A STRUCTURALLY DAMPED σ-EVOLUTION EQUATION WITH NONLINEAR MEMORY

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Abstract. In this paper we investigate the global existence of small data solutions for the following structurally damped σ-evolution model with nonlinear memory term

$$u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^{\frac{\sigma}{2}}u_t=\int_0^t(1+\tau)^{-\gamma}|u_t(\tau,\cdot)|^p\,d\tau,$$

with $\sigma>0$. In particular, for $\gamma\in((n-\sigma)/n,1)$ we find the sharp critical exponent, under the assumption of small data in L^1. Dropping the L^1 smallness assumption of initial data, we show how the critical exponent is consequently modified for the problem. In particular, we obtain a new interplay between the fractional order of integration $1-\gamma$ in the nonlinear memory term, and the assumption that initial data are small in L^m, for some $m>1$.

1. Introduction

We consider the nonlinear Cauchy problem

$$\begin{cases}u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^{\frac{\sigma}{2}}u_t=F(t,u_t), & x\in\mathbb{R}^n,\ t>0,\\ u(0,x)=0,\ \ u_t(0,x)=u_1(x), & \end{cases}\tag{1}$$

where $\sigma>0$, μ is a positive constant and the right-hand side is defined as

$$F(t,u_t)=\int_0^t(1+\tau)^{-\gamma}|u_t(\tau,\cdot)|^p\,d\tau,\tag{2}$$

for some $\gamma\in(0,1)$ and $p>1$. More in general, we may assume that

$$F(t,u_t)=\int_0^t(t-s)^{-\gamma}g(u_t(s,\cdot))\,ds,$$

where $g:\mathbb{R}\to\mathbb{R}$ is a locally Lipshitz function satisfying

$$g(0)=0,\quad |g(u)-g(v)|\lesssim|u-v|(|u|^{p-1}+|v|^{p-1}),\qquad\text{for some } p>1.\tag{3}$$

The linear part of the equation in (1), i.e.

$$u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^{\frac{\sigma}{2}}u_t=0\tag{4}$$

is a special case of a more general class of σ-evolution equations with structural damping

$$u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^\delta u_t=0,\qquad\sigma,\,\delta>0.\tag{5}$$

For any $\delta>0$ the equation in (5) is a dissipative σ-evolution equation; in particular, its energy

$$E(t)=\frac12\|u_t(t,\cdot)\|_{L_2}^2+\frac12\||D|^\sigma u(t,\cdot)\|_{L^2}^2,$$

is non-increasing, due to

$$E'(t)=-\mu\,\||D|^\delta u_t(t,\cdot)\|_{L^2}^2.$$

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Moreover, in the case $2\delta \in (0,\sigma)$ (effective damping, according to the definition in [5]) the solution to a Cauchy problem associated to (5) can be written as the sum of two terms, each one asymptotically behaving as the solution to a different diffusion problems (see [6] for more details). Instead, in the case $2\delta > \sigma$ in the asymptotic profile of the solution the wave structure appears and oscillations come into play (the case $\delta = \sigma = 1$ has been studied in details by R. Ikehata [12]). In [6] the authors studies equation (5); at first they obtain estimates for the solution to the linear Cauchy problem associated to (5) and then they apply those to study the critical exponent for the two corresponding nonlinear problems

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u|^p, \\ u(0,x) = 0, \quad u_t(0,x) = u_1(x), \end{cases} \tag{6}$$

and

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u_t|^p \\ u(0,x) = 0, \quad u_t(0,x) = u_1(x). \end{cases} \tag{7}$$

The critical exponents are

$$p_0(n,\sigma,\delta) := 1 + \frac{2\sigma}{(n-2\delta)_+}, \tag{8}$$

for Cauchy problem (6) and, respectively,

$$p_1(n,\sigma,\delta) := 1 + \frac{2\delta}{n}, \tag{9}$$

for Cauchy problem (7). In both the cases some restriction on the dimension appear: in fact, it is required $n < n_0 = n_0(\sigma,\delta)$ for (6) (and, respectively, $n < n_1 = n_1(\sigma,\delta)$ for (7)). Here, for $j = 0, 1$ the integer n_j is proportional to $1/(\sigma - 2\delta)$. In particular, in the limit case $2\delta = \sigma$ it holds $n_0 = n_1 = \infty$, i.e. (8) and (9) are critical exponents for any dimension $n \geq 1$ (see also [16]).

If the L^1 smallness assumption on the initial data is replaced by L^m smallness assumption, for some $m \in (1,2]$, then the two critical exponents in (8) and (9) become $p_0(n/m,\sigma,\delta)$ and $p_1(n/m,\sigma,\delta)$ (see [6, Section 2.5]).

In general, by critical exponent $\bar{p}$, in this paper we mean that

- if $p > \bar{p}$, then there exist global in time small data solutions for a suitable choice of data and solution spaces;
- if $1 < p < \bar{p}$, there exist arbitrarily small initial data, such that there exists no global in time weak solution.

The difficulty in treating the higher space dimensions $n > n_j$ is related to the loss of regularity which appears when one deals with $L^q - L^q$ estimates, with $q \in (1,2)$. Indeed, these estimates come into play when one considers power nonlinearities $|u|^p$ or $|u_t|^p$, when $p \in (1,2)$, and the critical exponent eventually becomes smaller than 2 in high space dimension n (for instance, Fujita exponent $1 + 2/n$ is smaller than 2 in space dimension $n \geq 3$).
The loss of regularity for $L^q - L^q$ estimates, with $q \in (1,2)$, is related to the wave structure of the equation at high frequencies, (this is studied in detail in [14] where the model (5) is studied in the case $\sigma = 1$ and $\delta = 0$). However, the presence of the structural damping in (5), when $\delta > 0$, generates a smoothing effect on the solution, which does not appear for the classical damping u_t (see later Proposition 2.1). This smoothing effect allows us to recover the additional regularity by using estimates which are singular at $t = 0$. The singularity order is proportional to $n(\sigma - 2\delta)/2\delta$, and it vanishes at $\sigma = 2\delta$. This effect explains, roughly speaking, the possibility to employ these estimates in higher space dimensions when $\sigma/(2\delta)$ tends to 1. This motivates our choice to fix $\delta = \sigma/2$ to better investigates the influence of the nonlinear memory term on the equation.

Recently, many authors investigated fractional PDEs from different points of view, since they are particularly interesting for the real world applications and the description of memory and hereditary process. In particular, it is of interest to understand how to treat nonlinear evolution problems in which the nonlinearity is represented by some memory term; for instance, one could consider a nonlinearity like $F(t, u_t)$

defined in (2), or even

$$G(t,u)=\int_0^t (1+\tau)^{-\gamma}|u(\tau,\cdot)|^p\,d\tau.$$

We remark that

$$F(t,u_t)=\Gamma(1-\gamma)J_{0|t}^{1-\gamma}|u_t(t,\cdot)|^p,\qquad G(t,u)=\Gamma(1-\gamma)J_{0|t}^{1-\gamma}|u(t,\cdot)|^p$$

where Γ is the Euler function and $J_{0|t}^{1-\gamma}(v)$ denotes the fractional Riemann-Liouville integral of a function v. Hence, we have

$$\lim_{\gamma\to 1}\Gamma(1-\gamma)F(t,u_t)=|u_t(t,\cdot)|^p$$

and, respectively

$$\lim_{\gamma\to 1}\Gamma(1-\gamma)G(t,u)=|u(t,\cdot)|^p;$$

thus, one can expect some relations with the case of a power nonlinearity $|u_t|^p$ and, respectively $|u|^p$, as $\gamma\to 1$.

In [1] the authors consider the Cauchy problem for the heat equation

$$\begin{cases} v_t-\Delta u=G(t,u) & t\geq 0,\ x\in\mathbb{R}^n,\\ v(0,x)=v_0(x), \end{cases}\tag{10}$$

and they prove that the critical exponent for (10) is given by

$$\bar{p}_{0,\gamma}(n)=\max\{\gamma^{-1},p_{0,\gamma}(n)\},\qquad \text{where}\qquad p_{0,\gamma}(n)=1+\frac{2(2-\gamma)}{n-2(1-\gamma)}.\tag{11}$$

Other diffusive models with nonlinear memory are treated in [10] and [17]; in particular, in this latter paper also fractional derivatives in time are considered in the linear part of the equation.

