Omega_\omega)} \leq C\left\|f_{1,\omega}\right\|_{L^2(\Omega_\omega)}. \quad (33)$$

Second, looking at the expression of $\lambda_{2,1}$ given by (30), one concludes by Cauchy-Schwartz inequality that

$$\left|\lambda_{2,1}\right| \leq \frac{C}{\sqrt{1-\frac{\pi}{\omega}}}\left\|f_{1,\omega}\right\|_{L^2(\Omega_\omega)}, \quad (34)$$

which implies that

$$\sqrt{1-\frac{\pi}{\omega}}\left\|\lambda_{2,1} r^{2+\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} \leq C\left\|f_{1,\omega}\right\|_{L^2(\Omega_\omega)}. \quad (35)$$

Now, since the weak solution

$$U_{1,\omega}(r,\theta) = \left(b_{1,\omega}(r) + a_{1,\omega}(r)\right)\sin\frac{\pi}{\omega}\theta = \left(b_{1,\omega}(r) + \lambda_{1,1} r^{2-\frac{\pi}{\omega}} + \lambda_{2,1} r^{2+\frac{\pi}{\omega}}\right)\sin\frac{\pi}{\omega}\theta$$

is uniformly bounded in $H^2$ hence in $L^2$ , i.e.,

$$\|U_{1,\omega}\|_{L^2(\Omega_\omega)} \leq C\,\|f_{1,\omega}\|_{L^2(\Omega_\omega)}\,. \tag{36}$$

then one concludes that

$$\left\|\lambda_{1,1} r^{2-\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} \leq \left(\|U_{1,\omega}\|_{L^2(\Omega_\omega)} + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} + \left\|\lambda_{2,1} r^{2+\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)}\right),$$

henceforth, inequalities (33), (35) and (36) imply that

$$\sqrt{1-\frac{\pi}{\omega}}\left\|\lambda_{1,1} r^{2-\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} \leq C\,\|f_{1,\omega}\|_{L^2(\Omega_\omega)}\,,$$

thus

$$\sqrt{1-\frac{\pi}{\omega}}\,\left|\lambda_{1,1}\right| \leq C\,\|f_{1,\omega}\|_{L^2(\Omega_\omega)}\,. \tag{37}$$

Therefore, we have in hand the following estimate:

$$\sqrt{1-\frac{\pi}{\omega}}\left(\left|\lambda_{1,1}\right| + \left|\lambda_{2,1}\right| + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^1(\Omega_\omega)}\right) \leq C\,\|f_{1,\omega}\|_{L^2(\Omega_\omega)}\,. \tag{38}$$

Hence, to complete the proof, it then remains for us to prove the uniform estimates of derivatives of order 2, 3 and 4.

So, if we calculate the hessian $\nabla^2\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)$, we obtain a function in $H^2(\Omega_\omega)$ which vanishes at $\theta = 0$ or $\theta = \omega$, then, Poincare's inequality allows us to go directly to the estimation of the third differential $\nabla^3\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)$, since one has

$$\left\|\nabla^2\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)} \leq C\left\|\nabla^3\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)}. \tag{39}$$

By definition of Sobolev's semi-norms *via* Fourier coefficients, as given by Lemma 2:

$$\begin{aligned}\left\|\nabla^3\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\int_0^1 \Bigg(&\left|b'''_{1,\omega}\right|^2 + 3\left(\frac{\pi}{\omega}\right)^2\left|\frac{b''_{1,\omega}}{r} - \frac{2b'_{1,\omega}}{r^2} + \frac{2b_{1,\omega}}{r^3}\right|^2 \\ &+ 3\left|\frac{b''_{1,\omega}}{r} - \left(1+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^2} + 2\left(\frac{\pi}{\omega}\right)^2\frac{b_{1,\omega}}{r^3}\right|^2 + \left(\frac{\pi}{\omega}\right)^2\left|\frac{3b'_{1,\omega}}{r^2} - \left(2+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b_{1,\omega}}{r^3}\right|^2\Bigg)\,rdr,\end{aligned}$$

where, after some calculus and simplifications,

$$\begin{aligned} b'''_{1,\omega}(r) = &\frac{1}{8}\left(2-\frac{\pi}{\omega}\right) r^{-1-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s) s^{1+\frac{\pi}{\omega}}\,ds \\ &+ \frac{1}{8}\left(2+\frac{\pi}{\omega}\right) r^{-1+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s) s^{1-\frac{\pi}{\omega}}\,ds \\ &+ \frac{1}{8}\left(2-\frac{\pi}{\omega}\right) r^{-3+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s) s^{3-\frac{\pi}{\omega}}\,ds \\ &+ \frac{1}{8}\left(2+\frac{\pi}{\omega}\right) r^{-3-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s) s^{3+\frac{\pi}{\omega}}\,ds, \end{aligned} \tag{40}$$

$$
\begin{aligned}
\frac{b''_{1,\omega}}{r} - \frac{2b'_{1,\omega}}{r^2} + \frac{2b_{1,\omega}}{r^3} = {} & \frac{1}{8} r^{-1-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1+\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} r^{-1+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1-\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} \left( \frac{2\omega}{\pi} - 1 \right) r^{-3+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3-\frac{\pi}{\omega}} ds \\
& - \frac{1}{8} \left( 1 + \frac{2\omega}{\pi} \right) r^{-3-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3+\frac{\pi}{\omega}} ds,
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\frac{b''_{1,\omega}}{r} - \left(1 + \left(\frac{\pi}{\omega}\right)^2\right) \frac{b'_{1,\omega}}{r^2} + 2\left(\frac{\pi}{\omega}\right)^2 \frac{b_{1,\omega}}{r^3} = {} & \frac{1}{8} \left(2 + \frac{\pi}{\omega}\right) r^{-1-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1+\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} \left(2 - \frac{\pi}{\omega}\right) r^{-1+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1-\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} \left(\frac{\pi}{\omega} - 2\right) r^{-3+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3-\frac{\pi}{\omega}} ds \\
& - \frac{1}{8} \left(\frac{\pi}{\omega} + 2\right) r^{-3-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3+\frac{\pi}{\omega}} ds,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\frac{3b'_{1,\omega}}{r^2} - \left(2 + \left(\frac{\pi}{\omega}\right)^2\right) \frac{b_{1,\omega}}{r^3} = {} & -\frac{1}{8} \left(1 + \frac{4\omega}{\pi}\right) r^{-1-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1+\frac{\pi}{\omega}} ds \\
& - \frac{1}{8} \left(1 - \frac{4\omega}{\pi}\right) r^{-1+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{1-\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} \left(1 - \frac{2\omega}{\pi}\right) r^{-3+\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3-\frac{\pi}{\omega}} ds \\
& + \frac{1}{8} \left(1 + \frac{2\omega}{\pi}\right) r^{-3-\frac{\pi}{\omega}} \int_0^r c_{1,\omega}(s) s^{3+\frac{\pi}{\omega}} ds,
\end{aligned} \tag{43}
$$

so that all the right hand sides of (40)...(43) are linear combinations of integrals of type $I^{\alpha,\beta}_{c_{1,\omega}}(r)$ as defined by Lemma 3 with $\alpha + \beta = 0$, thus those with $\beta = 1 - \frac{\pi}{\omega}$ will induce the estimate

$$
\sqrt{1 - \frac{\pi}{\omega}} \left\| \nabla^3 \left( b_{1,\omega} \sin \frac{\pi}{\omega}\theta \right) \right\|_{L^2(\Omega_\omega)} \leq C \left\| f_{1,\omega} \right\|_{L^2(\Omega_\omega)} . \tag{44}
$$

Hence, combining this last estimate with other estimations (39) and (38), we have just proven the inequality

$$
\sqrt{1 - \frac{\pi}{\omega}} \left( \left| \lambda_{1,1} \right| + \left| \lambda_{2,1} \right| + \left\| b_{1,\omega} \sin \frac{\pi}{\omega}\theta \right\|_{H^3(\Omega_\omega)} \right) \leq C \left\| f_{1,\omega} \right\|_{L^2(\Omega_\omega)} . \tag{45}
$$

