The general bilinear techniques for studying the propagation of mixed-type periodic and lump-type solutions in a homogenous-dispersive medium

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Abstract

This paper aims to construct new mixed-type periodic and lump-type solutions via the dependent variable transformation and the Hirota's bilinear form (general bilinear techniques). This study will be investigated by considering the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation which describes the weakly dispersive waves in a homogenous medium in fluid dynamics. The obtained solutions contain abundant physical structure. Consequently, the dynamical behaviors of these solutions are graphically discussed for different choices of the free parameters through 3D- and contour plots.

Keywords: Mixed-type solutions; lump-type solutions; (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation; Hirota's bilinear form

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1 Introduction

Nonlinear phenomena are investigated in many disciplines of the science, such as the marine engineering, fluid dynamics, plasma physics, chemistry, applied mathematics and so on [1-6]. With the development of nonlinear dynamics, the research of nonlinear partial differential equations (NPDEs) become more and more important. To further understand these phenomena, solving NPDEs plays a significant role in nonlinear sciences [7-16]. In the past few decades, many efficient and powerful techniques have been introduced to obtain the analytical solutions of these equations [17-28].

In this paper, a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (BKP) equation is considered as follows [29]:

$$u_{yt} + 3\,u_{xz} - 3u_x\,u_{xy} - 3\,u_y\,u_{xx} - u_{xxxy} = 0. \tag{1}$$

Eq.(1) is an exceedingly used model for assaying the dynamics of nonlinear waves and solitons in various fields of science especially in plasma physics, weakly dispersive environment, and fluid dynamics. Multiple-soliton solutions are generated and discussed by Ma [29]. Ma and Zhu [30] derived multiple wave solutions by using the multiple exp-function algorithm. Tang [31] obtained new analytical solutions which contain different wave structures such as periodic soliton, kinky periodic solitary, and periodic soliton solutions by using the extended homoclinic test approach. By employing the improved (G'/G)-expansion method with the aid of symbolic computation, Liu and Zeng [32] obtain new soliton solutions of the Eq. (1).

The organization of this paper will be arranged as: Section 2 gives the new mixed-type periodic solutions for the (3+1)-dimensional generalized BKP equation based on the dependent variable transformation and Hirota's bilinear form. Section 3 presents the lump-type solutions and illustrates the dynamical behaviors of the obtained solutions through 3D- and contour plots. Section 4 makes the conclusions.

2 New mixed-type periodic solutions

In Substituting the transformation $u = 2\left[\ln \xi(x,y,z,t)\right]_x$ into Eq.(1), we have the following Hirota's bilinear form [33-37]

$$(D_t D_y - D_x^3 D_y + 3 D_x D_z) f \cdot f = 0 \tag{2}$$

Equivalently, we have

$$-\xi_t \xi_y + \xi_{xxx}\xi_y - 3\xi_z \xi_x - 3\xi_{xy}\xi_{xx} + 3\xi_x \xi_{xxy} + \xi\left(\xi_{yt} + 3\xi_{xz} - \xi_{xxxy}\right) = 0. \tag{3}$$

In order to obtain the new mixed-type periodic solutions, a direct test function is written as

$$\xi = k_1\, e^{\zeta_1} + e^{-\zeta_1} + k_2\, \tan\left(\zeta_2\right) + k_3\, \tanh\left(\zeta_3\right), \tag{4}$$

where $\zeta_i = \eta_i\, x + \mu_i\, y + \gamma_i\, z + \nu_i\, t, i = 1,2,3$ and η_i, μ_i, γ_i, ν_i are unknown constants. Substituting Eq.(4) into Eq.(3), we have

Case(1)

$$k_2 = \mu_1 = \gamma_1 = \eta_3 = \nu_3 = 0, \nu_1 = \frac{\eta_1^3\mu_3 - 3\eta_1\gamma_3}{\mu_3}, \tag{5}$$

where η_1, γ_3, μ_3, k_1 and k_3 are arbitrary constants. Then

$$\xi = e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} k_1 + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} + k_3 \tanh\left(y\mu_3 + z\gamma_3\right). \tag{6}$$

Substituting Eq.(6) into $u = 2\left[\ln \xi\right]_x$, the first mixed-type periodic solution is read as

$$u_1 = \frac{2[e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} k_1\eta_1 - e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} \eta_1]}{e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} k_1 + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_3 - 3\eta_1\gamma_3\right)}{\mu_3}} + k_3 \tanh\left(y\mu_3 + z\gamma_3\right)}. \tag{7}$$

The physical structure of Eq.(7) is exhibited in Fig. 1.