In [4, 9] the authors study the nonlinear Cauchy problem

$$\begin{cases} u_{tt}-\Delta u+\mu u_t=G(t,u)\\ u(0,x)=0,\ u_t(0,x)=u_1(x), \end{cases}\tag{12}$$

and they prove that the critical exponent is again (11), as for the Cauchy problem (10) for any $n\leq 5$; this is reasonable since the solution to the linear Cauchy problem associated to (12), behaves asymptotically like the solution the linear problem associated to (10) with a suitable initial datum v_0 (see, for instance [13, 14, 15]).

Furthermore, in [3] the first author considers the Cauchy problem

$$\begin{cases} u_{tt}-\Delta u+\mu(-\Delta)^{\frac12}u_t=\int_0^t(t-s)^{-\gamma}|u(\tau,\cdot)|^p\,d\tau,\\ u(0,x)=0,\ u_t(0,x)=u_1(x), \end{cases}\tag{13}$$

and he proves that global small data solutions exist for $p>\bar{p}_{1,\gamma}(n)$ where

$$\bar{p}_{1,\gamma}(n)=\max\{\gamma^{-1},p_{1,\gamma}(n)\},\qquad \text{where}\qquad p_{1,\gamma}(n)=1+\frac{3-\gamma}{n+\gamma-2}\tag{14}$$

for any $n\geq 2$; moreover, this exponent is optimal.

We remark that the threshold $(n-2)/n$ which denotes the transition to the critical exponent γ^{-1} is the same for both the Cauchy problems (12) and (13). Moreover, it holds $\bar{p}_{1,\gamma}(n)>\bar{p}_{0,\gamma}(n)$ only when γ belongs to the interval $((n-2)/n,1)$; instead, if $\gamma\in(0,(n-2)/n)$ it holds $\bar{p}_{1,\gamma}(n)=\bar{p}_{0,\gamma}(n)=\gamma^{-1}$. This suggests that if the nonlinear memory term is sufficiently strong in space dimension $n\geq 3$, then the influence from classical and structural damping is the same. Otherwise, the critical exponent is larger in the structural damping case, as it happens for the power nonlinearity $|u|^p$.

Finally, we recall that the beam equation has been investigated in the case of a nonlinear memory term in [7].

In this paper we want to investigate the critical exponent for the Cauchy problem (1). As discussed before, the special structure of equation (4) gives some benefit in the estimates for the solution to the linear problem; in particular, it is possible to get non-singular estimates also for u_t which can be easily applied to study the nonlinear problem. We will prove that, under L^1 smallness assumption for the initial datum u_1, global solutions to (1) exist for any $p > \bar{p}_{\gamma,1}(\sigma, n)$, where

$$p > \bar{p}_{\gamma,1}(\sigma,n) := \begin{cases} p_\gamma(\sigma,n) & \text{if } \gamma > [(n-\sigma)/n, 1) \\ \gamma^{-1} & \text{if } \gamma \in (0,(n-\sigma)/n) \end{cases}, \qquad p_\gamma(\sigma,n) := 1 + \frac{(2-\gamma)\sigma}{n-\sigma(1-\gamma)}. \tag{15}$$

We are ready to state our first result.

Theorem 1.1. *Let $\sigma > 0$ and $\gamma \in (0,1)$; let us assume $p > \bar{p}_{\gamma,1}(\sigma, n)$.*
Then, there exists $\epsilon > 0$, sufficiently small, such that for any

$$u_1 \in L^1 \cap L^\infty \quad \textit{with} \quad \|u_1\|_{L^1\cap L^\infty} \leqslant \epsilon, \tag{16}$$

there is a uniquely determined energy solution

$$u \in \mathcal{C}^1\left([0,\infty), L^1 \cap L^\infty\right) \cap C([0,\infty), H^\sigma) \tag{17}$$

to (1)*. Furthermore, for $j = 0, 1$, the solution satisfies the following estimates:*

$$\|\partial_t^j u(t,\cdot)\|_{L^q} \lesssim \begin{cases} C(1+t)^{2-j-\gamma-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\|u_1\|_{L^1\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) < 2-j, \\ C(1+t)^{-\gamma}\log(2+t)\|u_1\|_{L^1\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) = 2-j, \\ C(1+t)^{-\gamma}\|u_1\|_{L^1\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) > 2-j. \end{cases} \tag{18}$$

Moreover, $\|(-\Delta)^{\frac{\sigma}{2}} u(t,\cdot)\|_{L^2}$ verifies the same estimate verified by $\|u_t(t,\cdot)\|_{L^2}$ in (18)*. Namely,*

$$E(t) \leq \begin{cases} (1+t)^{2(1-\gamma)-\frac{n}{\sigma}}\|u_1\|^2_{L^1\cap L^\infty} & \textit{if } n < 2\sigma, \\ (1+t)^{-2\gamma}(\log(2+t))^2\|u_1\|^2_{L^1\cap L^\infty} & \textit{if } n = 2\sigma, \\ (1+t)^{-2\gamma}\|u_1\|^2_{L^1\cap L^\infty} & \textit{if } n > 2\sigma. \end{cases}$$

Remark 1.1. In estimates (18) it does not appear a decay rate better than $(1+t)^{-\gamma}$. This restriction comes from the influence of the nonlinear memory term (see Lemma 4.2). In the best case scenario, a loss of decay $(1+t)^{1-\gamma}$ appears with respect to the corresponding linear estimates (see Proposition 2.1). This phenomenon has been described in [4].

If we drop the L^1 smallness assumption for the initial data u_1, replacing it by L^m smallness, for some $m > 1$, we expect that this loss of information on the datum does not influence the critical exponent for (1), for "sufficiently small" γ, with respect to m. This expectation is motivated by the fact that the presence of the nonlinear memory term induces a loss of decay rate which is $(1+t)^{1-\gamma}$, or even larger, with respect to the corresponding linear estimate with L^1 data. On the other hand, assuming initial data in L^m, with $m > 1$, leads to a loss of decay rate which is $(1+t)^{\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}$, with respect to assume initial data in L^1. Consequently, the critical exponent is modified only when this latter loss is greater than the loss related to the nonlinear memory term.

In particular, if initial data are assumed to be small in L^m, for some

$$1 < m < \frac{n^2}{(n^2-\sigma^2)_+},$$

then we show that global solutions exist for any $p > \bar{p}_{\gamma,m}(\sigma, n)$, where

$$\bar{p}_{\gamma,m}(\sigma,n) := \max\{\gamma^{-1},\, p_\gamma(\sigma,n),\, \tilde{p}_m(\sigma,n)\}, \quad \tilde{p}_m(\sigma,n) := 1 + \frac{(2-\gamma)\sigma m}{n}.$$

Explicitly, we may compute

$$\bar{p}_{\gamma,m}(\sigma,n) = \begin{cases} \tilde{p}_m(\sigma,n) & \text{if } \gamma \in [\bar{\gamma}, 1), \\ p_\gamma(\sigma,n) & \text{if } \gamma \in [(n-\sigma)/n, \bar{\gamma}), \\ \gamma^{-1} & \text{if } \gamma \in (0,(n-\sigma)/n), \end{cases}$$

where

$$\bar{\gamma} = \bar{\gamma}(m) = 1 - \frac{n}{\sigma}\Big(1 - \frac{1}{m}\Big).$$

We notice that $\bar{\gamma} > (n-\sigma)/n$ due to the assumption $m < n^2/(n^2-\sigma^2)$.

