Hierarchic control of a linear heat equation with missing data

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Abstract

The paper is devoted to the Stackelberg control of a linear parabolic equation with missing initial condition. The strategy involves two controls called follower and leader. The objective of the follower is to bring the state to a desired state while the leader has to bring the system to rest at the final time. The results are obtained by means of Fenchel-Legendre transform and appropriate Carleman inequalities.

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1 Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded subset of $\mathbb{R}^N$ with boundary Γ of class $\mathcal{C}^2$. Let ω and $\mathcal{O}$ be two non-empty open subsets of Ω with $\mathcal{O} \subsetneq \omega$. For $T > 0$, we set $Q = \Omega \times (0,T)$, $\omega_T = \omega \times (0,T)$, $\mathcal{O}_T = \mathcal{O} \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider the following controlled parabolic problem

$$\left\{ \begin{array}{rcll} \frac{\partial y}{\partial t} - \triangle y + a_0 y &=& h\chi_\omega + v\chi_{\mathcal{O}} & \text{in} \quad Q \\ y &=& 0 & \text{on} \quad \Sigma \\ y(\cdot,0;\cdot) &=& g & \text{in} \quad \Omega, \end{array} \right. \tag{1}$$

where the potential $a_0 \in L^\infty(Q)$, χ_X is the characteristic function of the set $X \subset \Omega$. The controls h and v belong to $L^2(Q)$. The initial condition, $g \in L^2(\Omega)$

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is unknown. Under these assumptions on the data, we have that $y = y(h; v, g) = y(x, t; h; v, g) = y(h; v, g) \in W(0, T)$ where

$$W(0,T) = \left\{\rho | \ \rho \in L^2((0,T), H_0^1(\Omega)) \text{ and } \frac{\partial \rho}{\partial t} \in L^2((0,T); H^{-1}(\Omega))\right\} \qquad (2)$$

is a Hilbert space (see [13]).

Remark 1 *Note that if $\rho \in W(0,T)$ then $\rho \in \mathcal{C}([0,T]; L^2(\Omega))$.*

Define the set

$$\Xi(Q) = L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0,T); L^2(\Omega)).$$

Then $\Xi \subset W(0,T)$.

Any environment phenomena in a bounded domain which can be described by linear reaction diffusion equation with missing initial condition can be modelled by (1). For instance, the diffusion of a pollutant in a lake. In this case, the state variable represents the concentration of the pollutant. The missing initial condition is explained by the fact of missing information on the beginning of a pollution. In such situation, it is important to control the phenomenon. In this paper we use a two-objective optimization approach proposed by H. Von Stackelberg in [22]. The goal of the first control (called follower) is to reduce the concentration of the pollutant to a desired state while the second control (called leader) has to clear the pollutant in the lake at a given final time T. More precisely, we are interested in the following problems:

Problem 1 *For a fixed $h \in L^2(\omega_T)$ and $\gamma > 0$, find the best control $v^\gamma = v^\gamma(h) \in L^2(\mathcal{O}_T)$ solution of*

$$\inf_{v \in L^2(\mathcal{O}_T)} \sup_{g \in L^2(\Omega)} \left[J(h; v, g) - J(0; 0, g) - \gamma \left|\left|g\right|\right|^2_{L^2(\Omega)}\right], \qquad (3)$$

where

$$J(h; v, g) = \left|\left|y(h; v, g) - z_d\right|\right|^2_{L^2(Q)} + N\left|\left|v\right|\right|^2_{L^2(\mathcal{O}_T)} \qquad (4)$$

with $z_d \in L^2(Q)$ and $N > 0$.

Problem 2 *Let $v^\gamma = v^\gamma(h)$ be the control obtained in the first objective and $y^\gamma = y(x, t; h; v^\gamma, 0)$ be the associated state. Find the control $\hat{h}_\gamma \in L^2(\omega_T)$ such that*

$$y(\cdot, T; \hat{h}_\gamma; v^\gamma(\hat{h}_\gamma), 0) = 0 \quad in \ \ \Omega. \qquad (5)$$

The optimization problem (3) is actually call Low-regret control problem. This kind of robust control was introduce by J.L. Lions [1] in the early nineties in order to solve problem with missing data. Since then many authors have used this notion to control models with incomplete data. We refer for instance to [3, 4, 5, 6, 7, 8, 9, 10] and the reference therein. Problem (3) with $\gamma = 0$ is called No-regret control problem [1]. It can also be obtained if possible (it is not always the case) as a limit of the Low-regret control. These notions of Low-regret and No-regret controls were combined to the notion of average control [12] to address a problem with missing boundary condition varying parameter in the coefficient of diffusion. We also refer to [11], where such a control has been consider for electromagnetic wave displacement with an unknown velocity of propagation.

Problem 2 is a null controllability problem. There are many works in the literature on null controllability of linear heat and nonlinear equation. Most of them are achieved by means of inequality of observability of type Carleman. We refer for instance to [16, 17, 19, 20] and the reference therein. Meanwhile, Stackelberg strategy for the control of partial differential equation have been studied by some authors. J. L. Lions [23] used the Stackelberg strategy on a system governed by a parabolic equation subjected to two controls. The follower acts on the system in order to bring the state not far from the desired state while the leader has to steer the state at the final time to a small neighborhood of a given state. O. Nakoulima [24] used this concept control for a backward heat equation involving two controls to determine: one of null controllability type with a constraint on the control, called follower, and the other of optimal control type, called leader. The results were achieved by means of a Carleman inequality adapted to the constraint. In [25, 26], M. Mercan revisited the notion of controllability in the sense of Stackelberg given by O. Nakoulima [24] by choosing the follower of the minimal norm. This new notion is then applied by O. Nakoulima *et al.* in [27] on the controllability of a two-stroke problem with the constraint on the states. The results were obtained by means of Carleman inequality adapted to the constraints. Recently G. Mophou *et al.* [28] considered the Stackelberg problem for coupled parabolic equations with a finite number of constraints on one of the states. The first control was supposed to bring the solution of the coupled system subjected to a finite number of constraints at rest at time T while the second expresses that the states do not move too far from a given state.

As far as we know there is no existing work on Stackelberg control for problem with missing data. So, using the decomposition of the state to eliminate the unknown initial condition, we prove using Fenchel-Legendre transform that problem (3) is equivalent to standard optimal control that we solve. To solve the null controllability problem in Problem 2, we established an appropriate Carleman inequality satisfying by the adjoint of system which characterises the optimal control. More precisely, we prove the following results.

Theorem 1.1 *Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 1$ with boundary Γ of class C^2. Let ω and $\mathcal{O}$ be nonempty subsets of Ω. Let also $\gamma > 0$. Then for any $h \in L^2(\omega_T)$ there exists $q^\gamma = q^\gamma(x,t;h) \in \Xi(Q)$, $p^\gamma = p^\gamma(x,t;h) \in W(0,T)$ and $\zeta^\gamma = \zeta^\gamma(x,t;h;v^\gamma) \in \Xi(Q)$ such that the optimization problem (3) has a unique solution $v^\gamma = v^\gamma(h)$ associate to the state $y^\gamma = y(x,t;h;v^\gamma(h),0)$ which* *is characterized by the following optimality systems:*

$$v^\gamma = -\frac{\hat{q}_\gamma}{N} \quad in \ \mathcal{O}_T, \tag{6}$$

$$\left\{\begin{array}{rclll} \dfrac{\partial y^\gamma}{\partial t} - \Delta y^\gamma + a_0 y^\gamma & = & h\chi_\omega + v^\gamma \chi_{\mathcal{O}} & in & Q, \\ y^\gamma & = & 0 & on & \Sigma, \\ y^\gamma(\cdot, 0; h; v^\gamma, 0) & = & 0 & in & \Omega, \end{array}\right. \tag{7}$$

$$\left\{\begin{array}{rclll} -\dfrac{\partial \zeta^\gamma}{\partial t} - \Delta \zeta^\gamma + a_0 \zeta^\gamma & = & y^\gamma & in & Q, \\ \zeta^\gamma & = & 0 & on & \Sigma, \\ \zeta^\gamma(\cdot, T; h; v^\gamma) & = & 0 & in & \Omega, \end{array}\right. \tag{8}$$

$$\left\{\begin{array}{rclll} \dfrac{\partial p^\gamma}{\partial t} - \Delta p^\gamma + a_0 p^\gamma & = & 0 & in & Q, \\ p^\gamma & = & 0 & on & \Sigma, \\ p^\gamma(\cdot, 0; h) & = & \dfrac{1}{\sqrt{\gamma}}\zeta^\gamma(\cdot, 0; h) & in & \Omega, \end{array}\right. \tag{9}$$

and

$$\left\{\begin{array}{rclll} -\dfrac{\partial q^\gamma}{\partial t} - \Delta q^\gamma + a_0 q^\gamma & = & y^\gamma - z_d + \dfrac{1}{\sqrt{\gamma}} p^\gamma & in & Q, \\ q^\gamma & = & 0 & on & \Sigma, \\ q^\gamma(\cdot, T; h) & = & 0 & in & \Omega. \end{array}\right. \tag{10}$$

Moreover, there exist two constants $C(a_0, T) > 0$ and $C(a_0, T, \gamma) > 0$ such that

$$\|v^\gamma\|_{L^2(\mathcal{O}_T)} \leq C(a_0, T)\left(\|z_d\|_{L^2(Q)} + C(a_0, T, \gamma)\|h\|_{L^2(\omega_T)}\right), \tag{11}$$

where from now on, $C(X)$ is used to denote a positive constant whose value varies from a line to another but depends on X.

Theorem 1.2 *under the assumption of Theorem 1.1, there exists a unique $\hat{h}_\gamma \in L^2(\omega_T)$ such that if $(\hat{v}_\gamma = v^\gamma(\hat{h}_\gamma), \hat{y}_\gamma = y(x, t; \hat{h}_\gamma; v^\gamma(\hat{h}_\gamma), 0), \hat{\zeta}_\gamma = \zeta^\gamma(x, t; \hat{h}_\gamma; v^\gamma(\hat{h}_\gamma)), \hat{p}_\gamma = p^\gamma(x, t; \hat{h}_\gamma), \hat{q}_\gamma = q^\gamma(x, t; \hat{h}_\gamma))$ satisfies (6)-(10) then $y(x, T; \hat{h}_\gamma; v^\gamma(\hat{h}_\gamma), 0) = 0$ in Ω. Moreover,*

$$\hat{h}_\gamma = \hat{\rho}_\gamma \quad in \ \omega_T, \tag{12}$$

where

$$\left\{\begin{array}{rclll} -\dfrac{\partial \hat{\rho}_\gamma}{\partial t} - \Delta \hat{\rho}_\gamma + a_0 \hat{\rho}_\gamma & = & \hat{\phi}_\gamma + \hat{\psi}_\gamma & in & Q, \\ \hat{\rho}_\gamma & = & 0 & on & \Sigma, \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{rclll} \dfrac{\partial \hat{\psi}_\gamma}{\partial t} - \Delta \hat{\psi}_\gamma + a_0 \hat{\psi}_\gamma & = & 0 & in & Q, \\ \hat{\psi}_\gamma & = & 0 & on & \Sigma, \\ \hat{\psi}_\gamma(\cdot, 0) & = & \frac{1}{\sqrt{\gamma}}\hat{\varsigma}_\gamma(\cdot, 0; \cdot) & in & \Omega, \end{array}\right. \tag{14}$$

$$\begin{cases} \dfrac{\partial \hat{\phi}_\gamma}{\partial t} - \Delta \hat{\phi}_\gamma + a_0 \hat{\phi}_\gamma &= -\frac{1}{N}\hat{\rho}_\gamma \chi_{\mathcal{O}} \quad in \quad Q, \\ \hat{\phi}_\gamma &= 0 \qquad\qquad on \quad \Sigma, \\ \hat{\phi}_\gamma(\cdot, 0) &= 0 \qquad\qquad in \quad \Omega, \end{cases} \tag{15}$$

and

$$\begin{cases} -\dfrac{\partial \hat{\varsigma}_\gamma}{\partial t} - \Delta \hat{\varsigma}_\gamma + a_0 \hat{\varsigma}_\gamma &= \frac{1}{\sqrt{\gamma}}\hat{\phi}_\gamma \quad in \quad Q, \\ \hat{\varsigma}_\gamma &= 0 \qquad on \quad \Sigma, \\ \hat{\varsigma}_\gamma(\cdot, T) &= 0 \qquad in \quad \Omega. \end{cases} \tag{16}$$

More over there exists $C = C(a_0, T, N, \omega, z_d) > 0$ *such that*

$$\|\hat{h}_\gamma\|_{L^2(\omega_T)} \leq C.$$

The rest of this paper is organized as follows. In Section 2, we prove using Fenchel-Legendre transform that the optimization problem (3) is equivalent to an optimal control problem that we solve and, give the optimality system that characterizes the optimal control. In Section 3, we start by establishing some inequalities of Carleman type associate to the adjoint states of the optimality system characterizing the optimal control of Problem 1 and, prove that null controllability problem stated in Problem 2 holds true. A conclusion is given in Section 4.

