Existence and multiple solutions for the critical fractional p-Kirchhoff type problems involving sign-changing weight functions

Jie Yang[1,2], Haibo Chen[1]*, Senli Liu[1],

1.School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China

2.Department of Mathematics, Huaihua University, Huaihua, Hunan 418008, PR China

Abstract

The aim of this paper is to study the existence and multiplicity of nonnegative solutions for the following critical Kirchhoff equation involving the fractional p-Laplace operator $(-\Delta)_p^s$. More precisely, we consider

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = \lambda f(x)|u|^{q-2}u + K(x)|u|^{p_s^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary $\partial\Omega$, $M(t) = a + bt^{m-1}$ with $m > 1, a > 0, b > 0$, dimension $N > sp$, $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent, and the parameters $\lambda > 0, 0 < s < 1 < q < p < \infty$. Applying Nehari manifold, fibering maps and Krasnoselskii genus theory, we investigate the existence and multiplicity of nonnegative solutions.

Keywords: fractional p-Laplace operator; Kirchhoff type problems; critical Sobolev exponent; Krasnoselskii genus

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1 Introduction and statement of results

The aim of this paper is to study the existence and multiplicity of nonnegative solutions for the following critical Kirchhoff equation involving the fractional p-Laplace operator $(-\Delta)_p^s$

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = \lambda f(x)|u|^{q-2}u + K(x)|u|^{p_s^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary $\partial\Omega$, $M(t) = a + bt^{m-1}$ with $m > 1, a, b > 0$, dimension $N > sp$, $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent, and the

*Corresponding author.
E-mail: dafeyang@163.com (J. Yang), math_chb@163.com (H. Chen), jasonliu0615@163.com(S. Liu).

parameters $\lambda > 0, 0 < s < 1 < q < p < \infty$. Here, the fractional p-Laplace operator $(-\Delta)_p^s$, up to normalization factors, is defined for every function $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \quad (x \in \mathbb{R}^N),$$

where $B_\epsilon(x)$ denotes the open ball in $\mathbb{R}^N$ with center x and radius ϵ.

The weight functions f and K are continuous on $\bar{\Omega}$ satisfying $f_\pm \not\equiv 0$,where $f = f_+ - f_-$ with $f_\pm = \max\{\pm f, 0\}$; Furthermore, we impose the following assumption on K:
(K) There exists a constant $0 < \rho < \frac{N}{p-1}$ such that $K(z) - K(x) = O(|x - z|^\rho)$ as $x \to z$ uniformly for $z \in \Pi = \{z \in \bar{\Omega} | K(z) = \max_{x \in \bar{\Omega}} K(x) \equiv 1\}$;
Remark 1.1. *Let $\Pi_\delta = \{x \in \mathbb{R}^N | dist(x, \Pi) \leq \delta\}$ for some $\delta > 0$. It follows from [39] that there exist three positive constants η_0, r_0 and σ_0 such that*

$$K(x) \geq \eta_0 \quad \text{for all } x \in \Pi_{r_0} \subset \Omega$$

and

$$K(z) - K(x) \leq \sigma_0 |x - z|^\rho \quad \text{for all } x \in B_{r_0}(z)$$

uniformly for $z \in \Pi$.

When $M(s) = 1, s = 1, p = 2$, the study on the semilinear critical elliptic problem began from Brezis and Nirenberg [20]. Henceforth, much attention has been given to all kinds of elliptic equations with critical growth in the bounded domain or in the whole space, see [21, 22, 23, 24, 25, 26]. Recently, many authors have been working on the solvability or multiplicity of Kirchhoff type equation with critical exponent, for example [3, 6, 14, 19, 21, 22, 23, 24, 25, 26, 28, 29, 30].

In [33], the authors first considered the following related problem in the bounded domain:

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

where $1 < q < 2 < p \leq 2^*(2^* = \frac{2N}{N-2}$ if $N \geq 3,\ 2^* = \infty$ if $N = 1, 2)$. By using the sub-supersolution method, they showed the existence and nonexistence results depending on a sharp constant λ_0.

In [31], Xie and Chen generalized (1.2) to the following Kirchhoff type equation:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2 dx)\Delta u = Q_\lambda(x)|u|^{q-2} + K(x)|u|^{2^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3), 1 < q < 2,\ M(s) = a + bs^\beta$ with $a > 0, b > 0$ and $\beta > 0$. The weight functions $Q_\lambda(x)$and K are continuous and changing-sign. Using the Nehari manifold, fibering maps and Ljusternik-Schnirelmann category, they proved that at least two positive solutions for (1.3) exist provided that $\beta = 1$ and $2^* \geq 4$. Furthermore, by the mountain pass theorem and Ekeland's variational principle, it was shown that (1.3) possessed at least three positive solutions whenever $\beta > \frac{2}{N-2}$, including the case that $\beta = 1$ and $2^* < 4$.

In the last decade, a lot of attention has been focused on the fractional Laplacian operator and non-local operator. In [32], the authors studied the Brezis-Nirenberg type results for the following elliptic equation involving the fractional Laplacian $(-\Delta)^s(0 < s < 1)$ in bounded domain:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0, s \in (0,1)$ is fixed, $2_s^* = \frac{2N}{N-2s}$, $\Omega \subset \mathbb{R}^N(N > 2s)$, is open, bounded and with Lipschitz boundary, $(-\Delta)^s$ is the fractional Laplace operator. The authors extended the classical Brezis-Nirenberg result to the case of non-local fractional operators through variational techniques. Soon afterwords, Mosconi et al. [35] obtained nontrivial solutions to the Brezis-Nirenberg problem for the fractional p-Laplacian operator, extending some results in the literature for the fractional Laplacian.

Inspired by the above papers, we consider the multiplicity of nonnegative solutions for (1.1) with the concave and convex nonlinearities. In particular, Mishra and Sreenadh [19], investigated the following fractional p-Kirchhoff equation

$$\begin{cases} M(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy)(-\Delta)_p^s u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $M(t) = a + bt, a, b > 0, p \geq 2, 1 < q < p < r \leq p_s^*, ps < N < 2ps$ with $s \in (0,1)$, $\lambda > 0$, $\Omega \subset \mathbb{R}^N$, is an open, bounded domain with Lipschitz boundary, and f, g are possibly sign changing on $\bar{\Omega}$. They examined the existence of multiple solutions for (1.5) by using Nehari manifold and fibering maps in the following three cases:(i) $2p < r$, (ii) $2p = r$, (iii) $2p > r$ when b and λ belongs to specific intervals.

Now, a natural question is whether the existence and multiplicity of solutions for the p-Kirchhoff equations in [19] can be generalized to the critical case?

Motivted by the above works, here we shall solve the above probelm. The goal of this paper is to consider the question related to the existence and multiplicity of nonnegative solutions for (1.1) in the following two cases:(i) $mp < p_s^*$, (ii) $mp = p_s^*$. Before stating our main results, we introduce some notations.

Notations and definitions. Throughout the article we assume that $\max_{x\in\bar{\Omega}} K(x) \equiv 1$, $q^* = \frac{p_s^*}{p_s^*-q}$, without further mentioning. The problem (1.1) has a variational structure and the natural space to look for solutions. It is the fractional Sobolev spaces $W_0^{s,p}(\Omega)$. We firstly recall some notations. The usual fractional Sobolev space $W^{s,p}(\Omega)$ endowed with the form

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \right)^{\frac{1}{p}}. \tag{1.6}$$

When $\Omega = \mathbb{R}^N$, we define $\mathcal{D}^{s,p}(\mathbb{R}^N)$ as the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy.$$

In [9], new function spaces E which take into account this boundary condition were introduced given as

$$E = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega), \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \right) \in L^p(Q) \right\},$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$. The space E is endowed with the norm defined by

$$\|u\|_E = \left(\|u\|^p_{L^p(\Omega)} + \iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}. \tag{1.7}$$

The function space E_0 denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in E. Subsequently, in [10], authors proved the space E_0 is a uniformly convex Banach space endowed with the norm

$$\|u\|_{E_0} = \left(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}} \tag{1.8}$$

which is equivalent to the natural one defined in (1.6). Since $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we note that the integral in (1.7) and (1.8) can be extended to $\mathbb{R}^{2N}$. The embedding $E_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$ and compact whenever $r \in [1, p_s^*)$(see[10, Theorem6.7]). As in [10], let S be the best constant of the fractional Sobolev embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ defined by

$$S = \inf_{u \in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy}{(\int_{\mathbb{R}^N} |u|^{p_s^*} dx)^{p/p_s^*}}, \tag{1.9}$$

which is well defined and strictly positive. For more details and properties on E and E_0, please refer to [10] and the references therein. From now on, we will denote $\|\cdot\|_{E_0}$ and $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|$ and $|\cdot|_p$ respectively.

Next, the energy functional associated to problem (1.1) is given by

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{mp}\|u\|^{mp} - \frac{1}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{1}{p_s^*}\int_\Omega K(x)|u|^{p_s^*} dx. \tag{1.10}$$

We now summarize our main results as follows.

Theorem 1.1. *Assume that $m < \frac{N}{N-ps}$, $f_\pm \not\equiv 0$ and (K) hold. Then there exist $0 < \lambda_* \le \lambda_0$ and $b_* > 0$ such that*
(i) for any $\lambda \in (0, \lambda_0)$, problem (1.1) admits at least one nonnegative ground state solution u_λ with $I_\lambda(u_\lambda) < 0$.
(ii) for any $\lambda \in (0, \lambda_*), b \in (0, b_*)$ problem (1.1) admits at least two nonnegative solutions u_λ and $u_{\lambda,b}$ satisfying $I_\lambda(u_\lambda) < 0 < I_\lambda(u_{\lambda,b})$, and u_λ is a ground state solution.

Theorem 1.2. *Assume that $m = \frac{N}{N-ps}$, $f_\pm \not\equiv 0$ and (K) hold. Then*
(i) for $b \ge S^{-m}$ and any $\lambda > 0$, problem (1.1) admits at least one nonnegative ground state solution;
(ii) for $b < S^{-m}$, there exist $0 < \tilde{\lambda}_* \le \lambda_0$, and $\tilde{b}_* > 0$ such that
(1) for any $\lambda \in (0, \lambda_0)$, problem (1.1) admits at least one nonnegative solution.
(2) for any $\lambda \in (0, \tilde{\lambda}_*)$ and $b \in (0, \tilde{b}_*)$, problem (1.1) admits at least two nonnegative solution u_λ and $u_{\lambda,b}$ satisfying $I_\lambda(u_\lambda) < 0 < I_\lambda(u_{\lambda,b})$, and u_λ is a ground state solution.

Remark 1.2. *When $m = 2$ and $K(x) \equiv 1$, then by Theorems 1.1-1.2, we get the existence and multiplicity of nonnegative solutions for the following problem:*

$$\begin{cases} \left(a + b\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)(-\Delta)^s_p u = \lambda f(x)|u|^{q-2}u + |u|^{p_s^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $ps < N < 2ps$, which respectively generalize Theorem 1.6 of [19] and Theorem 1.1 of [36].

