

On the integro-differential equations with reflection

Elhadi Ait Dads¹. Safoua Khelifi² and Mohsen Miraoui³

Abstract

There are a few purely periodic phenomena in nature, which allows one to consider several other generalizations, such as almost automorphic and measure pseudo almost automorphic oscillations. In this paper, by developing important properties on the composition of functions with reflection, using some exponential dichotomy properties and an application of the fixed-point theorem, several new sufficient conditions for the existence and the uniqueness of an pseudo almost automorphic solutions with measure for some general type reflection integro-differential equations. We suppose that the nonlinear part is measure pseudo almost automorphic and in which we distinguish the two constant and variable cases for the lipschitz coefficients of the functions associated with this part. It is assumed that the linear part of the equation considered admits an exponential dichotomy. Finally, an application is given on the very interesting model of Markus and Yamabe.

Keywords Exponential Dichotomy . Pseudo-almost automorphic . Reflection . Integro-differential equations

Mathematics Subject Classification 34K14. 35B15. 47D06

1 Introduction

Many physical, chemical, biological, economic phenomena, epidemiological findings may be more or less periodic. Different types of equations can be used to model these phenomena: these include integral equations, abstract operational equations, partial differential equations, difference equations, functional equations, piecewise constant argument differential equations, to name but a few. The study of these phenomena requires concepts that go beyond the concept of periodicity, that are relevant to the taking into account the fact that these phenomena are not entirely periodic. In the framework of this work, we consider a generalization of periodic functions: the functions almost automorphic. The concept of almost periodicity was generalized by the concept of almost automorphy in 1964 by Bochner in [7]. For more details on these last themes, one can read the two references of Veech [21] and N'Guérékata [20]. The

¹ Elhadi Ait Dads
aitdads@uca.ac.ma
Cadi Ayyad University, Morocco, UMI 209 UMMISCO, IRD Bondy France, Unité Associée au CNRST URAC 02 Morocco.

² Safoua Khelifi
khelifisafoua@gmail.com
LR11ES53, University of Sfax, Tunisia

³ Mohsen Miraoui
miraoui.mohsen@yahoo.fr
University of Kairouan, LR11ES53, University of Sfax, Tunisia

notion of measure pseudo almost periodicity and automorphy [2, 3, 5, 8, 10, 17, 18, 19], is a generalization of pseudo periodicity and weighted periodicity (see [13, 22]).

In this paper, motivated by above mentioned works, we will consider the following semilinear equation:

$$\begin{aligned} V'(s) = & A(s)V(s) + B(s)V(-s) + f(s, V(s), V(-s)) \\ & + \int_{-s}^s K(s+y)g(y, V(y), V(-y))dy \\ & + \int_s^{+\infty} [K(y-s) + K(s+y)]g(y, V(y), V(-y))dy. \end{aligned} \quad (1.1)$$

Equation (1.1) can be rewritten in another equivalent way of the following form:

$$\begin{aligned} V'(s) = & A(s)V(s) + B(s)V(-s) + f(s, V(s), V(-s)) \\ & + \int_s^{+\infty} [K(y-s) + K(y+s)]g(y, V(y), V(-y))dy + \int_{-s}^s K(y+s)g(y, V(y), V(-y))dy \\ = & A(s)V(s) + B(s)V(-s) + f(s, V(s), V(-s)) + \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy \\ & + \int_s^{+\infty} K(y+s)g(y, V(y), V(-y))dy + \int_{-s}^s K(y+s)g(y, V(y), V(-y))dy \\ = & A(s)V(s) + B(s)V(-s) + f(s, V(s), V(-s)) + \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy \\ & + \int_{-s}^{+\infty} K(y+s)g(y, V(y), V(-y))dy, \end{aligned}$$

where the two operators A and B are square matrix of order n in \mathbb{N}^* , $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$ are continuous functions. During the past years, there has been an increasing interest in the integro-differential equation which arise in some practical problem such as semilinear logistic. Some results about pseudo almost automorphic solutions associated with this type of equation (see [1, 14, 15]).

The plan of this work is as follows: Section 2 begins by giving definitions of exponential dichotomy and almost automorphic functions with measure. In addition to giving examples with figures and useful results in the following sections. Let us describe the content of this paper. In Sections 3, we give some results on the existence and the uniqueness of μ -pseudo almost automorphic solutions of system (1.1). In Section 4, we study the case when the Lipschitz coefficients of the functions are variable. At last, an example is given on the very interesting model of Markus and Yamabe in Section 5.

2 Preliminaries

2.1 Exponential dichotomy

Knowing that $\mathbb{M}_n(\mathbb{R})$ is the space of the square matrix of order $n \in \mathbb{N}^*$ with real coefficients, and J is an interval of \mathbb{R} , it is assumed that A is a matrix defined on J and with values in $\mathbb{M}_n(\mathbb{R})$.

Definition 2.1. [2] If $X(t)$ is the fundamental matrix of the equation:

$$x'(s) = A(s)x(s), \quad s \in J, \quad (2.1)$$

such that $X(0) = I$, then Eq. (2.1) has an exponential dichotomy on J , if there exist two real coefficients $\alpha > 0$, $\kappa > 1$ and a matrix Λ verifying $\Lambda^2 = \Lambda$, such that for all $s, t \in J$ we have:

$$\|X(s)\Lambda X^{-1}(t)\| \leq \kappa e^{\alpha(t-s)} \text{ if } t \leq s;$$

and

$$\|X(s)(I - \Lambda)X^{-1}(t)\| \leq \kappa e^{\alpha(s-t)} \text{ if } s \leq t.$$

Let $(\Lambda, \kappa, \alpha)$ the three associated coefficients with exponential dichotomy.