2 Resolution of Problem 1

Solving Problem 1 is equivalent to prove Theorem 1.1. But before going further, we need to transform the optimization problem (3) into an equivalent optimal control type problem.

Lemma 2.1 *Let* $h \in L^2(\omega_T)$ *and* $\gamma > 0$. *Then, the optimization problem* (3) *is equivalent to the following standard optimal control problem: find* $v^\gamma = v^\gamma(h) \in L^2(\mathcal{O}_T)$ *such that*

$$J^\gamma(h; v^\gamma) = \inf_{v \in L^2(\mathcal{O}_T)} J^\gamma(h; v), \tag{17}$$

where

$$J^\gamma(h; v) = J(h; v, 0) - \|z_d\|^2_{L^2(Q)} + \frac{1}{\gamma}\,||\zeta(\cdot, 0; h; v)||^2_{L^2(\Omega)}\,, \tag{18}$$

with the functional J *given by* (4) *and* $\zeta := \zeta(x, t; h; v) \in W(0, T)$, *solution of*

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \Delta \zeta + a_0 \zeta &= y(h; v, 0) \quad in \quad Q, \\ \zeta &= 0 \qquad\qquad on \quad \Sigma, \\ \zeta(\cdot, T; h; v) &= 0 \qquad\qquad in \quad \Omega. \end{cases} \tag{19}$$

Proof. Let $y = y(h; v, g) := y(x, t; h; v, g)$ be the solution of (1). Then,

$$y(h; v, g) = y(h; v, 0) + y(0; 0, g), \tag{20}$$

where $y(h;v,0)$ and $y(0;0,g)$ are respectively solutions of

$$\left\{\begin{array}{rcll} \dfrac{\partial y(h;v,0)}{\partial t} - \Delta y(h;v,0) + a_0 y(h;v,0) & = & h\chi_\omega + v\chi_{\mathcal{O}} & \text{in} \quad Q, \\ y(h;v,0) & = & 0 & \text{on} \quad \Sigma, \\ y(\cdot,0;h;v,0) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{21}$$

and

$$\left\{\begin{array}{rcll} \dfrac{\partial y(0;0,g)}{\partial t} - \Delta y(0;0,g) + a_0 y(0;0,g) & = & 0 & \text{in} \quad Q, \\ y(0;0,g) & = & 0 & \text{on} \quad \Sigma, \\ y(\cdot,0;0;0,g) & = & g & \text{in} \quad \Omega. \end{array}\right. \tag{22}$$

Then in view of the data, we have that $y(h;v,0) = y(x,t;h;v,0) \in \Xi(Q)$ and $y(0;0,g) = y(x,t;0;0,g) \in W(0,T)$.
Using the decomposition (20) and the fact that

$$J(h;v,0) = \|y(h;v,0) - z_d\|^2_{L^2(Q)} + N||v||^2_{L^2(\mathcal{O}_T)}$$

and

$$J(0;0,g) = \|y(0;0,g) - z_d\|^2_{L^2(Q)},$$

we have that,

$$\begin{aligned} J(h;v,g) &= \|y(;h;v,0) + y(0;0,g) - z_d\|^2_{L^2(Q)} + N||v||^2_{L^2(\mathcal{O}_T)} \\ &= \|y(h;v,0) - z_d\|^2_{L^2(Q)} + \|y(0;0,g)\|^2_{L^2(Q)} \\ &\quad + N||v||^2_{L^2(\mathcal{O}_T)} + 2\int_Q (y(h;v,0) - z_d)y(0;0,g)dxdt \\ &= J(h;v,0) + J(0;0,g) - \|z_d\|^2_{L^2(Q)} + 2\int_Q y(h;v,0)y(0;0,g)dxdt. \end{aligned}$$

Hence,

$$J(h;v,g) - J(0;0,g) = J(h;v,0) - \|z_d\|^2_{L^2(Q)} + 2\int_Q y(h;v,0)y(0;0,g)dxdt. \tag{23}$$

Now, if we multiply the first equation of (19) by $y(0;0,g)$ and integrate by parts over Q, we have that

$$\int_Q y(h;v,0)y(0;0,g)dxdt = \int_\Omega \zeta(\cdot,0;h;v)gdx. \tag{24}$$

Combining (24) and (23), we obtain:

$$J(h;v,g) - J(0;0,g) = J(h;v,0) - \|z_d\|^2_{L^2(Q)} + 2\int_\Omega \zeta(\cdot,0;h;v)gdx. \tag{25}$$

Thus, for any $\gamma > 0$, we have

$$\begin{aligned} &\sup_{g\in L^2(\Omega)} \left[J(h;v,g) - J(0;0,g) - \gamma||g||^2_{L^2(\Omega)}\right] \\ &= J(h;v,0) - \|z_d\|^2_{L^2(Q)} + 2\sup_{g\in L^2(\Omega)} \left[\int_\Omega \zeta(\cdot,0;h;v)gdx - \frac{\gamma}{2}||g||^2_{L^2(\Omega)}\right]. \end{aligned}$$

Using the Fenchel-Legendre transform, we have that

$$2 \sup_{g\in L^2(\Omega)} \left(\langle \zeta(\cdot,0;h;v), g\rangle_{L^2(\Omega)} - \frac{\gamma}{2} ||g||^2_{L^2(\Omega)} \right) = \frac{1}{\gamma} ||\zeta(\cdot,0;h;v)||^2_{L^2(\Omega)} .$$

Consequently,

$$\begin{aligned} & \sup_{g\in L^2(\Omega)} \left[J(h;v,g) - J(0;0,g) - \gamma ||g||^2_{L^2(\Omega)} \right] \\ = \; & J(h;v,0) - \|z_d\|^2_{L^2(Q)} + \frac{1}{\gamma} ||\zeta(\cdot,0;h;v)||^2_{L^2(\Omega)} \\ = \; & J^\gamma(h;v), \end{aligned}$$

and the optimization problem (3) is equivalent to the following optimal control problem: *Let $h \in L^2(\omega_T)$. For any $\gamma > 0$, find $v^\gamma = v^\gamma(h) \in L^2(\mathcal{O}_T)$ such that* (17) *holds true.* ■

Remark 2 *If we consider problem* (3) *with $\gamma = 0$:*

$$\inf_{u\in L^2(\mathcal{O}_T)} \sup_{g\in L^2(\Omega)} \left[J(h;v,g) - J(0;0,g) \right], \tag{26}$$

then, we deal with the No-regret control problem. Therefore, in view of (25), *the No-regret control has a sense if it belongs to the set*

$$\mathcal{U} = \left\{ v \in L^2(\mathcal{O}_T) \;\; \textit{such that} \;\; \int_\Omega \zeta(\cdot,0;h;v) g dx = 0 \;\; \forall\, g \in L^2(\Omega) \right\}.$$

2.1 Proof of Theorem 1.1.

We proceed in three steps.

Step 1. We prove that for any $h \in L^2(\omega_T)$ and $\gamma > 0$, the optimization problem (3) has a unique solution $v^\gamma = v^\gamma(h) \in L^2(\mathcal{O}_T)$.

In view of Lemma 2.1, the optimization problem (3) is equivalent to the optimal control problem (17). Thus, we need to prove that (17) has a unique solution $v^\gamma = v^\gamma(h) \in L^2(\mathcal{O}_T)$.

As for any $v \in L^2(\mathcal{O}_T)$, we have $J^\gamma(h;v) \geq -\|z_d\|^2_{L^2(Q)}$, it follows that $\inf\limits_{v\in L^2(\mathcal{O}_T)} J^\gamma(h;v)$ exists. So, let $(v^n)_n \subset L^2(\mathcal{O}_T)$ be a minimizing sequence so that

$$\lim_{n\to+\infty} J^\gamma(h;v^n) = \inf_{v\in L^2(\mathcal{O}_T)} J^\gamma(h;v). \tag{27}$$

Consequently, there exists a positive constant $C(\gamma) > 0$ depending on γ such that

$$J^\gamma(h;v^n) \leq C(\gamma). \tag{28}$$

Set $y^n = y^n(x,t) = y(x,t;h;v^n,0)$. Then (v^n, y^n) satisfies:

$$\left\{ \begin{array}{rcll} \dfrac{\partial y^n}{\partial t} - \Delta y^n + a_0 y^n & = & h\chi_\omega + v^n \chi_{\mathcal{O}} & \text{in} \quad Q \\ y^n & = & 0 & \text{on} \quad \Sigma \\ y^n(\cdot,0) & = & 0 & \text{in} \quad \Omega, \end{array} \right. \tag{29}$$

and it follows from (28) and the definition of J^γ given by (18) that

$$\|y^n - z_d\|^2_{L^2(Q)} + N||v^n||^2_{L^2(\mathcal{O}_T)} + \frac{1}{\gamma}||\zeta(\cdot,0;h;v^n)||^2_{L^2(\Omega)} \leq C(\gamma) + \|z_d\|^2_{L^2(Q)} = C(\gamma, z_d).$$

Hence, we deduce that

$$||v^n||_{L^2(\mathcal{O}_T)} \leq \quad C(\gamma, z_d), \tag{30a}$$
$$\|\zeta(\cdot,0;h;v^n)\|_{L^2(\Omega)} \leq \sqrt{\gamma} C(\gamma, z_d), \tag{30b}$$
$$||y^n||_{L^2(Q)} \leq \quad C(\gamma, z_d). \tag{30c}$$

It follows from (29) that there exists a constant $C(\gamma, z_d, h) > 0$ such that ,

$$||y^n||_{W(0,T)} \leq C(\gamma, z_d, h). \tag{31}$$

Since $\zeta^n = \zeta^n(x,t) = \zeta(x,t;h;v^n)$ satisfies:

$$\left\{\begin{array}{rcll} -\dfrac{\partial \zeta^n}{\partial t} - \Delta\zeta^n + a_0\zeta^n & = & y(h;v^n,0) & \text{in} \quad Q, \\ \zeta^n & = & 0 & \text{on} \quad \Sigma, \\ \zeta^n(\cdot,T) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{32}$$

using (30c), we prove that

$$||\zeta^n||_{W(0,T)} \leq C(\gamma, z_d). \tag{33}$$

From (30), (31) and (33), we have that there exist $v^\gamma \in L^2(\mathcal{O}_T)$, $\alpha^\gamma \in L^2(\Omega)$, $y^\gamma \in W(0,T)$, $\zeta^\gamma \in W(0,T)$ and sub-sequences of $(v^n)_n$, $(y^n(T))_n$, $(y^n)_n$, $(\zeta^n)_n$ still denoted $(v^n)_n$, $(\zeta^n(0))_n$, $(y^n)_n$, $(\zeta^n)_n$ respectively such that

$$v^n \rightharpoonup v^\gamma \quad \text{weakly in} \quad L^2(\mathcal{O}_T), \tag{34a}$$
$$\zeta(\cdot,0;h;v^n) = \zeta^n(0) \rightharpoonup \quad \alpha^\gamma \quad \text{weakly in} \quad L^2(\Omega), \tag{34b}$$
$$y^n \rightharpoonup y^\gamma \quad \text{weakly in} \quad W(0,T), \tag{34c}$$
$$\zeta^n \rightharpoonup \zeta^\gamma \quad \text{weakly in} \quad W(0,T). \tag{34d}$$