All we are left with now is the estimate of the fourth order differential $\nabla^4\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)$. So, by definition of Sobolev's semi-norms *via* Fourier coefficients given by Lemma 2, one has

$$\begin{aligned}\left\|\nabla^4\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\int_0^1 \Bigg( &\left|b^{(4)}_{1,\omega}\right|^2 + 4\left(\frac{\pi}{\omega}\right)^2\left|\frac{b'''_{1,\omega}}{r} - \frac{3b''_{1,\omega}}{r^2} + \frac{6b'_{1,\omega}}{r^3} - \frac{6b_{1,\omega}}{r^4}\right|^2 \\ &+ 6\left|\frac{b'''_{1,\omega}}{r} - \left(2+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b''_{1,\omega}}{r^2} + \left(2+4\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} - 6\left(\frac{\pi}{\omega}\right)^2\frac{b_{1,\omega}}{r^4}\right|^2 \\ &+ 4\left(\frac{\pi}{\omega}\right)^2\left|3\frac{b''_{1,\omega}}{r^2} - \left(8+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} + \left(6+3\left(\frac{\pi}{\omega}\right)^2\right)\frac{b_{1,\omega}}{r^4}\right|^2 \\ &+ \left|3\frac{b''_{1,\omega}}{r^2} - \left(3+6\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} + \left(\frac{\pi}{\omega}\right)^2\left(8+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b_{1,\omega}}{r^4}\right|^2\Bigg) rdr\end{aligned}$$

and, after some calculus and simplifications, one obtains

$$\begin{aligned}b^{(4)}_{1,\omega}(r) = &\frac{1}{8}\left(\frac{\pi}{\omega}+1\right)\left(\frac{\pi}{\omega}-2\right)r^{-2-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds \\ &+\frac{1}{8}\left(\frac{\pi}{\omega}-1\right)\left(\frac{\pi}{\omega}+2\right)r^{-2+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds \\ &-\frac{1}{8}\left(\frac{\pi}{\omega}-2\right)\left(\frac{\pi}{\omega}-3\right)r^{-4+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}ds \\ &-\frac{1}{8}\left(\frac{\pi}{\omega}+2\right)\left(\frac{\pi}{\omega}+3\right)r^{-4-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}ds \\ &+c_{1,\omega}(r)\end{aligned} \tag{46}$$

$$\begin{aligned}&\frac{b'''_{1,\omega}}{r} - \frac{3b''_{1,\omega}}{r^2} + \frac{6b'_{1,\omega}}{r^3} - \frac{6b_{1,\omega}}{r^4} \\ &= -\frac{1}{8}\left(\frac{\pi}{\omega}+1\right)r^{-2-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds \\ &+\frac{1}{8}\left(\frac{\pi}{\omega}-1\right)r^{-2+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds \\ &-\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}-2\right)\left(\frac{\pi}{\omega}-3\right)r^{-4+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}ds \\ &+\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}+2\right)\left(\frac{\pi}{\omega}+3\right)r^{-4-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}ds\end{aligned} \tag{47}$$

$$\begin{aligned}
&\frac{b'''_{1,\omega}}{r} - \left(2+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b''_{1,\omega}}{r^2} + \left(2+4\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} - 6\left(\frac{\pi}{\omega}\right)^2\frac{b_{1,\omega}}{r^4} \\
&= -\frac{1}{8}\left(\frac{\pi}{\omega}+1\right)\left(\frac{\pi}{\omega}+2\right) r^{-2-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds \\
&-\frac{1}{8}\left(\frac{\pi}{\omega}-1\right)\left(\frac{\pi}{\omega}-2\right) r^{-2+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds \\
&+\frac{1}{8}\left(\frac{\pi}{\omega}-3\right)\left(\frac{\pi}{\omega}-2\right) r^{-4+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}ds \\
&+\frac{1}{8}\left(\frac{\pi}{\omega}+3\right)\left(\frac{\pi}{\omega}+2\right) r^{-4-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}ds
\end{aligned} \tag{48}$$

$$\begin{aligned}
&3\frac{b''_{1,\omega}}{r^2} - \left(8+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} + \left(6+3\left(\frac{\pi}{\omega}\right)^2\right)\frac{b_{1,\omega}}{r^4} \\
&=\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}+1\right)\left(\frac{\pi}{\omega}+4\right) r^{-2-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds \\
&-\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}-1\right)\left(\frac{\pi}{\omega}-4\right) r^{-2+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds \\
&+\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}-2\right)\left(\frac{\pi}{\omega}-3\right) r^{-4+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}ds \\
&+\frac{1}{8}\frac{\omega}{\pi}\left(\frac{\pi}{\omega}+2\right)\left(\frac{\pi}{\omega}+3\right) r^{-4-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}ds
\end{aligned} \tag{49}$$

$$\begin{aligned}
&3\frac{b''_{1,\omega}}{r^2} - \left(3+6\left(\frac{\pi}{\omega}\right)^2\right)\frac{b'_{1,\omega}}{r^3} + \left(\frac{\pi}{\omega}\right)^2\left(8+\left(\frac{\pi}{\omega}\right)^2\right)\frac{b_{1,\omega}}{r^4} \\
&=\frac{1}{8}\left(\frac{\pi}{\omega}+1\right)\left(\frac{\pi}{\omega}+6\right) r^{-2-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds \\
&+\frac{1}{8}\left(\frac{\pi}{\omega}-1\right)\left(\frac{\pi}{\omega}-6\right) r^{-2+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds \\
&-\frac{1}{8}\left(\frac{\pi}{\omega}-2\right)\left(\frac{\pi}{\omega}-3\right) r^{-4+\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3-\frac{\pi}{\omega}}ds \\
&-\frac{1}{8}\left(\frac{\pi}{\omega}+2\right)\left(\frac{\pi}{\omega}+3\right) r^{-4-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{3+\frac{\pi}{\omega}}ds
\end{aligned} \tag{50}$$

so that all the right hand sides of (46)...(50) are linear combinations of integrals of type $I^{\alpha,\beta}_{c_{1,\omega}}(r)$ as defined by Lemma 3 with $\alpha+\beta=-1$. However, those with the bad power $\beta = 1-\dfrac{\pi}{\omega}$ are compensated by the same coefficient such that they will induce the uniform estimate

$$\left\|\nabla^4\left(b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)} \le C\,\|f_{1,\omega}\|_{L^2(\Omega_\omega)}. \tag{51}$$

Combining this last estimate with (38), we obtain the inequality (31).

**Proof of sharpness of estimate (31)**: Since $\frac{\pi}{\omega}<1$, one can find, for example, a r.h.s such as

$$f_{1,\omega}(r,\theta) = \frac{2}{\omega}\sqrt{\omega-\pi}\,r^{-\frac{\pi}{\omega}}\sin\left(\frac{\pi}{\omega}\theta\right),$$

$$\|f_{1,\omega}\|_{L^2(\Omega_\omega)} = 1,\ \forall\omega\in(\pi,2\pi),$$

$$c_{1,\omega}(r) = \frac{2}{\omega}\sqrt{\omega-\pi}\,r^{-\frac{\pi}{\omega}},$$

which gives after computation of $b_{1,\omega}(r), \lambda_{1,1}, \lambda_{2,1}$ from their expressions given respectively by (27), (29) and (30),

$$b_{1,\omega}(r) = \frac{\omega r^{4-\frac{\pi}{\omega}}}{16(2\omega-\pi)\sqrt{\omega-\pi}},$$

$$\lambda_{1,1} = \frac{\omega(\omega-2\pi)\sqrt{\omega-\pi}}{16\pi^2(\pi-2\omega)},$$

$$\lambda_{2,1} = \frac{\omega(3\omega-\pi)}{32\pi(\pi-2\omega)\sqrt{\omega-\pi}},$$

the asymptotic relation (32). The proof of the theorem is ended. □

## 3.2 | Second frequency term $k = 2$ and determination of $\lambda_{3,2}$

According to Remark 2, and the fact that $\frac{3}{4} < \frac{\pi}{\omega} < 1$, thus $1 - \frac{2\pi}{\omega} < 0$, and all the powers $\beta \neq 1 - \frac{2\pi}{\omega}$ in the other integrals in expression of $b_{2,\omega}(r)$ given by (26) have positive sign, hence the integral's limits are such that $a = c = d = 0$ and $b = 1$, i.e.