Theorem 2.1. [2] *If $(\Lambda, \kappa, \alpha)$ the three associated coefficients with exponential dichotomy of Eq. (2.1) on \mathbb{R} and $B : \mathbb{R} \rightarrow \mathbb{M}_n(\mathbb{R})$ is bounded and continuous such that $\delta = \sup_{s \in \mathbb{R}} |B(s)| < \frac{\alpha}{4\kappa^2}$, then the system*

$$y'(s) = [A(s) + B(s)]y(s), \quad s \in \mathbb{R}, \quad (2.2)$$

admits an exponential dichotomy with coefficients

$$\left(\mathfrak{Q}, \kappa_1 = \frac{5\kappa^2}{2}, -2\kappa\delta + \alpha \right),$$

where \mathfrak{Q} denotes a projection, with the same kernel as the one of Λ . Moreover, let $Y(t)$ the fundamental matrix of Eq. (2.2) verifying $Y(0) = I$. Then for all $s \in \mathbb{R}$, we have:

$$|Y(s)\mathfrak{Q}Y^{-1}(s) - X(s)\Lambda X^{-1}(s)| \leq \frac{4}{\alpha}\delta\kappa^3.$$

Let V be a solution of equation (1.1) and $Z(s) = \begin{pmatrix} V(s) \\ V(-s) \end{pmatrix}$. Then the function Z checks the equation

$$Z'(s) = \mathfrak{M}(s)Z(s) + \mathfrak{N}(s, Z(s)),$$

where

$$\mathfrak{M}(s) = \begin{pmatrix} A(s) & B(s) \\ -B(-s) & -A(-s) \end{pmatrix},$$

$$\mathfrak{N}(s, Z(s)) = \begin{pmatrix} f(s, V(s), V(-s)) \\ f(s, V(-s), V(s)) \end{pmatrix} + \mathfrak{G}(s, Z(s)),$$

and

$$\mathfrak{G}(s, Z(s)) = \begin{pmatrix} \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy + \int_{-s}^{+\infty} K(y+s)g(y, V(y), V(-y))dy \\ \int_{-s}^{+\infty} K(y+s)g(y, V(y), V(-y))dy + \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy \end{pmatrix}.$$

2.2 μ -Pseudo almost automorphic functions

In this section, let $\mathcal{BC}(\mathbb{R}, \mathbb{R}^n) = \{f : \mathbb{R} \rightarrow \mathbb{R}^n; f \text{ continuous and bounded}\}$.

Definition 2.2. [7] (Bohr, 1924) $f : \mathbb{R} \mapsto \mathbb{R}^n$ (continuous) is almost periodic (or in $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$, or Bohr almost periodic) if $\forall \varepsilon > 0, \exists l > 0; \forall \delta \in \mathbb{R}, \exists \tau \in [\delta, \delta + l]$ verifying

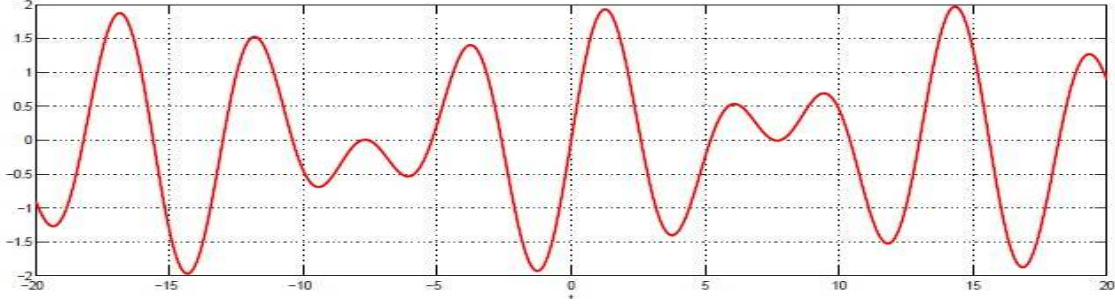
$$\|f(t + \delta) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Definition 2.3. [7] (Bochner, 1927) $f : \mathbb{R} \mapsto \mathbb{R}^n$ (continuous) is almost periodic (or Bochner almost periodic) if from every sequence of real numbers $(h'_n)_{n \in \mathbb{N}}$ one can extract a subsequence $(h_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} f(s + h_n) = g(s)$ exists uniformly on \mathbb{R} .

Theorem 2.2. [11] *A continuous function is almost periodic in the Bochner sense if it is almost periodic in the Bohr sense and vice versa.*

Example 2.3.

$$[t \rightarrow f(t) := \sin \sqrt{2}t + \sin t] \in \mathcal{AP}(\mathbb{R}, \mathbb{R}).$$

Figure 1: Curve of the almost periodic function f

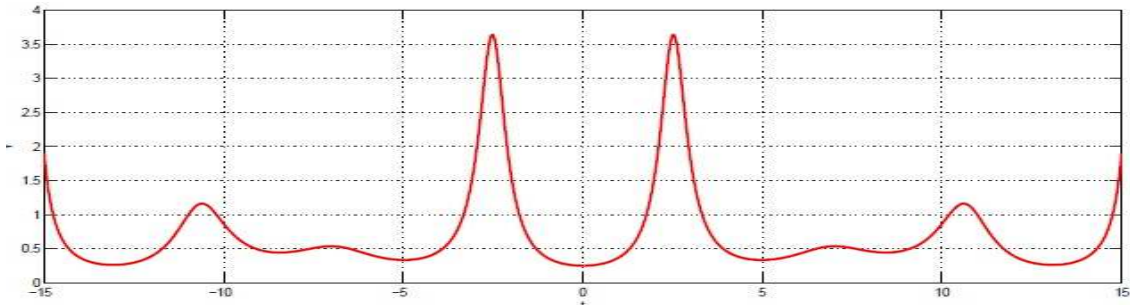
Remark 2.4. It is easy to check the following points:

- $t \rightarrow f(t) = \sin \sqrt{2}t + \sin t$ is quasi-periodic and in $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$, but it is not periodic.
- f is almost periodic $\Rightarrow f$ is uniformly continuous.
- Provided with the infinite norm, $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$ is a Banach space.
- The product of two functions of $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$ remains in $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.4. [7] $f : \mathbb{R} \rightarrow \mathbb{R}^n$ (continuous) is almost automorphic (or in $\mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$), if for all real sequence (s_n) , there exists a subsequence (τ_n) such that we have $g(s) := \lim_{n \rightarrow \infty} f(s + \tau_n)$ is well defined for all s in \mathbb{R} and $\lim_{n \rightarrow \infty} g(s - \tau_n) = f(s)$ for all s in \mathbb{R} .