If we multiply the first equation in (29) by $\Phi \in C^\infty(\bar{Q})$ such that $\Phi = 0$ on Σ, $\Phi(T) = 0$ in Ω and integrate by parts over Q, we have

$$\begin{array}{rcl} \displaystyle\int_Q \left(-\frac{\partial \Phi}{\partial t} - \Delta\Phi + a_0\Phi\right) y^n dx\, dt & = & \displaystyle\int_{\omega_T} h\Phi dx\, dt \\ & + & \displaystyle\int_{\mathcal{O}_T} v^n \Phi dx\, dt. \end{array} \tag{35}$$

Taking in (35), $\Phi \in D(Q)$, then passing to the limit when $n \to \infty$ while (30a) and (30c), we obtain that

$$\begin{array}{rcl} \displaystyle\int_Q \left(-\frac{\partial \Phi}{\partial t} - \Delta\Phi + a_0\Phi\right) y^\gamma dx\, dt & = & \displaystyle\int_{\omega_T} h\Phi dx\, dt \\ & + & \displaystyle\int_{\mathcal{O}_T} v^\gamma \Phi dx\, dt, \quad \forall \Phi \in D(Q), \end{array}$$

which after an integration by parts gives

$$\int_Q \left(\frac{\partial y^\gamma}{\partial t} - \Delta y^\gamma + a_0 y^\gamma\right) \Phi dx\, dt = \int_{\omega_T} h\Phi dx\, dt + \int_{\mathcal{O}_T} v^\gamma \Phi dx\, dt, \quad \forall \Phi \in D(Q).$$

Hence,

$$\frac{\partial y^\gamma}{\partial t} - \Delta y^\gamma + a_0 y^\gamma = h\chi_\omega + v^\gamma \chi_{\mathcal{O}} \text{in } Q. \tag{36}$$

As $y^\gamma \in L^2((0,T); H_0^1(\Omega))$, we have that

$$y^\gamma = 0 \text{ on } \Sigma. \tag{37}$$

Since y^γ and ζ^γ belong to $W(0,T)$, we have from Remark 1 that $y^\gamma(\cdot,T)$, $y^\gamma(\cdot,0)$, $\zeta^\gamma(\cdot,T)$ and $\zeta^\gamma(\cdot,0)$ exist and belong to $L^2(\Omega)$. Passing again to the limit in (35) when $n \to \infty$ while using (30a) and (30c), we obtain that

$$\int_Q \left(-\frac{\partial \Phi}{\partial t} - \Delta \Phi + a_0 \Phi\right) y^\gamma dx\, dt = \int_{\omega_T} h\Phi dx\, dt + \int_{\mathcal{O}_T} v^\gamma \Phi dx\, dt,$$
$$\forall \Phi \in C^\infty(\bar{Q}) \text{ such that } \Phi = 0 \text{ on } \Sigma \text{ and } \Phi(T) = 0 \text{ in } \Omega,$$

which after an integration by parts gives

$$\int_Q \left(\frac{\partial y^\gamma}{\partial t} - \Delta y^\gamma + a_0 y^\gamma\right) \Phi dx\, dt + \int_\Omega y^\gamma(0)\Phi(0) = \int_{\omega_T} h\Phi dx\, dt + \int_{\mathcal{O}_T} v^\gamma \Phi dx\, dt,$$
$$\forall \Phi \in C^\infty(\bar{Q}) \text{ such that } \Phi = 0 \text{ on } \Sigma \text{ and } \Phi(T) = 0 \text{ in } \Omega$$

because $y^\gamma = 0$ on Σ. In view of (36), it follows from this latter identity that

$$\int_\Omega y^\gamma(0)\Phi(0) = 0 \qquad \forall \Phi \in C^\infty(\bar{Q}) \text{ such that } \Phi = 0 \text{ on } \Sigma \text{ and } \Phi(T) = 0 \text{ in } \Omega.$$

Consequently

$$y^\gamma(0) = 0 \text{ in } \Omega. \tag{38}$$

From (36)-(38), we have that $y^\gamma = y^\gamma(x,t) = y(x,t;h;v^\gamma,0)$ is such that (v^γ, y^γ) satisfies:

$$\left\{\begin{array}{rcll} \dfrac{\partial y^\gamma}{\partial t} - \Delta y^\gamma + a_0 y^\gamma &=& h\chi_\omega + v^\gamma \chi_{\mathcal{O}} & \text{in} \quad Q, \\ y^\gamma &=& 0 & \text{on} \quad \Sigma, \\ y^\gamma(\cdot,0) &=& 0 & \text{in} \quad \Omega. \end{array}\right. \tag{39}$$

Proceeding as above for y^n, we prove that $\zeta^\gamma = \zeta^\gamma(x,t) = \zeta(x,t;h;v^\gamma) \in W(0,T)$ is solution to

$$\left\{\begin{array}{rcll} -\dfrac{\partial \zeta^\gamma}{\partial t} - \Delta\zeta^\gamma + a_0\zeta^\gamma &=& y(h;v^\gamma,0) & \text{in} \quad Q, \\ \zeta^\gamma &=& 0 & \text{on} \quad \Sigma, \\ \zeta^\gamma(\cdot,T;h;v^\gamma) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{40}$$

and

$$\zeta^n(.,0) \rightharpoonup \zeta^\gamma(\cdot,0;h;v^\gamma) = \alpha^\gamma \text{ weakly in } L^2(\Omega). \tag{41}$$

Using (34a), (34b), (34c), (41), the convexity and the lower semi-continuity (l.s.c) of J^γ, we have:

$$J^\gamma(h;v^\gamma) \leq \liminf_{n\to+\infty} J^\gamma(h;v^n) = \lim_{n\to+\infty} J^\gamma(h;v^n) = \inf_{v\in L^2(\mathcal{O}_T)} J^\gamma(h;v),$$

which implies that

$$J^\gamma(h;v^\gamma) = \inf_{v\in L^2(\mathcal{O}_T)} J^\gamma(h;v). \tag{42}$$

In addition, the strictly convexity of J^γ allows us to conclude that for any $\gamma > 0$, the control v^γ is unique.

Step 2. We prove that the optimal control v^γ, solution to the optimization problem (3) (equivalently (17)) is characterized by (6)-(10).

We already have (7) and (8) in the Step 1 (see (39) and (40)). To prove (9), (10) and (6), we write the Euler-lagrange optimality condition which characterizes v^γ:

$$\lim_{\lambda\longrightarrow 0} \frac{J^\gamma(h;v^\gamma+\lambda u) - J^\gamma(h;v^\gamma)}{\lambda} = 0 \quad \forall\, u \in L^2(\mathcal{O}_T). \tag{43}$$

We have after some calculations,

$$\begin{aligned} &\int_Q \overline{y}^\gamma(y(h;v^\gamma,0) - z_d)dxdt + N\int_{\mathcal{O}_T} v^\gamma u dxdt \\ &+\frac{1}{\gamma}\int_\Omega \bar{\zeta}^\gamma(\cdot,0;h;u)\zeta(\cdot,0;h;v^\gamma)dx = 0 \quad \forall\, u \in L^2(\mathcal{O}_T), \end{aligned} \tag{44}$$

where

$$\overline{y}^\gamma := \overline{y}^\gamma(x,t) = \frac{y(h;v^\gamma+\lambda u,0) - y(h;v^\gamma,0)}{\lambda}$$

and

$$\overline{\zeta}^\gamma := \overline{\zeta}^\gamma(x,t) = \frac{\zeta(x,t;h;v^\gamma+\lambda u) - \zeta(x,t;h;v^\gamma)}{\lambda}$$

are respectively solutions of

$$\left\{\begin{array}{rcll} \frac{\partial \overline{y}^\gamma}{\partial t} - \Delta\overline{y}^\gamma + a_0\overline{y}^\gamma &=& u\chi_{\mathcal{O}} & \text{in} \quad Q, \\ \overline{y}^\gamma &=& 0 & \text{on} \quad \Sigma, \\ \overline{y}^\gamma(\cdot,0) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{45}$$

and

$$\left\{\begin{array}{rcll} -\frac{\partial \overline{\zeta}^{\gamma}}{\partial t} - \Delta \overline{\zeta}^{\gamma} + a_0 \overline{\zeta}^{\gamma} & = & \overline{y}^{\gamma} & \text{in} \quad Q, \\ \overline{\zeta}^{\gamma} & = & 0 & \text{on} \quad \Sigma, \\ \overline{\zeta}^{\gamma}(\cdot, T) & = & 0 & \text{in} \quad \Omega. \end{array}\right. \tag{46}$$

To interpret (44), we consider $p^{\gamma} := p^{\gamma}(x, t; h)$ and $q^{\gamma} := q^{\gamma}(x, t; h)$ respectively solutions of (9) and (10).
If we multiply the first equation of (45) by q^{γ} and the first equation of (46) by $\frac{1}{\sqrt{\gamma}} p^{\gamma}$ and integrate by parts over Q we have respectively

$$\int_{\mathcal{O}_T} u q^{\gamma} dx dt = \int_Q \overline{y}^{\gamma} \left(y(h; v^{\gamma}, 0) - z_d + \frac{1}{\sqrt{\gamma}} p^{\gamma} \right) dx dt \tag{47}$$

and

$$0 = -\int_Q \frac{1}{\sqrt{\gamma}} p^{\gamma} \overline{y}^{\gamma} dx dt + \int_{\Omega} \overline{\zeta}^{\gamma}(., 0) \frac{1}{\gamma} \zeta(\cdot, 0; h; v^{\gamma}) dx \tag{48}$$

Combining (44), (47) and (48), we obtain

$$\int_{\mathcal{O}_T} u q^{\gamma} dx dt + N \int_{\mathcal{O}_T} v^{\gamma} u dx dt = 0 \quad \forall\, u \in L^2(\mathcal{O}_T)$$

which is equivalent to

$$\int_{\mathcal{O}_T} u (q^{\gamma} + N v^{\gamma}) dx dt = 0 \quad \forall\, u \in L^2(\mathcal{O}_T).$$

Therefore, we deduce (6).
Step 3. We prove (11).

