The exponential behavior of 3D stochastic primitive equations driven by fractional noise

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Abstract

In this article, we study the exponential behavior of 3D stochastic primitive equations driven by fractional noise. Since fractional Brownian motion is essentially different from Brownian motion, the standard method via classic stochastic analysis tools is not available. Here, we develop a method which is close to the method from dynamic system to show that the weak solutions to 3D stochastic primitive equations driven by fractional noise converge exponentially to the unique stationary solution of primitive equations. This method may be applied to other stochastic hydrodynamic equations and other noises including Brownian motion and Lévy noise.

Keywords: Stochastic primitive equations; fractional Brownian motion; exponential behavior

1 Introduction

In this article, we study the exponential behavior of solutions to 3D stochastic primitive equations (SPEs) driven by fractional Brownian motion. We first define a cylindrical domain $\mho = M \times (-h, 0) \subset \mathbb{R}^3$, where $M \subset \mathbb{R}^2$ is a smooth bounded domain, then formulate 3D stochastic PEs of Geophysical Fluid Dynamics as follows:

$$\begin{aligned}
&\partial_t v + L_1 v + (v\cdot\nabla)v + w\partial_z v + f v^{\perp} + \nabla p = G_1(t)\dot{W}_1^H,\\
&\partial_t T + L_2 T + (v\cdot\nabla)T + w\partial_z T = Q_2 + G_2(t)\dot{W}_2^H,\\
&\nabla\cdot v + \partial_z w = 0,\\
&\partial_z p + T = 0.
\end{aligned}$$

The unknowns are the fluid velocity field $(v, w) = (v_1, v_2, w) \in \mathbb{R}^3$, with $v = (v_1, v_2), v^{\perp} = (-v_2, v_1)$ being horizontal, the temperature T and the pressure function p.

The Coriolis parameter f is defined by $f = f_0 + \beta y$ and Q_2 is a given heat sourse, the viscocity and the heat diffusion operators L_1, L_2 are given by

$$L_i = -\nu_i\Delta - \mu_i\partial_{zz},\ i = 1, 2.$$

Here the positive constants ν_1, μ_1 are the horizontal and vertical Reynolds numbers, and positive constants ν_2, μ_2 stand for the horizontal and vertical heat diffusivity.

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Throughout this paper, $\nabla, \Delta, \mathrm{div}$ represent the horizontal gradient, Laplacian and divergence, respectively. $\dot{W}_i^H(t,x,y,z), i=1,2$ stands for the informal derivative for the fractional Wiener process W_i^H that will be introduced later in subsection 2.3.

For the cylindrical domain, the boundary can be partitioned into three parts: $\partial\mho := \Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

$$\begin{aligned}
&\Gamma_u = \{(x,y,z) \in \bar{\mho} : z = 0\},\\
&\Gamma_b = \{(x,y,z) \in \bar{\mho} : z = -h\},\\
&\Gamma_s = \{(x,y,z) \in \bar{\mho} : (x,y) \in \partial M, -h \le z \le 0\}.
\end{aligned}$$

We now supplement the SPE model with initial and boundary conditions:

$$\begin{aligned}
&v(x,y,z,0) = v_0(x,y,z),\ T(x,y,z,0) = T_0(x,y,z),\\
&\partial_z v = \eta, w = 0, \partial_z T = -\alpha(T-\tau) \text{ on } \Gamma_u,\\
&\partial_z v = 0, w = 0, \partial_z T = 0 \text{ on } \Gamma_b,\\
&v \cdot \vec{n} = 0, \partial_{\vec{n}} v \times \vec{n} = 0, \partial_{\vec{n}} T = 0 \text{ on } \Gamma_s,
\end{aligned}$$

where $\eta(x,y)$ represents the wind stres on the surface of the ocean, $\alpha > 0$, τ is the typical temperature distribution on the ocean surface, and $\vec{n}$ is the norm vector on Γ_s. For the sake of simplicity, we assume Q_2 is independent of time, and $\eta = \tau = 0$. We would like to mention that the results presented in this paper can be extended to the general case provided with simple modifications.