Indeed, it is easy to show that for a fixed $m \in (1, \frac{n^2}{(n^2-\sigma^2)_+})$, the statement of Theorem 1.1 remains valid if $\gamma \in (0, \bar{\gamma}]$ and $p > \bar{p}_{\gamma,1}(\sigma, n)$, as in (15), replacing the initial data assumption (16) by the weaker condition

$$u_1 \in L^m \cap L^\infty \quad \text{with} \quad \|u_1\|_{L^m\cap L^\infty} \leqslant \epsilon, \tag{19}$$

and the solution space (17) by

$$u \in \mathcal{C}^1\left([0,\infty), L^m \cap L^\infty\right) \cap C([0,\infty), H^\sigma) \tag{20}$$

if $m \in (1, 2]$, and by

$$u \in \mathcal{C}^1\left([0,\infty), L^m \cap L^\infty\right) \tag{21}$$

if $m > 2$. The proof also requires only minor modifications with respect to the proof of Theorem 1.1. In particular, we stress that $p > m$, due to

$$p > p_\gamma(\sigma, n) \geq \tilde{p}_m(\sigma, n) = 1 + \frac{(2-\gamma)\sigma m}{n} \geq 1 + \frac{(2-\bar{\gamma})\sigma m}{n} = m\left(1 + \frac{\sigma}{n}\right) > m,$$

which follows from the assumption $\gamma \leq \bar{\gamma}$.

Having this in mind, we may focus to the case $\gamma \in (\bar{\gamma}, 1)$, in which the critical exponent $\tilde{p}_m(\sigma, n)$ comes into play.

Theorem 1.2. *Let $\sigma > 0$, $m \in (1, n^2/(n^2-\sigma^2)^+)$ and $\gamma \in (\bar{\gamma}, 1)$; let us assume $p \geq \tilde{p}_m(\sigma, n)$. Then, there exists $\epsilon > 0$, sufficiently small, such that for any*

$$u_1 \in L^m \cap L^\infty \quad \textit{with} \quad \|u_1\|_{L^m\cap L^\infty} \leqslant \epsilon,$$

there is a uniquely determined energy solution

$$u \in \mathcal{C}^1\left([0,\infty), L^m \cap L^\infty\right) \cap \mathcal{C}([0,\infty), H^\sigma)$$

to (1)*, if $m \leq 2$, or a uniquely determined Sobolev solution*

$$u \in \mathcal{C}^1\left([0,\infty), L^m \cap L^\infty\right)$$

to (1)*, if $m > 2$. Furthermore, the solution satisfies the following estimates for $j = 0, 1$ and $q \in [m, \infty]$:*

$$\|\partial_t^j u(t,\cdot)\|_{L^q} \lesssim \begin{cases} C(1+t)^{1-j-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)}\|u_1\|_{L^m\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) < 2-j, \\ C(1+t)^{-1+\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\log(2+t)\|u_1\|_{L^m\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) = 2-j, \\ C(1+t)^{-1+\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\|u_1\|_{L^m\cap L^\infty} & \textit{if } \frac{n}{\sigma}\left(1-\frac{1}{q}\right) > 2-j. \end{cases} \tag{22}$$

Moreover, $\|(-\Delta)^{\frac{\sigma}{2}}u(t,\cdot)\|_{L^2}$ verifies the same estimate verified by $\|u_t(t,\cdot)\|_{L^2}$ in (22)*, if $m \leq 2$. Namely,*

$$E(t) \leq \begin{cases} (1+t)^{-\frac{n}{\sigma}\left(\frac{2}{m}-1\right)}\|u_1\|^2_{L^m\cap L^\infty} & \textit{if } n < 2\sigma, \\ (1+t)^{-2+\frac{2n}{\sigma}\left(1-\frac{1}{m}\right)}(\log(2+t))^2\|u_1\|^2_{L^m\cap L^\infty} & \textit{if } n = 2\sigma, \\ (1+t)^{-2+\frac{2n}{\sigma}\left(1-\frac{1}{m}\right)}\|u_1\|^2_{L^m\cap L^\infty} & \textit{if } n > 2\sigma. \end{cases}$$

Remark 1.2. We remark that the assumption $m < n^2/(n^2-\sigma^2)^+$ also implies that $m < n/(n-\sigma)_+$, so that $p > m$ as a consequence of

$$p \geq \tilde{p}_m(\sigma, n) = 1 + \frac{(2-\gamma)\sigma m}{n} > 1 + \frac{\sigma m}{n} > m.$$

Indeed, the latter is trivial if $\sigma \geq n$, whereas it is equivalent to $m < n/(n-\sigma)$ otherwise.

Remark 1.3. The main difference in the decay rate profile in Theorem 1.2 with respect to Theorem 1.1 is that no loss of decay rate is caused by the presence of the nonlinear memory term in (22), when

$$\frac{n}{\sigma}\left(1-\frac{1}{q}\right) < 2-j,$$

with respect to the linear problem (see Proposition 2.1).

Finally, we will employ the test function method to prove that no weak solution exists if $1 < p < \max\{p_\gamma(\sigma, n), \tilde{p}_m(\sigma, n)\}$, under suitable sign assumption on initial data in L^m.

This allows us to conclude that $\bar{p}_{\gamma,m}(\sigma, n)$ is really the critical exponent for any $\gamma \in ((n-\sigma)/n, 1)$. It remains an open problem to prove a satisfying nonexistence result when γ belongs to the interval $(0, (n-\sigma)/n)$.

Theorem 1.3. *Let us assume $u_1 \in L^1_{loc}$, u_1 non-negative. Then, there exists no global weak solution to* (1) *in the following cases:*

- *if $n \leq (1-\gamma)\sigma$ for any $\gamma \in (0,1)$ and $p > 1$;*
- *if $n > (1-\gamma)\sigma$, for any $\gamma \in (0,1)$ and $1 < p < p_\gamma(\sigma, n)$.*

Moreover, if $\sigma/2 \in \mathbb{N}$, no global weak solution exists also in the case

- *$n > (1-\gamma)\sigma$, for any $\gamma \in ((n-\sigma)/n, 1)$ and $p = p_\gamma(\sigma, n)$.*

Theorem 1.4. *Let us assume $u_1 \in L^m$ such that*

$$u_1(x) \geq c\,|x|^{-\frac{n}{m}}(\log(|x|))^{-1}, \;\; \forall |x| \gg 1. \tag{23}$$

Then, there exists no global weak solution to (1) *in the following cases:*

- *if $n \leq (1-\gamma)\sigma$ for any $\gamma \in (0,1)$ and $p > 1$;*
- *if $n > (1-\gamma)\sigma$, for any $\gamma \in (0,1)$ and $1 < p < \max\{p_\gamma(\sigma, n), \tilde{p}_m(\sigma, n)\}$.*

In view of the previous result we can conclude that $\bar{p}_{\gamma,m}(\sigma, n)$ is the critical exponent for any $\gamma \in ((n-\sigma)/n, 1)$. In particular we remark that, as we expected, for $\gamma \to 1$ the critical exponent $p_\gamma(\sigma, n)$ tends to $p_1(\sigma, \sigma/2, n)$, as defined in (9), and the critical exponent $\tilde{p}_m(\sigma, n)$ tends to $p_1(\sigma, \sigma/2, n/m)$, as $\gamma \to 1$.

However, it is interesting to remark that the critical exponent for the L^m theory, when $m > 1$, is not obtained by dividing the space dimension n by m, as it happens for the structurally damped evolution equation with power nonlinearity $|u_t|^p$ in [6], but it is smaller than this latter, due to

$$\tilde{p}_m(\sigma, n) = 1 + \frac{(2-\gamma)\sigma m}{n} < 1 + \frac{(2-\gamma)\sigma m}{n - m\sigma(1-\gamma)} = p_\gamma(\sigma, n/m).$$

This phenomenon is due to the interesting interplay between the loss of decay rate due to the nonlinear memory, and the loss of decay rate due to the different space assumption for the initial data. This interplay has a lesser negative influence on the decay rate profile of the solution, than one may expect.

2. Proof of Theorem 1.1

In order to prove our existence results, we need to employ decay estimates for the linear Cauchy problem associated to (1), that is

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\sigma}{2}} u_t = 0 \\ u(0,x) = 0, \;\; u_t(0,x) = u_1(x), . \end{cases} \tag{24}$$

In [6, 8, 16] the authors prove the following decay estimates for (24).