We consider the linear an continuous operators

$$\begin{array}{cccll} \mathcal{L} & : & L^2(\mathcal{O}_T) & \to & L^2(Q) \\ & & u & \mapsto & z \end{array}$$

and

$$\begin{array}{cccll} \mathcal{T} & : & L^2(\mathcal{O}_T) & \to & L^2(\Omega) \\ & & u & \mapsto & \vartheta(., 0), \end{array}$$

where z and ϑ are respectively solutions to

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} - \Delta z + a_0 z & = & u \chi_{\mathcal{O}} & \text{in} \quad Q \\ z & = & 0 & \text{on} \quad \Sigma \\ z(\cdot, 0) & = & 0 & \text{in} \quad \Omega \end{array}\right. \tag{49}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial \vartheta}{\partial t} - \Delta \vartheta + a_0 \vartheta & = & z & \text{in} \quad Q \\ \vartheta & = & 0 & \text{on} \quad \Sigma \\ \vartheta(\cdot, T) & = & 0 & \text{in} \quad \Omega. \end{array}\right. \tag{50}$$

Then y^γ and ζ^γ respectively solution of (39) and (40) can be decomposed as

$$y^\gamma = z^\gamma + w$$

and

$$\zeta^\gamma = \vartheta^\gamma + \pi$$

where $z^\gamma = \mathcal{L}(v^\gamma)$, $\vartheta^\gamma = \mathcal{T}(v^\gamma)$, w and π are solution to

$$\left\{\begin{array}{rcll} \dfrac{\partial w}{\partial t} - \Delta w + a_0 w & = & h\chi_\omega & \text{in} \quad Q \\ w & = & 0 & \text{on} \quad \Sigma \\ w(\cdot, 0) & = & 0 & \text{in} \quad \Omega \end{array}\right. \tag{51}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial \pi}{\partial t} - \Delta \pi + a_0 \pi & = & w & \text{in} \quad Q \\ \pi & = & 0 & \text{on} \quad \Sigma \\ \pi(\cdot, T) & = & 0 & \text{in} \quad \Omega. \end{array}\right. \tag{52}$$

Remark 3 *Note that in view of* (49),(50), (51) *and* (52), *we have that there exists a constant* $C(a_0, T) > 0$ *such that*

$$\|\mathcal{L}u\|_{L^2((0,T);H_0^1(\Omega))} = \|z\|_{L^2((0,T);H_0^1(\Omega))} \leq C(a_0,T)\|u\|_{L^2(\mathcal{O}_T)}, \tag{53a}$$

$$\|w\|_{L^2((0,T);H_0^1(\Omega))} \leq C(a_0,T)\|h\|_{L^2(\omega_T)}, \tag{53b}$$

$$\|\mathcal{T}u\|_{L^2((0,T);H_0^1(\Omega))} = \|\vartheta(.,0)\|_{L^2(\Omega)} \leq C(a_0,T)\|\mathcal{L}u\|_{L^2(Q)}, \tag{53c}$$

$$\|\pi(.,0)\|_{L^2(\Omega)} \leq C(a_0,T)\|w\|_{L^2(\omega_T)}. \tag{53d}$$

Hence the Euler-Lagrange condition (44)can be rewritten as

$$\begin{array}{l} \displaystyle\int_Q \mathcal{L}u \, (\mathcal{L}v^\gamma + w - z_d)\, dxdt + N\int_{\mathcal{O}_T} v^\gamma \, udxdt \\ \displaystyle+\frac{1}{\gamma}\int_\Omega \mathcal{T}u \, (\mathcal{T}v^\gamma + \pi(0))\, dx = 0 \;\; \forall \; u \in L^2(\mathcal{O}_T). \end{array}$$

Taking $u = v^\gamma$ in this latter identity, we deduce that

$$\begin{array}{l} \displaystyle\|\mathcal{L}v^\gamma\|^2_{L^2(Q)} + N\|v^\gamma\|^2_{L^2(\mathcal{O}_T)} + \frac{1}{\gamma}\|\mathcal{T}v^\gamma\|^2_{L^2(\Omega)} = \\ \displaystyle\int_Q \mathcal{L}v^\gamma(z_d - w)dxdt - \frac{1}{\gamma}\int_\Omega \mathcal{T}v^\gamma \, \pi(0)dx. \end{array} \tag{54}$$

Hence using Cauchy-Schwarz inequality, we obtain from (54) that there exist constants $C(a_0,T) > 0$ and $C(a_0,T,\gamma) > 0$ such that

$$\begin{array}{rcl} N\|v^\gamma\|^2_{L^2(\mathcal{O}_T)} & \leq & \|\mathcal{L}v^\gamma\|_{L^2(Q)}\|(z_d - w)\|_{L^2(Q)} + \frac{1}{\gamma}\|\mathcal{T}v^\gamma\|_{L^2(\Omega)}\|\pi(0)\|_{L^2(\Omega)} \\ & \leq & C(a_0,T)\|v^\gamma\|_{L^2(\mathcal{O}_T)}\left(\|z_d\|_{L^2(Q)} + C(a_0,T,\gamma)\|h\|_{L^2(\omega_T)}\right). \end{array}$$

It then follows from this latter inequality that

$$\|v^\gamma\|_{L^2(\mathcal{O}_T)} \leq C(a_0,T,N)\left(\|z_d\|_{L^2(Q)} + C(a_0,T,\gamma)\|h\|_{L^2(\omega_T)}\right).$$

■

3 Resolution of Problem 2

In this section we are concerned with the prof of Theorem 1.2. More precisely, we study the null controllability problem: *For any $\gamma > 0$, find $h \in L^2(\omega_T)$ such that if $(v^\gamma, y^\gamma, q^\gamma, p^\gamma, \zeta^\gamma)$ is solution to* (6)-(10) *then*

$$y(.,T;h;v^\gamma(h),0) = 0 \text{ in } \Omega. \tag{55}$$

To solve this null controllability problem, we use a penalization method (see [14]). So, we consider the optimal control problem

$$\inf_{h \in L^2(\omega_T)} \mathcal{J}_\varepsilon^\gamma(h), \tag{56}$$

where

$$\mathcal{J}_\varepsilon^\gamma(h) = \frac{1}{2\varepsilon}\|y(.,T;h;v^\gamma(h),0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|h\|^2_{L^2(\omega_T)} \tag{57}$$

with y^γ solution of (7).

Using minimizing sequence, we prove that there exists a unique optimal control $h_\varepsilon^\gamma \in L^2(\omega_T)$ solution to (56). Writing the Euler Lagrange first order optimality condition that characterizes the optimal control , we prove that there exists $(\rho_\varepsilon^\gamma, \phi_\varepsilon^\gamma, \psi_\varepsilon^\gamma, \varsigma_\varepsilon^\gamma)$ such that

$$h_\varepsilon^\gamma = \rho_\varepsilon^\gamma \quad \text{in } \omega_T, \tag{58}$$

$$\left\{\begin{array}{rcll} -\dfrac{\partial \rho_\varepsilon^\gamma}{\partial t} - \Delta\rho_\varepsilon^\gamma + a_0\rho_\varepsilon^\gamma &=& \phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma & \text{in} \quad Q,\\ \rho_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \rho_\varepsilon^\gamma(\cdot,T) &=& -\dfrac{1}{\varepsilon}y(.,T;h_\varepsilon^\gamma;v^\gamma(h_\varepsilon^\gamma),0) & \text{in} \quad \Omega, \end{array}\right. \tag{59}$$

$$\left\{\begin{array}{rcll} \dfrac{\partial \psi_\varepsilon^\gamma}{\partial t} - \Delta\psi_\varepsilon^\gamma + a_0\psi_\varepsilon^\gamma &=& 0 & \text{in} \quad Q,\\ \psi_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \psi_\varepsilon^\gamma(\cdot,0) &=& \frac{1}{\sqrt{\gamma}}\varsigma_\varepsilon^\gamma(\cdot,0;\cdot) & \text{in} \quad \Omega, \end{array}\right. \tag{60}$$

$$\left\{\begin{array}{rcll} \dfrac{\partial \phi_\varepsilon^\gamma}{\partial t} - \Delta\phi_\varepsilon^\gamma + a_0\phi_\varepsilon^\gamma &=& -\frac{1}{N}\rho_\varepsilon^\gamma\chi_{\mathcal{O}} & \text{in} \quad Q,\\ \phi_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \phi_\varepsilon^\gamma(\cdot,0) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{61}$$

$$\left\{\begin{array}{rcll} -\dfrac{\partial \varsigma_\varepsilon^\gamma}{\partial t} - \Delta\varsigma_\varepsilon^\gamma + a_0\varsigma_\varepsilon^\gamma &=& \frac{1}{\sqrt{\gamma}}\phi_\varepsilon^\gamma & \text{in} \quad Q,\\ \varsigma_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \varsigma_\varepsilon^\gamma(\cdot,T) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{62}$$

where $y_\varepsilon^\gamma(x,t) = y(x,t;h_\varepsilon^\gamma;v^\gamma(h_\varepsilon^\gamma),0)$ is such that $(y_\varepsilon^\gamma, h_\varepsilon^\gamma, v_\varepsilon^\gamma, q_\varepsilon^\gamma, p_\varepsilon^\gamma, \zeta_\varepsilon^\gamma)$ satisfies

$$\left\{\begin{array}{rcll} \dfrac{\partial y_\varepsilon^\gamma}{\partial t} - \Delta y_\varepsilon^\gamma + a_0 y_\varepsilon^\gamma &=& h_\varepsilon^\gamma\chi_\omega + v_\varepsilon^\gamma\chi_{\mathcal{O}} & \text{in} \quad Q,\\ y_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ y_\varepsilon^\gamma(0,.) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{63}$$

$$v_\varepsilon^\gamma = -\frac{q_\varepsilon^\gamma}{N} \ \text{ in } \mathcal{O}_T, \tag{64}$$

$$\left\{\begin{array}{rcll} -\dfrac{\partial q_\varepsilon^\gamma}{\partial t} - \Delta q_\varepsilon^\gamma + a_0 q_\varepsilon^\gamma &=& y_\varepsilon^\gamma - z_d + \dfrac{1}{\sqrt{\gamma}} p_\varepsilon^\gamma & \text{in} \quad Q, \\ q_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma, \\ q_\varepsilon^\gamma(\cdot, T; h_\varepsilon^\gamma) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{65}$$

$$\left\{\begin{array}{rcll} \dfrac{\partial p_\varepsilon^\gamma}{\partial t} - \Delta p_\varepsilon^\gamma + a_0 p_\varepsilon^\gamma &=& 0 & \text{in} \quad Q, \\ p_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma, \\ p_\varepsilon^\gamma(\cdot, 0; h_\varepsilon^\gamma) &=& \dfrac{1}{\sqrt{\gamma}} \zeta^\gamma(\cdot, 0; h_\varepsilon^\gamma; v_\varepsilon^\gamma) & \text{in} \quad \Omega, \end{array}\right. \tag{66}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial \zeta_\varepsilon^\gamma}{\partial t} - \Delta \zeta_\varepsilon^\gamma + a_0 \zeta_\varepsilon^\gamma &=& y_\varepsilon^\gamma & \text{in} \quad Q, \\ \zeta_\varepsilon^\gamma &=& 0 & \text{on} \quad \Sigma, \\ \zeta_\varepsilon^\gamma(\cdot, T; h_\varepsilon^\gamma; v_\varepsilon^\gamma) &=& 0 & \text{in} \quad \Omega, \end{array}\right. \tag{67}$$

with $v_\varepsilon^\gamma = v^\gamma(h_\varepsilon^\gamma)$, $\zeta^\gamma = \zeta^\gamma(x, t; h_\varepsilon^\gamma; v^\gamma(h_\varepsilon^\gamma))$, $p_\varepsilon^\gamma = p^\gamma(x, t; h_\varepsilon^\gamma)$ and $q_\varepsilon^\gamma = q^\gamma(x, t; h_\varepsilon^\gamma)$.

In order to pass to the limit when $\varepsilon \to 0$ in (58)-(67), we need some a priori estimates on the variables $h_\varepsilon^\gamma, \rho_\varepsilon^\gamma, \phi_\varepsilon^\gamma, \psi_\varepsilon^\gamma, \varsigma_\varepsilon^\gamma, y_\varepsilon^\gamma, v_\varepsilon^\gamma, q_\varepsilon^\gamma, p_\varepsilon^\gamma$ and ζ_ε^γ. To this end, we use the so-called Carleman inequality [16, 17]. So, let ω_0 be an open subset of Ω. We know [17] that there exits a function $\Psi \in \mathcal{C}^2(\overline{\Omega})$ such that

$$\left\{\begin{array}{rcll} \Psi(x) &>& 0 & \forall x \in \Omega, \\ \Psi(x) &=& 0 & \forall x \in \Gamma, \\ |\nabla \Psi(x)| &\neq& 0 & \forall x \in \overline{\omega} \backslash \omega_0. \end{array}\right. \tag{68}$$

For any $\lambda > 0$, we consider the weight function φ and η defined by

$$\varphi(x, t) = \frac{e^{\lambda(\Psi(x) + m_1)}}{t(T - t)}, \tag{69a}$$

$$\eta(x, t) = \frac{e^{\lambda(||\Psi||_\infty + m_2)} - e^{\lambda(\Psi(x) + m_1)}}{t(T - t)}, \tag{69b}$$

where m_1 and m_2 are two reals such that $m_2 > m_1$. For any $f \in L^2(Q)$ and $a_0 \in L^\infty(Q)$, we consider the following system

$$\left\{\begin{array}{rcll} \dfrac{\partial \rho}{\partial t} - \Delta \rho + a_0 \rho &=& f & \text{in} \quad Q, \\ \rho &=& 0 & \text{on} \quad \Sigma. \end{array}\right. \tag{70}$$

Then the following result holds true [16, 17].