By easy calculations, we have that

$$\begin{aligned}
w(x,y,z,t) &= -\int_{-1}^{z} \nabla \cdot v(x,y,\xi,t)d\xi,\\
p(x,y,z,t) &= p_s(x,y,z,t) - \int_{-1}^{z} T(x,y,\xi,t)d\xi.
\end{aligned}$$

Hence, the stochastic model, together with initial and boundary conditions, can be reformulated into the following equivalent form:

$$\begin{aligned}
&\partial_t v + L_1 v + (v\cdot\nabla)v - \left(\int_{-1}^{z} \nabla\cdot v(x,y,\xi,t)d\xi\right)\partial_z v\\
&\qquad + f v^{\perp} + \nabla p_s(x,y,z,t) - \int_{-1}^{z} \nabla T(x,y,\xi,t)d\xi = G_1(t)\dot{W}_1^H, \qquad (1.1)\\
&\partial_t T + L_2 T + (v\cdot\nabla)T - \left(\int_{-1}^{z} \nabla\cdot v(x,y,\xi,t)d\xi\right)\partial_z T = Q_2 + G_2(t)\dot{W}_2^H, \qquad (1.2)\\
&\partial_z v|_{\Gamma_u} = \partial_z v|_{\Gamma_b} = 0,\ v\cdot\vec{n}|_{\Gamma_s} = 0,\ \partial_{\vec{n}} v \times \vec{n}|_{\Gamma_s} = 0, \qquad (1.3)\\
&(\partial_z T + \alpha T)|_{\Gamma_u} = \partial_z T|_{\Gamma_b} = 0,\ \partial_{\vec{n}} T|_{\Gamma_s} = 0, \qquad (1.4)\\
&v(x,y,z,0) = v_0(x,y,z),\ T(x,y,z,0) = T_0(x,y,z). \qquad (1.5)
\end{aligned}$$

The Primitive equations are the basic model used in the study of climate and weather prediction, which describe the motion of the atmosphere when the hydrostatic assumption is enforced [20, 29, 30]. As far as we know, their mathematical study was initiated by J. L. Lions, R. Teman and S. Wang([33] − [36]). And this research field has developed and has received considerable attention from the mathematical community over the last two decades. Taking advantage of the fact that the pressure is essentially two-dimensional in the PEs, Cao and Titi [9] proved the global results for the existence of strong solutions of the full three-dimensional PEs. Independently, I. Kukavica and M. Ziane [32] developed a different proof which allows one to treat non-rectangular domains as well as different, physically realistic, boundary conditions. The existence of the global attractor is given by Ju [31]. For the PEs with partial dissipation, we refer the reader to the papers [3, 4, 5, 6, 10]. There are also other good works about the global well-posedness theory of PEs, we do not intend to mention each of them here.

The breakthroughs in the deterministic case motivated the development of the theory for the stochastic PEs. B. Ewald, M. Petcu, R. Teman [17] and N. Glatt-Holtz, M. Ziane [24] considered a two-dimensional stochastic PEs. Then N. Glatt-Holtz and R. Temam [22, 23] extended the case to the greater generality of physically relevant boundary conditions and nonlinear multiplicative noise. Following the methods similar to [9], Boling Guo and Daiwen Huang [21] studied the global well-posedness and long-time behavior of the three-dimensional system with additive noise. Using an approach similar to [32], A. Debussche, N. Glatt-Holtz, R. Temam and M. Ziane [13] considered three-dimensional system with multiplicative noise. In a subsequent paper, N. Glatt-Holtz, I. Kukavica, V. Vicol, and M. Ziane [25] established the existence and regularity of invariant measure for the SPEs. The ergodic theory and large deviations for the 3D SPEs were obtained by Dong, Zhai and Zhang in [15, 16]. Some analytical properties of weak solutions of 3D stochastic primitive equations with periodic boundary conditions were obtained in [14], in which the martingale problem associated to this model is shown to have a family of solutions satisfying the Markov property. Concerning the exponential behavior of SPEs, T. T. Medjo established the stability result for SPEs in [40] when the noise is Brownian motion. H. J. Gao and C. F. Sun also studied the long-time behavior, asymptotic and regular properties of SPEs, see [26, 27, 28].