To prove the above result, we apply the technique of Nehari manifold and fibering maps. The approach is motivated by [6]. Compared with [6], the main difficulty lies in the lack of compactness due to the nonlinearity with the critical growth. For this, adapting some calculations performed in [35] and applying the optimal asymptotic behavior of p-minimizers proved in [16], we are able to prove a compactness result for the energy functional under a suitable critical threshold. To prove the existence of the first solution, we obtain a minimizing sequence whose energy functional converge to a negative number, which is less than some critical level c_λ^* such that the (PS) condition holds. Meanwhile, for the second solution, we extract a minimizing sequence with energy functional converging to a positive mountain pass level, which also falls into the range of validity of the (PS) condition.

Since we deal with the multiplicity of solutions to (1.1), we use the idea borrowed from [17] to show negative values of an associated energy functional via genus. Moreover, we require that $f_- \equiv 0$. We point out that as far as we know, there appear only few papers on fractional p Laplacian probelms [6, 9, 19], but no results on the multiplicity of solutions for problem (1.1) are available. So the aim of this work is to give a first result in this direction.

Theorem 1.3. *Assume that $m \leq \frac{N}{N-ps}$ and $f_- \equiv 0$ hold. Then there exists $\lambda_3 > 0$ such that for any $\lambda \in (0, \lambda_3)$, problem (1.1) possesses infinitely many nontrivial solutions.*

The rest of this paper is organized as follows. In section 2, we introduce the Nehari manifold and fibering maps analysis for problem (1.1). In section 3, we prove the Theorems 1.1 and 1.2. The section 4 is devoted to the infinitely many solutions of problem (1.1).

2 The Nehari manifold and fibering maps analysis

The lemma below will be very useful in our work.

Lemma 2.1. *If $\{u_n\}$ is a bounded sequence in E_0, then*

$$\|u_n - u\|^p = \|u_n\|^p - \|u\|^p + o_n(1).$$

Proof. The proof of this result is obtained following the argument developed by Brezis and Lieb, see for example [13, Lemma 2.7]. $\square$

We have that I_λ is of class C^1 in E_0 and for any $v \in E_0$

$$\begin{aligned}\langle I_\lambda'(u), v\rangle = M(\|u\|^p) \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} dxdy \\ - \int_\Omega \lambda f(x)|u|^{q-2}uvdx - \int_\Omega K(x)|u|^{p_s^*-2}uvdx.\end{aligned}$$

Thus, we can define the Nehari manifold as

$$N_\lambda = \{u \in E_0 \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\}. \tag{2.1}$$

The Nehari manifold is closely linked to the behavior of the fiber map $\phi_u : t \in \mathbb{R}^+ \to I_\lambda(tu)$, which is introduced in [11] and discussed in [12]. Therefore, we have

$$\phi_u'(t) = at^{p-1}\|u\|^p + bt^{mp-1}\|u\|^{mp} - t^{q-1}\int_\Omega \lambda f(x)|u|^q dx - t^{p_s^*-1}\int_\Omega K(x)|u|^{p_s^*}dx.$$

Then $u \in N_\lambda$ if and only if $\phi_u^{'}(1) = 0$ holds. Furthermore, for $u \in N_\lambda$, we get

$$\phi_u^{''}(1) = a(p-q)\|u\|^p + b(mp-q)\|u\|^{mp} - (p_s^* - q)\int_\Omega K(x)|u|^{p_s^*}dx, \tag{2.2}$$

or

$$\phi_u^{''}(1) = a(p-p_s^*)\|u\|^p + b(mp-p_s^*)\|u\|^{mp} - (q - p_s^*)\int_\Omega \lambda f(x)|u|^q dx. \tag{2.3}$$

Like [12], we split N_λ into three components as

$$N_\lambda^{\pm} = \{u \in N_\lambda | \phi_u^{''}(1) \gtrless 0\} \quad \text{and} \quad N_\lambda^0 = \{u \in N_\lambda | \phi_u^{''}(1) = 0\}.$$

Finally, we define

$$H^+ = \{u \in E_0 | \int_\Omega f(x)|u|^q dx > 0\}, \quad H^- = \{u \in E_0 | \int_\Omega f(x)|u|^q dx \le 0\},$$

$$G^+ = \{u \in E_0 | \int_\Omega K(x)|u|^{p_s^*} dx > 0\}, \quad G^- = \{u \in E_0 | \int_\Omega K(x)|u|^{p_s^*} dx \le 0\},$$

According to $mp \le p_s^*$, we have the following Lemmas.

Lemma 2.2. *For any $\lambda > 0$, the functional I_λ is coercive and bounded below on N_λ.*

Proof. For any $u \in N_\lambda$, using (1.9) and Hölder inequality, we deduce that

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{p_s^*}\langle I_\lambda^{'}(u), u\rangle \\ &= (\frac{1}{p} - \frac{1}{p_s^*})a\|u\|^p + (\frac{1}{mp} - \frac{1}{p_s^*})b\|u\|^{mp} - (\frac{1}{q} - \frac{1}{p_s^*})\int_\Omega \lambda f(x)|u|^q dx \\ &\ge (\frac{1}{p} - \frac{1}{p_s^*})a\|u\|^p - (\frac{1}{q} - \frac{1}{p_s^*})\lambda |f_+|_{q^*} S^{-\frac{q}{p}}\|u\|^q. \end{aligned} \tag{2.4}$$

Thus I_λ is coercive and bounded below on N_λ due to $1 < q < p$. □

Lemma 2.3. *If u is a minimizer for I_λ on N_λ such that $u \in N_\lambda^0$, then $I_\lambda^{'}(u) = 0$ in E_0^{-1}.*

Proof. The details of the proof can be found in [12, Theorem 2.3]. □

Let

$$\lambda_1 = \left[\frac{a(p-q)}{p_s^* - q}\right]^{\frac{p-q}{p_s^*-p}} \frac{a(p_s^*-p)S^{\frac{p_s^*-q}{p_s^*-p}}}{(p_s^*-q)|f_+|_{q^*}}. \tag{2.5}$$

Then we have the following result.

Lemma 2.4. *For any $\lambda \in (0, \lambda_1)$, there exists $\lambda_1 > 0$ such that $N_\lambda^0 = \emptyset$.*

Proof. Suppose by contradiction that $u \in N_\lambda^0$ for all $\lambda \in (0, \lambda_1)$. Then by (2.2) and (2.3), we have

$$a(p-q)\|u\|^p + b(mp-q)\|u\|^{mp} - (p_s^*-q)\int_\Omega K(x)|u|^{p_s^*}dx = 0; \tag{2.6}$$

$$a(p-p_s^*)\|u\|^p + b(mp-p_s^*)\|u\|^{mp} - (q-p_s^*)\int_\Omega \lambda f(x)|u|^q dx = 0; \tag{2.7}$$

Hence, from (2.6) and (1.9), we get

$$\|u\|^p \le \frac{p_s^*-q}{a(p-q)}|u|_{p_s^*}^{p_s^*} \le \frac{p_s^*-q}{a(p-q)}S^{-\frac{p_s^*}{p}}\|u\|^{p_s^*}.$$

On the other hand, from (2.7), Hölder inequality and (1.9), we obtain that

$$\|u\|^p \le \frac{p_s^*-q}{a(p_s^*-p)}\lambda\int_\Omega f_+|u|^q dx \le \frac{p_s^*-q}{a(p_s^*-p)}\lambda|f_+|_{q^*}S^{-\frac{q}{p}}\|u\|^q.$$

These yield that

$$\left[\frac{a(p-q)}{p_s^*-q}S^{\frac{p_s^*}{p}}\right]^{\frac{1}{p_s^*-p}} \le \|u\| \le \left[\frac{(p_s^*-q)\lambda|f_+|_{q^*}S^{-\frac{q}{p}}}{a(p_s^*-p)}\right]^{\frac{1}{p-q}}.$$

Therefore,

$$\lambda \ge \left[\frac{a(p-q)}{p_s^*-q}\right]^{\frac{p-q}{p_s^*-p}}\frac{a(p_s^*-p)S^{\frac{p_s^*-q}{p_s^*-p}}}{(p_s^*-q)|f_+|_{q^*}} = \lambda_1,$$

which gives the contradiction because $\lambda < \lambda_1$. □

3 Proofs of Theorems 1.1-1.2

In this section, we give the proof of the existence and multiplicity results for the case $mp \le p_s^*$. We have the following Lemmas. Firstly, we show the component sets N_λ^+ and N_λ^- are nonempty. Now, we define

$$\lambda_2 = \begin{cases} \lambda_1, & m < \frac{N}{N-ps} \\ \left(\frac{1}{1-bS^m}\right)^{\frac{p-q}{p_s^*-p}}\lambda_1, & m = \frac{N}{N-ps} \end{cases}. \tag{3.1}$$

Hence, we obtain the following lemmas.

Lemma 3.1. *Let $m < \frac{N}{N-ps}$. Then*
(i) for any $u \in H^+ \cap G^+, \lambda \in (0, \lambda_2)$, there exist $0 < t^+ = t^+(u) < t_{max} = t_{max}(u) < t^- = t^-(u)$, such that $t^+u \in N_\lambda^+, t^-u \in N_\lambda^-$, and

$$I_\lambda(t^+u) = \min_{0\le t\le t^-} I_\lambda(tu), \quad I_\lambda(t^-u) = \max_{t\ge t_{max}} I_\lambda(tu).$$

(ii) for any $u \in H^+ \cap G^-$, $\lambda > 0$, there exists a unique $t^+ > 0$ such that $t^+u \in N_\lambda^+$ and

$$I_\lambda(t^+u) = \min_{t\ge 0} I_\lambda(tu).$$

Proof. For a given $u \in E_0\backslash\{0\}$, define $\psi_u(t) : \mathbb{R}^+ \to \mathbb{R}$ as

$$\psi_u(t) = at^{p-q}\|u\|^p + bt^{mp-q}\|u\|^{mp} - t^{p_s^*-q}\int_\Omega K(x)|u|^{p_s^*}dx. \tag{3.2}$$

We note that $tu \in N_\lambda$ if and only if $\psi_u(t) = \int_\Omega \lambda f(x)|u|^q dx$.
(i)Let $u \in H^+ \cap G^+$, by (3.2), it is easy to see $\psi_u(0) = 0, \psi_u(t) \to -\infty$ as $t \to \infty$. Moreover, since

$$\psi_u^{'}(t) = a(p-q)t^{p-q-1}\|u\|^p + (mp-q)bt^{mp-q-1}\|u\|^{mp} - (p_s^*-q)t^{p_s^*-q-1}\int_\Omega K(x)|u|^{p_s^*}dx \tag{3.3}$$

it is obvious that $\lim_{t\to 0^+}\psi_u'(t)>0$ and $\lim_{t\to\infty}\psi_u'(t)<0$. Let $\psi_u'(t)=t^{p-q-1}h_u(t)$, where

$$h_u(t)=a(p-q)\|u\|^p+(mp-q)bt^{mp-p}\|u\|^{mp}-(p_s^*-q)t^{p_s^*-p}\int_\Omega K(x)|u|^{p_s^*}dx.$$

Then it is easy to see that there exists a unique $t_0>0$ such that $h_u'(t_0)=0$. Indeed,

$$t_0=\left(\frac{(mp-q)(mp-p)b\|u\|^{mp}}{(p_s^*-q)(p_s^*-p)\int_\Omega K(x)|u|^{p_s^*}dx}\right)^{\frac{1}{p_s^*-mp}}.$$

Moreover, since $mp<p_s^*$, we have $h_u(0)=0,\ \lim\limits_{t\to\infty}h_u(t)=-\infty$, which indicates that there is a unique $t_{max}>t_0$ such that $h_u(t_{max})=0$. Hence, we obtain t_{max} as a unique critical point of $\psi_u(t)$ such that $\psi_u(t)$ achieves its maximum at t_{max}. Namely, $\psi_u(t)$ is increasing on $(0,t_{max})$, decreasing on (t_{max},∞) and $\psi_u'(t_{max})=0$.