$$b_{2,\omega}(r) = r^{2-\frac{2\pi}{\omega}} \int_0^r \frac{c_{2,\omega}(s)s^{1+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds + r^{2+\frac{2\pi}{\omega}} \int_1^r \frac{c_{2,\omega}(s)s^{1-\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 + \frac{2\pi}{\omega}\right)} ds$$
$$- r^{\frac{2\pi}{\omega}} \int_0^r \frac{c_{2,\omega}(s)s^{3-\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds - r^{-\frac{2\pi}{\omega}} \int_0^r \frac{c_{2,\omega}(s)s^{3+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 + \frac{2\pi}{\omega}\right)} ds. \tag{52}$$

It follows that the second term $U_{2,\omega}(r,\theta)$ in the Fourier series of the global solution is obtained by (recall that $a_{2,\omega}(r) = \lambda_{3,2} r^{\frac{2\pi}{\omega}}$ is the second Fourier coefficient of the singular part $u_{\omega,s}$):

$$U_{2,\omega}(r,\theta) = \left(b_{2,\omega}(r) + a_{2,\omega}(r)\right) \sin\frac{2\pi}{\omega}\theta = \left(r^{2-\frac{2\pi}{\omega}} \int_0^r \frac{c_{2,\omega}(s)s^{1+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds + r^{2+\frac{2\pi}{\omega}} \int_1^r \frac{c_{2,\omega}(s)s^{1-\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 + \frac{2\pi}{\omega}\right)} ds\right.$$
$$\left. -r^{\frac{2\pi}{\omega}} \left(\lambda_{3,2} + \int_0^r \frac{c_{2,\omega}(s)s^{3-\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds\right) - r^{-\frac{2\pi}{\omega}} \int_0^r \frac{c_{2,\omega}(s)s^{3+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 + \frac{2\pi}{\omega}\right)} ds\right) \sin\frac{2\pi}{\omega}\theta \tag{53}$$

hence applying the boundary conditions, $U_{2,\omega} = \Delta U_{2,\omega} = 0$ at $r = 1$, one obtains

$$\lambda_{3,2} = \int_0^1 \frac{c_{2,\omega}(s)s^{1+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds - \int_0^1 \frac{c_{2,\omega}(s)s^{3+\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 + \frac{2\pi}{\omega}\right)} ds - \int_0^1 \frac{c_{2,\omega}(s)s^{3-\frac{2\pi}{\omega}}}{8\left(\left(\frac{2\pi}{\omega}\right)^2 - \frac{2\pi}{\omega}\right)} ds. \tag{54}$$

**Theorem 4.** *$U_{2,\omega}(r,\theta)$ given by (53) is solution of (1) with r.h.s $f_{2,\omega}(r,\theta) = c_{2,\omega}(r)\sin\frac{2\pi}{\omega}\theta$. Moreover, there exists $C > 0$ uniform in $\omega \in (\pi, 4\pi/3)$, such that:*

$$\left|\lambda_{3,2}\right| + \left\| b_{2,\omega} \sin\frac{2\pi}{\omega}\theta \right\|_{H^4(\Omega_\omega)} \leq C \left\|f_{2,\omega}\right\|_{L^2(\Omega_\omega)} \tag{55}$$

*Proof.* As in the previous section, let us denote by $C > 0$ a generic constant uniform in $\omega$ which is not necessarily the same for all the inequalities which follow.

First, looking at the expression of $\lambda_{3,2}$ given by (54), one concludes directly by Cauchy-Schwartz inequality that

$$\left|\lambda_{3,2}\right| \leq C \left\|f_{2,\omega}\right\|_{L^2(\Omega_\omega)}, \tag{56}$$

which implies also that

$$\left\| \lambda_{3,2} r^{\frac{2\pi}{\omega}} \sin\frac{2\pi}{\omega}\theta \right\|_{H^2(\Omega_\omega)} \leq C \left\|f_{2,\omega}\right\|_{L^2(\Omega_\omega)}. \tag{57}$$

But, since the variational solution

$$U_{2,\omega}(r,\theta) = \left(b_{2,\omega}(r) + a_{2,\omega}(r)\right) \sin\frac{2\pi}{\omega}\theta = \left(b_{2,\omega}(r) + \lambda_{3,2} r^{\frac{2\pi}{\omega}}\right) \sin\frac{2\pi}{\omega}\theta$$

is uniformly bounded in $H^2$, i.e.,

$$\|U_{2,\omega}\|_{H^2(\Omega_\omega)} \leq C\,\|f_{2,\omega}\|_{L^2(\Omega_\omega)}\,. \tag{58}$$

then, by a triangular inequality, one has

$$\left\|b_{2,\omega}\sin\frac{2\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} \leq \left(\|U_{2,\omega}\|_{H^2(\Omega_\omega)} + \left\|\lambda_{3,2}r^{\frac{2\pi}{\omega}}\sin\frac{2\pi}{\omega}\theta\right\|_{H^2(\Omega_\omega)}\right),$$

henceforth, we obtain in hand the following estimate (in $H^2$):

$$\left|\lambda_{3,2}\right| + \left\|b_{2,\omega}\sin\frac{2\pi}{\omega}\theta\right\|_{H^2(\Omega_\omega)} \leq C\,\|f_{2,\omega}\|_{L^2(\Omega_\omega)} \tag{59}$$

Hence, to complete the proof, it then remains for us to prove the uniform estimates of derivatives of order 3 and 4.

By definition of Sobolev's semi-norms *via* Fourier coefficients, as given by Lemma 2, one has:

$$\begin{aligned}\left\|\nabla^3\left(b_{2,\omega}\sin\frac{2\pi}{\omega}\theta\right)\right\|^2_{L^2(\Omega_\omega)} &= \frac{\omega}{2}\int_0^1\left(\left|b'''_{2,\omega}\right|^2 + 3\left(\frac{2\pi}{\omega}\right)^2\left|\frac{b''_{2,\omega}}{r} - \frac{2b'_{2,\omega}}{r^2} + \frac{2b_{2,\omega}}{r^3}\right|^2\right.\\
&\left. + 3\left|\frac{b''_{2,\omega}}{r} - \left(1+\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^2} + 2\left(\frac{2\pi}{\omega}\right)^2\frac{b_{2,\omega}}{r^3}\right|^2 + \left(\frac{2\pi}{\omega}\right)^2\left|\frac{3b'_{2,\omega}}{r^2} - \left(2+\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^3}\right|^2\right) rdr,\end{aligned}$$

where, after some calculus and simplifications,

$$\begin{aligned} b'''_{2,\omega} &= \frac{1}{8}\left(2-\frac{2\pi}{\omega}\right)r^{-1-\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{1+\frac{2\pi}{\omega}}\,ds\\
&+\frac{1}{8}\left(2+\frac{2\pi}{\omega}\right)r^{-1+\frac{2\pi}{\omega}}\int_1^r c_{2,\omega}(s)s^{1-\frac{2\pi}{\omega}}\,ds\\
&+\frac{1}{8}\left(2-\frac{2\pi}{\omega}\right)r^{-3+\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{3-\frac{2\pi}{\omega}}\,ds\\
&+\frac{1}{8}\left(2+\frac{2\pi}{\omega}\right)r^{-3-\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{3+\frac{2\pi}{\omega}}\,ds\end{aligned} \tag{60}$$

$$\begin{aligned}&\frac{b''_{2,\omega}}{r} - \frac{2b'_{2,\omega}}{r^2} + \frac{2b_{2,\omega}}{r^3} =\\
&\frac{1}{8}r^{-1-\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{1+\frac{2\pi}{\omega}}\,ds\\
&+\frac{1}{8}r^{-1+\frac{2\pi}{\omega}}\int_1^r c_{2,\omega}(s)s^{1-\frac{2\pi}{\omega}}\,ds\\
&+\frac{1}{8}\left(\frac{2\omega}{2\pi}-1\right)r^{-3+\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{3-\frac{2\pi}{\omega}}\,ds\\
&-\frac{1}{8}\left(1+\frac{2\omega}{2\pi}\right)r^{-3-\frac{2\pi}{\omega}}\int_0^r c_{2,\omega}(s)s^{3+\frac{2\pi}{\omega}}\,ds\end{aligned} \tag{61}$$