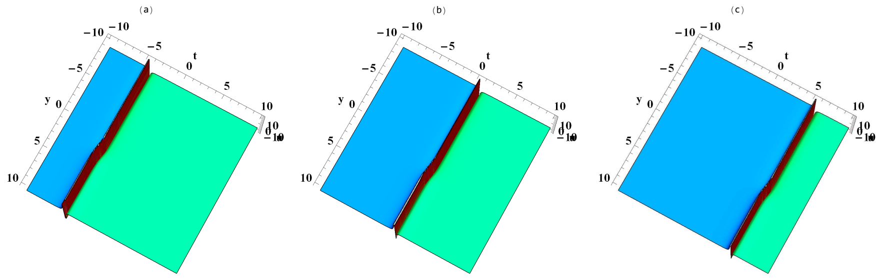

Fig. 1. Solution (7) at $k_1 = k_3 = -0.5$, $\eta_1 = -1$, $\mu_3 = 1$ $\gamma_3 = 1$, $z = -2$, (a) $x = -10$, (b) $x = 0$, (c) $x = 10$.

Case(2)

$$k_3 = \mu_1 = \gamma_1 = \eta_2 = \nu_2 = 0, \nu_1 = \frac{\eta_1^3\mu_2 - 3\eta_1\gamma_2}{\mu_2}, \tag{8}$$

where η_1, γ_2, μ_2, k_1 and k_2 are free real constants. Then

$$\xi = e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} k_1 + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} + k_2 \tan\left(y\mu_2 + z\gamma_2\right). \tag{9}$$

Substituting Eq.(9) into $u = 2\left[\ln \xi\right]_x$, the second mixed-type periodic solution is read as

$$u_2 = \frac{2[e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} k_1\eta_1 - e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} \eta_1]}{e^{x\eta_1 + \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} k_1 + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} + k_2 \tan\left(y\mu_2 + z\gamma_2\right)}. \tag{10}$$

Case(3)

$$k_1 = \mu_1 = \gamma_1 = \eta_2 = \nu_2 = \eta_3 = \nu_3 = 0, \nu_1 = \frac{\eta_1^3\mu_2 - 3\eta_1\gamma_2}{\mu_2}, \gamma_3 = \frac{\mu_3\gamma_2}{\mu_2}, \tag{11}$$

where η_1, γ_2, μ_2, μ_3, k_2 and k_3 are free real constants. Then

$$\xi = k_2 \tan\left(y\mu_2 + z\gamma_2\right) + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_2\mu_3}{\mu_2}\right). \tag{12}$$

Substituting Eq.(12) into $u = 2\left[\ln \xi\right]_x$, the third mixed-type periodic solution is read as

$$u_3 = \frac{-2e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} \eta_1}{k_2 \tan\left(y\mu_2 + z\gamma_2\right) + e^{-x\eta_1 - \frac{t\left(\eta_1^3\mu_2 - 3\eta_1\gamma_2\right)}{\mu_2}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_2\mu_3}{\mu_2}\right)}. \tag{13}$$

The physical structure of Eq.(13) is revealed in Figs. 2-4.

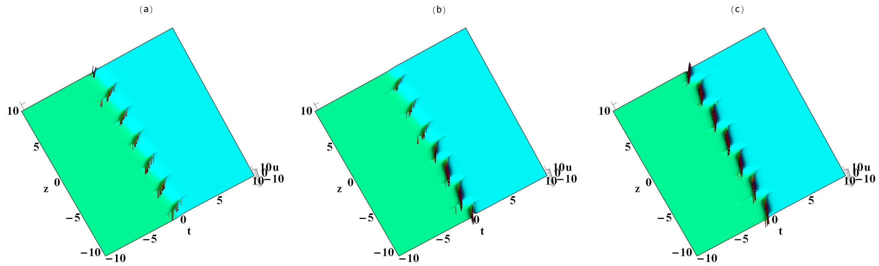

Fig. 2. Solution (13) at $k_3 = k_2 = -0.5$, $\eta_1 = \mu_2 = \mu_3 = 1$, $\gamma_2 = -1$, $x = 0$, (a) $y = -20$, (b) $y = 0$, (c) $y = 20$.

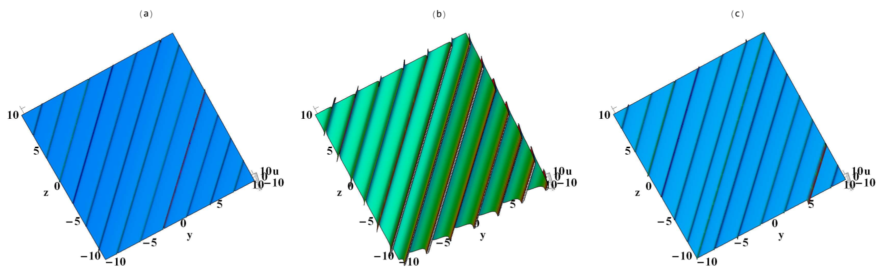

Fig. 3. Solution (13) at $k_3 = k_2 = -0.5$, $\eta_1 = \mu_2 = \mu_3 = 1$, $\gamma_2 = -1$, $x = 0$, (a) $t = -1$, (b) $t = 0$, (c) $t = 1$.