Furthermore, $\psi_u(t_{max})=\max\limits_{t>0}\psi_u(t)\geq\max\limits_{t>0}\bar\psi_u(t)$, where $\bar\psi_u(t)=at^{p-q}\|u\|^p-t^{p_s^*-q}\int_\Omega K(x)|u|^{p_s^*}dx$. Then, by (1.9), we have

$$\begin{aligned}\max_{t>0}\bar\psi_u(t)&=\|u\|^q\frac{a(p_s^*-p)}{p_s^*-q}\left(\frac{(p-q)a\|u\|^{p_s^*}}{(p_s^*-q)\int_\Omega K(x)|u|^{p_s^*}dx}\right)^{\frac{p-q}{p_s^*-p}}\\&\geq\|u\|^q\frac{a(p_s^*-p)}{p_s^*-q}\left(\frac{(p-q)aS^{\frac{p_s^*}{p}}}{(p_s^*-q)}\right)^{\frac{p-q}{p_s^*-p}}.\end{aligned}$$

For $u\in H^+$,

$$\psi_u(0)=0<\int_\Omega\lambda f(x)|u|^qdx\leq\lambda\int_\Omega f_+|u|^qdx\leq\lambda|f_+|_{q^*}S^{-\frac{q}{p}}\|u\|^q.$$

Therefore, if

$$\lambda<\lambda_1=\frac{a(p_s^*-p)S^{\frac{q}{p}}}{(p_s^*-q)|f_+|_{q^*}}\left(\frac{(p-q)aS^{\frac{p_s^*}{p}}}{(p_s^*-q)}\right)^{\frac{p-q}{p_s^*-p}},$$

then there exist unique $t^+=t^+(u)<t_{max}$ and $t^-=t^-(u)>t_{max}$ such that

$$\psi_u(t^+)=\int_\Omega\lambda f(x)|u|^qdx=\psi_u(t^-),\quad\psi_u'(t^+)>0,\quad\psi_u'(t^-)<0,$$

which implies that $t^+u,t^-u\in N_\lambda$. Meanwhile, we can conclude that $t^+u\in N_\lambda^+$ and $t^-u\in N_\lambda^-$, noting the relation $\phi_{tu}''(1)=t^{q+1}\psi_u'(t)$. Moreover, because $\phi_u'(t)=t^{q-1}(\psi_u(t)-\int_\Omega\lambda f(x)|u|^qdx)$, we obtain that $\phi_u'(t)<0$ for all $t\in[0,t^+)$ and $\phi_u'(t)>0$ for all $t\in(t^+,t^-)$. Thus $I_\lambda(t^+u)=\min\limits_{0\leq t\leq t^-}I_\lambda(tu)$. Similarly, $\phi_u'(t)>0$ for all $t\in(t^+,t^-)$ and $\phi_u'(t)<0$ for all $t\in(t^-,\infty)$ yield that $I_\lambda(t^-u)=\max\limits_{t\geq t_{max}}I_\lambda(tu)$.
(ii) For $u\in H^+\cap G^-$. From (3.2), we conclude that $\psi_u(t)\to\infty$ as $t\to\infty$, and $\psi_u'(t)>0$ for all $t>0$. Therefore, for all $\lambda>0$ there exists $t^+>0$ such that $t^+u\in N_\lambda^+$ and $I_\lambda(t^+u)=\min\limits_{t\geq0}I_\lambda(tu)$. $\square$

Lemma 3.2. *Let $m=\frac{N}{N-ps}$, and $b\geq 1/S^m$, then there exists a unique $0<t^+<t_{max}$ such that $t^+u\in N_\lambda$, when $u\in H^+$. Also, $I_\lambda(t^+u)=\min_{t\geq0}I_\lambda(tu)$.*

Proof. For $m=\frac{N}{N-ps}$, using equation (3.2) and (3.3), we get

$$\psi_u(t)=at^{p-q}\|u\|^p+t^{mp-q}(b\|u\|^{mp}-\int_\Omega K(x)|u|^{p_s^*}dx),\tag{3.4}$$

$$\psi_u'(t) = a(p-q)t^{p-q-1}\|u\|^p + (mp-q)t^{mp-q-1}(b\|u\|^{mp} - \int_\Omega K(x)|u|^{p_s^*}dx). \tag{3.5}$$

Since $b \geq 1/S^m$, one has $\psi_u(t) > 0, \psi_u'(t) > 0$ for all $t > 0$. One can easily see that $N_\lambda^+ = N_\lambda$. Furthermore, for $u \in H^+$, there exists a unique $t^+(u) > 0$ such that $t^+u \in N_\lambda = N_\lambda^+$. Hence, $I_\lambda(t^+u) = \min_{t\geq 0} I_\lambda(tu)$. This completes the proof. $\square$

Remark 3.1. *If $m = \frac{N}{N-ps}$, and $b \geq 1/S^m$, then $N_\lambda^+ = N_\lambda$ for any $\lambda > 0$.*

Lemma 3.3. *Let $m = \frac{N}{N-ps}$, $b < 1/S^m$ hold, then there exists a unique $t_{max}(u) > 0$ such that*
(i) for $u \in H^+$ with $\int_\Omega K(x)|u|^{p_s^*}dx - b\|u\|^{mp} > 0$ and $\lambda \in (0, \lambda_2)$, there exist unique $t^\pm$, $t^+(u) < t_{max} < t^-(u)$, such that $t^\pm u \in N_\lambda^\pm$, and

$$I_\lambda(t^+u) = \min_{0\leq t\leq t^-} I_\lambda(tu), \quad I_\lambda(t^-u) = \max_{t\geq t_{max}} I_\lambda(tu).$$

(ii) for any $u \in H^-$ with $\int_\Omega K(x)|u|^{p_s^*}dx - b\|u\|^{mp} > 0$ and $\lambda > 0$, there exists a unique $t^- > 0$ such that $t^-u \in N_\lambda^-$ and

$$I_\lambda(t^-u) = \max_{t\geq 0} I_\lambda(tu).$$

Proof. For each $u \in E_0\backslash\{0\}$ with $\int_\Omega K(x)|u|^{p_s^*}dx - b\|u\|^{p_s^*} > 0$, from $b < 1/S^m$, (3.4) and (3.5), we can easily see that there exists $t_{max} > 0$ satisfying

$$t_{max} = \left(\frac{a(p-q)\|u\|^p}{(p_s^* - q)(\int_\Omega K(x)|u|^{p_s^*}dx - b\|u\|^{p_s^*})}\right)^{\frac{1}{p_s^*-p}}$$

such that

$$\psi_u(t_{max}) \geq \frac{a(p_s^*-p)}{p_s^*-q}\left(\frac{a(p-q)S^m}{(p_s^*-q)(1-bS^m)}\right)^{\frac{p-q}{p_s^*-p}}\|u\|^q.$$

Therefore, if

$$\lambda < \lambda_2 = \frac{a(p_s^*-p)S^{\frac{q}{p}}}{(p_s^*-q)|f_+|_{q^*}}\left(\frac{a(p-q)S^m}{(p_s^*-q)(1-bS^m)}\right)^{\frac{p-q}{p_s^*-p}},$$

there exist unique $t^+ = t^+(u) < t_{max}$ and $t^- = t^-(u) > t_{max}$ such that

$$\psi_u(t^+) = \int_\Omega \lambda f(x)|u|^q dx = \psi_u(t^-), \quad \psi_u'(t^+) > 0, \quad \psi_u'(t^-) < 0.$$

Then, arguing similarly as in Lemma 3.1, we complete the proof. $\square$

Remark 3.2. *From Lemmas 2.4, 3.1 and 3.3, it is immediate to see that if $m < \frac{N}{N-ps}$ or $m = \frac{N}{N-ps}$, $b < 1/S^m$ and $\lambda \in (0, \lambda_1)$ hold, then $N_\lambda = N_\lambda^+ \cup N_\lambda^-$.*

Lemma 3.4. *Let $\lambda \in (0, \lambda_1)$, then there exists a gap structure in N_λ such that*

$$\|v\| > B_0 > B_\lambda > \|u\| \quad for\ any\ u \in N_\lambda^+, v \in N_\lambda^-.$$

Proof. If $u \in N_\lambda^+ \subset N_\lambda$, from (2.3), Hölder inequality and (1.9), we get

$$a(p_s^*-p)\|u\|^p < (p_s^*-q)\int_\Omega \lambda f(x)|u|^q dx \leq (p_s^*-q)\lambda S^{-\frac{q}{p}}|f_+|_{q^*}\|u\|^q.$$

Therefore,

$$\|u\| < \left(\frac{(p_s^* - q)\lambda S^{-\frac{q}{p}} |f_+|_{q^*}}{a(p_s^* - p)} \right)^{\frac{1}{p-q}} = B_\lambda.$$

Similarly, if $v \in N_\lambda^- \subset N_\lambda$, then from (2.2), we have

$$a(p-q)\|v\|^2 < (p_s^* - q) \int_\Omega K(x)|v|^{p_s^*} dx \le (p_s^* - q) S^{-p_s^*/2} \|v\|^{p_s^*}.$$

Hence,

$$\|v\| > \left(\frac{a(p-q) S^{\frac{p_s^*}{p}}}{(p_s^* - q)} \right)^{\frac{1}{p_s^* - p}} = B_0.$$

After direct calculations, it is easy to see that $B_0 > B_\lambda$ for $\lambda \in (0, \lambda_1)$, where λ_1 is given in (2.5). This ends the proof. □

Now, we are ready to study the infimum of I_λ on the $N_\lambda^\pm$. Let us define $c_\lambda^\pm = \inf_{u \in N_\lambda^\pm} I_\lambda(u)$ and $\tilde{\lambda} = \frac{q}{p}\lambda_1$.