$$
\begin{aligned}
&\frac{b''_{2,\omega}}{r} - \left(1 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^2} + 2\left(\frac{2\pi}{\omega}\right)^2\frac{b_{2,\omega}}{r^3} = \\
&\frac{1}{8}\left(2 + \frac{2\pi}{\omega}\right) r^{-1-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}} ds \\
&+ \frac{1}{8}\left(2 - \frac{2\pi}{\omega}\right) r^{-1+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}} ds \\
&+ \frac{1}{8}\left(\frac{2\pi}{\omega} - 2\right) r^{-3+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}} ds \\
&- \frac{1}{8}\left(\frac{2\pi}{\omega} + 2\right) r^{-3-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}} ds
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
&\frac{3b'_{2,\omega}}{r^2} - \left(2 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^3} = \\
&- \frac{1}{8}\left(1 + \frac{4\omega}{2\pi}\right) r^{-1-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}} ds \\
&- \frac{1}{8}\left(1 - \frac{4\omega}{2\pi}\right) r^{-1+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}} ds \\
&+ \frac{1}{8}\left(1 - \frac{2\omega}{2\pi}\right) r^{-3+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}} ds \\
&+ \frac{1}{8}\left(1 + \frac{2\omega}{2\pi}\right) r^{-3-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}} ds
\end{aligned} \tag{63}
$$

so that all the right hand sides of (60)...(63) are linear combinations of integrals of type $I^{\alpha,\beta}_{c_{2,\omega}}(r)$ as defined by Lemma 3 or $J^{\alpha,\beta}_{c_{2,\omega}}(r)$ as defined by Lemma 9, with $\alpha + \beta = 0$, thus the following estimate holds:

$$
\left\| \nabla^3 \left( b_{2,\omega} \sin \frac{2\pi}{\omega}\theta \right) \right\|_{L^2(\Omega_\omega)} \le C \left\| f_{2,\omega} \right\|_{L^2(\Omega_\omega)}. \tag{64}
$$

Similarly, with the help of Lemma 2, one has

$$
\begin{aligned}
\left\| \nabla^4 \left( b_{2,\omega} \sin \frac{2\pi}{\omega}\theta \right) \right\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\int_0^1 \Bigg( & \left| b^{(4)}_{2,\omega} \right|^2 + 4\left(\frac{2\pi}{\omega}\right)^2 \left| \frac{b'''_{2,\omega}}{r} - \frac{3b''_{2,\omega}}{r^2} + \frac{6b'_{2,\omega}}{r^3} - \frac{6b_{2,\omega}}{r^4} \right|^2 \\
&+ 6\left| \frac{b'''_{2,\omega}}{r} - \left(2 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b''_{2,\omega}}{r^2} + \left(2 + 4\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^3} - 6\left(\frac{2\pi}{\omega}\right)^2\frac{b_{2,\omega}}{r^4} \right|^2 \\
&+ 4\left(\frac{2\pi}{\omega}\right)^2 \left| 3\frac{b''_{2,\omega}}{r^2} - \left(8 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^3} + \left(6 + 3\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^4} \right|^2 \\
&+ \left| 3\frac{b''_{2,\omega}}{r^2} - \left(3 + 6\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^3} + \left(\frac{2\pi}{\omega}\right)^2\left(8 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^4} \right|^2 \Bigg) rdr
\end{aligned}
$$

and, after some calculus and simplifications, one obtains

$$\begin{aligned}
b_{2,\omega}^{(4)} = {} & \frac{1}{8}\left(\frac{2\pi}{\omega}+1\right)\left(\frac{2\pi}{\omega}-2\right) r^{-2-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}}\, ds \\
& + \frac{1}{8}\left(\frac{2\pi}{\omega}-1\right)\left(\frac{2\pi}{\omega}+2\right) r^{-2+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}}\, ds \\
& - \frac{1}{8}\left(\frac{2\pi}{\omega}-2\right)\left(\frac{2\pi}{\omega}-3\right) r^{-4+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}}\, ds \\
& - \frac{1}{8}\left(\frac{2\pi}{\omega}+2\right)\left(\frac{2\pi}{\omega}+3\right) r^{-4-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}}\, ds \\
& + c_{2,\omega}(r)
\end{aligned} \tag{65}$$

$$\begin{aligned}
& \frac{b_{2,\omega}'''}{r} - \frac{3b_{2,\omega}''}{r^2} + \frac{6b_{2,\omega}'}{r^3} - \frac{6b_{2,\omega}}{r^4} = \\
& - \frac{1}{8}\left(\frac{2\pi}{\omega}+1\right) r^{-2-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}}\, ds \\
& + \frac{1}{8}\left(\frac{2\pi}{\omega}-1\right) r^{-2+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}}\, ds \\
& - \frac{1}{8}\frac{\omega}{2\pi}\left(\frac{2\pi}{\omega}+1\right)\left(\frac{2\pi}{\omega}-6\right) r^{-4+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}}\, ds \\
& + \frac{1}{8}\frac{\omega}{2\pi}\left(\frac{2\pi}{\omega}+2\right)\left(\frac{2\pi}{\omega}+3\right) r^{-4-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}}\, ds
\end{aligned} \tag{66}$$

$$\begin{aligned}
& \frac{b_{2,\omega}'''}{r} - \left(2+\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}''}{r^2} + \left(2+4\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}'}{r^3} - 6\left(\frac{2\pi}{\omega}\right)^2\frac{b_{2,\omega}}{r^4} = \\
& - \frac{1}{8}\left(\frac{2\pi}{\omega}+1\right)\left(\frac{2\pi}{\omega}+2\right) r^{-2-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}}\, ds \\
& - \frac{1}{8}\left(\frac{2\pi}{\omega}-1\right)\left(\frac{2\pi}{\omega}-2\right) r^{-2+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}}\, ds \\
& + \frac{1}{8}\left(\frac{2\pi}{\omega}-3\right)\left(\frac{2\pi}{\omega}-2\right) r^{-4+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}}\, ds \\
& + \frac{1}{8}\left(\frac{2\pi}{\omega}+3\right)\left(\frac{2\pi}{\omega}+2\right) r^{-4-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}}\, ds
\end{aligned} \tag{67}$$

$$3\frac{b''_{2,\omega}}{r^2} - \left(8 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^3} + \left(6 + 3\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^4} = \tag{68}$$
$$+\frac{1}{8}\left(\frac{2\pi}{\omega}+1\right)\left(\frac{2\pi}{\omega}+6\right) r^{-2-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}} ds$$
$$+\frac{1}{8}\left(\frac{2\pi}{\omega}-1\right)\left(\frac{2\pi}{\omega}-6\right) r^{-2+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}} ds$$
$$+\frac{1}{8}\left(\frac{2\pi}{\omega}-3\right)\left(\frac{2\pi}{\omega}-2\right) r^{-4+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}} ds$$
$$-\frac{1}{8}\left(\frac{2\pi}{\omega}+2\right)\left(\frac{2\pi}{\omega}+3\right) r^{-4-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}} ds$$