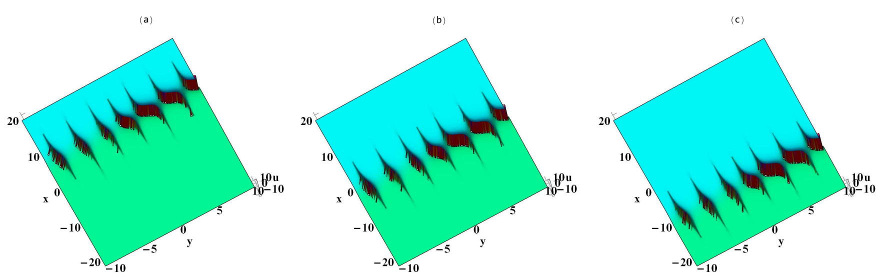

Fig. 4. Solution (13) at $k_3 = k_2 = -0.5$, $\eta_1 = \mu_2 = \mu_3 = 1$, $\gamma_2 = -1$, $z = 0$, (a) $t = -2$, (b) $t = 0$, (c) $t = 2$.

Case(4)

$$k_1 = \eta_2 = \nu_2 = \eta_3 = \nu_3 = 0, \nu_1 = \frac{\eta_1^3 \mu_1 - 3\eta_1\gamma_1}{\mu_1}, \gamma_2 = \frac{\mu_2\gamma_1}{\mu_1}, \gamma_3 = \frac{\mu_3\gamma_1}{\mu_1}, \tag{14}$$

where η_1, μ_1, γ_1, μ_2, μ_3, k_2 and k_3 are free real constants. Then

$$\begin{aligned} \xi &= k_2 \tan\left(y\mu_2 + \frac{z\gamma_1\mu_2}{\mu_1}\right) + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_1\mu_3}{\mu_1}\right) \\ &+ e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}. \end{aligned} \tag{15}$$

Substituting Eq.(15) into $u = 2\left[\ln \xi\right]_x$, the fourth mixed-type periodic solution is read as

$$\begin{aligned} u_4 &= [-2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}\eta_1]/[k_2 \tan\left(y\mu_2 + \frac{z\gamma_1\mu_2}{\mu_1}\right) \\ &+ e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}} + k_3 \tanh\left(y\mu_3 + \frac{z\gamma_1\mu_3}{\mu_1}\right)]. \end{aligned} \tag{16}$$

The physical structure for the solution in Eq.(16) is similar to that one given by Eq.(13).

Case(5)

$$k_2 = k_3 = 0, \nu_1 = \frac{4\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \tag{17}$$

where η_1, γ_1, μ_1 and k_1 are free real constants. Substituting these results into (4), we have

$$\xi = e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}} k_1 + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}. \tag{18}$$

Substituting Eq.(18) into $u = 2\left[\ln \xi\right]_x$, the fifth mixed-type periodic solution is read as

$$\begin{aligned} u_5 &= 2[e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}} k_1\eta_1 - e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}\eta_1] \\ &/\ [e^{x\eta_1 + y\mu_1 + z\gamma_1 + \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}} k_1 + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(4\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}]. \end{aligned} \tag{19}$$

The physical structure of Eq.(19) is demonstrated in Fig. 5.

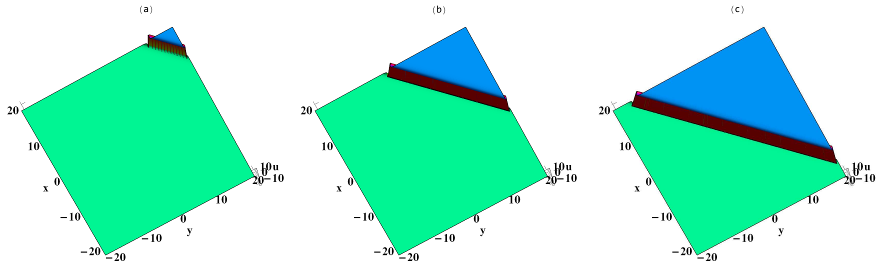

Fig. 5. Solution (19) at $\gamma_1 = -1$, $\eta_1 = \mu_1 = 1$, $k_1 = -2$, $z = 20$, (a) $t = -2$, (b) $t = 0$, (c) $t = 2$.

Case(6)

$$k_1 = k_3 = \eta_2 = \mu_2 = 0, \nu_1 = \frac{\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \nu_2 = -\frac{3\eta_1\gamma_2}{\mu_1}, \tag{20}$$

where η_1, γ_1, γ_2, μ_1 and k_2 are free real constants. Then

$$\xi = k_2 \tan\left(z\gamma_2 - \frac{3t\eta_1\gamma_2}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}. \tag{21}$$

Substituting Eq.(21) into $u = 2\left[\ln \xi\right]_x$, the sixth mixed-type periodic solution is read as

$$u_6 = -\frac{2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}\eta_1}{k_2 \tan\left(z\gamma_2 - \frac{3t\eta_1\gamma_2}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}}. \tag{22}$$

The physical structure of Eq.(22) is shown in Fig. 6.