Proposition 2.1. Let $j = 0, 1$. Then, for any $1 \leq m \leq q \leq \infty$ the solution $u = u(t,x)$ to (24) satisfies the following estimates:

$$\|\partial_t^j u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)+1-j} \|u_1\|_{L^m}, \tag{25}$$

and, as a consequence,

$$\|\partial_t^j u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)+1-j} \|u_1\|_{L^m \cap L^q}. \tag{26}$$

Moreover, we have

$$\|(-\Delta)^{\frac{\sigma}{2}} u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u_1\|_{L^m\cap L^q}. \tag{27}$$

Remark 2.1. We remark that in estimate (26) the assumption of additional regularity L^q for the datum allows to avoid the singularity which appears in (25) when $m < q$.

Let us now introduce some notation for the proof of the global (in time) existence of small data solutions. Throughout this section, we denote by $K_1(t,x)$ the fundamental solution to the linear Cauchy problem (1) with initial data $u_0 = 0$ and $u_1 = \delta_0$, where δ_0 is the Dirac distribution in $x = 0$ with respect to the spatial variable. As a consequence, we may represent the solution to the Cauchy problem (24) in the form

$$u^{\mathrm{lin}}(t,x) = K_1(t,x) *_{(x)} u_1(x).$$

We may introduce the operator

$$P:\, u \in X(T) \to Pu(t,x) := u^{\mathrm{lin}}(t,x) + Nu(t,x),$$

where $X(T)$ is an evolution space which we will define in a suitable way. Then, we define Nu an integral operator with the following representation

$$Nu(t,x) \doteq \int_0^t K_1(t-s,x) *_{(x)} F(s,u_t(\cdot,x))\,ds. \tag{28}$$

According to the Duhamel's principle, we can consider a global (in time) solution to (1) as a fixed point of the operator P. Hence, in order to get the global (in time) existence and uniqueness of the solution in $X(T)$, we need to prove the following two crucial estimates:

$$\|Pu\|_{X(T)} \lesssim \|u_1\|_{L^m\cap L^\infty} + \|u\|_{X(T)}^p, \tag{29}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right), \tag{30}$$

uniformly with respect to T. Thus, we show the desired property for the operator P providing that $\|u_1\|_{L^m\cap L^\infty} = \epsilon$ is sufficiently small. As a consequence of the Banach fixed point theorem, the conditions (29) and (30) guarantee the existence of a uniquely determined solution u to (1) that is $u = u^{lin} + Nu$. We simultaneously gain a local and a global existence result. In the proof of our global existence result, it will be useful the following straightforward estimates (see, for instance, [2]).

Proposition 2.2. Let $\alpha \in \mathbb{R}$, $\beta > 1$ and $\gamma \in (0,1)$. Then,

$$\int_0^t (1+t-s)^{-\alpha} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-\beta}\,d\tau\,ds \lesssim \begin{cases} (1+t)^{-\gamma} & \text{if } \alpha > 1, \\ (1+t)^{-\gamma}\log(2+t) & \text{if } \alpha = 1, \\ (1+t)^{1-\alpha-\gamma} & \text{if } \alpha < 1. \end{cases}$$

Proposition 2.3. Let $\alpha \in \mathbb{R}$ and β, $\gamma \in (0,1)$. Then,

$$\int_0^t (1+t-s)^{-\alpha} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-\beta}\,d\tau\,ds \lesssim \begin{cases} (1+t)^{1-\beta-\gamma} & \text{if } \alpha > 1, \\ (1+t)^{1-\beta-\gamma}\log(2+t) & \text{if } \alpha = 1, \\ (1+t)^{2-\alpha-\beta-\gamma} & \text{if } \alpha < 1. \end{cases}$$

We are now ready to prove Theorem 1.1.

Proof. [Theorem 1.1] If $n > \sigma$, we define

$$q_1 = \frac{n}{n-\sigma},$$

that is, the solution to

$$\frac{n}{\sigma}\left(1-\frac{1}{q}\right) = 1.$$

If $n > 2\sigma$, we also define

$$q_0 = \frac{n}{n-2\sigma},$$

that is, the solution to

$$\frac{n}{\sigma}\left(1-\frac{1}{q}\right)=2.$$

We now consider the solution space

$$X(T)=\mathcal{C}^1([0,T],L^1\cap L^\infty)\cap\mathcal{C}([0,T],H^\sigma),$$

equipped with the norm

$$\|u\|_{X(T)}:=\|u\|_{X_0(T)}+\sup_{s\in[0,T]}\big\{(1+s)^{\gamma-1}\big(\|u_t(s,\cdot)\|_{L^1}+(1+s)^{\frac{n}{\sigma}}\|u_t(s,\cdot)\|_{L^\infty}\big)\big\}$$

if $n<\sigma$, or

$$\|u\|_{X(T)}:=\|u\|_{X_0(T)}+\sup_{s\in[0,T]}\big\{(1+s)^{\gamma-1}\|u_t(s,\cdot)\|_{L^1}+(1+s)^{\gamma}(\ell(s)^{-1}\|u_t(s,\cdot)\|_{L^{q_1}}+\|u_t(s,\cdot)\|_{L^\infty})\big\}$$

if $n\geq\sigma$, where $\ell(t):=\log(2+t)$ and $X_0(t)$ denotes the evolution space $C([0,T],L^1\cap L^\infty\cap H^\sigma)$ equipped with the norm

$$\|u\|_{X_0(t)}:=\sup_{s\in[0,T]}\big\{(1+s)^{\gamma-2}\big(\|u(s,\cdot)\|_{L^1}+(1+s)^{\frac{n}{\sigma}}\|u(s,\cdot)\|_{L^\infty}+(1+s)^{\frac{n}{2\sigma}+1}\|u(s,\cdot)\|_{\dot H^\sigma}\big)\big\}\,,$$

if $n<2\sigma$, or

$$\|u\|_{X_0(t)}:=\sup_{s\in[0,T]}\big\{(1+s)^{\gamma-2}\|u(s,\cdot)\|_{L^1}+(1+s)^{\gamma}\,\ell(s)^{-1}\big(\|u(s,\cdot)\|_{L^\infty}+\|u(s,\cdot)\|_{\dot H^\sigma}\big)\big\}\,,$$

if $n=2\sigma$, or

$$\begin{aligned}\|u\|_{X_0(t)}:=\sup_{s\in[0,T]}&\big\{(1+s)^{\gamma-2}\|u(s,\cdot)\|_{L^1}+(1+s)^{\gamma}\,\ell(s)^{-1}\|u(s,\cdot)\|_{L^{q_0}}\\&+(1+s)^{\gamma}\left(\|u(s,\cdot)\|_{L^\infty}+\|u(s,\cdot)\|_{\dot H^\sigma}\right)\big\}\,,\end{aligned}$$

if $n>2\sigma$.

Applying Proposition 2.1, it follows immediately

$$\|u^{\rm lin}\|_{X(T)}\lesssim\|u_1\|_{L^1\cap L^\infty},\tag{31}$$

and so we conclude $u^{\rm lin}\in X(T)$.

In the remaining part of the proof we will estimate Nu in the $X(T)$ norm. To do this, we first remark that for any $t\in[0,T]$ it is possible to estimate by interpolation

$$\|u_t(t,\cdot)\|_{L^q}\lesssim(1+t)^{\left(1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)\right)^+-\gamma}\ell(t)\|u\|_{X(T)},\tag{32}$$

for any $q\in[1,\infty]$. In particular, the decay becomes $(1+t)^{-\gamma}$ for any $q>q_1$, when $n>\sigma$.