$$3\frac{b''_{2,\omega}}{r^2} - \left(3 + 6\left(\frac{2\pi}{\omega}\right)^2\right)\frac{b'_{2,\omega}}{r^3} + \left(\frac{2\pi}{\omega}\right)^2\left(8 + \left(\frac{2\pi}{\omega}\right)^2\right)\frac{b_{2,\omega}}{r^4} = \tag{69}$$
$$+\frac{1}{8}\frac{\omega}{2\pi}\left(\frac{2\pi}{\omega}+1\right)\left(\frac{2\pi}{\omega}+4\right) r^{-2-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{1+\frac{2\pi}{\omega}} ds$$
$$+\frac{1}{8}\frac{\omega}{2\pi}\left(\frac{2\pi}{\omega}-1\right)\left(\frac{2\pi}{\omega}-4\right) r^{-2+\frac{2\pi}{\omega}} \int_1^r c_{2,\omega}(s) s^{1-\frac{2\pi}{\omega}} ds$$
$$-\frac{1}{8}\left(\frac{2\pi}{\omega}-3\right)\left(\frac{2\pi}{\omega}-2\right) r^{-4+\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3-\frac{2\pi}{\omega}} ds$$
$$-\frac{1}{8}\frac{\omega}{2\pi}\left(\frac{2\pi}{\omega}+2\right)\left(\frac{2\pi}{\omega}+3\right) r^{-4-\frac{2\pi}{\omega}} \int_0^r c_{2,\omega}(s) s^{3+\frac{2\pi}{\omega}} ds$$

so that all the right hand sides of (65)...(69) are linear combinations of $c_{2,\omega}$ and of integrals of type $I^{\alpha,\beta}_{c_{2,\omega}}(r)$ as defined by Lemma 3 or $J^{\alpha,\beta}_{c_{2,\omega}}(r)$ as defined by Lemma 9, with $\alpha + \beta = -1$, thus the following estimate holds:

$$\left\|\nabla^4\left(b_{2,\omega}\sin\frac{2\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)} \leq C\left\|f_{2,\omega}\right\|_{L^2(\Omega_\omega)}. \tag{70}$$

Finally, combining this last estimate with (64) and (59), we obtain the inequality (55). The proof of the theorem is achieved. □

*Remark 3.* Note that here we could have the same problem in the neighborhood of $2\pi$ as that encountered in the neighborhood of $\pi$ in the case $k = 1$. However, as we assumed that $\omega$ is in $(\pi, 4\pi/3)$ then the problem does not arise.

## 3.3 | Third frequency term $k = 3$ and determination of $\lambda_{3,3}$

According to Remark 2, and the fact that $\frac{3}{4} < \frac{\pi}{\omega} < 1$, thus $1 - \frac{3\pi}{\omega} < 0$, and all the powers $\beta \neq 1 - \frac{3\pi}{\omega}$ in the other integrals in expression of $b_{3,\omega}(r)$ given by (26) have positive sign, hence the integral's limits are the same as in the case $k = 2$, i.e., $a = c = d = 0$ and $b = 1$, and we have

$$b_{3,\omega}(r) = r^{2-\frac{3\pi}{\omega}} \int_0^r \frac{c_{3,\omega}(s)s^{1+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds + r^{2+\frac{3\pi}{\omega}} \int_1^r \frac{c_{3,\omega}(s)s^{1-\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 + \frac{3\pi}{\omega}\right)} ds \tag{71}$$
$$- r^{\frac{3\pi}{\omega}} \int_0^r \frac{c_{3,\omega}(s)s^{3-\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds - r^{-\frac{3\pi}{\omega}} \int_0^r \frac{c_{3,\omega}(s)s^{3+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 + \frac{3\pi}{\omega}\right)} ds.$$

It follows that the third term $U_{3,\omega}(r,\theta)$ in the Fourier series of the global solution is obtained by (recall that $a_{3,\omega}(r) = \lambda_{3,3} r^{\frac{3\pi}{\omega}}$ is the third Fourier coefficient of the singular part $u_{\omega,s}$):

$$U_{3,\omega}(r,\theta) = \big(b_{3,\omega}(r) + a_{3,\omega}(r)\big)\sin\frac{3\pi}{\omega}\theta = \left( r^{2-\frac{3\pi}{\omega}} \int_0^r \frac{c_{3,\omega}(s) s^{1+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds + r^{2+\frac{3\pi}{\omega}} \int_1^r \frac{c_{3,\omega}(s) s^{1-\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 + \frac{3\pi}{\omega}\right)} ds \right.$$
$$\left. - r^{\frac{3\pi}{\omega}} \left( \lambda_{3,3} + \int_0^r \frac{c_{3,\omega}(s) s^{3-\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds \right) - r^{-\frac{3\pi}{\omega}} \int_0^r \frac{c_{3,\omega}(s) s^{3+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 + \frac{3\pi}{\omega}\right)} ds \right) \sin\frac{3\pi}{\omega}\theta \quad (72)$$

hence applying the boundary conditions, $U_{3,\omega} = \Delta U_{3,\omega} = 0$ at $r = 1$, one obtains

$$\lambda_{3,3} = \int_0^1 \frac{c_{3,\omega}(s) s^{1+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds - \int_0^1 \frac{c_{3,\omega}(s) s^{3+\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 + \frac{3\pi}{\omega}\right)} ds - \int_0^1 \frac{c_{3,\omega}(s) s^{3-\frac{3\pi}{\omega}}}{8\left(\left(\frac{3\pi}{\omega}\right)^2 - \frac{3\pi}{\omega}\right)} ds. \quad (73)$$

**Theorem 5.** *$U_{3,\omega}(r,\theta)$ given by (72) is solution of (1) with r.h.s $f_{3,\omega}(r,\theta) = c_{3,\omega}(r)\sin\frac{3\pi}{\omega}\theta$. Moreover, there exists $C > 0$ uniform in $\omega \in (\pi, 4\pi/3)$, such that:*

$$\left|\lambda_{3,3}\right| + \left\| b_{3,\omega} \sin\frac{3\pi}{\omega}\theta \right\|_{H^4(\Omega_\omega)} \le C \left\| f_{3,\omega} \right\|_{L^2(\Omega_\omega)}. \quad (74)$$

*Proof.* Similar to the proof Theorem 4. □

## 3.4 | Regular frequency terms $k \ge 4$ and determination of $A_{k,\omega}$ and $B_{k,\omega}$

According to Remark 2, and the fact that $\frac{3}{4} < \frac{\pi}{\omega} < 1$ and $k \ge 4$, there are now two negative powers $\beta = 1 - \frac{k\pi}{\omega}$ and $\beta = 3 - \frac{k\pi}{\omega}$, and the two other powers in the expression of $b_{k,\omega}(r)$ given by (26) have positive sign. Hence, the integral's limits defining $b_{k,\omega}$ are such that $a = d = 0$ and $b = c = 1$, and we have

$$b_{k,\omega}(r) = r^{2-\frac{k\pi}{\omega}} \int_0^r \frac{c_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)} ds + r^{2+\frac{k\pi}{\omega}} \int_1^r \frac{c_{k,\omega}(s) s^{1-\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 + \frac{k\pi}{\omega}\right)} ds \quad (75)$$
$$- r^{\frac{k\pi}{\omega}} \int_1^r \frac{c_{k,\omega}(s) s^{3-\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)} ds - r^{-\frac{k\pi}{\omega}} \int_0^r \frac{c_{k,\omega}(s) s^{3+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 + \frac{k\pi}{\omega}\right)} ds.$$

It follows that the $k^{th}$– term $U_{k,\omega}(r,\theta)$ in the Fourier series of the global solution can be written as follows,

$$U_{k,\omega}(r,\theta) = \left( b_{k,\omega}(r) + A_{k,\omega} r^{2+\frac{k\pi}{\omega}} + B_{k,\omega} r^{\frac{k\pi}{\omega}} \right) \sin\frac{k\pi}{\omega}\theta \quad (76)$$

where the constants $A_{k,\omega}$ and $B_{k,\omega}$ are determined by the Dirichlet conditions $U_{k,\omega} = \Delta U_{k,\omega} = 0$ at $r = 1$. Solving these two equations, one obtains:

$$\begin{cases} A_{k,\omega} = \dfrac{k\pi - \omega}{k\pi + \omega} \int_0^1 \dfrac{c_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)} ds, \\ B_{k,\omega} = \int_0^1 \dfrac{c_{k,\omega}(s) s^{3+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 + \frac{k\pi}{\omega}\right)} ds - \dfrac{2k\pi}{k\pi + \omega} \int_0^1 \dfrac{c_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}}{8\left(\left(\frac{k\pi}{\omega}\right)^2 - \frac{k\pi}{\omega}\right)} ds. \end{cases} \quad (77)$$

Instead of being singularity coefficients as in the previous cases, we recall here that the coefficients $A_{k,\omega}$ and $B_{k,\omega}$ participate rather in the regular part as coefficients of $r$ power $2 + \frac{k\pi}{\omega}$ and $\frac{k\pi}{\omega}$ which are both greater than 3 .