Example 2.5.

$$[t \rightarrow g(t) := \frac{1}{2 + \cos t + \cos \sqrt{2}t}] \in \mathcal{AA}(\mathbb{R}, \mathbb{R}).$$

Figure 2: Curve of the almost automorphic function g

Remark 2.6. 1. $(\mathcal{AA}(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space.

2. $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n) \subsetneq \mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$, since $g \in \mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$ and g is not in $\mathcal{AP}(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.5. [7] A function f from a set $\mathbb{R} \times \mathbb{R}^n$ to a set \mathbb{R}^n is called almost automorphic in t uniformly respecting $x \in \mathbb{R}^n$ (or in $\mathcal{AAU}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$), if and only if points (i) and (ii) are true, where:

- (i) $\forall x \in \mathbb{R}^n, t \rightarrow f(t, x)$ is in $\mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$.
- (ii) The function f is uniformly continuous on all K (compact) in \mathbb{R}^n respecting $x \in \mathbb{R}^n$.

Let \mathfrak{d} the set of each positive measures on \mathcal{B} , where \mathcal{B} denotes the Lebesgue σ -field of \mathbb{R} . Then, we pose

$$\mathcal{M} = \left\{ \mu \in \mathfrak{d}; \mu(\mathbb{R}) = +\infty \text{ and } \mu([m, n]) < +\infty, \forall m, n \in \mathbb{R}, m \leq n \right\}.$$

Let $\mu \in \mathcal{M}$. We define the following ergodic space:

$$\mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu) := \left\{ \varphi \in \mathcal{BC}(\mathbb{R}, \mathbb{R}^n) : \lim_{z \rightarrow \infty} \frac{1}{\mu([-z, z])} \int_{-z}^z \|\varphi(s)\| d\mu(s) = 0 \right\}.$$

Definition 2.6. [6] A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is μ -pseudo almost automorphic (or in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu)$), if we have

$$f = a_0 + e_0,$$

where $a_0 \in \mathcal{AA}(\mathbb{R}, \mathbb{R}^n)$ and $e_0 \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu)$.

Example 2.7.

$$[s \rightarrow h(s) := \sin \frac{1}{2 - \sin \pi s - \sin s} + \frac{1}{\sqrt{1 + s^2}}] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu), \text{ where } \mu \in \mathcal{M}.$$

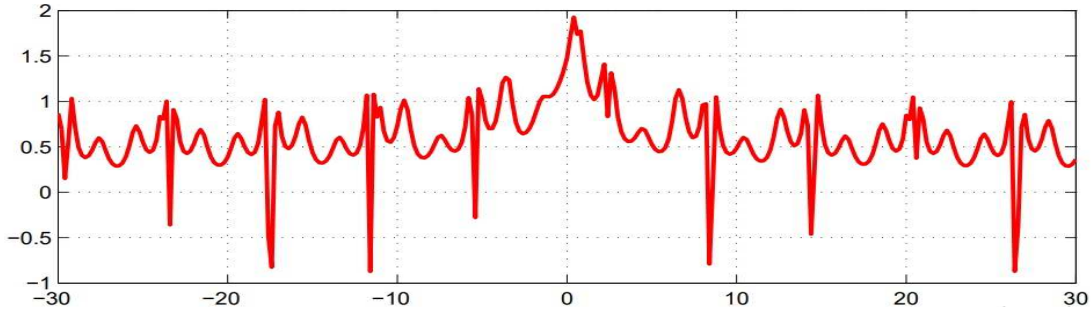


Figure 3: Curve of the measure pseudo almost automorphic function h

Definition 2.7. [5] If $\mu \in \mathcal{M}$, $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is μ -ergodic in t uniformly with respect to $x \in \mathbb{R}^n$ (or in $\mathcal{EU}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$), if and only if points (i) and (ii) are true, where

- (i) $\forall x \in \mathbb{R}^n, t \rightarrow f(t, x)$ is in $\mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu)$.
- (ii) f is uniformly continuous on all K (compact) in \mathbb{R}^n respecting $x \in \mathbb{R}^n$.

Definition 2.8. [5] If $\mu \in \mathcal{M}$, $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ (continuous) is μ -pseudo almost automorphic (or in $\mathcal{PAAU}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$), if we have:

$$f = a_1 + e_1,$$

where $a_1 \in \mathcal{AAU}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $e_1 \in \mathcal{EU}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu)$.

Remark 2.8. (i) $\mathcal{AA}(\mathbb{R}, \mathbb{R}^n) \subset \mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu) \subset \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$.

(ii) The paper of Blot and collaborators [5], is the best reference to show the translation invariance and the completeness of the space of $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu)$.

In this paper and in order to establish our results, we formulate the condition **(H.1)**, where:

(H.1) $\forall \tau \in \mathbb{R}$, there exist I (bounded interval) and $\beta > 0$; if $E \in \mathcal{B}$ and $E \cap I = \emptyset$, we have

$$\mu(\{a + \tau : a \in E\}) \leq \beta \mu(E).$$

Lemma 2.9. [5] *If μ in \mathcal{M} and **(H.1)** trues, then we have:*

- 1) *The decomposition in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu)$ is unique.*
- 2) *$(\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu), \|\cdot\|_\infty)$ is a Banach space.*
- 3) *The space $\mathcal{PAA}(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant.*

3 μ -Pseudo almost automorphic solution

We also assume that the conditions **(H.2)**- **(H.5)** hold, where:

(H.2) $\exists m, n > 0$, such that $\forall A \in \mathcal{B}$, we have $m + n\mu(A) - \mu(-A) \geq 0$.