By using the derived $L^1\cap L^q-L^q$ estimates stated in (26), applying the Minkowski's integral inequality, we get

$$\|\partial_tNu(t,\cdot)\|_{L^q}\lesssim\int_0^t(1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\int_0^s(s-\tau)^{-\gamma}\||u_t(\tau,\cdot)|^p\|_{L^1\cap L^q}\,d\tau\,ds,$$

for $q=1,\infty$, and for $q=q_1$ if $n>\sigma$. By (32) we know that

$$\||u_t(\tau,\cdot)|^p\|_{L^1}=\|u_t(\tau,\cdot)\|_{L^p}^p\lesssim(1+t)^{\left(p-\frac{n}{\sigma}(p-1)\right)^+-p\gamma}\ell(t)^p\|u\|_{X(T)}^p;\tag{33}$$

moreover, for any $q\geq q_1$ since $qp>q_1$ we get

$$\||u_t(\tau,\cdot)|^p\|_{L^q}=\|u_t(\tau,\cdot)\|_{L^{qp}}^p\lesssim(1+t)^{-p\gamma}\ell(t)^p\|u\|_{X(T)}^p.\tag{34}$$

If, and only if, $p>\bar p_{\gamma,1}(\sigma,n)$, then from estimate (33) we may deduce that there exists an exponent $\beta_\gamma(n,\sigma,p)>1$ such that

$$\|u_t(\tau,\cdot)\|_{L^p}^p\lesssim(1+t)^{-\beta_\gamma(n,\sigma,p)}\|u\|_{X(T)}^p.\tag{35}$$

In particular, for any $q > q_1$ from estimate (34) we see that $\|u_t(\tau,\cdot)\|_{L^{qp}}^p$ also satisfies estimate (35). Thus, for any $q = 1,\, q_1,\, \infty$ we get

$$\|\partial_t N u(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^p \int_0^t (1+t-s)^{-\frac{n}{\sigma}(1-\frac{1}{q})} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-\beta_\gamma(n,\sigma,p)}\, d\tau\, ds.$$

Therefore, for any $p > \bar{p}_{\gamma,1}(n)$ we can apply Proposition 2.2 to get the following desired estimates

$$\|\partial_t N u(t,\cdot)\|_{L^1} \lesssim (1+t)^{1-\gamma}\|u\|_{X(T)}^p;$$

moreover, for $n \neq \sigma$ we get

$$\|\partial_t N u(t,\cdot)\|_{L^\infty} \lesssim (1+t)^{(1-\frac{n}{\sigma})^+-\gamma}\|u\|_{X(T)}^p;$$

finally, for $n \geq \sigma$ we find

$$\|\partial_t N u(t,\cdot)\|_{L^{q_1}} \lesssim (1+t)^{-\gamma}\ell(t)\|u\|_{X(T)}^p.$$

In a similar way, we estimate $\|Nu(t,\cdot)\|_{L^q}$ for $q = 1, \infty$, and q_0 if $n > 2\sigma$. We apply (26) to get

$$\|Nu(t,\cdot)\|_{L^q} \lesssim \int_0^t (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{q})} \int_0^s (s-\tau)^{-\gamma}\||u_t(\tau,\cdot)|^p\|_{L^1\cap L^q}\, d\tau\, ds.$$

Thus, using estimate (35) we find

$$\|Nu(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^p \int_0^t (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{q})} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-\beta_\gamma(n,\sigma,p)}\, d\tau\, ds.$$

Again, for $p > \bar{p}_{\gamma,1}(n)$, we can apply Proposition 2.2 to get the following desired estimates of $\|Nu\|_{L^q}$:

$$\|Nu(t,\cdot)\|_{L^1} \lesssim (1+t)^{2-\gamma}\|u\|_{X(T)}^p,$$

moreover, for $n \neq 2\sigma$ we get

$$\|Nu(t,\cdot)\|_{L^\infty} \lesssim (1+t)^{(2-\frac{n}{\sigma})^+-\gamma}\|u\|_{X(T)}^p;$$

furthermore, for any $n \geq 2\sigma$ we have

$$\|Nu(t,\cdot)\|_{L^{q_0}} \lesssim (1+t)^{-\gamma}\ell(t)\|u\|_{X(T)}^p.$$

Finally, we have

$$\|(-\Delta)^{\frac{\sigma}{2}} Nu(t,\cdot)\|_{L^2} \lesssim \|u\|_{X(T)}^p \int_0^t (1+t-s)^{-\frac{n}{2\sigma}} \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-\beta_\gamma(n,\sigma,p)}\, d\tau\, ds,$$

which implies

$$\|(-\Delta)^{\frac{\sigma}{2}} Nu(t,\cdot)\|_{L^2} \lesssim (1+t)^{(1-\frac{n}{2\sigma})^+-\gamma}\ell(t)^b\|u\|_{X(T)}^p,$$

where $b = 0$ for any $n \neq 2\sigma$ and $b = 1$ if $n = 2\sigma$.

Collecting the above derived estimates and (31) we have proved

$$\|u^{\mathrm{lin}}\|_{X(T)} + \|u^{\mathrm{non}}\|_{X(T)} \lesssim \|u_1\|_{L^1\cap L^\infty} + \|u\|_{X(T)}^p,$$

which allows us to get the desired estimate (29).

In order to prove (30) we may rewrite it as

$$\|Nu - Nv\|_{X(T)} \lesssim \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right).$$

Then, we use (3) together with Hölder inequality to get

$$\|g(u)-g(v)\|_{L^1\cap L^q} \lesssim \|g(u)-g(v)\|_{L^1\cap L^q}\left(\|u\|_{L^1\cap L^q}^{p-1} + \|v\|_{L^1\cap L^q}^{p-1}\right),$$

and we proceed as in the proof of (29) to get the desired result.

The proof of the decay estimates in (18) follows by straightforward calculations, thanks to the choice of the norm of $X(T)$. $\square$

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall modify the norm of $X(T)$ to take into account of the different profile of the decay rate of the solution, due to the influence of the L^m regularity of the initial datum.

Proof. [Theorem 1.2] Now, for a given $T>0$, we fix

$$X(T)=\mathcal{C}^1([0,T],L^m\cap L^\infty)\cap\mathcal{C}([0,T],H^\sigma),$$

if $m\leq 2$, and

$$X(T)=\mathcal{C}^1([0,T],L^m\cap L^\infty),$$

if $m>2$.

Assume first that $n<\sigma$. Then we set

$$\|u\|_{X_0(t)}:=\sup_{s\in[0,T]}\left\{(1+s)^{-1}\|u(s,\cdot)\|_{L^m}+(1+s)^{-1+\frac{n}{m\sigma}}\|u(s,\cdot)\|_{L^\infty}+(1+s)^{\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(-\Delta)^{\frac{\sigma}{2}}u(s,\cdot)\|_{L^2}\right\},$$

if $m\leq 2$ and

$$\|u\|_{X_0(t)}:=\sup_{s\in[0,T]}\left\{(1+s)^{-1}\|u(s,\cdot)\|_{L^m}+(1+s)^{-1+\frac{n}{m\sigma}}\|u(s,\cdot)\|_{L^\infty}\right\},$$

if $m>2$. Then we define

$$\|u\|_{X(T)}:=\|u\|_{X_0(t)}+\sup_{s\in[0,T]}\left\{\|u_t(s,\cdot)\|_{L^m}+(1+s)^{\frac{n}{m\sigma}}\|u_t(s,\cdot)\|_{L^\infty}\right\}.$$

If $n\geq\sigma$, we modify the norm into

$$\begin{aligned}\|u\|_{X(T)}:=\|u\|_{X_0(t)}+\sup_{s\in[0,T]}&\Big\{\|u_t(s,\cdot)\|_{L^m}+(1+s)^{\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma,n)}\right)}\|u_t(s,\cdot)\|_{L^{\tilde{p}_m(\sigma,n)}}\\&+(1+s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\big(\ell(s)^{-1}\|u_t(s,\cdot)\|_{L^{q_1}}+\|u_t(s,\cdot)\|_{L^\infty}\big)\Big\},\end{aligned}$$

where $q_1=n/(n-\sigma)$ is the solution to

$$\frac{n}{\sigma}\left(1-\frac{1}{q}\right)=1,$$

as in the proof of Theorem 1.1. Moreover, if $n\geq 2\sigma$, we replace $(1+s)^{-1+\frac{n}{m\sigma}}\|u(s,\cdot)\|_{L^\infty}$ in $\|u\|_{X_0(T)}$ by

$$(1+s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\big(\ell(s)^{-1}\|u(s,\cdot)\|_{L^{q_0}}+\|u(s,\cdot)\|_{L^\infty}\big),$$

where $q_0=n/(n-2\sigma)$ is the solution to

$$\frac{n}{\sigma}\left(1-\frac{1}{q_0}\right)=2,$$

as in the proof of Theorem 1.1. We proceed similarly to modify $(1+s)^{\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(-\Delta)^{\frac{\sigma}{2}}u(s,\cdot)\|_{L^2}$ in $\|u\|_{X_0(T)}$.