**Theorem 6.** *For any integer $k \ge 4$, $U_{k,\omega}(r,\theta)$ given by (76) is solution of (1) with r.h.s $f_{k,\omega}(r,\theta) = c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta$. Moreover, there exists $C_k > 0$ uniform in $\omega \in (\pi, 4\pi/3)$ such that:*

$$\left\| U_{k,\omega} \right\|_{H^4(\Omega_\omega)} \le C_k \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)}. \quad (78)$$

*As a result, the single Fourier coefficient of $U_{k,\omega}$ which is defined by*

$$d_{k,\omega}(r) := b_{k,\omega}(r) + A_{k,\omega} r^{2+\frac{k\pi}{\omega}} + B_{k,\omega} r^{\frac{k\pi}{\omega}}$$

*can be obtained recursively as follows:*

$$g_{k,\omega}(r) = \frac{\omega}{2k\pi}\left( r^{\frac{k\pi}{\omega}} \left( \int_1^r c_{k,\omega}(s) s^{1-\frac{k\pi}{\omega}}\, ds + \int_0^1 c_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}\, ds \right) - r^{-\frac{k\pi}{\omega}} \int_0^r c_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}\, ds \right), \tag{79}$$

*where*

$$d_{k,\omega}(r) = \frac{\omega}{2k\pi}\left( r^{\frac{k\pi}{\omega}} \left( \int_1^r g_{k,\omega}(s) s^{1-\frac{k\pi}{\omega}}\, ds + \int_0^1 g_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}\, ds \right) - r^{-\frac{k\pi}{\omega}} \int_0^r g_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}\, ds \right). \tag{80}$$

*Proof.* Using the fact that

$$U_{k,\omega}(r,\theta) = b_{k,\omega}(r) \sin\frac{k\pi}{\omega}\theta + \left( A_{k,\omega} r^{2+\frac{k\pi}{\omega}} + B_{k,\omega} r^{\frac{k\pi}{\omega}} \right) \sin\frac{k\pi}{\omega}\theta, \tag{81}$$

one can already prove, following the same argument as in the proof of Theorem 4 by using properly lemmas 3 and 4, that

$$\left\| b_{k,\omega}(r) \sin\frac{k\pi}{\omega}\theta \right\|_{H^4(\Omega_\omega)} \leq C_k^{'} \|f_{k,\omega}\|_{L^2(\Omega_\omega)}.$$

In addition, since powers $2 + \frac{k\pi}{\omega}$ and $\frac{k\pi}{\omega}$ are both greater than 3 , it is straightforward that the same estimate holds for the right term in the r.h.s of (81), i.e.,

$$\left\| \left( A_{k,\omega} r^{2+\frac{k\pi}{\omega}} + B_{k,\omega} r^{\frac{k\pi}{\omega}} \right) \sin\frac{k\pi}{\omega}\theta \right\|_{H^4(\Omega_\omega)} \leq C_k^{''} \|f_{k,\omega}\|_{L^2(\Omega_\omega)}.$$

Henceforth, putting $C_k = \max(C_k^{'}, C_k^{''})$, we obtain Inequality (78).

Next, one can check by a simple calculus that $g_{k,\omega}(r)$ is nothing but the single Fourier coefficient of $V_{k,\omega} = \Delta U_{k,\omega}$ the laplacian of $U_{k,\omega}$. Henceforth, $V_{k,\omega} \in H^2(\Omega_\omega)$ and consequently on can use integration by parts to obtain the recursive formula (79)-(80) and the proof of the theorem is ended. □

*Remark 4.* The proof of the recursive formula (79)-(80) in the previous theorem can also be obtained by solving, in polar coordinates, the boundary value problem $V_{k,\omega} = \Delta U_{k,\omega}$ in $\Omega_\omega$ with $U_{k,\omega} = 0$ on the boundary $\partial\Omega_\omega$, which leads to a second order differential equation in terms of Fourier coefficients:

$$d_{k,\omega}^{''}(r) + \frac{1}{r} d_{k,\omega}^{'}(r) - \frac{k^2\pi^2}{\omega^2}\frac{d_{k,\omega}(r)}{r^2} = g_{k,\omega}(r),$$

whose general solution is given by

$$d_{k,\omega}(r) = \frac{\omega}{2k\pi}\left( r^{\frac{k\pi}{\omega}} \int_1^r g_{k,\omega}(s) s^{1-\frac{k\pi}{\omega}}\, ds - r^{-\frac{k\pi}{\omega}} \int_0^r g_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}}\, ds \right) + C_1 r^{\frac{k\pi}{\omega}} + C_2 r^{-\frac{k\pi}{\omega}},$$

where $C_2$ must be equal 0 since $d_{k,\omega}$ is at least $L^2(rdr)$. The constant $C_1$ is found by the boundary condition at $r = 1$, $d_{k,\omega}(1) = 0$. We find thus exactly the formula(79).

**Lemma 7.** *Let $v_\omega \in H^2(\Omega_\omega)$ a periodic distribution such that $v_\omega(.,0) = v_\omega(.,\omega) = 0$ and let $v_{k,\omega}(r)$ its $k^{th}-$ partial Fourier coefficient w.r.t $\theta$. There exist a constant $C > 0$ independent of $\omega \in (\pi, 2\pi)$ and of $v_\omega$ such that:*

$$\sum_{k\geq 4}\left( \left\| v_{k,\omega}^{''} \right\|_{L^2(rdr)} + k \left\| \frac{v_{k,\omega}^{'}}{r} \right\|_{L^2(rdr)} + k^2 \left\| \frac{v_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \right) \leq C\|\nabla^2 v_\omega\|_{L^2(\Omega_\omega)}.$$

*Proof.* A direct consequence of the the following straightforward identities (since $k\pi/\omega \neq 1$ for $k \geq 4$)

$$\frac{v_{k,\omega}'}{r} = \frac{\left( \frac{V_{k,\omega}'}{r} - \left(\frac{k\pi}{\omega}\right)^2 \frac{V_{k,\omega}}{r^2} \right) - \left(\frac{k\pi}{\omega}\right)^2 \left( \frac{V_{k,\omega}'}{r} - \frac{V_{k,\omega}}{r^2} \right)}{1 - \left(\frac{k\pi}{\omega}\right)^2},$$

$$\frac{v_{k,\omega}}{r^2} = \frac{\left(\frac{V'_{k,\omega}}{r} - \left(\frac{k\pi}{\omega}\right)^2 \frac{V_{k,\omega}}{r^2}\right) - \left(\frac{V'_{k,\omega}}{r} - \frac{V_{k,\omega}}{r^2}\right)}{1 - \left(\frac{k\pi}{\omega}\right)^2},$$

and the expression of the semi-norm $\|\nabla^2 v_\omega\|_{L^2(\Omega_\omega)}$ as given by Lemma 2, lead to:

$$\sum_{k\geq 4}\left(\int_0^1 \left|v''_{k,\omega}(r)\right|^2 rdr + k^2\int_0^1 \left|\frac{v'_{k,\omega}(r)}{r}\right|^2 rdr + k^4\int_0^1 \left|\frac{v_{k,\omega}(r)}{r^2}\right|^2 rdr\right) \leq C\|\nabla^2 v_\omega\|^2_{L^2(\Omega_\omega)},$$

with a constant $C > 0$ independent of $\omega \in (\pi, 2\pi)$, which allows one to conclude the proof. □

We have now the main theorem for this section which gives normal convergence in $H^4$ of the series $\sum_{k\geq 4} U_{k,\omega}(r,\theta)$:

**Theorem 7.** *Let $\omega \in (\pi, 4\pi/3)$. For any integer $k \geq 4$, let $U_{k,\omega}(r,\theta)$ defined as in Theorem 6. Then the series*

$$U_\omega(r,\theta) := \sum_{k\geq 4} U_{k,\omega}(r,\theta)$$

*is solution of (1) with r.h.s*

$$f_\omega(r,\theta) := \sum_{k\geq 4} c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta$$