As it is pointed out in [44, 41]that studies from climate show that the complex multi-scale nature of the earth's climate system results in many uncertainties that should be accounted for in the basic dynamical models of atmospheric and oceanic processes. It is further suggested in [48] that the uncertainties prefer to be **Non-Markovian**. This is the motivation for us to study the stability of SPEs driven by fractional Brownian motion.

Since the noise here is fractional Brownian motion which has memory for the increment and is essentially different from Brownian motion, the stochastic integral is not an Itô integral. We instead define stochastic integrals via pathwise generalized Stieltjes integrals as is the case in [37, 43, 47], for the details, one can refer to Section 2 of this article. As a result, the method studying the exponential behavior of the stationary solutions to SPEs in the present work is different from the standard method, see [7, 8, 40] and other references. In the following, we will illustrate the differences more clearly.

As we know if one tries to obtain the moment stability for the stochastic equations with nonlinear multiplicative Wiener noise, Itô formula will play an important role, see Theorem 3.2 in [40] and other references. Furthermore, if one tries to establish stability of the sample paths, then the moment stability and Borel-Cantelli lemma are key tools. One can refer to Theorem 3.3 in [40] and other references. If the noise is a spacial linear noise as considered in Theorem 4.2 of [40], Medjo takes advantage of the polynomial growth of Brownian motion to show that the stationary solution to SPEs is the almost surely exponentially stable.

In our article, we consider the case of additive fractional Brownian motion, as is the case in [7, 8] and [39]. A common method to deal with this type of noise is to introduce a fractional Ornstein-Uhlenback process to convert the SPEs into primitive equations with random coefficients. Here, we find that under the assumptions on the diffusion coefficients of the fractional Brownian motion, the fractional Ornstein-Uhlenback process is uniformly bounded with respect to time t which belongs to $[0, \infty)$, please see Proposition 2.2 for details. This is a key result which helps us to show that the stationary solution to SPEs is exponentially stable, which is stated in Theorem 3.5.

It is known that one other specialty about Ornstein-Uhlenback processes driven by Brownian motion is the polynomial growth of sample paths. Therefore, the uniform Gronwall inequality can not be applied to stochastic equations. For this reason, the random attractor obtained for the stochastic equations are pullback random attractor. There are many literatures about the topic of random attractor, we list some of them here for the convenience of readers, [1, 2, 11, 12, 19]. In this article, we find that the uniform boundedness of the fractional Ornstein-Uhlenback process may open a way to obtain forward attractor (not the pullback random attractor) for the stochastic equations. By virtue of the uniform boundedness of the fractional Ornstein-Uhlenback process, we establish the uniform estimate for the strong solution to SPEs via uniform Gronwall inequality, see Proposition 3.4. In a forthcoming article, we establish the existence of forward attractor for SPEs.

To study the long-time behavior of stochastic dynamic system, here, we introduce a method which is close to the method from dynamic system to study the exponentially stable behavior. In view of the method, the ergodicity of fractional Ornstein-Uhlenback process is an important tool which can help us to obtain the uniform *a priori* estimates with respect to time t for the solutions. But the dissipation of SPEs is not enough (see the proof of Proposition 3.4), one can not directly use the ergodicity of fractional Ornstein-Uhlenback process to establish the desired energey estimates. Therefore, to overcome the difficulty, we introduce another fractional Ornstein-Uhlenback process depending on the parameter β to obtain uniform estimates, please see (2.9) for the definition of this spacial fractional Ornstein-Uhlenback process and detailed calculations below (3.5).