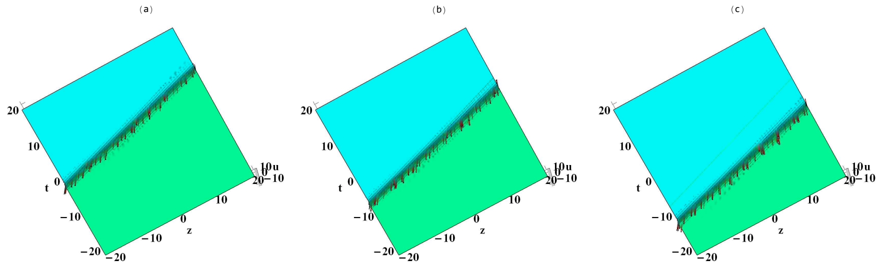

Fig. 6. Solution (22) at $\eta_1 = \mu_1 = 1$, $\gamma_1 = -1$, $\gamma_2 = k_2 = -2$, $y = 2$,
(a) $x = -20$, (b) $x = 0$, (c) $x = 20$.

Case(7)

$$k_1 = k_2 = \eta_3 = \mu_3 = 0, \nu_1 = \frac{\eta_1^3\mu_1 - 3\eta_1\gamma_1}{\mu_1}, \nu_3 = -\frac{3\eta_1\gamma_3}{\mu_1}, \tag{23}$$

where η_1, γ_1, γ_3, μ_1 and k_3 are free real constants. Then

$$\xi = k_3 \tanh\left(z\gamma_3 - \frac{3t\eta_1\gamma_3}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}. \tag{24}$$

Substituting Eq.(24) into $u = 2\left[\ln \xi\right]_x$, the seventh mixed-type periodic solution is read as

$$u_7 = -\frac{2e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}\eta_1}{k_3 \tanh\left(z\gamma_3 - \frac{3t\eta_1\gamma_3}{\mu_1}\right) + e^{-x\eta_1 - y\mu_1 - z\gamma_1 - \frac{t\left(\eta_1^3\mu_1 - 3\eta_1\gamma_1\right)}{\mu_1}}}. \tag{25}$$

The physical structure of Eq.(25) is listed in Fig. 7.

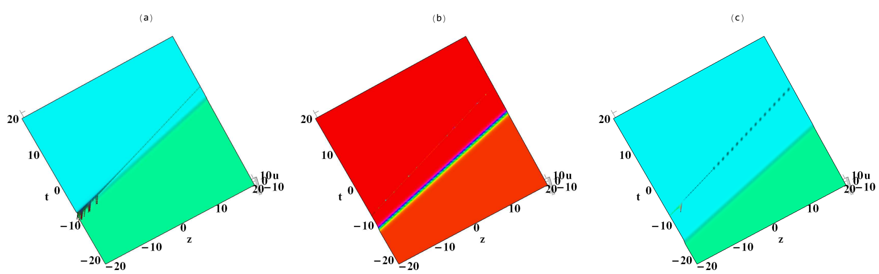

Fig. 7. Solution (25) at $\eta_1 = \mu_1 = 1$, $\gamma_1 = -1$, $\gamma_3 = k_3 = -2$, $x = 20$,
(a) $y = -20$, (b) $y = 0$, (c) $y = 20$.

3 Lump-type solutions

To derive the lump-type solutions of the Eq.(1), we have

$$\begin{aligned}\xi =& \left(t\hbar_4 + x\hbar_1 + y\hbar_2 + z\hbar_3 + \hbar_5\right)^2 + \left(t\hbar_9 + x\hbar_6 + y\hbar_7 + z\hbar_8 + \hbar_{10}\right)^2 \\ &+ \hbar_{11} + \kappa_1 e^{t\Xi_4+\Xi_5+\Xi_1 x+\Xi_2 y+\Xi_3 z} + \kappa_2 e^{-t\Xi_4-\Xi_5-\Xi_1 x-\Xi_2 y-\Xi_3 z},\end{aligned} \tag{26}$$

where $\hbar_i(i = 1, \cdots, 11)$, $\kappa_i(i = 1, 2)$ and $\Xi_i(1, \cdots, 5)$ are unknown constants. Substituting Eq.(26) into Eq.(3), the values of the unknown parameters in Eq.(26) are obtained as follows

$$\begin{aligned}(I): \hbar_7 =& \frac{\hbar_2\hbar_6}{\hbar_1}, \hbar_8 = \frac{\hbar_3\hbar_6}{\hbar_1}, \Xi_3 = \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1}, \kappa_2 = \frac{\left(\hbar_1^2 + \hbar_6^2\right)^2}{\kappa_1\Xi_1^4}, \\ \Xi_4 =& -\frac{\Xi_1^3}{2} - \frac{3\Xi_1\hbar_3}{\hbar_2}, \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2}, \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \Xi_2 = -\frac{\Xi_1\hbar_2}{\hbar_1},\end{aligned} \tag{27}$$

with $\hbar_1 \neq 0$, $\hbar_2 \neq 0$, $\Xi_1 \neq 0$, $\kappa_1 \neq 0$.