*Moreover, there exists $C > 0$ uniform in $\omega \in (\pi, 4\pi/3)$ such that:*

$$\|U_\omega\|_{H^4(\Omega_\omega)} \leq C\,\|f_\omega\|_{L^2(\Omega_\omega)}.$$

*Proof.* By Theorem 6, we know that for any $k \geq 4$, $U_{k,\omega} \in H^4(\Omega_\omega)$ and is a weak solution of Problem (1) with r.h.s $f_{k,\omega}(r,\theta) = c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta$. Thus, one has already, by Theorem 1, the uniform estimate ($C$ independent of $k \geq 4$),

$$\|U_{k,\omega}\|_{H^2(\Omega_\omega)} \leq C\,\|f_{k,\omega}\|_{L^2(\Omega_\omega)},$$

which implies, by taking the series (over $k \geq 4$), that:

$$\|U_\omega\|_{H^2(\Omega_\omega)} \leq C\,\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{82}$$

We then have to show this last estimate in $H^4$ norm. So it is enough to show it for the two semi-norms $\|\nabla^3 U_\omega\|_{L^2(\Omega_\omega)}$ and $\|\nabla^4 U_\omega\|_{L^2(\Omega_\omega)}$. More precisely, it will be sufficient to do it only for each Fourier series term $U_{k,\omega}$ but with a uniform constant independent of $k \geq 4$. Furthermore, and without restriction, we will demonstrate this only for derivatives of order 4 w.r.t $r$. Derivatives of order 3 are simpler to treat and the other derivatives defining $\nabla^3$ and $\nabla^4$ can be treated in a similar manner. Thanks to Theorem 6, let $g_{k,\omega}(r)$ and $d_{k,\omega}(r)$ the single Fourier coefficients of $U_{k,\omega}$ and $V_{k,\omega} = \Delta U_{k,\omega}$, respectively, as defined by the recursive formula (79)-(80). Thus, one has on the one hand:

$$\begin{aligned} d^{(4)}_{k,\omega}(r) = \frac{1}{2\omega^3}\Bigg\{&(k\pi - 3\omega)(k\pi - 2\omega)(k\pi - \omega)r^{-4+\frac{k\pi}{\omega}}\left(\int_1^r g_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}\,ds + \int_0^1 g_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}\,ds\right)\\ &- (k\pi + 3\omega)(k\pi + 2\omega)(k\pi + \omega)r^{-4-\frac{k\pi}{\omega}}\int_0^r g_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}\,ds + 2\omega\left(\omega^2\left(g''_{k,\omega}(r) - \frac{g'_{k,\omega}(r)}{r}\right) + \frac{g_{k,\omega}(r)}{r^2}\left(\pi^2k^2 + 3\omega^2\right)\right)\Bigg\}, \end{aligned} \tag{83}$$

where, we can see that (using the notations given by the two fundamental lemmas 3 and 4):

$$r^{-4+\frac{k\pi}{\omega}}\int_1^r g_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}\,ds = r^{-4+\frac{k\pi}{\omega}}\int_1^r \frac{g_{k,\omega}(s)}{s^2}s^{3-\frac{k\pi}{\omega}}\,ds = J^{-4+\frac{k\pi}{\omega},3-\frac{k\pi}{\omega}}_{\left(\frac{g_{k,\omega}}{r^2}\right)}$$

$$\Longrightarrow \left\|r^{-4+\frac{k\pi}{\omega}}\int_1^r g_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}\,ds\right\|_{L^2(rdr)} \leq \frac{\omega}{k\pi - 3\omega}\left\|\frac{g_{k,\omega}}{r^2}\right\|_{L^2(rdr)},$$

$$r^{-4-\frac{k\pi}{\omega}}\int_0^r g_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}\,ds = r^{-4-\frac{k\pi}{\omega}}\int_0^r \frac{g_{k,\omega}(s)}{s^2}s^{3+\frac{k\pi}{\omega}}\,ds = I^{-4-\frac{k\pi}{\omega},3+\frac{k\pi}{\omega}}_{\left(\frac{g_{k,\omega}}{r^2}\right)}$$

$$\implies \left\| r^{-4-\frac{k\pi}{\omega}} \int_0^r g_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}} ds \right\|_{L^2(rdr)} \leq \frac{\omega}{k\pi + 3\omega} \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)},$$

and, since $\frac{3}{4} < \frac{\pi}{\omega} < 1$ and $k \geq 4$, thus $3 < \frac{k\pi}{\omega} < k$ which implies that $r \mapsto r^{-4+\frac{k\pi}{\omega}}$ is $L^2(rdr)$, and one obtains, with the help of Cauchy-Schwarz inequality:

$$\begin{aligned}
\left\| r^{-4+\frac{k\pi}{\omega}} \int_0^1 g_{k,\omega}(s) s^{1+\frac{k\pi}{\omega}} ds \right\|^2_{L^2(rdr)} &= \left( \int_0^1 \left( g_{k,\omega}(s) s^{\frac{1}{2}} \right) s^{\frac{1}{2}+\frac{k\pi}{\omega}} ds \right)^2 \frac{\omega}{2\,(k\pi - 3\omega)}, \\
&\leq \|g_{k,\omega}\|^2_{L^2(rdr)} \left( \int_0^1 s^{1+\frac{2k\pi}{\omega}} ds \right)^2 \frac{\omega}{2\,(k\pi - 3\omega)}, \\
&\leq \frac{\omega^2}{4\left(k^2\pi^2 - 9\omega^2\right)} \|g_{k,\omega}\|^2_{L^2(rdr)}. \\
&\leq \frac{\omega^2}{4\left(k^2\pi^2 - 9\omega^2\right)} \left\| \frac{g_{k,\omega}}{r^2} \right\|^2_{L^2(rdr)},
\end{aligned}$$

since $r \in (0, 1)$. So, using the expression of $d^{(4)}_{k,\omega}(r)$ given by (83), we arrived at the following estimate:

$$\begin{aligned}
\left\| d^{(4)}_{k,\omega} \right\|_{L^2(rdr)} \leq \frac{1}{2\omega^3} \Bigg\{ &(k\pi - 3\omega)(k\pi - 2\omega)(k\pi - \omega) \left( \frac{\omega}{k\pi - 3\omega} + \frac{\omega}{2\sqrt{k^2\pi^2 - 9\omega^2}} \right) \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \\
&+ (k\pi + 3\omega)(k\pi + 2\omega)(k\pi + \omega) \frac{\omega}{k\pi + 3\omega} \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \\
&+2\omega \left( \omega^2 \left( \left\| g''_{k,\omega} \right\|_{L^2(rdr)} + \left\| \frac{g'_{k,\omega}}{r} \right\|_{L^2(rdr)} \right) + \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \left(\pi^2 k^2 + 3\omega^2\right) \right) \Bigg\},
\end{aligned}$$

which, for $k$ sufficiently large ($k \geq k_0 \geq 4$), becomes

$$\left\| d^{(4)}_{k,\omega} \right\|_{L^2(rdr)} \leq C \left( \left\| g''_{k,\omega} \right\|_{L^2(rdr)} + \left\| \frac{g'_{k,\omega}}{r} \right\|_{L^2(rdr)} + k^2 \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \right),$$

and which implies, with the help of Lemma (7), that

$$\begin{aligned}
\sum_{k\geq k_0} \left\| d^{(4)}_{k,\omega} \right\|_{L^2(rdr)} &\leq \sum_{k\geq k_0} \left( \left\| g''_{k,\omega} \right\|_{L^2(rdr)} + k \left\| \frac{g'_{k,\omega}}{r} \right\|_{L^2(rdr)} + k^2 \left\| \frac{g_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \right) \\
&\leq C |||\nabla^2 V_\omega|||_{L^2(\Omega_\omega)}, \qquad (84)
\end{aligned}$$

here we recall that $g_{k,\omega}$ is the $k^{th}$- Fourier series coefficient of

$$V_\omega(r, \theta) = \sum_{k\geq 1} g_{k,\omega}(r) \sin \frac{k\pi}{\omega}\theta = \sum_{k\geq 1} V_{k,\omega}(r, \theta).$$