(H.3) $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$ are μ -pseudo almost automorphic in t .

(H.4)(i) There exists $L_f^1, L_f^2 > 0$; $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\left| f(t, x_1, x_2) - f(t, y_1, y_2) \right| < L_f^1 |x_1 - y_1| + L_f^2 |x_2 - y_2|.$$

(ii) There exists $L_g^1, L_g^2 > 0$; $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\left| g(t, x_1, x_2) - g(t, y_1, y_2) \right| < L_g^1 |x_1 - y_1| + L_g^2 |x_2 - y_2|.$$

(H.5) $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$\int_0^{+\infty} K(y) dy := c \in \mathbb{R}.$$

(H.6) The Eq. $x'(s) = A(s)x(s)$ has an exponential dichotomy with coefficients $(\Lambda, \alpha, \kappa)$.

(H.7) The operator $B : \mathbb{R} \rightarrow \mathbb{M}_n(\mathbb{R})$ is uniformly bounded in $t \in \mathbb{R}$ and continuous. In addition, one of the following two conditions is assumed.

$$\lim_{z \rightarrow +\infty} \frac{1}{2z} \int_{-z}^z \|B(s)\| ds = 0 \quad \text{or} \quad \sup_{s \in \mathbb{R}} \|B(s)\| < \frac{\alpha}{4\kappa^2}.$$

Lemma 3.1. [2] *Let $f \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$. If $\mu \in \mathcal{M}$ such that **(H.1)**-**(H.2)** are true, then $[t \mapsto f(-t)] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$.*

Lemma 3.2. *If **(H.1)**-**(H.4)** are true and $V \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then $[t \mapsto f(t, V(t), V(-t))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$.*

Proof. By using **(H.3)**, $f \in \mathcal{PAA}(\mathbb{R}^3, \mathbb{R}, \mu)$, then $f = \varphi + h$, where: $\varphi \in \mathcal{EU}(\mathbb{R}^3, \mathbb{R}, \mu)$ and $h \in \mathcal{AAU}(\mathbb{R}^3, \mathbb{R})$ such that

$$\lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |\varphi(s, u)| d\mu(s) = 0,$$

uniformly for $u \in \mathbb{R}^2$. Since $V \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then $V = V_1 + V_2$ where $V_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, and $V_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. Now let us rewrite

$$\begin{aligned} f(t, V(t), V(-t)) &= h(t, V_1(t), V_1(-t)) + f(t, V(t), V(-t)) - h(t, V_1(t), V_1(-t)) \\ &= h(t, V_1(t), V_1(-t)) + f(t, V(t), V(-t)) - f(t, V_1(t), V_1(-t)) + \varphi(t, V_1(t), V_1(-t)). \end{aligned}$$

Consider the function $H(t) := h(t, V_1(t), V_1(-t))$, suppose that $\{s_n\}$ is a sequence of \mathbb{R} , since $h \in \mathcal{AAU}(\mathbb{R}^3, \mathbb{R})$, then there exists a subsequence $\{\tau_n\}$ of $\{s_n\}$ such that:

- (1) $\lim_{n \rightarrow \infty} h(t + \tau_n, v, u) = \phi(t, v, u)$, for all $t, v, u \in \mathbb{R}$;
- (2) $\lim_{n \rightarrow \infty} \phi(t - \tau_n, v, u) = h(t, v, u)$, for all $t, v, u \in \mathbb{R}$;
- (3) $\lim_{n \rightarrow \infty} V_1(t + \tau_n) = U_1(t)$, for all $t \in \mathbb{R}$;
- (4) $\lim_{n \rightarrow \infty} U_1(t - \tau_n) = V_1(t)$, for all $t \in \mathbb{R}$.

If we define $\Phi(t) : \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi(t) = \phi(t, U_1(t), U_1(-t))$, then for $t \in \mathbb{R}$ we obtain:

$$\lim_{n \rightarrow \infty} H(t + \tau_n) = \Phi(t); \quad \lim_{n \rightarrow \infty} \Phi(t - \tau_n) = H(t).$$

Obviously, we have

$$\begin{aligned} |H(t + \tau_n) - \Phi(t)| &\leq |h(t + \tau_n, V_1(t + \tau_n), V_1(-t + \tau_n)) - h(t + \tau_n, U_1(t), U_1(-t))| \\ &\quad + |h(t + \tau_n, U_1(t), U_1(-t)) - \phi(t, U_1(t), U_1(-t))|. \end{aligned}$$

Since $V_1(t)$ is almost automorphic, $V_1(t)$, and $U_1(t)$ are bounded. By (3), we have $h(t, V_1(t), V_1(-t))$ is uniformly continuous on each compact subset $K \subset \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} |h(t + \tau_n, V_1(t + \tau_n), V_1(-t + \tau_n)) - h(t + \tau_n, U_1(t), U_1(-t))| = 0.$$

Since $h \in \mathcal{AA}(\mathbb{R}^3, \mathbb{R})$, then $\lim_{n \rightarrow \infty} H(t + \tau_n) = \Phi(t)$. Using the same argument, we obtain

$$\lim_{n \rightarrow \infty} \Phi(t - \tau_n) = H(t), \text{ for all } t \in \mathbb{R}.$$

The last result implies that H is almost automorphic. It only remains to show that

$$s \rightarrow [f(s, V(s), V(-s)) - f(s, V_1(s), V_1(-s))] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu).$$

Consider now the function $\Omega(s) = f(s, V(s), V(-s)) - f(s, V_1(s), V_1(-s))$. Clearly $\Omega \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$. Since

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f^1 |u_1 - v_1| + L_f^2 |u_2 - v_2|.$$