Applying Proposition 2.1, it follows immediately that

$$\|u^{\mathrm{lin}}\|_{X(T)}\lesssim\|u_1\|_{L^m\cap L^\infty},\tag{36}$$

and so we conclude $u^{\mathrm{lin}}\in X(T)$. It remains to estimate Nu to prove (29)-(30).

If $u\in X(T)$, then for any $p\geq\tilde{p}_m(\sigma,n)$, we may estimate

$$\|u(t,\cdot)\|_{L^p}\leq\|u\|_{X(T)}\,(1+t)^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma,n)}\right)}.\tag{37}$$

By using the derived $L^1\cap L^q-L^q$ estimates stated in (26), applying the Minkowski's integral inequality, we get for $q\in[1,\infty]$:

$$\|\partial_t Nu(t,\cdot)\|_{L^q}\lesssim\|u\|_{X(T)}^p\int_0^t(1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\int_0^s(s-\tau)^{-\gamma}\||u_t(\tau,\cdot)|^p\|_{L^1\cap L^p}\,d\tau\,ds.$$

Thus, employing estimates (37), due to $p\geq\tilde{p}_m(\sigma,n)$, we get

$$\|\partial_t Nu(t,\cdot)\|_{L^q}\lesssim\|u\|_{X(T)}^p\int_0^t(1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\int_0^s(s-\tau)^{-\gamma}(1+\tau)^{\frac{n}{\sigma}\left(1-\frac{1}{m}\right)+\gamma-2}\,d\tau\,ds.$$

Indeed, for $p \geq \tilde{p}_m(\sigma, n)$, we get

$$(1+\tau)^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{\tilde{p}_m(\sigma,n)}\right)p} \leq (1+\tau)^{-\frac{n}{\sigma}\left(\frac{\tilde{p}_m(\sigma,n)}{m}-1\right)} = (1+\tau)^{\frac{n}{\sigma}\left(1-\frac{1}{m}\right)+\gamma-2}.$$

In particular,

$$2-\gamma-\frac{n}{\sigma}\left(1-\frac{1}{m}\right) < 1,$$

due to $\gamma > \bar{\gamma} = 1 - n(1-1/m)/\sigma$.

We shall now distinguish two cases. If $n < \sigma$ or $q < q_1$, that is,

$$\frac{n}{\sigma}\left(1-\frac{1}{q}\right) < 1,$$

by applying Proposition 2.3, due to

$$2-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)-\gamma-\left(2-\gamma-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)\right) = \frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right),$$

we obtain:

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim \|u\|^p_{X(T)} \lesssim (1+t)^{-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u\|^p_{X(T)}.$$

If $n \geq \sigma$ and $q = q_1$, or, respectively, $q > q_1$, then, by applying Proposition 2.3, due to

$$1-\gamma-\left(2-\gamma-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)\right) = -1+\frac{n}{\sigma}\left(1-\frac{1}{m}\right),$$

we obtain:

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim \|u\|^p_{X(T)} \lesssim (1+t)^{-1+\frac{n}{\sigma}\left(1-\frac{1}{m}\right)} \ell(t) \|u\|^p_{X(T)},$$

or, respectively,

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim \|u\|^p_{X(T)} \lesssim (1+t)^{-1+\frac{n}{\sigma}\left(1-\frac{1}{m}\right)} \|u\|^p_{X(T)}.$$

We proceed similarly to estimate $\|Nu(t,\cdot)\|_{L^q}$ and $\|(-\Delta)^{\frac{\sigma}{2}} Nu(t,\cdot)\|_{L^2}$ if $m \leq 2$.

This concludes the proof of estimate (29). Finally, with the same approach used in the proof of Theorem 1.1 we get estimate (30). □

4. Proof of Theorem 1.3

In order to prove our non-existence result, we employ the test functions method. In particular, to treat the non-local differential operators $(-\Delta)^\sigma$ and $(-\Delta)^{\sigma/2}$ we apply the following fundamental lemma:

Lemma 4.1 ([11]). *Let $\langle x\rangle := (1+|x|^2)^{1/2}$ and $q > n$. Fixed $\theta \notin \mathbb{N}$, we define $m_\theta = [\theta]$, the greatest integer which is smaller than θ, and $s_\theta = \theta - m_\theta$. Then, there exists a positive constant $A_{n,q,\theta}$ depending only on n, q and θ such that for any $x \in \mathbb{R}^n$,*

$$|((-\Delta)^\theta \langle\cdot\rangle^{-q})(x)| \leq \langle x\rangle^{-n-2s_\theta}. \tag{38}$$

Proof. If $\theta = s_\theta \in (0,1)$, then the proof follows by straightforward calculations as in Lemma 1.5 in [11]*. In the case $\theta = m_\theta + s_\theta$ with $m_\theta \geq 1$ we use the representation*

$$((-\Delta)^\theta \langle\cdot\rangle^{-q})(x) = ((-\Delta)^{s_\theta}((-\Delta)^{m_\theta}\langle\cdot\rangle^{-q}))(x),$$

and we remark that $((-\Delta)^{m_\theta}\langle\cdot\rangle^{-q})(x)$ can be written as a linear combination of functions $\varphi_i(x) := \langle x\rangle^{-q_i}$, with $q_i > n$. Then, we apply the already known result to $((-\Delta)^{s_\theta}\langle\cdot\rangle^{-q_i})(x)$ to conclude the proof. □

Remark 4.1. If $\theta \in \mathbb{N}$ it is immediate the following estimate:

$$|(-\Delta)^\theta \langle x\rangle^{-q}| \lesssim \langle x\rangle^{-q-2\theta},$$

for any $q > n$.

4.1. **The general case $\sigma > 0$ and $p < p_\gamma(\sigma, n)$, $m \geq 1$.** For any $\alpha \in (0,1)$ and for a fixed $T > 0$ we denote by $J^\alpha_{0|t}$, $J^\alpha_{t|T}$ the fractional integral operators defined by

$$J^\alpha_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)} f(s)\, ds,$$

$$J^\alpha_{t|T} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{-(1-\alpha)} f(s)\, ds,$$

and the following fractional differential operators

$$D^\alpha_{0|t} := \partial_t J^{1-\alpha}_{0|t}, \quad D^\alpha_{t|T} := \partial_t J^{1-\alpha}_{t|T}.$$

The following properties are satisfied

$$\int_0^T (D^\alpha_{0|t} f)(t) g(t) dt = \int_0^T f(t) (D^\alpha_{t|T} g)(t) dt, \tag{39}$$

$$D^\alpha_{0|t} J^\alpha_{0|t} f(t) = f(t). \tag{40}$$

Let us introduce the function $\omega := \omega(t) \in C_c([0,\infty))$ defined by

$$\omega(t) = \begin{cases} (1 - t/T) & \text{if } t \in [0,T], \\ 0 & \text{if t>T.} \end{cases} \tag{41}$$

Then, it holds $\operatorname{supp} \omega = [0,T]$ and $\omega(t)^\beta \in C^k_c([0,\infty))$ for any $\beta > k \geq 0$. Moreover, the following useful lemma is satisfied.