Proposition 3.1 (Global Carleman's inequality) *Let Ψ, φ and η be the functions defined respectively as in (68)-(69b). Let also ω' be such that $\omega_0 \subset \omega' \subset \omega$ Then, there exist numbers $\lambda_0 > 1$ and $s_0 = s_0(\Omega, a_0, T) > 1$ and there* *exists some number $C=C(\Omega,T)>0$ such that, for any $\lambda\geq\lambda_0$, for any $s\geq s_0$ and for any ρ solution of* (70) *the following inequality holds:*

$$\begin{aligned}&\int_Q \frac{e^{-2s\eta}}{s\varphi}\left|\frac{\partial\rho}{\partial t}\right|^2 dxdt+\int_Q \frac{e^{-2s\eta}}{s\varphi}\left|\Delta\rho\right|^2 dxdt+\\ &\int_Q s\lambda^2\varphi e^{-2s\eta}\left|\nabla\rho\right|^2 dxdt+\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|\rho\right|^2 dxdt\leq\\ &C\left(\int_Q e^{-2s\eta}\left|f\right|^2 dxdt+\int_{\omega'}\int_0^T s^3\lambda^4\varphi^3 e^{-2s\eta}\left|\rho\right|^2 dxdt\right).\end{aligned}\tag{71}$$

Remark 4 *Note that by the change of variable $t\mapsto T-t$, inequality* (71) *holds also true for ρ solution to*

$$\left\{\begin{array}{rcll}-\dfrac{\partial\rho}{\partial t}-\Delta\rho+a_0\rho &=& f & in\quad Q,\\ \rho &=& 0 & on\quad \Sigma\end{array}\right.$$

with $f\in L^2(Q)$ and $a_0\in L^\infty(Q)$.

Proposition 3.2 *Under the assumptions of Proposition 3.1, there exist $s_1=\max\{s_0,2C(\Omega,T)\}>0$, $\lambda_1=\max\{\lambda_0,2C(\Omega,T)\}>0$ and $C=C(\Psi,N,T,\Omega)>0$ such that for any $\lambda\geq\lambda_1$ and $s\geq s_1$, we have*

$$\begin{aligned}&\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho_\varepsilon^\gamma}{\partial t}\right|^2+|\Delta\rho_\varepsilon^\gamma|^2\right)dxdt+\int_Q s\lambda^2\varphi e^{-2s\eta}\left|\nabla\rho_\varepsilon^\gamma\right|^2 dxdt+\\ &\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dxdt\leq C\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt,\end{aligned}\tag{72}$$

for all ρ_ε^γ which satisfies (59)-(62),

Proof. Let ω' be such that $\omega'\subset\omega\subset\Omega$. We consider as in [18], the function $\theta\in C_0^\infty(\Omega)$ and such that

$$\begin{array}{l}0\leq\theta\leq 1\text{ on }\omega,\ \theta=1\text{ on }\omega',\ \theta=0\text{ on }\Omega\setminus\omega\\ \dfrac{\Delta\theta}{\sqrt{\theta}}\in L^\infty(\omega),\ \dfrac{\nabla\theta}{\sqrt{\theta}}\in\left\{L^\infty(\omega)\right\}^N.\end{array}$$

We set $u=s^3\lambda^4\varphi^3e^{-2s\eta}$. Then it follows from the definition of the functions η and φ given by (69) that

$$u(x,0)=u(x,T)=0,$$

$$\nabla u=\quad u(3\lambda+2s\lambda\varphi)\nabla\Psi,\tag{73a}$$

$$\frac{\partial u}{\partial t}=u\left[3\varphi^{-1}\frac{\partial\varphi}{\partial t}-2s\frac{\partial\eta}{\partial t}\right],\tag{73b}$$

and

$$\begin{aligned}\Delta(u\theta) &= u\theta\left(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2\right)|\nabla\Psi|^2 + u\Delta\theta \\ &\quad + u\theta\left(3\lambda + 2s\lambda\varphi\right)\Delta\Psi + u\left(6\lambda + 4s\lambda\varphi\right)\nabla\Psi.\nabla\theta.\end{aligned} \tag{74}$$

If we multiply the first equation in (59) by $\theta u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)$, where ϕ_ε^γ and ψ_ε^γ are respectively solution of (61) and (60) and integrate by parts over Q, we have

$$\begin{aligned}\int_Q u\theta\left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right)^2 dx\,dt &= -\frac{1}{N}\int_Q \theta u(\rho_\varepsilon^\gamma)^2\chi_{\mathcal{O}} dx\,dt \\ &+ \int_Q \theta\left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right)\rho_\varepsilon^\gamma \frac{\partial u}{\partial t} dx\,dt \\ &- \int_Q \left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right)\rho_\varepsilon^\gamma \Delta\left(\theta u\right) dx\,dt \\ &- 2\int_Q \rho_\varepsilon^\gamma \nabla\left(\theta u\right).\nabla\left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right) dx\,dt \\ &= K_1 + K_2 + K_3 + K_4,\end{aligned}$$

where

$$\begin{aligned}K_1 &= -\frac{1}{N}\int_Q \theta u(\rho_\varepsilon^\gamma)^2\chi_{\mathcal{O}} dx\,dt, \\ K_2 &= \int_Q \theta\left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right)\rho_\varepsilon^\gamma \frac{\partial u}{\partial t}) dx\,dt, \\ K_3 &= -\int_Q \left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right)\rho_\varepsilon^\gamma \Delta\left(\theta u\right) dx\,dt, \\ K_4 &= -2\int_Q \rho_\varepsilon^\gamma \nabla\left(\theta u\right).\nabla\left(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right) dx\,dt.\end{aligned}$$

So,

$$\int_{\omega_T} u(\phi^\gamma + \psi^\gamma)^2 dx\,dt = K_1 + K_2 + K_3 + K_4. \tag{75}$$

$$\begin{aligned}K_1 &\leq \frac{1}{N}\int_{\omega_T} u(\rho_\varepsilon^\gamma)^2 dx\,dt \\ &\leq C(N)\int_{\omega_T} s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho_\varepsilon^\gamma|^2 dx\,dt.\end{aligned}$$

Using (73b), (74) and Young inequality, we obtain that

$$K_2 \leq \frac{\delta_1}{2}\int_{\omega_T} u\left|\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma\right|^2 dx\,dt + C(\Psi,T)\int_{\omega_T} s^5\lambda^4\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.$$

$$
\begin{aligned}
K_3 &= -\int_Q (\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma \Delta(\theta u)\,dx\,dt \\
&= -\int_Q \theta u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma \left(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2\right)|\nabla\Psi|^2\,dx\,dt \\
&- \int_Q u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma \Delta\theta dx\,dt \\
&- \int_Q \theta u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho^\gamma (3\lambda + 2s\lambda\varphi)\,\Delta\Psi dx\,dt \\
&- 2\int_Q u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma (3\lambda + 2s\lambda\varphi)\,\nabla\Psi.\nabla\theta dx\,dt \\
&= K_{31} + K_{32} + K_{33} + K_{34},
\end{aligned}
$$

where

$$
\begin{aligned}
K_{31} &= -\int_Q \theta u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma \left(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2\right)|\nabla\Psi|^2\,dx\,dt \\
&= \int_Q \left\{\theta^{1/2}u^{1/2}(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\right\}\left\{-\theta^{1/2}u^{1/2}\rho_\varepsilon^\gamma \left(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2\right)|\nabla\Psi|^2\right\}dx\,dt \\
&\leq \frac{\delta_2}{2}\int_{\omega_T} u|\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma|^2 dx\,dt + C(\Psi)\int_{\omega_T} s^7\lambda^8\varphi^7 e^{-2s\eta}|\rho_\varepsilon^\gamma|^2 dx\,dt,
\end{aligned}
$$

$$
\begin{aligned}
K_{32} &= -\int_Q u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma \Delta\theta dx\,dt \\
&= \int_Q \left\{\theta^{1/2}u^{1/2}(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\right\}\left\{-u^{1/2}\rho_\varepsilon^\gamma \frac{\Delta\theta}{\theta^{1/2}}\right\}dx\,dt \\
&\leq \frac{\delta_3}{2}\int_{\omega_T} u|\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma|^2 dx\,dt \\
&+ C\int_{\omega_T} s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt,
\end{aligned}
$$

$$
\begin{aligned}
K_{33} &= -\int_Q \theta u(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma (3\lambda + 2s\lambda\varphi)\,\Delta\Psi dx\,dt \\
&= \int_Q \left\{\theta^{1/2}u^{1/2}(\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma)\right\}\left\{-\theta^{1/2}u^{1/2}\rho_\varepsilon^\gamma (3\lambda + 2s\lambda\varphi)\,\Delta\Psi\right\}dx\,dt \\
&\leq \frac{\delta_4}{2}\int_{\omega_T} u\,|\phi_\varepsilon^\gamma + \psi_\varepsilon^\gamma|^2\,dx\,dt \\
&+ C(\Psi)\int_{\omega_T} s^5\lambda^6\varphi^5 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt,
\end{aligned}
$$

$$
\begin{aligned}
K_{34} &= -2\int_Q u(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\rho_\varepsilon^\gamma\,(3\lambda+2s\lambda\varphi)\,\nabla\Psi.\nabla\theta dx\,dt\\
&= \int_Q \left\{\theta^{1/2}u^{1/2}(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right\}\left\{-2u^{1/2}\rho_\varepsilon^\gamma\,(3\lambda+2s\lambda\varphi)\,\nabla\Psi.\frac{\nabla\theta}{\theta^{1/2}}\right\}dx\,dt\\
&\le \frac{\delta_5}{2}\int_{\omega_T} u\,|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2\,dx\,dt\\
&+ C(\Psi)\int_{\omega_T} s^5\lambda^6\varphi^5 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt.
\end{aligned}
$$

Therefore

$$
K_3 \le \sum_{i=2}^{5}\frac{\delta_i}{2}\int_{\omega_T} u|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2 dx\,dt + C(\Psi)\int_{\omega_T} s^7\lambda^8\varphi^7 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt.
$$

Now we compute the term K_4. Using (73a) and Young inequality, we have

$$
\begin{aligned}
K_4 &= -2\int_Q \rho_\varepsilon^\gamma\nabla\left(\theta u\right).\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt\\
&= -2\int_Q \theta u\rho_\varepsilon^\gamma\,(3\lambda+2s\lambda\varphi)\,\nabla\Psi.\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt - 2\int_Q u\rho^\gamma\nabla\theta.\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt\\
&= K_{41}+K_{42},
\end{aligned}
$$

where

$$
\begin{aligned}
K_{41} &= -2\int_Q \theta u\rho_\varepsilon^\gamma\,(3\lambda+2s\lambda\varphi)\,\nabla\Psi.\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt\\
&= \int_Q \left\{s^{1/2}\varphi^{1/2}\theta^{1/2}e^{-s\eta}\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right\}\left\{-2s^{5/2}\lambda^4\varphi^{5/2}\theta^{1/2}e^{-s\eta}\rho_\varepsilon^\gamma\,(3\lambda+2s\lambda\varphi)\,\nabla\Psi\right\}dx\,dt\\
&\le \frac{1}{4}\int_{\omega_T} s\varphi e^{-2s\eta}\,|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)|^2\,dx\,dt\\
&+ C(\Psi)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
K_{42} &= -2\int_Q u\rho^\gamma\nabla\theta.\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt\\
&= \int_Q \left\{s^{1/2}\varphi^{1/2}\theta^{1/2}e^{-s\eta}\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right\}.\left\{-2s^{5/2}\lambda^4\varphi^{5/2}e^{-s\eta}\rho_\varepsilon^\gamma\frac{\nabla\theta}{\theta^{1/2}}\right\}dx\,dt\\
&\le \frac{1}{4}\int_{\omega_T} s\varphi e^{-2s\eta}\,|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)|^2\,dx\,dt\\
&+ C\int_{\omega_T} s^5\lambda^8\varphi^5 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt.
\end{aligned}
$$