Lemma 3.5. *Let* $m < \frac{N}{N-ps}$ *or* $m = \frac{N}{N-ps}$, $b < 1/S^m$. *Then*
(i) For any $\lambda \in (0, \lambda_1)$, *we have* $c_\lambda^+ = \inf_{u \in N_\lambda^+} I_\lambda(u) < 0$.
(ii) $c_\lambda^- \ge \alpha_0 > 0$ *for any* $\lambda \in (0, \tilde{\lambda})$. *In particular, if* $\lambda \in (0, \tilde{\lambda})$, *then*

$$c_\lambda^+ = \inf_{u \in N_\lambda} I_\lambda(u).$$

Proof. (i) For $u \in N_\lambda^+$, together with (2.3), we observe that

$$\int_\Omega \lambda f(x)|u|^q dx \ge \left(\frac{p_s^* - p}{p_s^* - q} \right) a\|u\|^p + \left(\frac{p_s^* - mp}{p_s^* - q} \right) b\|u\|^{mp}. \tag{3.6}$$

Hence, it follows from (2.4) and (3.6) that

$$\begin{aligned} c_\lambda^+ \le I_\lambda(u) &= (\frac{1}{p} - \frac{1}{p_s^*}) a\|u\|^p + (\frac{1}{mp} - \frac{1}{p_s^*}) b\|u\|^{mp} - (\frac{1}{q} - \frac{1}{p_s^*}) \int_\Omega \lambda f(x)|u|^q dx \\ &\le -\left(\frac{1}{q} - \frac{1}{p} \right) \left(1 - \frac{p}{p_s^*} \right) a\|u\|^p - \left(\frac{1}{q} - \frac{1}{mp} \right) \left(1 - \frac{mp}{p_s^*} \right) b\|u\|^{mp} < 0. \end{aligned}$$

(ii) For $m < \frac{N}{N-ps}$ or $m = \frac{N}{N-ps}$, $b < 1/S^m$ and $u \in N_\lambda^-$, in view of Lemma 3.4 and $\lambda \in (0, \tilde{\lambda})$, we conclude that

$$\begin{aligned} I_\lambda(u) &= (\frac{1}{p} - \frac{1}{p_s^*}) a\|u\|^p + (\frac{1}{mp} - \frac{1}{p_s^*}) b\|u\|^{mp} - (\frac{1}{q} - \frac{1}{p_s^*}) \int_\Omega \lambda f(x)|u|^q dx \\ &\ge \|u\|^q \left[(\frac{1}{p} - \frac{1}{p_s^*}) a \left(\frac{a(p-q)}{p_s^* - q} S^{\frac{p_s^*}{p}} \right)^{\frac{p-q}{p_s^* - p}} - \lambda (\frac{1}{q} - \frac{1}{p_s^*}) |f_+|_{q^*} S^{-\frac{q}{p}} \right] \\ &\ge \frac{(p_s^* - q)|f_+|_{q^*} \|u\|^q}{q p_s^* S^{\frac{q}{p}}} (\tilde{\lambda} - \lambda) \ge \alpha_0 > 0. \end{aligned}$$

This completes the proof. □

The following technical lemma is necessary for studying the compactness of I_λ.

Lemma 3.6. *For a given $u \in N_\lambda^+$ and $\lambda \in (0, \lambda_1)$, there is a number ϵ and a differentiable function $\zeta : B(0,\epsilon) \subseteq E \to \mathbb{R}^+$ such that $\zeta(0) = 1$, the function $\zeta(v)(u+v) \in N_\lambda^+$ and*

$$\langle \zeta'(0), v\rangle = \frac{pa\langle u,v\rangle + mpb\|u\|^{(m-1)p}\langle u,v\rangle - q\int_\Omega \lambda f(x)|u|^{q-2}uvdx - p_s^*\int_\Omega K(x)|u|^{p_s^*-2}uvdx}{(p-q)a\|u\|^p + (mp-q)b\|u\|^{mp} - (p_s^*-q)\int_\Omega K(x)|u|^{p_s^*}dx} \tag{3.7}$$

or

$$\langle \zeta'(0), v\rangle = \frac{pa\langle u,v\rangle + mpb\|u\|^{(m-1)p}\langle u,v\rangle - q\int_\Omega \lambda f(x)|u|^{q-2}uvdx - p_s^*\int_\Omega K(x)|u|^{p_s^*-2}uvdx}{a(p-p_s^*)\|u\|^p + b(mp-p_s^*)\|u\|^{mp} - (q-p_s^*)\int_\Omega \lambda f(x)|u|^{q}dx} \tag{3.8}$$

where

$$\langle u,v\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy$$

for all $v \in B_\epsilon(0) = \{v \in E_0 : \|v\| \leq \epsilon\}$.

Proof. For a given $u \in N_\lambda^+$, define $F : \mathbb{R}^+ \times E_0 \to \mathbb{R}$ as follows

$$F(t,v) = t^p a\|u+v\|^p + t^{mp}b\|u+v\|^{mp} - t^q\int_\Omega \lambda f(x)|u+v|^q dx - t^{p_s^*}\int_\Omega K(x)|u+v|^{p_s^*}dx.$$

Hence,

$$F(1,0) = a\|u\|^p + b\|u\|^{mp} - \int_\Omega \lambda f(x)|u|^q dx - \int_\Omega K(x)|u|^{p_s^*}dx = 0.$$

Moreover, we have

$$\frac{\partial F}{\partial t}(1,0) = pa\|u\|^p + mpb\|u\|^{mp} - q\int_\Omega \lambda f(x)|u|^q dx - p_s^*\int_\Omega K(x)|u|^{p_s^*}dx > 0. \tag{3.9}$$

Using the implicit function theorem at $(1,0)$, we obtain that there exists $\tilde{\epsilon} > 0$ and a differentialable function $\zeta : B_{\tilde{\epsilon}}(0) \subseteq E_0 \to \mathbb{R}$ such that $\zeta(0) = 1$, (3.7), (3.8) hold and $F(\zeta(v), v) = 0$ for all $v \in B_{\tilde{\epsilon}}(0)$. Therefore, we get

$$a\|\zeta(v)(u+v)\|^p + b\|\zeta(v)(u+v)\|^{mp} - \int_\Omega \lambda f(x)|\zeta(v)(u+v)|^q dx - \int_\Omega K(x)|\zeta(v)(u+v)|^{p_s^*}dx = 0$$

that is, $\zeta(v)(u+v) \in N_\lambda$ for any $v \in E_0$ with $\|v\| < \tilde{\epsilon}$. By (3.9), we can choose $0 < \epsilon < \tilde{\epsilon}$ such that for any $v \in E_0$ with $\|v\| < \epsilon$, we get $\frac{\partial F}{\partial t}(\zeta(v), v) > 0$ which implies that

$$pa\|\zeta(v)(u+v)\|^p + mpb\|\zeta(v)(u+v)\|^{mp} - q\int_\Omega \lambda f(x)|\zeta(v)(u+v)|^q dx - p_s^*\int_\Omega K(x)|\zeta(v)(u+v)|^{p_s^*}dx > 0,$$

that is $\zeta(v)(u+v) \in N_\lambda^+$ for all $v \in B_\epsilon(0)$. □

Lemma 3.7. *If $\lambda \in (0, \lambda_1)$, then there exists a minimizing sequence $\{u_k\} \subset N_\lambda$ such that*

$$I_\lambda(u_k) = c_\lambda + o_k(1), \quad \text{and } I_\lambda'(u_k) = o_k(1),$$

with $c_\lambda = \inf_{u\in N_\lambda} I_\lambda(u)$.

Proof. Since by Lemma 2.2 and Ekeland's variational principle [15], there exists a minimizing sequence $\{u_k\} \subset N_\lambda$ for I_λ such that

$$c_\lambda < I_\lambda(u_k) < c_\lambda + \frac{1}{k}, \tag{3.10}$$

and also

$$I_\lambda(u_k) < I_\lambda(u) + \frac{1}{k}\|u - u_k\|, \quad \text{for any } u \in N_\lambda. \tag{3.11}$$

From (3.10) and Lemma 2.2, we get $\sup_k \|u_k\| < \infty$. Next, we want to show $\|I_\lambda^{'}(u_k)\| \to 0$ as $k \to \infty$. By Lemma 3.6, we obtain the differentiable functions $\zeta_k : B_{\epsilon_k}(0) \to \mathbb{R}$ for some $\epsilon_k > 0$ such that $\zeta_k(v)(u_k - v) \in N_\lambda$ for all $v \in B_{\epsilon_k}(0)$. For fixed k, choose $0 < \varrho < \epsilon_k$ and define $v_\varrho = \varrho u/\|u\|$ with $u \in E_0, u \not\equiv 0$. We set $\omega_\varrho = \zeta_k(v_\varrho)(u_k - v_\varrho)$. Then it is clear that $\omega_\varrho \in N_\lambda$. From (3.11), we observe that

$$I_\lambda(\omega_\varrho) - I_\lambda(u_k) \geq -\frac{1}{k}\|\omega_\varrho - u_k\|.$$

Applying the mean value theorem, we have

$$\langle I_\lambda^{'}(u_k), \omega_\varrho - u_k\rangle + o_k(\|\omega_\varrho - u_k\|) \geq -\frac{1}{k}\|\omega_\varrho - u_k\|.$$

Therefore,

$$-\langle I_\lambda^{'}(u_k), v_\varrho\rangle + (\zeta_k(v_\varrho) - 1)\langle I_\lambda^{'}(u_k), u_k - v_\varrho\rangle \geq -\frac{1}{k}\|\omega_\varrho - u_k\| + o_k(\|\omega_\varrho - u_k\|)$$

Considering $\langle I_\lambda^{'}(\omega_\varrho), u_k - v_\varrho\rangle = 0$, we get

$$-\varrho\langle I_\lambda^{'}(u_k), \frac{u}{\|u\|}\rangle + (\zeta_k(v_\varrho) - 1)\langle I_\lambda^{'}(u_k) - I_\lambda^{'}(\omega_\varrho), u_k - v_\varrho\rangle \geq -\frac{1}{k}\|\omega_\varrho - u_k\| + o_k(\|\omega_\varrho - u_k\|).$$

Hence

$$\langle I_\lambda^{'}(u_k), \frac{u}{\|u\|}\rangle \leq \frac{1}{k\varrho}\|\omega_\varrho - u_k\| + \frac{o_k(\|\omega_\varrho - u_k\|)}{\varrho} + \frac{(\zeta_k(v_\varrho) - 1)}{\varrho}\langle I_\lambda^{'}(u_k) - I_\lambda^{'}(\omega_\varrho), u_k - v_\varrho\rangle. \tag{3.12}$$