$$\begin{aligned}(II): \hbar_7 =& -\frac{\hbar_1\hbar_2}{\hbar_6}, \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \Xi_1 = \Xi_4 = 0, \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \\ \Xi_3 =& \frac{\Xi_2\hbar_3}{\hbar_2}, \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2},\end{aligned} \tag{28}$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$.

$$\begin{aligned}(III): \hbar_8 =& \frac{3\hbar_1\hbar_2\hbar_3 + 3\hbar_6\hbar_7\hbar_3 + \hbar_4\left(\hbar_2^2 + \hbar_7^2\right)}{3\hbar_2\hbar_6 - 3\hbar_1\hbar_7}, \Xi_2 = \frac{\Xi_1\epsilon_1\sqrt{\hbar_2^2 + \hbar_7^2}}{\sqrt{\hbar_1^2 + \hbar_6^2}}, \\ \hbar_9 =& \frac{3\hbar_3\hbar_1^2 + \hbar_2\hbar_4\hbar_1 + \hbar_6\left(3\hbar_3\hbar_6 + \hbar_4\hbar_7\right)}{\hbar_1\hbar_7 - \hbar_2\hbar_6}, \Xi_3 = \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1}, \\ \hbar_{11} =& \frac{3\kappa_1\kappa_2\Xi_2\Xi_1^3 + 3\left(\hbar_1^2 + \hbar_6^2\right)\left(\hbar_1\hbar_2 + \hbar_6\hbar_7\right)}{3\hbar_1\hbar_3 + \hbar_2\hbar_4}, \\ \Xi_4 =& \frac{\frac{3\Xi_1^4\left(\Xi_1\hbar_2\left(\hbar_2^2+\hbar_7^2\right)+\Xi_2\left(\hbar_1\left(\hbar_7^2-\hbar_2^2\right)-2\hbar_2\hbar_6\hbar_7\right)\right)}{\Xi_1^2\left(\hbar_2^2+\hbar_7^2\right)-2\Xi_2\Xi_1\left(\hbar_1\hbar_2+\hbar_6\hbar_7\right)+\Xi_2^2\left(\hbar_1^2+\hbar_6^2\right)} - 2\Xi_1^3\hbar_2 - 3\Xi_1\hbar_3}{\hbar_2}, \\ \hbar_4 =& \frac{3\left(\frac{\Xi_1^3\Xi_2\left(\hbar_2\hbar_6-\hbar_1\hbar_7\right)^2}{\Xi_1^2\left(\hbar_2^2+\hbar_7^2\right)-2\Xi_2\Xi_1\left(\hbar_1\hbar_2+\hbar_6\hbar_7\right)+\Xi_2^2\left(\hbar_1^2+\hbar_6^2\right)} - \hbar_1\hbar_3\right)}{\hbar_2},\end{aligned} \tag{29}$$

with $\hbar_2 \neq 0$, $\Xi_1 \neq 0$, $3\,\hbar_1\hbar_3 + \hbar_2\hbar_4 \neq 0$, $\epsilon_1 = \pm 1$, $\hbar_2\hbar_6 - \hbar_1\hbar_7 \neq 0$, $\hbar_1^2 + \hbar_6^2 \neq 0$.

$$
\begin{aligned}
(IV): \hbar_7 &= -\frac{\hbar_1\hbar_2}{\hbar_6}, \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \hbar_4 = \frac{3\Xi_1^3\Xi_2\hbar_2\left(\hbar_1^2+\hbar_6^2\right)}{\Xi_1^2\hbar_2^2+\Xi_2^2\hbar_6^2} - \frac{3\hbar_1\hbar_3}{\hbar_2},\\
\Xi_3 &= \frac{\Xi_2\Xi_1^3 + 2\Xi_2\Xi_4}{3\Xi_1}, \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2}, \hbar_{11} = \frac{2\kappa_1\kappa_2\Xi_1^2}{\hbar_1^2+\hbar_6^2},\\
\Xi_4 &= -2\Xi_1^3 + \frac{3\Xi_1^4\hbar_2\left(\Xi_1\hbar_2+\Xi_2\hbar_1\right)}{\Xi_1^2\hbar_2^2+\Xi_2^2\hbar_6^2} - \frac{3\Xi_1\hbar_3}{\hbar_2}, \Xi_2 = \frac{\epsilon_2\Xi_1\hbar_2}{\hbar_6},
\end{aligned} \tag{30}
$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$, $\hbar_1^2 + \hbar_6^2 \neq 0$, $\epsilon_2 = \pm 1$.

$$
\begin{aligned}
(V): \hbar_7 &= -\frac{\hbar_1\hbar_2}{\hbar_6}, \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \Xi_2 = \Xi_3 = 0, \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2},\\
\Xi_4 &= \Xi_1^3 - \frac{3\Xi_1\hbar_3}{\hbar_2}, \hbar_4 = -\frac{3\hbar_1\hbar_3}{\hbar_2},
\end{aligned} \tag{31}
$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$.