Thus,

$$\begin{aligned}K_4 &\leq \frac{1}{2}\int_{\omega_T} s\varphi e^{-2s\eta}\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|^2 dx\,dt\\ &+ C(\Psi)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.\end{aligned}$$

Finally, in view of (75), we have that

$$\begin{aligned}\int_{\omega_T} u|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2 dx\,dt &\leq \sum_{i=1}^{5}\frac{\delta_i}{2}\int_{\omega_T} u|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2 dx\,dt\\ &+ \frac{1}{2}\int_{\omega_T} s\varphi e^{-2s\eta}\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|^2 dx\,dt\\ &+ C(\Psi,N,T)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.\end{aligned}$$

Choose in this latter identity δ_i, $1\leq i\leq 5$ such that $\displaystyle\sum_{i=1}^{5}\frac{\delta_i}{2}=\frac{1}{2}$, then using the fact that $\omega'\subset\omega$, we obtain that

$$\begin{aligned}&\int_0^T\int_{\omega'} u|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2 dx\,dt \leq \int_{\omega_T} s\varphi e^{-2s\eta}\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|^2 dx\,dt+\\ &C(\Psi,N,T)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.\end{aligned} \tag{76}$$

Now, applying (71) to $\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma$ where ϕ_ε^γ and ψ_ε^γ are respectively solution of (61) and (60), we have that there exist $\lambda>\lambda_0>1$, $s>s_0(a_0,\Omega,T)>1$ and $C=C(\Omega,T)>0$ such that

$$\begin{aligned}&\int_Q e^{-2s\eta}\left(s\lambda^2\varphi|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)|^2+s^3\lambda^4\varphi^3\left|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma\right|^2\right)dx\,dt\leq\\ &C\frac{1}{N^2}\int_{\mathcal{O}_T} e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt+C\int_0^T\int_{\omega'} s^3\lambda^4\varphi^3 e^{-2s\eta}\left|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma\right|^2 dx\,dt,\end{aligned}$$

which in view of (76), the fact that $\varphi^{-1}\in L^\infty(Q)$ and $\lambda>1$ gives

$$\begin{aligned}&\int_Q e^{-2s\eta}\left(s\lambda^2\varphi\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|+s^3\lambda^4\varphi^3\left|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma\right|^2\right)dx\,dt\leq\\ &C(\Omega,T)\int_Q s\lambda\varphi e^{-2s\eta}\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|^2 dx\,dt+C(\Psi,N)\int_Q s^2\lambda^4 e^{-2s\eta}\varphi^3\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt+\\ &C(\Psi,N,T,\Omega)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.\end{aligned} \tag{77}$$

Using the fact that $s,\lambda>1$ and φ^{-1} is bounded, then choosing $\lambda\geq\lambda_1=\max\{\lambda_0,2C(\Omega,T\}$ in (77) and we obtain that

$$\begin{aligned}&\int_Q e^{-2s\eta}\left(s\lambda^2\varphi\left|\nabla(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)\right|+s^3\lambda^4\varphi^3\left|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma\right|^2\right)dx\,dt\leq\\ &C(\Psi,N,T,\Omega)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt.\end{aligned} \tag{78}$$

Taking into account the Remark 4 and applying (71) to ρ_ε^γ solution of (59),

$$\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho_\varepsilon^\gamma}{\partial t}\right|^2 + |\Delta\rho_\varepsilon^\gamma|^2\right)dxdt + \int_Q s\lambda^2\varphi e^{-2s\eta}\,|\nabla\rho_\varepsilon^\gamma|^2\,dxdt +$$
$$\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dxdt \leq$$
$$C(\Omega,T)\int_Q e^{-2s\eta}\,|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2\,dx\,dt + C(\Omega,T)\int_0^T\int_{\omega'} s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt.$$

Using the fact that $\varphi^{-1}\in L^\infty(Q)$ and $\omega'\subset\omega$, we deduce that

$$\begin{aligned}&\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho_\varepsilon^\gamma}{\partial t}\right|^2 + |\Delta\rho_\varepsilon^\gamma|^2\right)dxdt + \int_Q s\lambda^2\varphi e^{-2s\eta}\,|\nabla\rho_\varepsilon^\gamma|^2\,dxdt +\\ &\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dxdt \leq\\ &C(\Omega,T)s^2\lambda^4\int_Q \varphi^3 e^{-2s\eta}\,|\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma|^2\,dx\,dt +\\ &C(\Omega,T)\int_{\omega_T} s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dx\,dt.\end{aligned} \tag{79}$$

Combining (78) and (79), then choosing $s\geq s_1=\max\{s_0, 2C(\Omega,T)\}$, we deduce that

$$\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho_\varepsilon^\gamma}{\partial t}\right|^2 + |\Delta\rho_\varepsilon^\gamma|^2\right)dxdt + \int_Q s\lambda^2\varphi e^{-2s\eta}\,|\nabla\rho_\varepsilon^\gamma|^2\,dxdt +$$
$$\int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}\,|\rho_\varepsilon^\gamma|^2\,dxdt + s^3\lambda^4\int_Q e^{-2s\eta}\varphi^3(\phi_\varepsilon^\gamma+\psi_\varepsilon^\gamma)dx\,dt \leq$$
$$C(\Psi,N,T,\Omega)\int_{\omega_T} s^7\lambda^{10}\varphi^7 e^{-2s\eta}\,|\rho^\gamma|^2\,dx\,dt,$$

which implies that there exist $s_1=\max\{s_0,2C(\Omega,T)\}>0$, $\lambda_1=\max\{\lambda_0,2C(\Omega,T)\}>0$ and $C=C(\Psi,N,T,\Omega)>0$ such that (72) holds true.

■

We fix $s=s_1=\max\{s_0,2C(\Omega,T)\}>0$ and $\lambda=\lambda_1=\max\{\lambda_0,2C(\Omega,T)\}>0$. Then, we consider the weight functions

$$\tilde\eta(x,t)=\begin{cases}\eta(x,\frac{T}{2}) & \text{if} \quad t\in\left]0,\frac{T}{2}\right],\\ \eta(x,t) & \text{if} \quad t\in\left[\frac{T}{2},T\right[,\end{cases} \tag{80}$$

and

$$\tilde\varphi(x,t)=\begin{cases}\varphi(x,\frac{T}{2}) & \text{if} \quad t\in\left]0,\frac{T}{2}\right],\\ \varphi(x,t) & \text{if} \quad t\in\left[\frac{T}{2},T\right[,\end{cases} \tag{81}$$

where the functions φ and η are given by (69). Then applying Proposition 3.2 with $s=s_1$ and $\lambda=\lambda_1$, we have that there exists $C=C(\Psi,N,T,\Omega)>0$ such that ρ_ε^γ solution of (59) satisfies

$$
\begin{aligned}
&\int_Q \frac{e^{-2s_1\tilde{\eta}}}{s_1\tilde{\varphi}}\left(\left|\frac{\partial\rho_\varepsilon^\gamma}{\partial t}\right|^2+|\Delta\rho_\varepsilon^\gamma|^2\right)dxdt+\int_Q s_1\lambda_1^2\tilde{\varphi}e^{-2s_1\tilde{\eta}}\left|\nabla\rho_\varepsilon^\gamma\right|^2\,dxdt+\\
&\int_Q s_1^3\lambda_1^4\tilde{\varphi}^3e^{-2s_1\tilde{\eta}}\left|\rho_\varepsilon^\gamma\right|^2\,dxdt\le C\int_{\omega_T} s_1^7\lambda_1^{10}\tilde{\varphi}^7e^{-2s_1\tilde{\eta}}\left|\rho_\varepsilon^\gamma\right|^2 dx\,dt,
\end{aligned}
\tag{82}
$$

from which we deduce that there exists $C=C(\Psi,N,T,\Omega)>0$ such that

$$
\int_Q \tilde{\varphi}^3e^{-2s_1\tilde{\eta}}\left|\rho_\varepsilon^\gamma\right|^2\,dxdt\le Cs_1^4\lambda_1^6\int_{\omega_T}\tilde{\varphi}^7e^{-2s_1\tilde{\eta}}\left|\rho_\varepsilon^\gamma\right|^2\,dxdt. \tag{83}
$$

We set

$$
\hat{\eta}(t)=\max_{x\in\Omega}\tilde{\eta}(x,t).
$$

Then $\hat{\eta}(t)>0$ and $\frac{d\hat{\eta}}{dt}(t)>0$ for $t\in(0,T)$. We define the weight function

$$
\kappa(t)=e^{-s_1\hat{\eta}(t)} \tag{84}
$$

and we have that

$$
\kappa^2(t)\le e^{-2s_1\tilde{\eta}}\text{ for }(x,t)\in Q. \tag{85}
$$

Proposition 3.3 *Let λ and s in Proposition 3.2 be such $s=s_1=\max\{s_0,2C(\Omega,T)\}>0$ and $\lambda=\lambda_1=\max\{\lambda_0,2C(\Omega,T)\}>0$. Then there exists $C=C(\Psi,N,T,\Omega,s_1,\lambda_1,a_0)>0$ such that for all ρ_ε^γ and ϕ_ε^γ which satisfy* (59)-(62),

$$
\int_Q \tilde{\varphi}^3e^{-2s_1\tilde{\eta}}\left|\rho_\varepsilon^\gamma\right|^2\,dxdt+\int_Q\kappa^2|\phi_\varepsilon^\gamma|^2dxdt\le C\int_{\omega_T}\left|\rho_\varepsilon^\gamma\right|^2\,dxdt. \tag{86}
$$

Proof. If we multiply the first equation in (61) by $\kappa^2\phi_\varepsilon^\gamma$ and integrate by part over Ω, we obtain that

$$
\begin{aligned}
&\int_\Omega\frac{1}{2}\frac{\partial}{\partial t}(\kappa\phi_\varepsilon^\gamma)^2dx+\int_\Omega s_1\frac{\partial\hat{\eta}}{\partial t}(\kappa\phi_\varepsilon^\gamma)^2dx+||\kappa\nabla\phi_\varepsilon^\gamma||^2_{L^2(\Omega)}\\
&\le||a_0||_{L^\infty(Q)}||\kappa\phi_\varepsilon^\gamma||^2_{L^2(\Omega)}+\frac{1}{2N^2}\|\kappa\rho_\varepsilon^\gamma\|_{L^2(\mathcal{O})}+\frac{1}{2}||\kappa\phi_\varepsilon^\gamma||^2_{L^2(\Omega)}
\end{aligned}
\tag{87}
$$

because

$$
\frac{1}{2}\frac{\partial}{\partial t}(\kappa\phi_\varepsilon^\gamma)^2 \quad=\quad -s_1\frac{\partial\hat{\eta}}{\partial t}(\kappa\phi_\varepsilon^\gamma)^2+\kappa^2\phi_\varepsilon^\gamma\frac{\partial\phi_\varepsilon^\gamma}{\partial t}.
$$