Since $\|\omega_\varrho - u_k\| \leq \rho|\zeta_k(v_\varrho)| + |\zeta_k(v_\varrho) - 1|\|u_k\|$ and

$$\lim_{\varrho\to 0} \frac{|\zeta_k(v_\varrho) - 1|}{\varrho} \leq \|\zeta_k^{'}(0)\|,$$

passing to the limit $\varrho \to 0^+$ in (3.12), we obtain

$$\langle I_\lambda^{'}(u_k), \frac{u}{\|u\|}\rangle \leq \frac{C}{k}(1 + \|\zeta_k^{'}(0)\|)$$

for some $C > 0$ independent of u. Now, we have to show that $\|\zeta_k^{'}(0)\|$ is bounded. Arguing by contradiction, we assume that $\langle \zeta'(0), v\rangle = \infty$. From (3.7) and Hölder inequality, we observe that

$$\langle \zeta_k'(0), v\rangle = \frac{C\|v\|}{(p-q)a\|u_k\|^p + (mp-q)b\|u_k\|^{mp} - (p_s^* - q)\int_\Omega K(x)|u_k|^{p_s^*}dx}$$

for some $C > 0$, which implies that there exists a subsequence $\{u_k\}$ such that

$$(p-q)a\|u_k\|^p + (mp-q)b\|u_k\|^{mp} - (p_s^* - q)\int_\Omega K(x)|u_k|^{p_s^*}dx = o_k(1). \tag{3.13}$$

In a similar way, from (3.8) and Hölder inequality we can also prove that

$$a(p - p_s^*)\|u_k\|^p + b(mp - p_s^*)\|u_k\|^{mp} - (q - p_s^*)\int_\Omega \lambda f(x)|u_k|^q dx = o_k(1). \tag{3.14}$$

Then using (3.13) and (3.14), following the proof of Lemma 2.4, we can see that $\lambda \geq \lambda_1$, which contradicts the assumption $\lambda < \lambda_1$. This ends the proof. $\square$

Thanks to Lemma 3.7, we can get a (PS) sequence for I_λ. Now we are ready to show the following compactness result, which is crucial for the existence of solution for problem (1.1). For this, define

$$c_\lambda^* := \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda^{\frac{p}{p-q}}, \tag{3.15}$$

where $D = \frac{(p-q)(p_s^*-q)|f_+|_{q^*}^{\frac{p}{p-q}}}{pqp_s^*}\left(\frac{p_s^*-q}{(p_s^*-p)S}\right)^{\frac{q}{p-q}}$.

Lemma 3.8. *I_λ satisfies the (PS) condition at level $c_\lambda < c_\lambda^*$, where c_λ^* is given in (3.15).*

Proof. Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence for I_λ with $c_\lambda < c_\lambda^*$, i.e.

$$I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \|I_\lambda'(u_n)\|_{E_0^{-1}} \to 0. \tag{3.16}$$

By (2.4), $\{u_n\}$ is bounded in E_0. Hence, up to a subsequence, there exists $u \in E_0$ such that

$$\begin{aligned} &u_n \to u, && \text{a. e. in } \Omega, \qquad \|u_n\| \to \beta, \\ &u_n \rightharpoonup u, && \text{weakly in } E_0, \\ &u_n \to u, && \text{strongly in } L^r(\Omega), 1 \le r < p_s^*. \end{aligned} \tag{3.17}$$

Meanwhile, there exists $\bar{g} \in L^p(\Omega)$ such that $|u_n(x)| \le \bar{g}(x)$ a.e. in Ω. Denote $v_n = u_n - u$, then we can assume that $\lim_{n\to\infty} \|v_n\| = d_1 > 0$. Otherwise, the conclusion follows. By Lemma 2.1 and (3.17) we obtain

$$\begin{aligned} \|u_n\|^p &= \|u_n - u\|^p + \|u\|^p + o_n(1), \\ \int_\Omega K(x)|u_n|^{p_s^*}dx &= \int_\Omega K(x)|u_n - u|^{p_s^*}dx + \int_\Omega K(x)|u|^{p_s^*}dx + o_n(1) \end{aligned} \tag{3.18}$$

as $n \to \infty$. Therefore, we deduce from (3.17)-(3.18) that

$$o_n(1) = \langle I_\lambda'(u_n), u_n\rangle = M(\|u_n\|^p)\|u_n\|^p - \int_\Omega \lambda f(x)|u|^q dx - \int_\Omega K(x)|u|^{p_s^*}dx - \int_\Omega K(x)|v_n|^{p_s^*}dx, \tag{3.19}$$

$$o_n(1) = \langle I_\lambda'(u_n), u\rangle = M(\|u_n\|^p)\|u\|^p - \int_\Omega \lambda f(x)|u|^q dx - \int_\Omega K(x)|u|^{p_s^*}dx, \tag{3.20}$$

which imply

$$M(\|u_n\|^p)\|v_n\|^p - \int_\Omega K(x)|v_n|^{p_s^*}dx = o_n(1).$$

Let us denote $\lim_{n\to\infty}\int_\Omega K(x)|v_n|^{p_s^*}dx = d_2$. Then we have the following key formula

$$(a + b\beta^{(m-1)p})d_1^p = d_2. \tag{3.21}$$

Therefore, from (3.21), it is clear that $d_2 > 0$. Additionally, by the definition of S in (1.9), we get

$$d_1^p \ge S d_2^{p/p_s^*}. \tag{3.22}$$

From (3.21)-(3.22), we get that

$$d_1^p \ge a^{\frac{N-ps}{ps}} S^{\frac{N}{ps}} \tag{3.23}$$

Now, using the Hölder inequality, we have

$$
\begin{aligned}
c_\lambda &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{p_s^*}\langle I_\lambda'(u_n), u_n\rangle\right)\\
&= \lim_{n\to\infty}\left\{(\frac{1}{p}-\frac{1}{p_s^*})a\|u_n\|^p + (\frac{1}{mp}-\frac{1}{p_s^*})b\|u_n\|^{mp} - (\frac{1}{q}-\frac{1}{p_s^*})\lambda\int_\Omega f|u_n|^q dx\right\}\\
&\geq (\frac{1}{p}-\frac{1}{p_s^*})ad_1^p + (\frac{1}{p}-\frac{1}{p_s^*})a\|u\|^p - (\frac{1}{q}-\frac{1}{p_s^*})\lambda|f_+|_{q^*}S^{-\frac{q}{p}}\|u\|^q.
\end{aligned}
$$

Therefore, let us set

$$
F_\lambda(t) = (\frac{1}{p}-\frac{1}{p_s^*})at^p - (\frac{1}{q}-\frac{1}{p_s^*})\lambda|f_+|_{q^*}S^{-\frac{q}{p}}t^q.
$$

By a direct computation, $F_\lambda(t)$ attains its minimum

$$
\min_{t\geq 0} F_\lambda(t) = -\frac{(p-q)(p_s^*-q)(\lambda|f_+|)^{\frac{p}{p-q}}}{pqp_s^*}\left(\frac{p_s^*-q}{(p_s^*-p)S}\right)^{\frac{q}{p-q}} = -D\lambda^{\frac{p}{p-q}},
$$

where $D = \frac{(p-q)(p_s^*-q)|f_+|_{q^*}^{\frac{p}{p-q}}}{pqp_s^*}\left(\frac{p_s^*-q}{(p_s^*-p)S}\right)^{\frac{q}{p-q}}$. Hence, we get

$$
c_\lambda \geq \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda^{\frac{p}{p-q}} = c_\lambda^*
$$

which contradicts the hypothesis $c_\lambda < c_\lambda^*$. This ends the proof. ☐

Proposition 3.9. *Assume that $m < \frac{N}{N-ps}$ or $m = \frac{N}{N-ps}, b < S^{-m}$. Then for $\lambda \in (0, \lambda_0)$, I_λ has a minimizer u_λ in N_λ, which is a nonnegative solution of (1.1) with $I_\lambda(u_\lambda) = c_\lambda^+$ and $\|u_\lambda\| \to 0$ as $\lambda \to 0$.*

Proof. Let us fix $\lambda < \lambda_0 = \min\{\lambda_1, \tilde{\lambda}, \lambda_2, \lambda_3\}$, where $\lambda_1, \tilde{\lambda}$ and λ_2 given respectively in (2.5), Lemma 3.5 and (3.1). Meanwhile, we set

$$
\lambda_3 := (\frac{s}{N}(aS)^{\frac{N}{ps}}/D)^{\frac{p-q}{p}}. \tag{3.24}
$$

For $0 < \lambda < \lambda_0$, putting together the definition of c_λ^* and Lemma 3.5, we have

$$
c_\lambda^+ < 0 < c_\lambda^*.
$$

As a consequence of Ekeland's varational principle [15], we can find a (PS) sequence $\{u_n\} \subset N_\lambda^+ \subset N_\lambda$ such that $I_\lambda(u_n) \to c_\lambda^+$ as $n \to \infty$. From Lemma 3.8, there exists $u_\lambda \in N_\lambda$ such that

$$
I_\lambda'(u_\lambda) = 0, \quad I_\lambda(u_\lambda) = c_\lambda^+ < 0.
$$

Now we claim that $u_\lambda \in N_\lambda^+$. We discuss the proof only for the case $m < \frac{N}{N-ps}$, while the case $m = \frac{N}{N-ps}, b < S^{-m}$ follows similarly. If not, then $u_\lambda \in N_\lambda^-$ because of the Remark 3.2. Together with (2.2), we get $u_\lambda \in G^+$. Also from $u_\lambda \in N_\lambda$ and $I_\lambda(u_\lambda) = c_\lambda^+ < 0$, we can see that $u_\lambda \in H^+$. Therefore, from Lemma 3.1, we obtain there exist $t^-(u_\lambda) > t^+(u_\lambda) > 0$ such that $t^-u_\lambda \in N_\lambda^-$ and $t^+u_\lambda \in N_\lambda^+$. This implies $t^- = 1$ and $t^+ < 1$. Hence, we can find $\tilde{t} \in (t^+, t^-)$ such that

$$
I_\lambda(t^+u_\lambda) = \min_{0\leq t\leq t^-} I_\lambda(tu_\lambda) < I_\lambda(\tilde{t}u_\lambda) < I_\lambda(t^-u_\lambda) = I_\lambda(u_\lambda) = c_\lambda^+,
$$

which gives the desired contradiction. Thus $u_\lambda \in N_\lambda^+$.