Thus we obtain

$$\begin{aligned} \frac{1}{\mu([-z, z])} \int_{-z}^z |\Omega(s)| d\mu(s) &= \frac{1}{\mu([-z, z])} \int_{-z}^z |f(s, V(s), V(-s)) - f(s, V_1(s), V_1(-s))| d\mu(s) \\ &\leq \frac{1}{\mu([-z, z])} \int_{-z}^z L_f^1 |V(s) - V_1(s)| + L_f^2 |V(-s) - V_1(-s)| d\mu(s) \\ &\leq \frac{L_f^1}{\mu([-z, z])} \int_{-z}^z |V_2(s)| d\mu(s) + \frac{L_f^2}{\mu([-z, z])} \int_{-z}^z |V_2(-s)| d\mu(s). \end{aligned}$$

From Lemma 3.1, we have

$$\lim_{z \rightarrow \infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |\Omega(s)| d\mu(s) = 0.$$

Then $\Omega \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. Since $[t \rightarrow \varphi(t, V_1(t), V_1(-t))] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ (see [5]), therefore $[t \rightarrow f(t, V(t), V(-t))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$. \square

Lemma 3.3. *If the conditions (H.1)–(H.5) are true and $V \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then*

$$[s \mapsto \int_s^{+\infty} K(y - s)g(y, V(y), V(-y))dy] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

Proof. From Lemma 3.2, $[y \mapsto g(y, V(y), V(-y))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$. Then

$$g(y, V(y), V(-y)) = g_1(y) + g_2(y),$$

where $g_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$ and $g_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. We pose $\Theta(t) = \int_t^{+\infty} K(s-t)g(s, V(s), V(-s))ds$. Then

$$\begin{aligned} \Theta(t) &= \int_t^{+\infty} K(s-t)g_1(s)ds + \int_t^{+\infty} K(s-t)g_2(s)ds \\ &= \theta_1(t) + \theta_2(t), \end{aligned}$$

where $\theta_1(t) = \int_t^{+\infty} K(s-t)g_1(s)ds$, and $\theta_2(t) = \int_t^{+\infty} K(s-t)g_2(s)ds$. Now, we prove that $\theta_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$. Since $g_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, then for every sequence $(\tau'_n)_{n \in \mathbb{N}}$ there exists a subsequence (τ_n) such that

$$u_1(t) := \lim_{n \rightarrow \infty} g_1(t + \tau_n) \quad (3.1)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} u_1(t - \tau_n) = g_1(t), \quad (3.2)$$

for each $t \in \mathbb{R}$. Set

$$\theta_1(t) = \int_t^{+\infty} K(s-t)g_1(s)ds \text{ and } M(t) = \int_t^{+\infty} K(s-t)u_1(s)ds.$$

Now, we have

$$\begin{aligned} |\theta_1(t + \tau_n) - M(t)| &= \left| \int_{t+\tau_n}^{+\infty} K(s-t-\tau_n)g_1(s)ds - \int_t^{+\infty} K(s-t)u_1(s)ds \right| \\ &= \left| \int_t^{+\infty} K(s-t)(g_1(s+\tau_n) - u_1(s))ds \right|. \end{aligned}$$

Using Eq. (3.1), hypotheses (H.4) and the L.D.C. theorem, then we obtain:

$$\int_t^{+\infty} K(s-t)(g_1(s+\tau_n) - u_1(s))ds \longrightarrow 0, \text{ as } n \rightarrow \infty, t \in \mathbb{R}.$$

Therefore, we have

$$M(t) = \lim_{n \rightarrow \infty} \theta_1(t + \tau_n), t \in \mathbb{R}.$$

Using the same argument, we obtain

$$\theta_1(t) = \lim_{n \rightarrow \infty} M(t - \tau_n), t \in \mathbb{R}.$$

Therefore, $\theta_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$. In order to prove that $\Theta(t) \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, it remains to be shown that $\theta_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$, as

$$\lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |\theta_2(s)|d\mu(s) = 0.$$

$$\begin{aligned} \lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |\theta_2(s)|d\mu(s) &\leq \lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z \int_s^{+\infty} |K(y-s)||g_2(y)|dyd\mu(s) \\ &\leq \lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z \int_0^{+\infty} |K(y)||g_2(y+s)|dyd\mu(s) \\ &= \lim_{z \rightarrow +\infty} \int_0^{+\infty} \frac{K(y)}{\mu([-z, z])} \int_{-z}^z |g_2(y+s)|d\mu(s)dy. \end{aligned}$$

By the L.D.C. theorem and Lemma 2.9, we have:

$$\lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |\theta_2(s)| d\mu(s) \leq \int_0^{+\infty} K(y) \lim_{z \rightarrow +\infty} \frac{1}{\mu([-z, z])} \int_{-z}^z |g_2(y+s)| d\mu(s) dy = 0.$$

This completes the proof. \square

Remark 3.4. In what precede we show that,

$$\left[s \rightarrow \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy \right] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu). \quad (3.3)$$

From (H.2) and Eq. (3.3), we obtain,

$$\left[s \rightarrow \int_{-s}^{+\infty} K(y+s)g(y, V(y), V(-y))dy \right] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

Knowing that A and B are two square matrix, in [12], Gupta has studied the equation:

$$V'(s) = A(s)V(s) + B(s)V(-s) + f(s, V(s), V(-s)). \quad (3.4)$$

In [1], Ait Dads et al. have shown very important results on the existence and uniqueness of solution of Eq. (3.4).