Observing that $s_1>0$ and $\dfrac{d\hat{\eta}}{dt}(t)>0$ for $t\in(0,T)$, we deduce from (87) that

$$
\frac{\partial}{\partial t}||\kappa\phi_\varepsilon^\gamma||^2_{L^2(\Omega)}\le(2||a_0||_\infty+1)\,||\kappa\phi_\varepsilon^\gamma||^2_{L^2(\Omega)}+\frac{1}{N^2}\kappa^2(t)\int_{\mathcal{O}}|\rho_\varepsilon^\gamma|^2dx,\quad\forall t\in(0,T).
$$

It then follows from Gronwall lemma that

$$
\begin{aligned}
||\kappa(t)\phi_\varepsilon^\gamma(t)||^2_{L^2(\Omega)} &\leq e^{(2||a_0||_\infty+1)t}\frac{t}{N^2}\kappa^2\int_{\mathcal{O}}|\rho_\varepsilon^\gamma|^2dx \\
&\leq e^{(2||a_0||_\infty+1)T}\frac{T}{N^2}\kappa^2(t)\int_{\mathcal{O}}|\rho_\varepsilon^\gamma|^2dx, \quad \forall t\in(0,T).
\end{aligned}
$$

Hence

$$
\int_Q \kappa^2(t)|\phi_\varepsilon^\gamma(x,t)|^2dxdt \leq C(a_0,T,N)\int_Q \kappa^2(t)|\rho_\varepsilon^\gamma|^2dxdt. \tag{88}
$$

Using the expression of κ given by (84), the fact that $\tilde{\varphi}^{-1}\in L^\infty(Q)$ and (85), we obtain that

$$
\int_Q \kappa^2(t)|\phi_\varepsilon^\gamma|^2dxdt \leq C(a_0,T,N)\int_Q \tilde{\varphi}^3e^{-2s_1\tilde{\eta}}|\rho_\varepsilon^\gamma|^2dxdt,
$$

which in view of (83) gives

$$
\int_Q \kappa^2(t)|\phi_\varepsilon^\gamma|^2dxdt \leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1)\int_{\omega_T}\tilde{\varphi}^7e^{-2s_1\tilde{\eta}}|\rho_\varepsilon^\gamma|^2dxdt, \tag{89}
$$

where $C=C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1)>0$.

Adding (83) to (89) and using the fact that $\tilde{\varphi}^7e^{-2s_1\tilde{\eta}}\in L^\infty(Q)$, we obtain that there exists $C=C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1)>0$ such that (86) holds true.

■

Proposition 3.4 *There exist positive constants $C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,z_d)$ and $C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d)$ such that*

$$
\begin{aligned}
||\rho_\varepsilon^\gamma||_{L^2(\omega_T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,z_d), &\text{(90a)}\\
||h_\varepsilon^\gamma||_{L^2(\omega_T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,z_d), &\text{(90b)}\\
||v_\varepsilon^\gamma||_{L^2(\mathcal{O}_T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d), &\text{(90c)}\\
||y_\varepsilon^\gamma||_{W(0,T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d), &\text{(90d)}\\
||\zeta_\varepsilon^\gamma||_{W(0,T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d), &\text{(90e)}\\
||p_\varepsilon^\gamma||_{W(0,T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d), &\text{(90f)}\\
||q_\varepsilon^\gamma||_{W(0,T)} &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma,z_d). &\text{(90g)}
\end{aligned}
$$

Proof. If we multiply the first equation of (63), (65), (66) and (67) respectively by ρ_ε^γ, ϕ_ε^γ, $\varsigma_\varepsilon^\gamma$ and ψ_ε^γ, then integrate by parts over Q, we successively obtain the following, using (64):

$$
\begin{array}{l}
\displaystyle\int_{\omega_T} h_\varepsilon^\gamma\rho_\varepsilon^\gamma dx\,dt+\frac{1}{\varepsilon}\|y(\cdot,T;h_\varepsilon^\gamma,v^\gamma(h_\varepsilon^\gamma),0)\|^2_{L^2(\Omega)}= \\
\displaystyle\int_Q y_\varepsilon^\gamma\phi_\varepsilon^\gamma dx\,dt+\int_Q y_\varepsilon^\gamma\psi_\varepsilon^\gamma dx\,dt+\frac{1}{N}\int_{\mathcal{O}_T}\rho_\varepsilon^\gamma q_\varepsilon^\gamma dx\,dt
\end{array}, \tag{91}
$$

$$
\int_Q \phi_\varepsilon^\gamma y_\varepsilon^\gamma dx\,dt-\int_Q z_d\phi_\varepsilon^\gamma dx\,dt+\frac{1}{\sqrt{\gamma}}\int_Q p_\varepsilon^\gamma\phi_\varepsilon^\gamma dx\,dt=\frac{1}{N}\int_{\mathcal{O}_T} q_\varepsilon^\gamma\rho_\varepsilon^\gamma dx\,dt, \tag{92}
$$

$$0 = \int_Q \frac{1}{\sqrt{\gamma}} p_\varepsilon^\gamma \phi_\varepsilon^\gamma dx\, dt - \int_\Omega \frac{1}{\sqrt{\gamma}} \zeta_\varepsilon^\gamma(\cdot,0;\cdot)\varsigma_\varepsilon^\gamma(\cdot,0)dx \tag{93}$$

and

$$0 = -\int_Q \psi_\varepsilon^\gamma y_\varepsilon^\gamma dx\, dt + \int_\Omega \frac{1}{\sqrt{\gamma}} \zeta_\varepsilon^\gamma(\cdot,0;\cdot)\varsigma_\varepsilon^\gamma(\cdot,0)dx \tag{94}$$

Adding the relations (91), (94), (92) and (93) together with $h_\varepsilon^\gamma = \rho_\varepsilon^\gamma \chi_\omega$, we obtain the following

$$\|\rho_\varepsilon^\gamma\|^2_{L^2(\omega_T)} + \frac{1}{\varepsilon}\|y(\cdot,T;h_\varepsilon^\gamma, v^\gamma(h_\varepsilon^\gamma),0)\|^2_{L^2(\Omega)} = \int_Q z_d \phi_\varepsilon^\gamma dx\, dt,$$

from which we deduce that

$$\begin{aligned}\|\rho_\varepsilon^\gamma\|^2_{L^2(\omega_T)} &\leq \left\|\frac{1}{\kappa} z_d\right\|_{L^2(Q)} \|\kappa\phi_\varepsilon^\gamma\|_{L^2(Q)} \\ &\leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1)\left\|\tfrac{1}{\kappa} z_d\right\|_{L^2(Q)} \|\rho_\varepsilon^\gamma\|_{L^2(\omega_T)}\end{aligned}$$

because of (86). It then follows from this latter inequality that

$$\|\rho_\varepsilon^\gamma\|_{L^2(\omega_T)} \leq C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1)\left\|\frac{1}{\kappa} z_d\right\|_{L^2(Q)},$$

and in view of (58), we have (90b).
From (11), we have that there exist two constants $C(a_0,T) > 0$ and $C(a_0,T,\gamma) > 0$ such that

$$|v_\varepsilon^\gamma\|_{L^2(\mathcal{O}_T)} \leq C(a_0,T)\left(\|z_d\|_{L^2(Q)} + C(a_0,T,\gamma)\|h_\varepsilon^\gamma\|_{L^2(\omega_T)}\right).$$

Therefore using (90b), we deduce that there exists $C = C(a_0,T,N,\Psi,\Omega,s_1,\lambda_1,\gamma) > 0$ such that (90c) holds true. Using (90b) and (90c) while computing the energy estimate of y_ε^γ solution of (63), we obtain (90d) and finally estimates (90e)-(90g).
■

3.1 Proof of Theorem 1.2

We proceed in three steps.
Step 1. We first pass to the limit in (58) and (63)-(67) when $\varepsilon \to 0$.

From (90), we have that there exist $\hat{\rho}_\gamma$, $\hat{h}_\gamma$, $\hat{v}_\gamma$, $\hat{y}_\gamma$, $\hat{\zeta}_\gamma$, $\hat{p}_\gamma$ and $\hat{q}_\gamma$ and subsequences of $(\rho_\varepsilon^\gamma)$, (h_ε^γ), (v_ε^γ), (y_ε^γ), $(\zeta_\varepsilon^\gamma)$, (p_ε^γ) and (q_ε^γ) still denoted $(\rho_\varepsilon^\gamma)$, (h_ε^γ), (v_ε^γ), (y_ε^γ), $(\zeta_\varepsilon^\gamma)$, (p_ε^γ) and (q_ε^γ) such that

$$\rho_\varepsilon^\gamma \rightharpoonup \hat{\rho}_\gamma \text{ weakly in } L^2(\omega_T), \tag{95a}$$

$$h_\varepsilon^\gamma \rightharpoonup \hat{h}_\gamma \text{ weakly in } L^2(\omega_T), \tag{95b}$$

$$v_\varepsilon^\gamma \rightharpoonup \hat{v}_\gamma \text{ weakly in } L^2(\mathcal{O}_T), \tag{95c}$$

$$y_\varepsilon^\gamma \rightharpoonup \hat{y}_\gamma \text{ weakly in } W(0,T), \tag{95d}$$

$$\zeta_\varepsilon^\gamma \rightharpoonup \hat{\zeta}_\gamma \text{ weakly in } W(0,T), \tag{95e}$$

$$p_\varepsilon^\gamma \rightharpoonup \hat{p}_\gamma \text{ weakly in } W(0,T), \tag{95f}$$

$$q_\varepsilon^\gamma \rightharpoonup \hat{q}_\gamma \text{ weakly in } W(0,T). \tag{95g}$$

Hence passing to the limit in (58) and (64) while using (95a), (95c) and (95g), have that

$$\hat{h}_\gamma = \hat{\rho}_\gamma \quad \text{in } \omega_T, \tag{96}$$

$$\hat{v}_\gamma = -\frac{\hat{q}_\gamma}{N} \quad \text{in } \mathcal{O}_T. \tag{97}$$

Because $\hat{y}_\gamma$, $\hat{\zeta}_\gamma$, $\hat{p}_\gamma$ and $\hat{q}_\gamma$ belong to $W(0,T)$, we know on the one hand that $(\hat{y}_\gamma(0), \hat{\zeta}_\gamma(0), \hat{p}_\gamma(0), \hat{q}_\gamma(0))$ and $(\hat{y}_\gamma(T), \hat{\zeta}_\gamma(T), \hat{p}_\gamma(T), \hat{q}_\gamma(T))$ exist and belong to $L^2(\Omega)$, and on the other the traces in space of $\hat{y}_\gamma$, $\hat{\zeta}_\gamma$, $\hat{p}_\gamma$ and $\hat{q}_\gamma$ exists and we have

$$\hat{y}_\gamma = \hat{\zeta}_\gamma = \hat{p}_\gamma = \hat{q}_\gamma = 0 \text{ in } \Sigma$$

because $\hat{y}_\gamma, \hat{\zeta}_\gamma, \hat{p}_\gamma, \hat{q}_\gamma \in L^2((0,T); H_0^1(\Omega))$. Consequently using standard argument, we prove while using (95) that $\hat{y}_\gamma(x,t) = y(x,t; \hat{h}_\gamma; \hat{v}_\gamma(\hat{h}_\gamma), 0)$, $\hat{\zeta}_\gamma(x,t) = \zeta^\gamma(x,t; \hat{h}_\gamma; \hat{v}_\gamma(\hat{h}_\gamma))$, $\hat{p}_\gamma(x,t) = p^\gamma(x,t; \hat{h}_\gamma)$ and $\hat{q}_\gamma(x,t) = q^\gamma(x,t; \hat{h}_\gamma)$ are respectively solution to

$$\left\{\begin{array}{rcll} \dfrac{\partial \hat{y}_\gamma}{\partial t} - \Delta \hat{y}_\gamma + a_0 \hat{y}_\gamma & = & \hat{h}_\gamma \chi_\omega + \hat{v}_\gamma \chi_{\mathcal{O}} & \text{in} \quad Q \\ \hat{y}_\gamma & = & 0 & \text{on} \quad \Sigma, \\ \hat{y}_\gamma(\cdot,0) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{98}$$