We point out that $I_\lambda(u) \neq I_\lambda(|u|)$, since $\|u\| \neq \||u|\|$ in E_0. To achieve our aim, we study the positive part of the problem (1.1) by defining

$$I_\lambda^+(u) = \frac{a}{p}\|u\|^p + \frac{b}{mp}\|u\|^{mp} - \frac{1}{q}\int_\Omega \lambda f(x)|u^+|^q dx - \frac{1}{p_s^*}\int_\Omega K(x)|u^+|^{p_s^*}dx.$$

Then arguing similarly as above, it is readily to show that there exists a critical point $u_\lambda \in N_\lambda^+$ for I_λ^+. That is, for any $v \in E_0$,

$$\begin{aligned} &M(\|u_\lambda\|^p)\iint_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy \\ &= \int_\Omega \lambda f(x)|u_\lambda^+|^{q-1}vdx - \int_\Omega K(x)|u_\lambda^+|^{p_s^*-1}vdx. \end{aligned} \tag{3.25}$$

Then, from (3.25) with test function $v = u_\lambda^-$, we get

$$M(\|u_\lambda\|^p)\iint_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(u_\lambda^-(x)) - u_\lambda^-(y))}{|x-y|^{N+ps}}dxdy = o(1).$$

From the last equation, together with the facts

$$|u_\lambda^-(x)-u_\lambda^-(y)|^p \leq |u_\lambda^-(x)-u_\lambda^-(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(u_\lambda^-(x)-u_\lambda^-(y))$$

and

$$|u_\lambda^-(x)-u_\lambda^-(y)| \leq |u_\lambda(x)-u_\lambda(y)|,$$

we obtain

$$(a+b\|u_\lambda^-\|^{(m-1)p})\iint_Q \frac{|u_\lambda^-(x)-u_\lambda^-(y)|^p}{|x-y|^{N+ps}}dxdy = o(1).$$

Moreover, by $a, b > 0$, we get $\|u_\lambda^-\| = 0$, which implies that u_λ is a nonnegative solution of (1.1). Furthermore, from lemma 3.5, we deduce that u_λ is a ground state solution of (1.1).

Now, we shall show that the solution u_λ is a local minimizer of I_λ in E_0. It follows from Lemmas 3.1 and 3.3 that $t^+(u_\lambda) = 1 < t_{max}(u_\lambda)$, noting that $u_\lambda \in N_\lambda^+$. Therefore, from continuity of $u \mapsto t_{max}(u)$, for fixed $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that $t_{max}(u_\lambda - u) > 1 + \epsilon$ for all $\|u\| < \delta_1$. On the other hand, by Lemma 3.12, it is easy to verify that for a given $\delta_2 > 0$, there is a C^1 map $\zeta : B_{\delta_2}(0) \to \mathbb{R}^+$ such that $\zeta(u)(u_\lambda - u) \in N_\lambda^+$ and $\zeta(0) = 1$. Thus, considering $0 < \delta = \min\{\delta_1, \delta_2\}$ and uniqueness of zeros of fibering map, we conclude $t^+(u_\lambda - u) = \zeta(u) < 1 + \epsilon < t_{max}(u_\lambda - u)$ for all $\|u\| < \delta$. From $t_{max}(u_\lambda - u) > 1$, we get $I_\lambda(u_\lambda) \leq I_\lambda(t^+(u_\lambda - u)(u_\lambda - u)) \leq I_\lambda(u_\lambda - u)$. It follows immediately that u_λ is a local minimizer of I_λ in E_0. Finally using Lemma 3.4, the proof is completed. □

Lemma 3.10. *There exists $\lambda_4 > 0$ and $r > 0$ such that for any $\lambda \in (0, \lambda_4)$*

$$\inf_{u\in E_0, \|u\|=r} I_\lambda(u) = \alpha > 0.$$

In particular, when $\lambda = 0$, there exists $r_1 > r$ such that $I_0(u) > 0$, for all $u \in B_{r_1}\backslash\{0\}$.

Proof. For $u \in E_0$, we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{mp}\|u\|^{mp} - \frac{1}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{1}{p_s^*}\int_\Omega K(x)|u|^{p_s^*}dx \\ &\geq \frac{a}{p}\|u\|^p - \frac{\lambda|f_+|_{q^*}}{qS^{q/p}}\|u\|^q - \frac{1}{p_s^* S^{p_s^*/p}}\|u\|^{p_s^*}. \end{aligned} \tag{3.26}$$

From (3.26) we obtain

$$I_\lambda(u) \geq \left(\frac{a}{p}\|u\|^{p-q} - \frac{\lambda|f_+|_{q^*}}{qS^{q/p}} - \frac{1}{p_s^* S^{p_s^*/p}}\|u\|^{p_s^*-q}\right)\|u\|^q.$$

Denote

$$\tilde{g}(t) = \frac{a}{p}t^{p-q} - \frac{1}{p_s^*}S^{-p_s^*/p}t^{p_s^*-q},$$

for all $t \geq 0$. In view of $p < p_s^*$, for each $u \in E_0$ with

$$\|u\| = r := \left[\frac{ap_s^* S^{p_s^*/p}(p-q)}{p(p_s^*-q)}\right]^{1/(p_s^*-p)},$$

we obtain $\max_{t\geq 0}\tilde{g}(t) = \tilde{g}(r) > 0$. Therefore, taking

$$\lambda < \lambda_4 = \frac{\tilde{g}(r)q}{S^{-q/p}|f_+|_{q^*}},$$

we can conclude that

$$I_\lambda(u) \geq (\tilde{g}(r) - \lambda\frac{|f_+|_{q^*}}{qS^{p/q}})r^q =: \alpha > 0.$$

The first part of proof is finished.

When $\lambda = 0$, Using (3.26), there exists $r_1 = [(p_s^*-q)/(p-q)]^{1/(p_s^*-p)}r > r$, such that $I_0(u) > 0$, for all $u \in B_{r_1}\backslash\{0\}$. □

Lemma 3.11. *Let r be as in Lemma 3.10. Then*
(i) for $m < \frac{N}{N-ps}, a > 0, b > 0$, there exists $e \in E_0$ with $\|e\| > r$ such that $I_\lambda(e) < 0$;
(ii) for $m = \frac{N}{N-ps}, a > 0, 0 < b < 1/S^m$, there exists $e \in E_0$ with $\|e\| > r$ such that $I_\lambda(e) < 0$.

Proof. (i) For $m < \frac{N}{N-ps}, a > 0, b > 0$. For a given $u \in E_0\backslash\{0\}$ with $\int_\Omega K(x)|u|^{p_s^*}dx > 0$, by Fatou's Lemma, we can deduce that

$$\begin{aligned}\lim_{t\to\infty}\frac{I_\lambda(tu)}{t^{p_s^*}} &= \frac{a}{p}\lim_{t\to\infty}\frac{\|u\|^p}{t^{p_s^*-p}} + \frac{b}{mp}\lim_{t\to\infty}\frac{\|u\|^{mp}}{t^{p_s^*-mp}} - \frac{1}{q}\lim_{t\to\infty}\frac{\int_\Omega \lambda f(x)|u|^q dx}{t^{p_s^*-q}} - \frac{1}{p_s^*}\int_\Omega K(x)|u|^{p_s^*}dx \\ &< 0.\end{aligned}$$

Clearly, there exists $T > 0$ large enough such that $\|e\| = \|Tu\| \geq r$ and $I_\lambda(e) < 0$.

(ii) For $m = \frac{N}{N-ps}, a > 0, 0 < b < 1/S^m$. For a given $u \in E_0 \setminus \{0\}$ with $\int_\Omega K(x)|u|^{p_s^*}dx - b\|u\|^{mp} > 0$, by Fatou's Lemma, we can deduce that

$$\lim_{t\to\infty}\frac{I_\lambda(tu)}{t^{p_s^*}} = \frac{a}{p}\lim_{t\to\infty}\frac{\|u\|^p}{t^{p_s^*-p}} - \frac{1}{q}\lim_{t\to\infty}\frac{\int_\Omega \lambda f(x)|u|^q dx}{t^{p_s^*-q}} + \frac{1}{mp}\left(b\|u\|^{mp} - \int_\Omega K(x)|u|^{mp}dx\right) < 0.$$

The rest proof is the same as that in (i). □

In Lemma 3.10 and 3.11, we show that I_λ satisfies the mountain pass geometry. Therefore, there exists a $(PS)_c$ sequence with

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], E_0) : \gamma(0) = 0, \gamma(1) = e\}.$$

In order to estimate the mountain pass level c, we recall some facts. The authors in [16] observed that the following critical fractional p-Laplacian problem:

$$\begin{cases}(-\Delta)_p^s u = S|u|^{p_s^*-2}u, & x \in \mathbb{R}^N,\\ u \in D^{s,p}(\mathbb{R}^N).\end{cases} \tag{3.27}$$

admitted a positive radially symmetric decreasing solution $U = U(r)$ satisfying

$$\lim_{|x|\to\infty} |x|^{\frac{N-sp}{p-1}} U(x) = U_\infty \in \mathbb{R} \setminus \{0\}. \tag{3.28}$$

For any $\epsilon > 0$, define

$$U_{\epsilon,y}(x) = \frac{1}{\epsilon^{\frac{N-ps}{p}}} U(\frac{|x-y|}{\epsilon}).$$

Moreover, the following estimates hold:

Lemma 3.12. *[16] There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{N-sp}{p-1}}} \tag{3.29}$$

and

$$\frac{U(\theta r)}{U(r)} \leq \frac{1}{2}. \tag{3.30}$$

Let θ be the universal constant in Lemma 3.12 that depends only on N, p and s. For $\epsilon, \delta > 0$, let

$$m_{\epsilon,y,\delta} := \frac{U_{\epsilon,y}(\delta)}{U_{\epsilon,y}(\delta) - U_{\epsilon,y}(\theta\delta)},$$

$$g_{\epsilon,y,\delta}(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq U_{\epsilon,y}(\theta\delta),\\ m_{\epsilon,y,\delta}^p(t - U_{\epsilon,y}(\theta\delta)), & \text{if } U_{\epsilon,y}(\theta\delta) \leq t \leq U_{\epsilon,y}(\delta),\\ t + U_{\epsilon,y}(\delta)(m_{\epsilon,y,\delta}^{p-1} - 1), & \text{if } t \geq U_{\epsilon,y}(\delta),\end{cases}$$

and

$$G_{\epsilon,y,\delta}(t) := \int_0^t (g'_{\epsilon,y,\delta}(\tau))^{\frac{1}{p}} d\tau = \begin{cases} 0, & \text{if } 0 \leq t \leq U_{\epsilon,y}(\theta\delta),\\ m_{\epsilon,y,\delta}(t - U_\epsilon(\theta\delta)), & \text{if } U_{\epsilon,y}(\theta\delta) \leq t \leq U_{\epsilon,y}(\delta),\\ t, & \text{if } t \geq U_{\epsilon,y}(\delta).\end{cases}$$

We conclude that $g_{\epsilon,y,\delta}$ and $G_{\epsilon,y,\delta}$ are nondecreasing and absolutely continuous functions. For any $z \in \Pi$, let

$$u_{\epsilon,z,\delta} = G_{\epsilon,z,\delta}(U_{\epsilon,z}(r)).$$

By the definition of $G_{\epsilon,z,\delta}$, we get

$$u_{\epsilon,z,\delta}(r) = \begin{cases} U_{\epsilon,z}(r), & \text{if } r \leq \delta,\\ 0, & \text{if } r \geq \theta\delta,\end{cases} \tag{3.31}$$

As in [35, Lemma 2.7], the following asymptotic estimations hold:

Lemma 3.13. *There exists $C = C(N,p,s) > 0$ such that for any $\epsilon \leq \frac{\delta}{2}$ the following estimates hold*

$$\| u_{\epsilon,z,\delta} \|^p \leq S^{\frac{N}{ps}} + C((\frac{\epsilon}{\delta})^{\frac{N-ps}{p-1}}),$$

$$|u_{\epsilon,z,\delta}|_{p_s^*}^{p_s^*} \geq S^{\frac{N}{ps}} - C((\frac{\epsilon}{\delta})^{\frac{N}{p-1}}).$$

Moreover, in view of the definition of $u_{\epsilon,z,\delta}$, the following result also holds. From now on, we assume that $\delta = \frac{r_0}{2}$ and let $u_\epsilon = u_{\epsilon,z,r_0/2}$.