Theorem 3.5. [2] Let $f \in \mathcal{PAA}(\mathbb{R}^3, \mathbb{R}, \mu)$. Assume that $M : \mathbb{R} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defined by

$$M(s) = \begin{pmatrix} A(s) & B(s) \\ -B(-s) & -A(-s) \end{pmatrix},$$

be continuous, non-singular and almost automorphic function such that $\{M^{-1}(s)\}_{s \in \mathbb{R}}$, is bounded. If conditions (H.1)–(H.4)(i), (H.6) and (H.7) are hold, then equation (3.4) has a unique solution in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, provided that

$$\max(L_f^1, L_f^2) < \frac{\alpha}{2k}.$$

Theorem 3.6. Assume that $M : \mathbb{R} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defined by

$$M(s) = \begin{pmatrix} A(s) & B(s) \\ -B(-s) & -A(-s) \end{pmatrix},$$

be continuous, non-singular and almost automorphic function such that $\{M^{-1}(s)\}_{s \in \mathbb{R}}$, is bounded. If conditions (H.1)–(H.7) are hold, then equation (1.1) has a unique solution in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, provided that

$$\max(L_f^1 + 2cL_g^1, L_f^2 + 2cL_g^2) < \frac{\alpha}{2k}.$$

Proof. Let Γ be the operator defined in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ by

$$\Gamma v(t) = \int_{-\infty}^{+\infty} \mathcal{G}(s, t) F(s, v(s), v(-s)) ds,$$

where

$$\mathcal{G}(s, t) = \begin{cases} X(s)\Lambda X^{-1}(t) & \text{if } s \geq t \\ -X(s)(I - \Lambda)X^{-1}(t) & \text{if } s \leq t. \end{cases}$$

and

$$F(s, v(s), v(-s)) = f(s, V(s), V(-s)) + \int_s^{+\infty} K(y-s)g(y, V(y), V(-y))dy + \int_{-s}^{+\infty} K(y+s)g(y, V(y), V(-y))dy.$$

For $y \in \mathbb{R}$, according to the Lemma 3.1, then $[y \mapsto V(-y)] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and using the Lemma 3.2 and Lemma 3.3, to conclude

$$[s \mapsto \int_s^{+\infty} K(y-s)g(y, v(y), v(-y))dy] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$$

and

$$[s \mapsto \int_{-s}^{+\infty} K(y+s)g(y, v(y), v(-y))dy] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

So, we obtain that

$$\Gamma : \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu) \rightarrow \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

The fact remains that Γ admits a simple fixed point which is a solution μ -pseudo almost automorphic of equation (1.1). Indeed, we have:

$$\begin{aligned} & \left| F(t, v(t), v(-t)) - F(t, u(t), u(-t)) \right| \\ \leq & \left| f(t, v(t), v(-t)) - f(t, u(t), u(-t)) \right| \\ & + \int_t^{+\infty} K(s-t) \left| g(s, v(s), v(-s)) - g(s, u(s), u(-s)) \right| ds \\ & + \int_{-t}^{+\infty} K(t+s) \left| g(s, v(s), v(-s)) - g(s, u(s), u(-s)) \right| ds \\ \leq & \left| f(t, v(t), v(-t)) - f(t, u(t), u(-t)) \right| \\ & + \int_0^{+\infty} K(s) \left| g((s+t), v(s+t), v(-(s+t))) - g((s+t), u(s+t), u(-(s+t))) \right| ds \\ & + \int_0^{+\infty} K(s) \left| g(s-t, v(s-t), v(-(s-t))) - g(s-t, u(s-t), u(-(s-t))) \right| ds \\ \leq & (L_f^1 + 2cL_g^1) \|v - u\|_\infty + (L_f^2 + 2cL_g^2) \|v - u\|_\infty. \end{aligned}$$

Then

$$\left| F(t, v(t), v(-t)) - F(t, u(t), u(-t)) \right| \leq L_F := \max(L_f^1 + 2cL_g^1, L_f^2 + 2cL_g^2) \|v - u\|_\infty.$$

Let $u, v \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and $L_F := \max(L_f^1 + 2kL_g^1, L_f^2 + 2kL_g^2)$, then we have:

$$\begin{aligned}
 |\Gamma v(t) - \Gamma u(t)| &\leq \left| \int_{-\infty}^{\infty} \mathcal{G}(s, t) F(s, v(s), v(-s)) ds - \int_{-\infty}^{\infty} \mathcal{G}(s, t) F(s, u(s), u(-s)) ds \right| \\
 &\leq \int_{-\infty}^{\infty} |\mathcal{G}(s, t)| |F(s, v(s), v(-s)) - F(s, u(s), u(-s))| ds \\
 &\leq \int_{-\infty}^{\infty} |\mathcal{G}(s, t)| L_F \|v - u\|_{\infty} ds \\
 &\leq \int_{-\infty}^t \kappa L_F e^{-\alpha(t-s)} \|v - u\|_{\infty} ds + \int_t^{+\infty} \kappa L_F e^{-\alpha(s-t)} \|v - u\|_{\infty} ds \\
 &\leq 2\kappa L_F \|v - u\|_{\infty} \int_0^{\infty} e^{-\alpha s} ds \\
 &\leq \frac{2\kappa L_F}{\alpha} \|v - u\|_{\infty}.
 \end{aligned}$$

Since $2k \max(L_f^1 + 2cL_g^1, L_f^2 + 2cL_g^2) < \alpha$, then the operator $\Gamma : \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu) \rightarrow \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ is a contraction. So Γ has a unique fixed point in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and equation (1.1) has a unique μ -pseudo almost automorphic solution. \square

4 The Lipschitz coefficients of the functions are variable [$L_f^i, L_g^i \in L^p(\mathbb{R}, \mathbb{R}, dt) \cap L^p(\mathbb{R}, \mathbb{R}, d\mu(t))$] for $p > 1$ and $i=1,2$

(H'.4)

- There exists $L_f^1, L_f^2 \in L^p(\mathbb{R}, \mathbb{R}^+, ds) \cap L^p(\mathbb{R}, \mathbb{R}^+, d\mu(s))$, $i = 1, 2$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ such that $\forall s, x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have:

$$|f(s, x_1, x_2) - f(s, y_1, y_2)| < L_f^1(s)|x_1 - y_1| + L_f^2(s)|x_2 - y_2|.$$

- There exists $L_g^1, L_g^2 \in L^p(\mathbb{R}, \mathbb{R}^+, ds) \cap L^p(\mathbb{R}, \mathbb{R}^+, d\mu(s))$, $i = 1, 2$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ such that $\forall s, x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have:

$$|g(s, x_1, x_2) - g(s, y_1, y_2)| < L_g^1(s)|x_1 - y_1| + L_g^2(s)|x_2 - y_2|.$$