$$
\begin{aligned}
(VI): \hbar_7 &= -\frac{\hbar_1\hbar_2}{\hbar_6}, \hbar_8 = \frac{\hbar_2\hbar_4}{3\hbar_6}, \Xi_3 = 0, \hbar_9 = -\frac{3\hbar_3\hbar_6}{\hbar_2},\\
\Xi_4 &= \Xi_1^3 - \frac{3\Xi_1\hbar_3}{\hbar_2}, \hbar_4 = -\frac{3\hbar_3\hbar_6\left(\hbar_1+\hbar_6\right)}{\hbar_2\left(\hbar_6-\hbar_1\right)},\\
\Xi_2 &= \frac{\Xi_1\hbar_2\hbar_3}{\Xi_1^2\hbar_2\left(\hbar_1-\hbar_6\right)-\hbar_3\hbar_6},\\
\Xi_1 &= \frac{\epsilon_3\sqrt{2}\sqrt{\hbar_1^2+\hbar_6^2}\sqrt{\frac{\hbar_3^2\hbar_6^2\left(\hbar_1^2+\hbar_6^2\right)}{\left(\hbar_1-\hbar_6\right)^2}}}{\sqrt{\frac{\hbar_2\hbar_3\hbar_6\left(\hbar_1^2+\hbar_6^2\right)^2}{\hbar_1-\hbar_6}}},
\end{aligned} \tag{32}
$$

with $\hbar_2 \neq 0$, $\hbar_6 \neq 0$, $\hbar_1 \neq \hbar_6$, $\epsilon_3 = \pm 1$. Substituting Eq.(27)-Eq.(32) into the variable substitution $u = 2\left[\ln\xi\right]_x$, six lump-type solutions can be derived.

As an example, substituting Eq.(26) and Eq.(27) into the variable substitution $u =$

$2\,[\ln\xi]_x$, the lump-type solution of Eq.(1) can be written as follows

$$
\begin{aligned}
u =\ & [2[-\frac{(\hbar_1^2+\hbar_6^2)^{\,2}\exp[\Xi_1\left(\frac{3t\hbar_3}{\hbar_2}-x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)+\frac{\Xi_1^3(t\hbar_1+z\hbar_2)}{2\hbar_1}-\Xi_5]}{\kappa_1\Xi_1^3} \\
&+\kappa_1\Xi_1\exp[\Xi_1\left(-\frac{3t\hbar_3}{\hbar_2}+x-\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)-\frac{\Xi_1^3\,(t\hbar_1+z\hbar_2)}{2\hbar_1}+\Xi_5] \\
&+2\hbar_1[\hbar_1\left(x-\frac{3t\hbar_3}{\hbar_2}\right)+y\hbar_2+z\hbar_3+\hbar_5]+2\hbar_6[\hbar_{10}+\hbar_6(-\frac{3t\hbar_3}{\hbar_2}+x \\
&+\frac{y\hbar_2+z\hbar_3}{\hbar_1})]]]/[\hbar_{11}+\kappa_1\exp[\Xi_1\left(-\frac{3t\hbar_3}{\hbar_2}+x-\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right) \\
&-\frac{\Xi_1^3\,(t\hbar_1+z\hbar_2)}{2\hbar_1}+\Xi_5]+[\left(\hbar_1^2+\hbar_6^2\right)^{\,2}\exp[\Xi_1\left(\frac{3t\hbar_3}{\hbar_2}-x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right) \\
&+\frac{\Xi_1^3\,(t\hbar_1+z\hbar_2)}{2\hbar_1}-\Xi_5]]/(\kappa_1\Xi_1^4)+[\hbar_1\left(x-\frac{3t\hbar_3}{\hbar_2}\right)+y\hbar_2+z\hbar_3+\hbar_5]^2 \\
&+[\hbar_6\left(-\frac{3t\hbar_3}{\hbar_2}+x+\frac{y\hbar_2+z\hbar_3}{\hbar_1}\right)+\hbar_{10}]^2],
\end{aligned} \tag{33}
$$

with the constraint $\hbar_1\neq 0$, $\hbar_2\neq 0$, $\Xi_1\neq 0$ and $\kappa_1\neq 0$.

To analyze the dynamical behaviors for solution (33), the values of parameters are selected as follows

$$
\begin{aligned}
&\hbar_1=\hbar_5=2, \hbar_2=\hbar_6=-1, \hbar_3=3, \\
&\hbar_{10}=\hbar_{11}=\Xi_5=0, \Xi_1=-2, \kappa_1=1.
\end{aligned} \tag{34}
$$

Substituting Eq.(34) into Eq.(33), the dynamical behaviors for solution (33) are shown in Fig. 8 and Fig. 9.