Lemma 3.14. *Let (K) hold, then there exists $C = C(N,p,s) > 0$ such that for any $\epsilon \leq \frac{\delta}{2}$,*

$$\int_{\Pi_{r_0}} (1-K)|u_\epsilon|^{p_s^*} dx \leq C\epsilon^\rho.$$

Proof. Using Remark 1.1, a change of variable, Lemma 3.12 and the fact that $\epsilon \leq \frac{\delta}{2}$, we get

$$\begin{aligned}
\int_{B_\delta(0)} (1-K)|u_\epsilon|^{p_s^*} dx &\leq \sigma_0 \int_{B_\delta(0)} |x|^\rho U_{\epsilon,z}^{p_s^*}(x)dx = C\epsilon^\rho \int_{B_{\frac{\delta}{\epsilon}}(0)} |x|^\rho U^{p_s^*} dx \\
&= C\epsilon^\rho \omega_{N-1} \int_1^{\frac{\delta}{\epsilon}} r^{\rho+N-1} U^{p_s^*}(r)dr \\
&\leq C\epsilon^\rho \omega_{N-1} \int_1^{\frac{\delta}{\epsilon}} r^{\rho+N-1-\frac{Np}{p-1}} dr \\
&= \frac{(p-1)C\epsilon^\rho}{N-(p-1)\rho} \left[1 - (\frac{\epsilon}{\delta})^{\frac{N}{p-1}-\rho}\right] \leq C\epsilon^\rho.
\end{aligned} \tag{3.32}$$

Since $U_{\epsilon,z}$ is radially nonincreasing, taking $\delta = \frac{r_0}{2}, \theta = 2$, then for $\delta \leq r \leq \theta\delta = r_0$, we see

$$0 \leq m_\epsilon(U_{\epsilon,z}(r) - U_{\epsilon,z}(\theta\delta)) = U_{\epsilon,z}(\delta)\frac{U_{\epsilon,z}(r) - U_{\epsilon,z}(\theta\delta)}{U_{\epsilon,z}(\delta) - U_{\epsilon,z}(\theta\delta)} \leq U_{\epsilon,z}(\delta).$$

Together with the definition of $U_{\epsilon,z}$, $\frac{\delta}{\epsilon} \geq 2$ and Lemma 3.12, we get

$$\begin{aligned}
\int_{B_{r_0}(0)\setminus B_\delta(0)} (1-K)|u_\epsilon|^{p_s^*} dx &\leq \int_{B_{r_0}(0)\setminus B_\delta(0)} |u_\epsilon|^{p_s^*} dx \\
&= \int_{B_{r_0}(0)\setminus B_\delta(0)} [m_\epsilon(U_{\epsilon,z}(r) - U_{\epsilon,z}(\theta\delta))]^{p_s^*} dx \\
&\leq \int_{B_{r_0}(0)\setminus B_\delta(0)} U_{\epsilon,z}^{p_s^*}(\delta)dx \\
&= |U_{\epsilon,z}(\delta)|^{p_s^*} |B_{r_0}(0) \setminus B_\delta(0)| \\
&\leq C\delta^{-\frac{Np}{p-1}} \epsilon^{\frac{N}{p-1}} \leq C\epsilon^{\frac{N}{p-1}}.
\end{aligned} \tag{3.33}$$

Using (3.32) and (3.33), we deduce that

$$\begin{aligned}
\int_{\Pi_{r_0}} (1-K)|u_\epsilon|^{p_s^*} dx &\leq \int_{B_\delta(0)} (1-K)|u_\epsilon|^{p_s^*} dx + \int_{B_{r_0}(0)\setminus B_\delta(0)} (1-K)|u_\epsilon|^{p_s^*} dx \\
&\leq C\epsilon^\rho + C\epsilon^{\frac{N}{p-1}} \leq C\epsilon^\rho,
\end{aligned}$$

because of $0 < \rho < \frac{N}{p-1}$. □

Lemma 3.15. *Let $m < \frac{N}{N-ps}$ or $m = \frac{N}{N-ps}, b < 1/S^m$, and (K) hold, then there exists $\lambda_* \in (0, \lambda_0), b_* > 0$ such that for any $\lambda \in (0, \lambda_*), b \in (0, b_*)$ the inequality*

$$c \le \sup_{t\ge 0} I_\lambda(tu_\epsilon) < \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda^{\frac{p}{p-q}}$$

holds.

Proof. To estimate c, we define the functions,

$$s(t) := I_\lambda(tu_\epsilon) = \frac{at^p}{p}\|u_\epsilon\|^p + \frac{bt^{mp}}{mp}\|u_\epsilon\|^{mp} - \frac{t^q}{q}\int_{\Pi_{r_0}} \lambda f(x)|u_\epsilon|^q dx - \frac{t^{p_s^*}}{p_s^*}\int_{\Pi_{r_0}} K(x)|u_\epsilon|^{p_s^*} dx$$

and

$$\tilde{s}(t) := \frac{at^p}{p}\|u_\epsilon\|^p - \frac{t^{p_s^*}}{p_s^*}\int_{\Pi_{r_0}} K(x)|u_\epsilon|^{p_s^*} dx$$

for all $t \ge 0$. Then for $m < \frac{N}{N-ps}$ or $m = \frac{N}{N-ps}, b < 1/S^m$, we can easily check that there exists $t_\epsilon > 0$ such that $s'(t_\epsilon) = 0$ and $\max_{t\ge 0} s(t) = s(t_\epsilon)$. It is clear that there exists $T > 0$ such that

$$0 < t_\epsilon \le T \quad \text{for } \epsilon \text{ sufficiently small,} \tag{3.34}$$

where T is given in Lemma 3.11. Gathering the estimates in Lemma 3.13 and Lemma 3.14, we get

$$\tilde{s}(t) \le \frac{s}{N}(aS)^{\frac{N}{sp}} + \frac{1}{p_s^*}\int_{\Pi_{r_0}} (1-K)|Tu_\epsilon|^{p_s^*} dx \le \frac{s}{N}(aS)^{\frac{N}{sp}} + O(\epsilon_0^\rho)$$

for some $\epsilon_0 > 0$. Let $\lambda_5 \le \min\{\lambda_0, \lambda_4\}$ such that $\frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda_5^{\frac{p}{p-q}} > 0$. Since

$$\lim_{t\to 0^+}\left(\frac{at^p}{p}S^{\frac{N}{sp}} + \frac{bt^{mp}}{mp}S^{\frac{Nm}{sp}}\right) = 0,$$

which implies that there exists $t_1 \in (0, t_\epsilon)$ such that for each $\lambda \in (0, \lambda_5)$ we get

$$\max_{0\le t\le t_1} s(t) \le \max_{0\le t\le t_1}\left(\frac{at^p}{p}S^{\frac{N}{sp}} + \frac{bt^{mp}}{mp}S^{\frac{Nm}{sp}}\right) \le \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda_5^{\frac{p}{p-q}} < \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda^{\frac{p}{p-q}}.$$

Clearly, there exist two positive numbers $b_* > 0$ and $\lambda_* \in (0, \lambda_5]$ such that for any $\lambda \in (0, \lambda_*), b \in (0, b_*)$ we get

$$\lambda\frac{t_1^q}{q}\int_{\Pi_{r_0}} f(x)|u_\epsilon|^q dx > D\lambda^{\frac{p}{p-q}} + Cb + C\epsilon_0^\rho.$$

Thus for all $\lambda \in (0, \lambda_*)$ and $b \in (0, b_*)$, we can conclude that

$$c \le \sup_{t\ge 0} I_\lambda(tu_\epsilon) = \sup_{t\ge 0} s(t) < \frac{s}{N}(aS)^{\frac{N}{ps}} - D\lambda^{\frac{p}{p-q}},$$

which completes the proof. □

Proof of Theorem 1.1. (i)The proof of (i) follows immediately by the Proposition 3.9.
(ii) By Lemma 3.10 and Lemma 3.11, it is easy to see I_λ has a mountain pass geometry. Thus there exists a bounded (PS) sequence $\{v_n\}$. Then from Lemmas 3.15 and 3.8, up to a subsequence, we get there exists $u_{\lambda,b} \in E_0$ such that $v_n \to u_{\lambda,b}$, a nontrivial and nonnegative solution of (1.1). Finally, since $I_\lambda(u_\lambda) < 0 < I_\lambda(u_{\lambda,b})$, u_λ and $u_{\lambda,b}$ are distinct. This ends the proof. □

Proof of Theorem 1.2. (i) By Remark 3.1, we see $N_\lambda^+ = N_\lambda$ and define $c_\lambda = \inf_{u \in N_\lambda} I_\lambda(u)$. It is clear that $c_\lambda < 0$. As done similarly in Proposition 3.9, there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset N_\lambda^+$ for I_λ. It follows from Lemma 2.2 $\{u_n\}$ is bounded in E_0. Hence, up to a subsequence, there exists $u \in E_0$ verifying (3.17). Denote $v_n = u_n - u$, for $b \geq S^{-m}$, we get

$$\int_\Omega K(x)|v_n|^{p_s^*}dx - b\|v_n\|^{mp} \leq 0.$$

Then combining this and (3.19)-(3.20), we get

$$a\|v_n\|^p \leq a\|v_n\|^p + b\|u\|^{(m-1)p}\|v_n\|^p + b\|v_n\|^{mp} - \int_\Omega K(x)|v_n|^{p_s^*}dx = o_n(1),$$

which implies that $u_n \to u$ in E_0. Following the argument used in Proposition 3.9, we get u is a nonnegative solution of (1.1).
(ii) The proof is similar to that of Theorem 1.1. □

4 Proof of Theorem 1.3

In this section, we will show you the proof of the last result by applying the Krasnoselskii genus theory. First of all, recall that in [2], let X be a real Banach space. Set

$$\sum = \{A \subset X \setminus \{0\} : \text{A is compact and } A = -A\}.$$

Definition 4.1. Let $A \in \sum$ and $X = \mathbb{R}^k$. The genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \min\{k \geq 1 : \text{there exists an odd continuous mapping } \phi : A \to \mathbb{R}^k \setminus \{0\}\}.$$

If such a mapping does not exist for any $k > 0$, we set $\gamma(A) = +\infty$. Moreover, from definition, $\gamma(\emptyset) = 0$.