Lemma 4.1. *If (H.1)–(H.3) and (H'.4) hold and $V \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then*

$$[s \mapsto f(s, V(s), V(-s))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

Proof. We repeat the same fakes as the proof of Lemma 3.2, only it remains to show that

$$[s \mapsto f(s, V(s), V(-s)) - f(s, V_1(s), V_1(-s))] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu).$$

Indeed, Using assumption (H'.4), then

$$|f(s, u_1, u_2) - f(s, v_1, v_2)| \leq L_f^1(s)|u_1 - v_1| + L_f^2(s)|u_2 - v_2|, \forall s, u_1, v_1 \in \mathbb{R}.$$

By Holder's inequality and some usual inequalities, we obtain:

$$\begin{aligned}
& \frac{1}{\mu([-z, z])} \int_{-z}^z \left| f(s, V(s), V(-s)) - f(s, V_1(s), V_1(-s)) \right| d\mu(s) \\
& \leq \frac{1}{\mu([-z, z])} \int_{-z}^z L_f^1(s) |V(s) - V_1(s)| + L_f^2(s) |V(-s) - V_1(-s)| d\mu(s) \\
& \leq \frac{\|V_2\|_\infty}{\mu([-z, z])} \int_{-z}^z |L_f^1(s)| d\mu(s) + \frac{\|V_2\|_\infty}{\mu([-z, z])} \int_{-z}^z |L_f^2(s)| d\mu(s) \\
& \leq \frac{\|V_2\|_\infty}{\mu([-z, z])} \left[\left(\int_{-z}^z |L_f^1(s)|^p d\mu(s) \right)^{\frac{1}{p}} \left(\int_{-z}^z d\mu(s) \right)^{\frac{1}{q}} + \left(\int_{-z}^z |L_f^2(s)|^p d\mu(s) \right)^{\frac{1}{p}} \left(\int_{-z}^z d\mu(s) \right)^{\frac{1}{q}} \right] \\
& \leq \text{cst.} \frac{\|V_2\|_\infty}{[\mu([-z, z])]^{\frac{1}{p}}} \rightarrow 0 \text{ if } z \rightarrow +\infty.
\end{aligned}$$

Therefore

$$[s \rightarrow f(s, V(s), V(-s))] \in \mathcal{PAP}(\mathbb{R}, \mathbb{R}, \mu).$$

□

In the following, we assume that:

(H'.5) $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that, for all $\tau > 1$, $\int_0^{+\infty} (K(y))^\tau dy < +\infty$.

In the remainder of this paragraph, it is assumed that $p > 1$. Since $q := \frac{p}{p-1} > 1$ then

$$C := \left(\int_0^{+\infty} (K(y))^q dy \right)^{\frac{1}{q}} < \infty.$$

Theorem 4.2. Assume that $M : \mathbb{R} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defined by

$$M(s) = \begin{pmatrix} A(s) & B(s) \\ -B(-s) & -A(-s) \end{pmatrix},$$

be continuous, almost automorphic and non-singular function such that $\{M^{-1}(s)\}_{s \in \mathbb{R}}$, is bounded. If conditions **(H.1)**-**(H.3)**, **(H'.4)**-**(H'.5)** and **(H.6)**-**(H.7)** are hold, then Eq. (1.1) has a unique solution in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, provided that

$$\|L_f^1\|_p + \|L_f^2\|_p + 2C\|L_g^1\|_p + 2C\|L_g^2\|_p < \frac{(\alpha q)^{\frac{1}{q}}}{2\kappa}.$$

Proof. We have:

$$\begin{aligned}
 & \left| F(s, v(s), v(-s)) - F(s, u(s), u(-s)) \right| \\
 \leq & \left| f(s, v(s), v(-s)) - f(s, u(s), u(-s)) \right| + \int_s^{+\infty} K(\tau - s) \left| g(\tau, v(\tau), v(-\tau)) - g(\tau, u(\tau), u(-\tau)) \right| d\tau \\
 & + \int_{-s}^{+\infty} K(s + \tau) \left| g(\tau, v(\tau), v(-\tau)) - g(\tau, u(\tau), u(-\tau)) \right| d\tau \\
 \leq & |L_f^1(s)| |v(s) - u(s)| + |L_f^2(s)| |v(-s) - u(-s)| \\
 & + \int_s^{+\infty} K(\tau - s) \left(L_g^1(\tau) |v(\tau) - u(\tau)| + L_g^2(\tau) |v(-\tau) - u(-\tau)| \right) d\tau \\
 & + \int_{-s}^{+\infty} K(\tau + s) \left(L_g^1(\tau) |v(\tau) - u(\tau)| + L_g^2(\tau) |v(-\tau) - u(-\tau)| \right) d\tau \\
 \leq & (|L_f^1(s)| + |L_f^2(s)|) \|v - u\|_\infty + \left[\left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} \|L_g^1\|_p + \left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} \|L_g^2\|_p \right] \|v - u\|_\infty \\
 & + \left[\left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} \|L_g^2\|_p + \left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} \|L_g^1\|_p \right] \|v - u\|_\infty \\
 \leq & \left[|L_f^1(s)| + 2\|L_g^1\|_p \left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} + |L_f^2(s)| + 2\|L_g^2\|_p \left(\int_0^{+\infty} (K(y))^q \right)^{\frac{1}{q}} \right] \|v - u\|_\infty.
 \end{aligned}$$

Then we have

$$\left| F(s, v(s), v(-s)) - F(s, u(s), u(-s)) \right| \leq \left[|L_f^1(s)| + |L_f^2(s)| + 2C(\|L_g^1\|_p + \|L_g^2\|_p) \right] \|v - u\|_\infty.$$