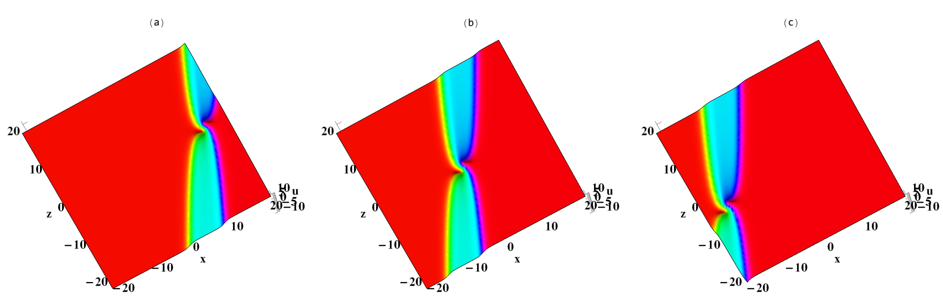

Fig. 8. Dynamical behaviors for solution (33) with $y=0$ when $t=-2$ in (a) (d), $t=0$ in (b) (e) and $t=2$ in (c) (f).

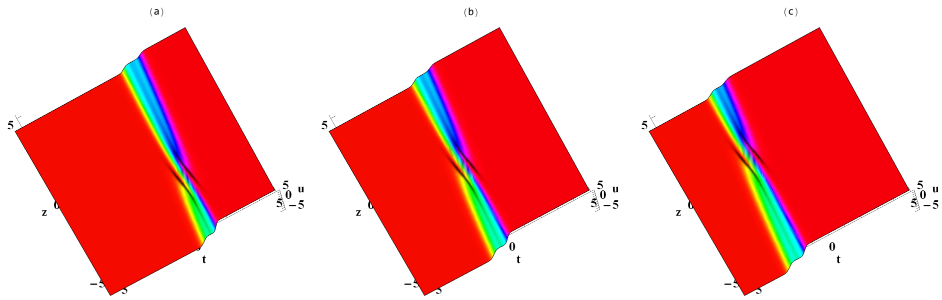

Fig. 9. Dynamical behaviors for solution (33) with $y = 0$ when $x = -10$ in (a) (d), $x = 0$ in (b) (e) and $x = 10$ in (c) (f).

In Fig. 8, the interaction behavior between two solitary waves and a lump wave can be found with $t = -2; 0; 2$ on the $x - z$ plane. The interaction solutions reveal the characteristic of "elastic collision", that is, two solitary waves and lump wave keep their shape and velocity invariant in the process of transmission. Fig. 9 demonstrates the interaction behavior between two solitary waves and a lump wave with $x = -10; 0; 10$ on the $t - z$ plane.

4 Conclusion

Based on the dependent variable transformation and Hirota's bilinear form, new mixed-type and lump-type solutions of the (3+1)-dimensional generalized BKP equation are presented. Moreover, Figs. 1-7 show the dynamical behaviors for the mixed-type periodic solution. Fig. 8 demonstrates the interaction behavior between two solitary waves and a lump wave on the $x - z$ plane, which describes the characteristic of "elastic collision". Fig. 9 reveals the interaction behavior between two solitary waves and a lump wave on the $t - z$ plane. As can be seen from the above solution process, the direct test function is very effective for solving the mixed-type periodic solutions of NPDEs.

References

[1] Ma WX. lumps and their interaction solutions of (3+1)-dimensional linear PDEs. J. Geom. Phys. 2018;133:10-16.

[2] Tang YN, Tao SQ, Guan Q. Lump solitons and the interaction phenomena of them for two classes of nonlinear evolution equations. Nonlinear Dyn. 2017;89:429-442.

[3] Wazwaz AM, Osman MS. Analyzing the combined multi-waves polynomial solutions in a two-layer-liquid medium. Comput. Math. Appl. 2018;76(2):276-283.

[4] Wazwaz AM. Two new integrable Kadomtsev-Petviashvili equations with time-dependent coefficients: multiple real and complex soliton solutions. Wave. Random. Complex. 2018. DOI: 10.1080/17455030.2018.1559962.

[5] Wazwaz AM. A variety of multiple-soliton solutions for the integrable (4+1)-dimensional Fokas equation. Wave. Random. Complex. 2018. DOI: 10.1080/17455030.2018.1560515.

[6] Wazwaz AM. Painlevé analysis for a new integrable equation combining the modified Calogero-Bogoyavlenskii-Schiff (MCBS) equation with its negative-order form. Nonlinear Dyn. 2018;91(2):877-883.

[7] Osman MS, Machado JAT. The dynamical behavior of mixed-type soliton solutions described by (2+ 1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients. J. Electromagnet. Wave. 2018;32(11):1457-1464.

[8] Wazwaz AM., Samir AET. Optical Gaussons for nonlinear logarithmic Schrödinger equations via the variational iteration method. Optik 2019;180:414-418.

[9] Li YZ, Liu JG. Multiple periodic-soliton solutions of the (3+1)-dimensional generalized shallow water Equation. Pramana 2018;90(6):71.

[10] Wazwaz AM. Construction of a hierarchy of negative-order integrable Burgers equations of higher orders. Math. Method. Appl. Sci. 2018;42(5):1553-1560.

[11] Lan ZZ. Multi-soliton solutions for a (2+1)-dimensional variable-coefficient nonlinear schrödinger equation. Appl. Math. Lett. 2018;86:243-248.

[12] Osman MS, Machado JAT. New nonautonomous combined multi-wave solutions for (2+1)-dimensional variable coefficients KdV equation. Nonlinear Dyn. 2018;93(2):733-740.