Lemma 4.2. *For any $\beta > 0$ it holds*

$$\int_t^\infty \omega(\tau)^\beta \, d\tau = \frac{T}{\beta+1} \omega(t)^{\beta+1}. \tag{42}$$

Furthermore, for any $\alpha \in (0,1)$ and $\beta > \alpha$ it holds

$$D^\alpha_{t|T} \omega(t)^\beta = C(\alpha,\beta) T^{-\alpha} \omega(t)^{\beta-\alpha}, \tag{43}$$

where

$$C(\alpha,\beta) = \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}.$$

Proof. The identity of (42) can be easily obtained by applying the definition of ω stated in (41). For the proof of (43) see for instance Lemma 4.1 in [3]. □

We set $\alpha := 1 - \gamma$ and we fix β such that

$$\beta > (\alpha+1) p' \tag{44}$$

where $p' = p/(p-1)$ is the Hölder conjugate of p. Let $\Psi = \Psi(x) \in C^\infty(\mathbb{R}^n)$ be defined by

$$\Psi(x) = \langle x \rangle^{-q}, \text{ for some } q > n, \tag{45}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Then, for any $R > 0$ we define $\Psi_R(x) = \Psi(x/R)$ and

$$\Phi_R(t,x) = \omega(t)^\beta \Psi_R(x).$$

Finally we introduce the functions

$$\varphi(t,x) := D^\alpha_{t|T} \Phi_R(t,x), \quad \tilde{\varphi}(t,x) := \int_t^\infty \varphi(\tau,x) d\tau,$$

defined for any $t \geq 0$ and $x \in \mathbb{R}^n$. It is clear that we have $\partial_t \tilde{\varphi} = -\varphi$; moreover, by Lemma 4.2 we can conclude that $\operatorname{supp} \varphi$, $\operatorname{supp} \tilde{\varphi} \subset [0,T] \times \mathbb{R}^n$; explicitely:

$$\varphi(t,x) = C(\alpha,\beta) T^{-\alpha} \omega(t)^{\beta-\alpha} \Psi_R(x); \tag{46}$$

$$\tilde{\varphi}(t,x) = \frac{C(\alpha,\beta)}{\beta+1-\alpha} T^{1-\alpha} \omega(t)^{\beta-\alpha+1} \Psi_R(x). \tag{47}$$

Let us suppose by contraddiction that there exists u a global weak solution to (1). Then, for any $R>0$ we can define the integral term

$$I_R := \Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^\alpha_{0|t}(|u_t|^p)\varphi\,dx\,dt.$$

By properties (39) and (40) we find on one hand the identity

$$I_R = \Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u_t|^p\Phi_R\,dx\,dt. \tag{48}$$

On the other hand, since u satisfies the Cauchy problem (1), applying integration by parts we get

$$I_R+\int_{\mathbb{R}^n}u_1(x)\varphi(0,x)\,dx = -\int_0^T\int_{\mathbb{R}^n}u_t\varphi_t\,dx\,dt+\int_0^T\int_{\mathbb{R}^n}u_t(\mu(-\Delta)^{\frac{\sigma}{2}}\varphi-(-\Delta)^\sigma\tilde{\varphi})\,dt\,dx.$$

By using Young inequality we can estimate

$$\int_0^T\int_{\mathbb{R}^n}|u_t||\varphi_t|\,dx\,dt\leq\epsilon\int_0^T\int_{\mathbb{R}^n}|u_t|^p\Phi_R\,dx\,dt+C_\epsilon\int_0^T\int_{\mathbb{R}^n}|\varphi_t|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt$$

By (46), since $\omega'(t)=-1/T$, we find

$$\varphi_t(t,x) = -C(\alpha,\beta)(\beta-\alpha)T^{-1-\alpha}\omega(t)^{\beta-\alpha-1}\Psi_R(x).$$

Thus, there exists $C>0$ such that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|\varphi_t|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt &= CT^{-(\alpha+1)p'}\int_0^T\int_{\mathbb{R}^n}\omega(t)^{\beta-(\alpha+1)p'}\Psi(x/R)\,dx\,dt\\ &\lesssim T^{-(\alpha+1)p'+1}R^n;\end{aligned}$$

in the last estimate we used that the exponent $\beta-(\alpha+1)p'$ is positive and the function $\Psi=\Psi(x)$ is integrable on $\mathbb{R}^n$. In order to treat the remaining integral terms we use Lemma 4.1; in particular, since $q>n$, there exists $\delta>0$ such that

$$|(-\Delta)^\sigma\Psi_R(x)|\lesssim\langle x\rangle^{-n-\delta},\qquad |(-\Delta)^{\sigma/2}\Psi_R(x)|\lesssim\langle x\rangle^{-n-\delta};$$

moreover, we recall that the following identity holds for any $\theta>0$:

$$(-\Delta)^\theta\Psi_R(x) = R^{-2\theta}((-\Delta)^\theta\Psi)(x/R). \tag{49}$$

Thus, applying again Young inequality we find

$$\int_0^T\int_{\mathbb{R}^n}|u_t||(-\Delta)^{\frac{\sigma}{2}}\varphi|\,dt\,dx\leq\epsilon\int_0^T\int_{\mathbb{R}^n}|u_t|^p\Phi_R\,dx\,dt+C_\epsilon\int_0^T\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\sigma}{2}}\varphi\right|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt.$$

By (38), (46) and (49) with $\theta=\sigma/2$ we find

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\sigma}{2}}\varphi\right|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt &\lesssim T^{-\alpha p'}R^{-\sigma p'}\int_0^T\int_{\mathbb{R}^n}\omega(t)^{\beta-\alpha p'}\langle x/R\rangle^{-n-\delta}\,dx\,dt\\ &\lesssim T^{-\alpha p'+1}R^{n-\sigma p'};\end{aligned}$$

in the last estimate we used that the exponent $\beta-\alpha p'$ is positive due to assumption (44) and the function $\langle x\rangle^{-n-\sigma}$ is integrable. In a similar way we estimate the last integral term:

$$\int_0^T\int_{\mathbb{R}^n}|u_t|\,|(-\Delta)^\sigma\tilde{\varphi}|\,dt\,dx\leq\epsilon\int_0^T\int_{\mathbb{R}^n}|u_t|^p\Phi_R\,dx\,dt+C_\epsilon\int_0^T\int_{\mathbb{R}^n}|(-\Delta)^\sigma\tilde{\varphi}|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt.$$

By (38), (47) and (49) with $\theta=\sigma$ we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|(-\Delta)^\sigma\tilde{\varphi}|^{p'}\Phi_R^{-\frac{1}{p-1}}\,dx\,dt &\lesssim T^{(1-\alpha)p'}R^{-2\sigma p'}\int_0^T\int_{\mathbb{R}^n}\omega(t)^{\beta-(\alpha-1)p'}\langle x/R\rangle^{-n-2\delta}\,dx\,dt\\ &\lesssim T^{(1-\alpha)p'+1}R^{n-2\sigma p'}.\end{aligned} \tag{50}$$

Collecting the previous estimates we obtain that there exists $C > 0$ such that

$$(1-3\epsilon)I_R + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx \leq CT^{1-\alpha p'}R^n(T^{-p'} + R^{-\sigma p'} + T^{p'}R^{-2\sigma p'}). \tag{51}$$

Being $\varphi \geq 0$, since u_1 is non-negative, we find

$$\int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx \geq 0.$$

Thus, fixing $\epsilon \in (0,1/3)$, by (51), we get

$$I_R \lesssim T^{1-\alpha p'}R^n(T^{-p'} + R^{-\sigma p'} + T^{p'}R^{-2\sigma p'}).$$

Therefore, if $p < p_\gamma(\sigma,n)$ it is sufficient to set $R = T^{\frac{1}{\sigma}}$ to get

$$I_R \lesssim T^{1+n/\sigma-(\alpha+1)p'}.$$

In fact, it holds $1 + n/\sigma - (\alpha+1)p' < 0$ if, and only if, $p < p_\gamma(\sigma,n)$, or $n < \alpha\sigma$. In this case, by Beppo Levi's theorem on monotone convergence, since $\Phi_{T^{1/\sigma}} \nearrow 1$ as $T \to \infty$, we derive

$$\lim_{T\to\infty} I_{T^{\frac{1}{\sigma}}} = \int_0^\infty \int_{\mathbb{R}^n} |u_t|^p\,dx\,dt \equiv 0;$$

hence, $u_t \equiv 0$. In order to complete the proof of Theorem 1.3 it remains to treat the critical case $p = p_\gamma(\sigma,n)$ in the case $\sigma/2 \in \mathbb{N}$.