Let u and v in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$; then for all $t \in \mathbb{R}$ we have:

$$\begin{aligned}
 |\Gamma_v(t) - \Gamma_u(t)| & \leq \left| \int_{-\infty}^{\infty} \mathcal{G}(s, t) F(s, v(s), v(-s)) ds - \int_{-\infty}^{\infty} \mathcal{G}(s, t) F(s, u(s), u(-s)) ds \right| \\
 & \leq \int_{-\infty}^{\infty} \mathcal{G}(s, t) \left| F(s, v(s), v(-s)) - F(s, u(s), u(-s)) \right| ds \\
 & \leq \int_{-\infty}^{\infty} \mathcal{G}(s, t) \left[|L_f^1(s)| + |L_f^2(s)| + 2C(\|L_g^1\|_p + \|L_g^2\|_p) \right] \|v - u\|_\infty ds \\
 & \leq \int_{-\infty}^t \left[|L_f^1(s)| + |L_f^2(s)| + 2C(\|L_g^1\|_p + \|L_g^2\|_p) \right] \kappa e^{-\alpha(t-s)} \|v - u\|_\infty ds \\
 & \quad + \int_t^{+\infty} \left[|L_f^1(s)| + |L_f^2(s)| + 2C(\|L_g^1\|_p + \|L_g^2\|_p) \right] \kappa e^{-\alpha(s-t)} \|v - u\|_\infty ds \\
 & \leq 2\kappa \left(\|L_f^1\|_p + \|L_f^2\|_p + 2C(\|L_g^1\|_p + \|L_g^2\|_p) \right) \left(\int_0^{+\infty} |e^{-\alpha qs}| ds \right)^{\frac{1}{q}} \|v - u\|_\infty \\
 & \leq 2\kappa \frac{\|L_f^1\|_p + \|L_f^2\|_p + 2C(\|L_g^1\|_p + \|L_g^2\|_p)}{(\alpha q)^{\frac{1}{q}}} \|v - u\|_\infty.
 \end{aligned}$$

Since $\|L_f^1\|_p + \|L_f^2\|_p + 2C(\|L_g^1\|_p + \|L_g^2\|_p) < \frac{(\alpha q)^{\frac{1}{q}}}{2\kappa}$, then the operator $\Gamma : \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu) \rightarrow \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ is a contraction. So Γ has a unique fixed point in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and equation (1.1) has a unique solution in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$. \square

5 Example

We consider the following model bearing the names of Markus and Yamabe (see [16]):

$$\begin{aligned} v'(s) &= A(s)v(s) + B(s)v(-s) + f(s, v(s), v(-s)) \\ &+ \int_s^{+\infty} [K(s+y) + K(y-s)]g(y, v(y), v(-y))dy + \int_{-s}^s K(s-y)g(y, v(y), v(-y))dy. \end{aligned} \quad (5.1)$$

$$A(s) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(s) & 1 - \frac{3}{2}\cos(s)\sin(s) \\ -1 - \frac{3}{2}\cos(s)\sin(s) & -1 + \frac{3}{2}\sin^2(s) \end{pmatrix},$$

A is periodic of period π and the eigenvalues of $A(s)$ are

$$\lambda_1(s) = \frac{i\sqrt{7}-1}{4} \text{ and } \lambda_2(s) = \frac{-i\sqrt{7}-1}{4}$$

From [16], the system $v'(s) = A(s)v(s)$ has an exponential dichotomy, then α and k exists. So the hypothesis **(H.6)** is satisfied.

We pose: $f(s, x, y) = \frac{\alpha b(s)(\sin(x)+\sin(y))}{4k}$, with $b(s) = \sin(\frac{1}{2-\sin(s)-\sin(\pi s)})$ and $g(s, x, y) = G(s)(e^{-x} + e^{-y})$, with $G(s) = \cos(\frac{1}{2-\cos(s)-\cos(\pi s)})$, for all $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^+$.
If we pose $K(s) = \frac{\alpha e^{-s}}{10k}$, then $c = \frac{\alpha}{10k}$ and $\max(L_f^1 + 2cL_g^1, L_f^2 + 2cL_g^2) = \frac{9\alpha}{20k} < \frac{\alpha}{2k}$.
Let

$$B(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then, we have

$$\sup_{t \in \mathbb{R}} |B(s)| = 0 < \frac{\alpha}{4\kappa^2}.$$

This implies that assumption **(H.7)** is true.

Let μ be the measure defined by the following weight: $\rho(s) = e^{\sin(s)}$, $\forall s \in \mathbb{R}$. Then, $\forall z > 0$, we have

$$\frac{2z}{e} \leq \mu([-z, z]) = \int_{-z}^z e^{\sin s} ds \leq 2ez.$$

therefore $\mu \in \mathcal{M}$ satisfies **(H.1)**. Indeed, for all $a \in A$ and $\tau \in \mathbb{R}$, we have $2 + \sin(a) \geq \sin(\tau + a)$, then $\mu(\tau + A) \leq e^2\mu(A)$.

Since for all $s \in A = [m, n]$, one has $\sin(-s) \leq 2 + \sin(s)$, where $m, n \in \mathbb{R}$ such that $m > n$, then we have $\mu(-A) \leq 1 + e^2\mu(A)$. Therefore condition **(H.2)** is true.

We deduce that all assumptions **(H.1)-(H.7)** of Theorem 3.6 are satisfied. Since $M : \mathbb{R} \rightarrow \mathbb{R}^4 \times \mathbb{R}^4$ defined by $M(s) = \begin{pmatrix} A(s) & B(s) \\ -B(-s) & -A(-s) \end{pmatrix}$ is continuous, non-singular and almost automorphic function such that $\{M^{-1}(s)\}_{s \in \mathbb{R}}$, is bounded, then equation (5.1) has a unique solution in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$.

6 Conclusion

At the term of this paper, we can say that we have brought our contribution in the theory of integral with reflection equations. Indeed, the originality here is to show the existence and the uniqueness of solution in the space $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ of a class of integral equations with delay to infinity. The approach we used is based primarily on techniques analyzes and the Banach fixed point theorem. In the end, we illustrated our theoretical result to the study of the existence and the uniqueness of solutions in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ of a logistic differential equation, and finally we give an example.

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