[13] Wazwaz AM, Lakhveer K. Complex simplified Hirota¡¯s forms and Lie symmetry analysis for multiple real and complex soliton solutions of the modified KdV-Sine-Gordon equation. Nonlinear Dyn. 2019;95(3):2209-2215.

[14] Abdel-Gawad HI, Tantawy M, Osman MS. Dynamic of DNA's possible impact on its damage, Math. Method Appl. Sci. 2016;39(2):168-176.

[15] Osman MS. Multi-soliton rational solutions for quantum Zakharov-Kuznetsov equation in quantum magnetoplasmas, Waves Random Complex. 2016;26(4):434-443.

[16] Osman MS. Multi-soliton rational solutions for some nonlinear evolution equations, Open Phys. 2016;14(1):26-36.

[17] Wazwaz AM. Abundant solutions of various physical features for the (2+1)-dimensional modified kdv-Calogero-Bogoyavlenskii-Schiff equation. Nonlinear Dyn. 2017;9(3):1727-1732.

[18] Wazwaz AM, Lakhveer K. Optical solitons and Peregrine solitons for Nonlinear Schrödinger equation by variational iteration method. Optik 2019;179:804-809.

[19] Qawasmeh A, Alquran M. Soliton and Periodic Solutions for (2+1)-Dimensional Dispersive Long Water-Wave System. Appl. Math. Sci. 2014;8(50):2455-2463.

[20] Alquran M, Qawasmeh A. Soliton solutions of shallow water wave equations by means of (G'/G)-expansion method. J. Appl. Anal. Comput. 2014;4(3):221-229.

[21] Osman MS, Wazwaz AM. An efficient algorithm to construct multi-soliton rational solutions of the (2+ 1)-dimensional kdv equation with variable coefficients. Appl. Math. Comput. 2018;321:282-289.

[22] Ma WX, Yong XL, Zhang HQ. Diversity of interaction solutions to the (2+1)-dimensional Ito equation. Comput. Math. Appl. 2018;75(1):289-295.

[23] Su JJ, Gao YT, Jia SL. Solitons for a generalized sixth-order variable-coefficient nonlinear schrödinger equation for the attosecond pulses in an optical fiber. Commun. Nonlinear. Sci. 2017;50:128-141.

[24] Wazwaz AM. Negative-order integrable modified kdv equations of higher orders. Nonlinear Dyn. 2018;93(3):1371-1376.

[25] Lan ZZ, Gao B. Lax pair, infinitely many conservation laws and solitons for a (2+1)-dimensional Heisenberg ferromagnetic spin chain equation with time-dependent coefficients, Appl. Math. Lett. 2018;79:6-12.

[26] Osman MS. One-soliton shaping and inelastic collision between double solitons in the fifth-order variable-coefficient Sawada-Kotera equation. Nonlinear Dyn. 2019;96(2):1491-1496.

[27] Osman MS, Ghanbari B, Machado JAT. New complex waves in nonlinear optics based on the complex Ginzburg-Landau equation with Kerr law nonlinearity. Eur. Phys. J. Plus 2019;134(1):20.

[28] Lu D, Osman MS, Khater MMA, Attia RAM, Baleanu D. Analytical and numerical simulations for the kinetics of phase separation in iron (Fe-Cr-X (X= Mo, Cu)) based on ternary alloys. Physica A 2020;537:122634.

[29] Ma WX, Fan EG. Linear superposition principle applying to Hirota bilinear equations. Commun. Theor. Phys. 2011;61:950-959.

[30] Ma WX, Zhu ZN. Solving the (3 + 1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm. Appl. Math. Comput. 2012;218:11871-11879.

[31] Tang YN, Zai WJ. New exact periodic solitary-wave solutions for the (3+1) -dimensional generalized KP and BKP equations, Comput. Math. Appl. 2015;70(10):2432-2441.

[32] Chen ST, Ma WX. Lump solutions of a generalized Calogero-Bogoyavlenskii-Schiff equation. Comput. Math. Appl. 2018;76(7):1680-1685.

[33] Ma WX, Yong XL, Zhang HQ. Diversity of interaction solutions to the (2+1)-dimensional Ito equation. Comput. Math. Appl. 2018;75(1):289-295.

[34] Liu SJ, Tang XY, Lou SY. Multiple Darboux-Bäcklund transformations via truncated Painlevé expansion and Lie point symmetry approach. Chin. Phys. B 2018;27(6):060201.

[35] Li YZ, Liu JG. New periodic solitary wave solutions for the new (2+1)-dimensional Korteweg-de Vries equation. Nonlinear Dyn. 2018;91(1):497-504.

[36] Ma WX. Riemann-Hilbert problems of a six-component fourth-order AKNS system and its soliton solutions. Comput. Appl. Math. 2018;37(5):6359-6375.

[37] Tan W., Dai ZD, Dai HP. Dynamical analysis of lump solution for the(2+1)-Ito equation. Therm. Sci. 2017;21(4):1673-1679.