4.2. **The critical exponent $p_\gamma(\sigma,n)$ in the case $\sigma/2 \in \mathbb{N}$.** In the critical case $\bar{p} := p_\gamma(\sigma,n)$, we set $R = T^{\frac{1}{\sigma}}K^{-\frac{1}{\sigma}}$ for some fixed $K > 1$. Then, from estimate (51) we get the existence of $C > 0$ such that

$$(1-3\epsilon)\int_0^T \int_{\mathbb{R}^n} |u_t|^p \Phi_{(T/K)^{1/\sigma}}\,dx\,dt \leq CK^{-\frac{n}{\sigma}}(1 + K^{\bar{p}'} + K^{2\bar{p}'});$$

thus, taking $T \to \infty$ we find

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t|^p\,dx\,dt \leq C(K), \tag{52}$$

for some $C(K) > 0$, i.e. $u_t \in L^p$.

In order to prove our result in the critical case $p = p_\gamma(\sigma,n)$ when $\sigma/2 \in \mathbb{N}$, it is sufficient tu use a classical test function; we will use the same notations as in Section 4.1. Let $\Psi \in C_c^\infty(\mathbb{R}^n)$ be a radial test function such that

- supp $\Psi = B_1$;
- $\Psi(x) = 1$, for any $x \in B_{1/2}$;
- Ψ is decreasing.

Then, for any integer $\theta \in \mathbb{N}$, it holds $(-\Delta)^\theta \Psi_R(x) = 0$ for any $|x| \leq R/2$.

Applying the same ideas used in Section 4.1 we can estimate

$$\Gamma(\alpha)\int_0^T \int_{\mathbb{R}^n} J^\alpha_{0|T}(|u_t|^{\bar{p}})\varphi\,dx\,dt \leq -\int_0^T \int_{\mathbb{R}^n} u_t\varphi_t\,dx\,dt + \int_0^T \int_{\mathbb{R}^n} u_t((-\Delta)^{\frac{\sigma}{2}}\varphi - (-\Delta)^\sigma\tilde{\varphi})\,dt\,dx. \tag{53}$$

Then, taking $R = T^{1/\sigma}K^{-1/\sigma}$, we find

$$\int_0^T \int_{\mathbb{R}^n} |u_t||\varphi_t|\,dx\,dt \leq \epsilon\int_0^T \int_{\mathbb{R}^n} |u_t|^{\bar{p}}\Phi_R\,dx\,dt + C_\epsilon K^{-\frac{n}{\sigma}}.$$

and

$$\int_0^T \int_{\mathbb{R}^n} |u_t||(-\Delta)^{\frac{\sigma}{2}}\varphi|\,dt\,dx \leq \epsilon\int_0^T \int_{\mathbb{R}^n} |u_t|^{\bar{p}}\Phi_R\,dx\,dt + C_\epsilon K^{-\frac{n}{\sigma}+\bar{p}'}.$$

In the critical case we can improve the estimate for the last integral term: we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|u_t||(-\Delta)^\sigma\tilde{\varphi}|\,dx\,dt &= \int_0^T\int_{|x|>(T/K)^{1/\sigma}}|u_t||(-\Delta)^\sigma\tilde{\varphi}|\,dx\,dt\\ &\leq \left(\int_0^T\int_{|x|>(T/K)^{1/\sigma}}|(-\Delta)^\sigma\tilde{\varphi}|^{\bar{p}'}\,\Phi_R^{-\frac{1}{\bar{p}-1}}\,dx\,dt\right)^{\frac{1}{\bar{p}'}}\\ &\times\left(\int_0^T\int_{|x|>(T/K)^{1/\sigma}}|u_t|^{\bar{p}}\Phi_R\,dx\,dt\right)^{\frac{1}{\bar{p}}}.\end{aligned}$$

In analogous way as in (50) we get

$$\int_0^T\int_{\mathbb{R}^n}|(-\Delta)^\sigma\tilde{\varphi}|^{\bar{p}'}\,\Phi_R^{-\frac{1}{\bar{p}-1}}\,dx\,dt \lesssim K^{-\frac{n}{\sigma}+2\bar{p}'},$$

and so by (53), taking account of (48), we obtain

$$\begin{aligned}(1-2\epsilon)\int_0^T\int_{\mathbb{R}^n}|u_t|^{\bar{p}}\Phi_R\,dx\,dt &\leq \bar{C}K^{-\frac{n}{\sigma}}(1+K^{\bar{p}'})\\ &+\tilde{C}K^{-\frac{n}{\sigma\bar{p}'}+2}\left(\int_0^T\int_{|x|>(T/K)^{1/\sigma}}|u_t|^{\bar{p}}\Phi_R\,dx\,dt\right)^{\frac{1}{\bar{p}}},\end{aligned}$$

for some constants $\bar{C}$, $\tilde{C}>0$. Since $u_t\in L^p$ by (52), for any fixed $K>0$ it holds,

$$\lim_{T\to\infty}\int_0^T\int_{|x|>(T/K)^{1/\sigma}}|u_t|^{\bar{p}}\Phi_R\,dx\,dt=0;$$

thus, we finally get

$$\int_0^\infty\int_{\mathbb{R}^n}|u_t|^{\bar{p}}\,dx\,dt\leq\bar{C}K^{-\frac{n}{\sigma}}(1+K^{\bar{p}'}).$$

The right hand side is arbitrary small since it holds

$$\bar{p}'=\frac{n+\sigma}{(\alpha+1)\sigma}<\frac{n}{\sigma},$$

for any $\gamma>(n-\sigma)/n$. Thus, we conclude $u_t\equiv 0$, and so for any $t\geq 0$ we have $u(t,x)=u(0,x)=0$. This concludes the proof of Theorem 1.3.

4.3. **The case $u_1\in L^m$, $m>1$.** By our assumption on u_1 in Theorem 1.4, we have

$$u_1(x)\gtrsim|x|^{-\frac{n}{m}}(\log(|x|))^{-1};$$

thus, we find

$$\begin{aligned}\int_{\mathbb{R}^n}u_1(x)\langle x/R\rangle^{-q}\,dx &\geq \int_{|x|\leq R}u_1(x)\langle x/R\rangle^{-q}\,dx\geq 2^{-\frac{q}{2}}\int_{|x|\leqslant R}u_1(x)\,\mathrm{d}x\\ &\gtrsim\int_{|x|\leqslant R}|x|^{-\frac{n}{m}}(\log(|x|))^{-1}\,dx\gtrsim\int_0^R r^{\frac{n(m-1)}{m}-1}(\log(r))^{-1}\,dr\\ &\gtrsim(\log(R))^{-1}R^{\frac{n(m-1)}{m}},\end{aligned}\tag{54}$$

where we used

$$\langle x/R\rangle^{-q}=\left(1+|x|^2/R^2\right)^{-\frac{q}{2}}\geqslant 2^{-\frac{q}{2}}\quad\text{for any }|x|\leqslant R.$$

Recalling that

$$\varphi(0,x)=C(\alpha,\beta)T^{-\alpha}\langle x/R\rangle^{-q},$$

the nonnegativity of u_1, togheter with estimates (51) and (54) implies:

$$C(\alpha,\beta)T^{-\alpha}(\log(R))^{-1}R^{\frac{n(m-1)}{m}} \lesssim C(\alpha,\beta)T^{-\alpha}\int_{\mathbb{R}^n} u_1(x)\langle x/R\rangle^{-q}\,dx$$
$$= \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx \lesssim T^{1-\alpha p'}R^n(T^{-p'}+R^{-\sigma p'}+T^{p'}R^{-2\sigma p'}).$$

Setting again $R = T^{\frac{1}{\sigma}}$ we get

$$(\log(R))^{-1}T^{-\alpha+\frac{n(m-1)}{\sigma m}} \lesssim T^{1-(\alpha+1)p'+\frac{n}{\sigma}},$$

which leads to a contradiction for $T\to\infty$ if $p<\tilde{p}_m(\sigma,n)$. This completes the proof of Theorem 1.4.

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