$$\left\{\begin{array}{rcll} -\dfrac{\partial \hat{q}_\gamma}{\partial t} - \Delta \hat{q}_\gamma + a_0 \hat{q}_\gamma & = & \hat{y}_\gamma - z_d + \dfrac{1}{\sqrt{\gamma}} \hat{p}_\gamma & \text{in} \quad Q, \\ \hat{q}_\gamma & = & 0 & \text{on} \quad \Sigma, \\ \hat{q}_\gamma(\cdot,T) & = & 0 & \text{in} \quad \Omega, \end{array}\right. \tag{99}$$

$$\left\{\begin{array}{rcll} \dfrac{\partial \hat{p}_\gamma}{\partial t} - \Delta \hat{p}_\gamma + a_0 \hat{p}_\gamma & = & 0 & \text{in} \quad Q, \\ \hat{p}_\gamma & = & 0 & \text{on} \quad \Sigma, \\ \hat{p}_\gamma(\cdot,0) & = & \dfrac{1}{\sqrt{\gamma}} \hat{\zeta}_\gamma(\cdot,0; \hat{h}_\gamma; \hat{v}_\gamma) & \text{in} \quad \Omega, \end{array}\right. \tag{100}$$

and

$$\left\{\begin{array}{rcll} -\dfrac{\partial \hat{\zeta}_\gamma}{\partial t} - \Delta \hat{\zeta}_\gamma + a_0 \hat{\zeta}_\gamma & = & \hat{y}_\gamma & \text{in} \quad Q, \\ \hat{\zeta}_\gamma & = & 0 & \text{on} \quad \Sigma, \\ \hat{\zeta}_\gamma(\cdot,T) & = & 0 & \text{in} \quad \Omega. \end{array}\right. \tag{101}$$

Step 2. We pass to the limit when $\varepsilon \to 0$ in (59)-(62).
Using (90a) and the fact that $\mathcal{O} \subsetneq \omega$, we have that there exists $C(a_0, T, N, \Psi, \Omega, s_1, \lambda_1, z_d) > 0$ such that

$$||\rho_\varepsilon^\gamma||_{L^2(\mathcal{O}_T)} \leq C(a_0, T, N, \Psi, \Omega, s_1, \lambda_1, z_d).$$

This latter estimation and (61), (62) and (60) allow us to prove that there exists $C = C(a_0, T, N, \Psi, \Omega, s_1, \lambda_1, z_d) > 0$ such that

$$||\phi_\varepsilon^\gamma||_{W(0,T)} \leq C, \tag{102a}$$
$$||\varsigma_\varepsilon^\gamma||_{W(0,T)} \leq C, \tag{102b}$$
$$||\psi_\varepsilon^\gamma||_{W(0,T)} \leq C. \tag{102c}$$

Hence there exist $\hat{\phi}_\gamma$, $\hat{\varsigma}_\gamma$ and $\hat{\psi}_\gamma$ such that

$$\phi_\varepsilon^\gamma \rightharpoonup \hat{\phi}_\gamma \text{ weakly in } W(0,T), \tag{103a}$$
$$\varsigma_\varepsilon^\gamma \rightharpoonup \hat{\varsigma}_\gamma \text{ weakly in } W(0,T), \tag{103b}$$
$$\psi_\varepsilon^\gamma \rightharpoonup \hat{\psi}_\gamma \text{ weakly in } W(0,T). \tag{103c}$$

Moreover using standard argument, we have that $(\hat{\psi}_\gamma, \hat{\phi}_\gamma, \hat{\varsigma}_\gamma)$ satisfies (14)-(16).
Step 3. We prove that when $\varepsilon \to 0$, we have $\rho_\varepsilon^\gamma \to \hat{\rho}_\gamma$ with $\hat{\rho}_\gamma$ solution of (13).
Set

$$\theta_1 = \min\left\{ \frac{e^{-2s_1\tilde{\eta}}}{s_1\tilde{\varphi}}, s_1\lambda_1^2\tilde{\varphi}e^{-2s_1\tilde{\eta}}, s_1^3\lambda_1^4\tilde{\varphi}^3e^{-2s_1\tilde{\eta}}, \right\}, \tag{104}$$

where $\tilde{\varphi}$ and $\tilde{\eta}$ is given by (81) and (80). Then it follows from (82) that there exists $C = C(\Psi, N, T, \Omega, s_1, \lambda_1) > 0$ such that

$$\int_Q \theta_1 \left(\left|\frac{\partial \rho_\varepsilon^\gamma}{\partial t}\right|^2 + |\Delta\rho_\varepsilon^\gamma|^2 + |\nabla\rho_\varepsilon^\gamma|^2 \, dxdt + |\rho_\varepsilon^\gamma|^2 \right) dxdt \leq C \int_{\omega_T} |\rho_\varepsilon^\gamma|^2 \, dx\, dt$$

because $\tilde{\varphi}^7 e^{-2s_1\tilde{\eta}}$ is bounded. Using (90a) in this latter inequality yields

$$\int_Q \theta_1 \left(\left|\frac{\partial \rho_\varepsilon^\gamma}{\partial t}\right|^2 + |\Delta\rho_\varepsilon^\gamma|^2 + |\nabla\rho_\varepsilon^\gamma|^2 \, dxdt + |\rho_\varepsilon^\gamma|^2 \right) dxdt \leq C \tag{105}$$

where $C = C(a_0, T, N, \Psi, \Omega, s_1, \lambda_1, z_d) > 0$. This implies that $\Delta\rho_\varepsilon^\gamma$, $\nabla\rho_\varepsilon^\gamma$ and ρ_ε^γ are bounded in $L^2(\theta_1, Q)$ where

$$L^2(\theta_1, Q) = \left\{ \rho | \quad \int_Q \theta_1 \rho^2 dx\, dt < \infty \right\}.$$

Therefore using the definitions of Ψ, $\tilde{\varphi}$ and $\tilde{\eta}$ given respectively by (68), (81) and (80), we have that there exists $C(a_0, T, N, \Psi, \Omega, s_1, \lambda_1, z_d) > 0$ such that

$$\|\rho_\varepsilon^\gamma\|_{L^2((0,T-\beta);H^2(\Omega))} \leq C, \tag{106}$$

for some $\beta > 0$. Hence $\rho_\varepsilon^\gamma \rightharpoonup \hat{\rho}_\gamma$ weakly in $D'(Q)$ and $\rho_\varepsilon^\gamma \rightharpoonup \hat{\rho}_\gamma$ weakly in $D'(\Sigma)$. Then passing to the limit in (59) when $\varepsilon \to 0$ we obtain that $\hat{\rho}_\gamma$ satisfies (13).
■

4 Conclusion

We prove that the linear heat equation with missing initial condition is Stackelberg null controllable provided that the follower control set is strictly included in the leader control set,i.e: $\mathcal{O} \subsetneq \omega$.

Notes

The authors declare that have no conflict of interest.

References

[1] J.L. Lions, Contrôle à moindres regrets des systèmes distribués, C. R. Acad. Sci. Paris Sér. I 315 (1992)1253-1257.

[2] J.L. Lions, Contrôle de Pareto de systèmes distribués. Le cas stationnaire, C. R. Acad. Sci. Paris Sér. I 302 (6)(1986) 223-227.

[3] J.L. Lions, No-regret and Low-regret control. Environment, economics and their mathematical models, 1994.

[4] O. Nakoulima, A. Omrane, J. Velin, Perturbations à moindres regrets dans les systèmes distribués à données manquantes, C. R. Acad. Sci. Paris Sr. I 330 (2000) 801806.

[5] O. Nakoulima, A. Omrane, R. Dorville. *Low-regret control of singular distributed systems: the ill-posed backwards heat problem.* Appl. Math. Lett. 17, 549–552 (2004).

[6] O. Nakoulima,A. Omrane, J. Velin. *No-regret control for nonlinear distributed systems with incomplete data.* J. Math. Pures Appl. 81, 1161–1189 (2002).

[7] B. Jacob, A. Omrane. *Optimal control for age-structured population dynamics of incomplete data.* J. Math. Anal. Appl.370(1), 42–48 (2010).

[8] O. Nakoulima, A. Omrane, J. Velin. *Low-regret perturbations in distributed systems with incomplete data.* SIAM J. Control Optim. 42(4), 1167–1184 (2003).

[9] G. Mophou. *Optimal control for fractional diffusion equations with incomplete data.* J. Optim. Theory Appl. (2015).

[10] D. Baleanu, C. Joseph and G. Mophou G. *Low-regret control for a fractional wave equation with incomplete data.* Adv Differ Equ (2016) 2016: 240. https://doi.org/10.1186/s13662-016-0970-8

[11] Abdelhak Hafdallah and Abdelhamid Ayadi. *Optimal control of electromagnetic wave displacement with an unknown velocity of propagation.* International Journal of Control, DOI: 10.1080/00207179.2018.1458157. 2018

[12] G. Mophou, R. Foko Tiomela and A. Seibou. *Optimal control of averaged state of a parabolic equation with missing boundary condition.* International Journal of Control. 2018. DOI: 10.1080/00207179.2018.1556810

[13] J.L. Lions. *Optimal control of systems governed partial differential equations.* Springer, NY, (1971).

[14] Lions J.L., Magenes E.: Problèmes aux limites non homogènes et applications. Dunod, Paris (1968)

[15] J. L. Lions. *tOptimal control of systems governed by partial differential equations problèmes aux limites.* Springer. 1971.

[16] A. Fursikov A. and O. Yu. Imanuvilov. *Controllability of evolution equations.* Lecture Notes, Research Institute of Mathematics, Seoul National University, Korea, (1996).

[17] O. Yu. Imanuvilov. *Controllability of parabolic equations.* Sbornik Mathematics 186 : 6, (1995), pp. 879-900.

[18] Lus de Teresa. *Insensitizing controls for a semilinear heat equation.* Comm. Partial Differential Equations 25 (2000), no. 1-2, 39-72.

[19] G. M. Mophou. *Null controllability with constraints on the state for the nonlinear heat equations.* Forum Mathematicum. Vol 23(2), pp. 285-319. 2011.

[20] G. M. Mophou and O. Nakoulima. *Null controllability with constraints on the state for the semilinear heat equation.* Journal of Optimization Theory and Applications. Vol 143, Issue 3, pp. 539-565. 2009.

[21] G. M. Mophou and J.-P. Puel. *Boundary sentinels with given sensitivity.* Revista Matematica Complutense. Vol 22, No 1, pp 165-185. 2009.

[22] H. von Stackelberg. *Markform undGleichgewicht.* Springer, Berlin, Germany 1934.

[23] J. L. Lions. *Some remarks on Stackelberg's optimization.* Math. Models Methods Appl. Sci. 4 (1994), no. 4, 477-487.

[24] O. Nakoulima, *Optimal control for distributed systems subject to null-controllability. Application to discriminating sentinels.* ESAIM Control Optim. Calc. Var. 13 (2007), no. 4, 623-638.

[25] M. Mercan. *Optimal control for distributed linear systems subjected to null-controllability.* Appl. Anal. 92 (2013), no. 9, 1928-1943.

[26] M. Mercan. *Optimal control for distributed linear systems subjected to null controllability with constraints on the state.* Advances in interdisciplinary mathematical research, 213-232, Springer Proc. Math. Stat., 37, Springer, New York, 2013.

[27] M. Mercan, O. Nakoulima. *Control of Stackelberg for a two-stroke problem.* Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms 22 (2015), no. 6, 441-463.

[28] M. Kéré, M. Mercan, G. Mophou. *Control of Stackelberg for coupled parabolic equations.* J. Dyn. Control Syst. 23 (2017), no. 4, 709-733.