Now, and on the other hand, since $V_{k,\omega} \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$, one can check that $\Delta V_{k,\omega} = f_{k,\omega}$ (the $k^{th}$- Fourier series coefficient of $f_\omega$) then by the "second fundamental inequality", cf. ([20], Corollary 2.3.6 p.31), see also the proof of Theorem 1, which states that

$$|||\nabla^2 V_{k,\omega}|||_{L^2(\Omega_\omega)} \leq |||\Delta V_{k,\omega}|||_{L^2(\Omega_\omega)} = \|f_{k,\omega}\|_{L^2(\Omega_\omega)},$$

we conclude that

$$|||\nabla^2 V_\omega|||_{L^2(\Omega_\omega)} \leq \sum_{k\geq 4} |||\nabla^2 V_{k,\omega}|||_{L^2(\Omega_\omega)} \leq \sum_{k\geq 4} \|f_{k,\omega}\|_{L^2(\Omega_\omega)}$$

and, consequently, by (84), one obtains

$$\sum_{k\geq k_0} \left\| d^{(4)}_{k,\omega} \right\|_{L^2(rdr)} \leq \sum_{k\geq 4} \|f_{k,\omega}\|_{L^2(\Omega_\omega)}.$$

Henceforth,

$$\left\|\sum_{k\geq k_0} d_{k,\omega}^{(4)}(r)\sin\frac{k\pi}{\omega}\theta\right\|_{L^2(\Omega_\omega)} \leq \sqrt{\frac{\omega}{2}}\sum_{k\geq k_0}\left\|d_{k,\omega}^{(4)}\right\|_{L^2(rdr)}$$
$$\leq \sqrt{\frac{\omega}{2}}\sum_{k\geq 4}\|f_{k,\omega}\|_{L^2(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}.$$

Thus, we obtained

$$\left\|U_\omega^{(4,0)}\right\|_{L^2(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}.$$

For the other derivatives to complete the $\nabla^4$, the arguments are similar. Finally, the proof of the theorem is achieved. □

## 4 | PROOF OF THE MAIN RESULT

In this section, we present the proof of the main Theorem 2 stated in Section 2.

***Proof of Theorem 2***. The proof now is direct as a consequence of theorems 3, 4, 5 and 7, and the former Fourier series analysis. In order to lighten the length of the manuscript, a rough outline can be written as follows:

We use four problems: we write the Fourier series expansion of $f_\omega$ separating the singular frequencies $k = 1, 2, 3$ from the regular ones $k \geq 4$, as follows:

$$f_\omega = f_{1,\omega} + f_{2,\omega} + f_{3,\omega} + \sum_{k\geq 4} f_{k,\omega}$$

where

$$f_{j,\omega}(r,\theta) = c_{j,\omega}(r)\sin\frac{j\pi}{\omega}\theta,\ j \geq 1. \tag{85}$$

By Theorem 3,

$$\left(\lambda_{1,1}r^{2-\frac{\pi}{\omega}} + \lambda_{2,1}r^{2+\frac{\pi}{\omega}}\right)\sin\frac{\pi}{\omega}\theta + b_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta$$

is the solution of Problem (1) with r.h.s $f_{1,\omega}$ corresponding to the singular frequency $k = 1$. Thus $u_{\omega,r}^{(1)}(r,\theta) = b_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta$ belongs to $H^4(\Omega_\omega)$, and the estimate (18) follows from (31).

By Theorem 4, and Theorem 5,

$$\lambda_{3,j}r^{\frac{j\pi}{\omega}}\sin\frac{j\pi}{\omega}\theta + b_{j,\omega}(r)\sin\frac{j\pi}{\omega}\theta$$

for $j = 2, 3$, is the solution of Problem (1) with r.h.s $f_{j,\omega}$ corresponding to a superposition of the singular frequencies $j = 2$ and 3. Therefore, $u_{\omega,r}^{(j)}(r,\theta) = b_{j,\omega}(r)\sin\frac{j\pi}{\omega}\theta$ belongs to $H^4(\Omega_\omega)$.

Next, by Theorem 7, the Fourier series of solution of Problem 1 with r.h.s $\sum_{k\geq 4} f_{k,\omega}$ corresponding to a superposition of all regular frequency $k \geq 4$ is given by

$$\sum_{k\geq 4}\left(b_{k,\omega}(r) + A_{k,\omega}r^{2+\frac{k\pi}{\omega}} + B_{k,\omega}r^{\frac{k\pi}{\omega}}\right)\sin\frac{k\pi}{\omega}\theta = \sum_{k\geq 4}U_{k,\omega}(r,\theta),$$

and the uniform estimate (19) follows from the three theorems 4,5 and 7 corresponding to the two singular frequencies $k = 2, 3$ and the regular ones $k \geq 4$ respectively.

Now, as discussed in the beginning of Section 3, a power function of $r$, $(r,\theta) \mapsto r^{\alpha_k}\sin\frac{k\pi}{\omega}\theta$, $\alpha_k$ non-integer, belongs to the Sobolev space $H^{2+\sigma}(\Omega_\omega)$ for all $\sigma < \alpha_k - 1$. Thus, the regularity $H^{2+\sigma}$ of the singular part

$$\left(\lambda_{1,1}r^{2-\frac{\pi}{\omega}} + \lambda_{2,1}r^{2+\frac{\pi}{\omega}}\right)\sin\frac{\pi}{\omega}\theta + +\lambda_{3,2}r^{\frac{2\pi}{\omega}}\sin\frac{2\pi}{\omega}\theta + \lambda_{3,3}r^{\frac{3\pi}{\omega}}\sin\frac{3\pi}{\omega}\theta,$$

is achieved for all $\sigma$ such that

$$\sigma < 2 - \frac{\pi}{\omega} - 1 = 1 - \frac{\pi}{\omega}.$$

So, $u_\omega \in H^{2+\sigma}(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ and the decomposition (14) follows with their explicit expressions as given in Section 3.

Finally, the sharpness Estimate (18) follows directly from theorems 3 - Inequality (31) and the three theorems 4,5 and 7 corresponding to the two singular frequencies $k = 2, 3$ and the regular ones $k \geq 4$ respectively. The proof of the theorem is achieved. □

## 5 | CONCLUSION AND OUTLOOK

Throughout this paper, we have given explicit extraction formulas via Fourier analysis of the coefficients of singularity and regular part of the solutions of a family of bi-harmonic equations with Navier's boundary conditions on a family of open non-convex planar sectors with opening angle $\omega \in (\pi, 4\pi/3)$. We have shown that explicit and sharp estimates can be obtained by highlighting the decomposition of the solution into regular/singular parts whose behavior in the vicinity of the critical angle $\omega = \pi$ is as follows:

- The regular part associated the the first Fourier frequency $k = 1$ is unstable in $H^4$ norm in the vicinity of $\pi$. Two coefficients of singularity $\lambda_{1,1}$ and $\lambda_{2,1}$ both are unbounded for $\omega$ close to $\pi^+$.
- The second and third frequencies $k = 2, 3$ produce bounded coefficients of singularity $\lambda_{3,2}$ and $\lambda_{3,3}$ w.r.t $\omega$ in the vicinity of $\pi$.
- A stable regular part in the norm $H^4$ corresponding to all frequencies higher than 1.
- The global solution remains stable in the $H^2$ norm from standard uniform estimates of the weak variational solution. This problem is actually quite similar to that of Babuška, cf.[15], when additional regularity on the source term $f_\omega$ is assumed at the origin. To the authors knowledge, question of existence of stable $H^4$ decomposition near a concave corner was never addressed in the literature and still an open problem.
- Possible extension of the results herein are envisaged for boundary value problems with general (mixed) boundary conditions.
- Another possible and open question is: Can we approach a nearly flat boundary by another completely flat one in the case of a non-convex opening angle? This will be of great interest in numerical approximation for fourth order elliptic problems when the error depends on the finite elements, the mesh used, and the regularity of the solutions.

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