For the functional defined in (1.10), we deduce from Hölder inequality and (1.9) that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{p}\|u\|^p + \frac{b}{mp}\|u\|^{mp} - \frac{\lambda}{q}|f|_{q^*}S^{-q/p}\|u\|^q - \frac{1}{p_s^*}S^{-p_s^*/p}\|u\|^{p_s^*} \\ &\geq C_1\|u\|^p - C_2\lambda\|u\|^q - C_3\|u\|^{p_s^*}, \end{aligned}$$

where $C_1 = \frac{a}{p}, C_2 = \frac{|f|_{q^*}S^{-q/p}}{q}, C_3 = \frac{1}{p_s^*}S^{-p_s^*/p}$. Define $l : \mathbb{R}^+ \to \mathbb{R}$ as

$$l(t) = C_1t^p - C_2\lambda t^q - C_3t^{p_s^*}.$$

It is easy to see that

$$I_\lambda(u) \geq l(\|u\|). \tag{4.1}$$

We note that $l(t)$ achieves its positive maximum. Similar to the argument in the proof of Theorem 1.1, one can readily show that there exists $\lambda_3 > 0$ as given in (3.24) such that for any $\lambda \in (0, \lambda_3)$, we get that $c_\lambda^* > 0$, where c_λ^* defined in (3.15). Also we can conclude that there are constants $0 < T_1 < T_2$, for $t < T_1$, $l(t) \leq 0$, for $t \in (T_1, T_2)$, $l(t) > 0$, and $l(t) < 0$ for $t > T_2$. Let $\kappa \in C_0^\infty(\mathbb{R}^+)$ be nonincreasing such that $0 \leq \kappa \leq 1$,$\kappa = 1$ if $t \leq T_1$ and $\kappa = 0$ if $t \geq T_2$. From now on, we set $\omega(u) = \kappa(\|u\|)$. We consider the following truncated functional

$$J(u) = \frac{a}{p}\|u\|^p + \frac{b}{mp}\|u\|^{mp} - \frac{1}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{1}{p_s^*}\int_\Omega K(x)|u|^{p_s^*}\omega(u)dx.$$

As (4.1), we obtain that $J(u) \geq \tilde{l}(\|u\|)$, where $\tilde{l}(t) = C_1 t^p - C_2\lambda t^q - C_3 t^{p_s^*}\kappa(t)$. It is immediate to see that $\tilde{l}(t) \geq l(t)$ for $t \geq 0$, $\tilde{l}(t) = l(t)$ if $0 \leq t \leq T_1$, $\tilde{l}(t) \geq l(t)$ if $T_1 < t \leq T_2$, and if $t > T_2$, $\tilde{l}(t) = C_1 t^p - C_2\lambda t^q$, which is strictly increasing. Hence, we immediately get $\tilde{l}(t) > 0$ if $t > T_2$. Furthermore, $\tilde{l}(t) > 0$ for $t > T_1$.

From this, we can state the next technical lemma.

Lemma 4.2. *(i)* $J \in C^1(E_0, \mathbb{R})$.
(ii) If $J(u) < 0$, *then* $\|u\| < T_1$ *and* $J(\tilde{u}) = I_\lambda(\tilde{u})$ *for all* $\tilde{u}$ *in a small enough neighbourhood of* u.
(iii) There is a $\lambda_3 > 0$ *such that if* $\lambda \in (0, \lambda_3)$, *then* J *satisfies the* $(PS)_c$ *condition for* $c < 0$.

Proof. (i) Since $\kappa \in \mathcal{C}^\infty$ and $\kappa = 1$ for u near 0, $J \in C^1(E_0, \mathbb{R})$ and completes the proof.

Let us prove (ii) by contradiction. Assume $J(u) < 0$ and $\|u\| \geq T_1$, then $0 > J(u) \geq \tilde{l}(\|u\|) \geq 0$, a contradiction. Thus (ii) holds true.

Next, we prove (iii). Let λ_3 be given in (3.24). If $\{u_n\}$ is a $(PS)_c$ sequence for J with $c < 0$, i.e.,

$$J(u_n) \to c \text{ and } \|J'(u_n)\|_{E_0^{-1}} \to 0 \text{ as } n \to \infty.$$

Then it follows from (ii) that $\|u\| < T_1$. Hence, $I_\lambda(u_n) = J(u_n)$ and $I'_\lambda(u_n) = J'(u_n)$. By Lemma 3.8, I_λ satisfies the $(PS)_c$ condition for $c < 0 < c_\lambda^*$. This indicates that J satisfies the $(PS)_c$ condition for $c < 0$. □

Now, we are ready to state the main result of this section constructing negative critical values of J via genus, as done similarly in [17].

Lemma 4.3. *Let* $f_- \equiv 0$ *and* $k \in \mathbb{N}$, *there exists an* $\epsilon = \epsilon(k) > 0$, *such that* $\gamma(\{u \in E_0 : J(u) \leq -\epsilon\}) \geq k$.

Proof. Fix k, let us set E_k be a k dimensional subspace of E_0. Let $u \in E_k$ be such that $\|u\| = 1$. For $0 < \theta < T_1$, we obtain

$$I_\lambda(\theta u) = J(\theta u) = \frac{a}{p}\theta^p + \frac{b}{mp}\theta^{mp} - \frac{\theta^q}{q}\int_\Omega \lambda f(x)|u|^q dx - \frac{\theta^{p_s^*}}{p_s^*}\int_\Omega K(x)|u|^{p_s^*}\omega(\theta)dx.$$

Obviously, all the norms in E_k are equivalent. It is easy to see that

$$\sigma_k = \inf\{\int_\Omega |u|^q dx : u \in E_k, \|u\| = 1\} > 0.$$

Thus

$$J(\theta u) \leq \frac{a\theta^p}{p} + \frac{b\theta^{mp}}{mp} - \lambda C\frac{\theta^q}{q}\sigma_k - \frac{\theta^{p_s^*}}{p_s^*}\int_\Omega K(x)|u|^{p_s^*}\omega(\theta)dx.$$

Then we conclude that for any $\epsilon > 0$ there exists a positive number $\theta < T_1$ such that for each $u \in E_k$ with $\|u\| = 1$, $J(\theta u) \leq -\epsilon$. Denote $S_\theta = \{u \in E_0 | \|u\| = \theta\}$. Clearly, $S_\theta \cap E_k \subset \{u \in E_0 : J(u) \leq -\epsilon\}$. Applying monotonicity property of genus and the fact that $\gamma(S_\theta \cap E_k) = k$, we get that $\gamma(\{u \in E_0 : J(u) \leq -\epsilon\}) \geq \gamma(S_\theta \cap E_k) = k$. □

Now we will prove the existence of infinitely many solutions for problem (1.1).

Proof of Theorem 1.3. Set

$$\sum_k = \{A \subset \sum : \gamma(A) \geq k\}, \quad c_k = \inf_{A\in\sum_k} \sup_{u\in A} J(u), \; k = 1, 2, \cdots.$$

Also, consider the following set

$$K_c = \{u \in E_0 : J(u) = c, J'(u) = 0\}, \quad J^{-\epsilon} = \{u \in E_0 : J(u) \leq -\epsilon\}$$

and assume that $0 < \lambda < \lambda_3$, with λ_3 given in Lemma 4.2 (iii). It follows from Lemma 4.3 that for any $k \in \mathbb{N}$, there is a $\epsilon(k) > 0$ such that $\gamma(J^{-\epsilon}) \geq k$. Together with the fact that J is continuous and even, we get that $J^{-\epsilon} \in \sum_k$ and $c_k \leq -\epsilon(k) < 0$. Furthermore, $c_k > -\infty$, as J is bounded from below.

Now, we claim that if there exist $k, i \in \mathbb{N}$ such that $c = c_k = c_{k+1} = \cdots = c_{k+i}$, then $\gamma(K_c) \geq i + 1$. Arguing by contradiction, we assume that $c = c_k = c_{k+1} = \cdots = c_{k+i} < 0$ and $\gamma(K_c) \leq i$. Hence, by Lemma 4.2 (iii), J satisfies the $(PS)_c$ condition, which implies that K_c is compact. Since $\gamma(K_c) \leq i$, from [2, Proposition 7.5], there exists a closed and symmetric set V with $K_c \subset V$ and $\gamma(V) \leq i$. On the other hand, because $c < 0$, we can also hypothesize that the closed set $V \subset J^0$. From [18, Lemma 1.3], there is an odd homomorphism $\tau : E_0 \to E_0$ such that

$$\tau(J^{c+\delta} - V) \subset J^{c-\delta} \tag{4.2}$$

for some $0 < \delta < -c$. Noting that

$$c = c_{k+i} = \inf_{A\in\sum_{k+i}} \sup_{u\in A} J(u),$$

there is an $A \in \sum_{k+i}$ such that $\sup_{u\in A} J(u) < c + \delta$, i. e. $A \subset J^{c+\delta}$. Together with (4.2), we get $\tau(A - V) \subset \tau(J^{c+\delta} - V) \subset J^{c-\delta}$, which implies

$$\sup_{u\in\tau(A-V)} J(u) \leq c - \delta. \tag{4.3}$$

On the other hand, by $\gamma(V) \leq i$, we get that $\gamma(\tau(\overline{A - V})) \geq \gamma(\overline{A - V}) \geq \gamma(A) - \gamma(V) \geq k$. from this, it is clearly that $\tau(\overline{A - V}) \in \sum_k$ and $\sup_{u\in\tau(A-V)} J(u) \geq c_k = c$, which contradicts (4.3). Thus the claim holds true.

Finally, if for all $k \in \mathbb{N}, \sum_{k+1} \subset \sum_k, c_k \leq c_{k+1} \leq 0$ and all c_k are distinct, then $\gamma(K_c) \geq 1$. We observe that there is a sequence of distinct negative critical values of J. If for some k_1, there exists a $i \geq 1$ such that $c = c_{k_1} = c_{k_1+1} = \cdots = c_{k_1+i}$, then from the claim, it is clear that $\gamma(K_c) \geq i + 1$, which implies that K_c contains infinitely many distinct elements. Then from Lemma 4.2 (ii), we know that there exist infinitely many critical points for I_λ, concluding the proof. □

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