Based on the foregoing results, we are going to establish our main result Theorem 3.5 which shows that the stationary solution to SPEs is exponentially stable. The first difficulty we will encounter is to deal with the stochastic term. As we know fractional Brownian motion is not a semi-martingale, thus, we can not use the classic tools, i.e., Itô formula and Burkholder-Davis-Gundy inequality to achieve the stability results. Our idea is to discretize the stochastic integral with respect to fractional Brownnian motion, which helps us to open the way to make full use of the stationary properties and polynomial growth properties of the increments of the fractional Brownian motion as well as regularities of the fractional Brownian motion. After delicate and careful estimates via the regularities and stationary properties of fractional Brownian motion, we establish the uniform estimate of the stochastic terms, please see the estimates of K_1, K_2, N_1 and N_2 in the proof of Theorem 3.5.

We would like to mention that the result and method presented in this article may be a basic tool to study the exponential behavior of stochastic partial differential equations driven by fractional Brownian motion.

The structure of the paper is as follows: In section 2, we give definitions of functional spaces and operators regards to the SPEs, and introduce the pathwise integral with fractional calculus techniques, with those techniques, under certain conditions on the forcing terms, we show that the O-U processes are uniformly bounded in H^3 norm. In section 3, we give the definition of exponential stability, then discuss the uniform bounded property for H^1 norm of solutions, and finally under a stronger condition for the forcing terms, we establish the exponential stability in the almost sure sense for the solutions of SPEs.

2 Preliminaries

2.1 Functional spaces

For $1 \le p \le \infty$, let $L^p(\mho), L^p(M)$ be the usual Lebesgue spaces with norms $|\cdot|_p$ and $|\cdot|_{L^p(M)}$, respectively. For $m > 0$, we denote by $(H^{m,p}(\mho), \|\cdot\|_{m,p})$ and $(H^{m,p}(M), \|\cdot\|_{H^{m,p}(M)})$ be the usual Sobolev spaces. When $p = 2$, we write them as $(H^m(\mho), \|\cdot\|_m)$ and $(H^m(M), \|\cdot\|_{H^m(M)})$ for short. For simplicity of notations, we sometimes directly write $|\cdot|_p, \|\cdot\|_m$ as norms in $L^p(M), H^m(M)$, if there is no confusion.

We now define the following function spaces:

$$V_1 = \{v \in (C^\infty(\mho))^2 : \partial_z v|_{\Gamma_u} = \partial_z v|_{\Gamma_b} = 0, v \cdot \vec{n}|_{\Gamma_s} = 0, \partial_{\vec{n}} v \times \vec{n}|_{\Gamma_s} = 0, \int_{-1}^{0} \nabla \cdot v dz = 0\};$$
$$V_2 = \{T \in C^\infty(\mho) : (\partial_z T + \alpha T)|_{\Gamma_u} = \partial_z T|_{\Gamma_b} = 0, \partial_{\vec{n}} T|_{\Gamma_s} = 0\}.$$

Denote by $\mathcal{V}_1, \mathcal{V}_2$ the closure spaces of V_1, V_2 under $(H^1(\mho))^2, H^1(\mho)$, respectively. Let $\mathcal{H}_1$ be the closure space of V_1 under the norm $|\cdot|_2$ and $\mathcal{H}_2 = L^2(\mho)$. Now set

$$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2,\ \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2.$$

For $U := (v, T), \widetilde{U} := (\widetilde{v}, \widetilde{T}) \in \mathcal{V}$, we equip $\mathcal{V}$ with the inner product

$$\begin{aligned}
&\langle U, \widetilde{U} \rangle_{\mathcal{V}} := \langle v, \widetilde{v} \rangle_{\mathcal{V}_1} + \langle T, \widetilde{T} \rangle_{\mathcal{V}_2}, \\
&\langle v, \widetilde{v} \rangle_{\mathcal{V}_1} := \int_{\mho} (\nabla v \cdot \nabla \widetilde{v} + \partial_z v \cdot \partial_z \widetilde{v}) dxdydz, \\
&\langle T, \widetilde{T} \rangle_{\mathcal{V}_2} := \int_{\mho} (\nabla T \cdot \nabla \widetilde{T} + \partial_z T \cdot \partial_z \widetilde{T}) dxdydz + \alpha \int_{\Gamma_u} T \widetilde{T} d\Gamma_u.
\end{aligned}$$

Consequently, the norm in $\mathcal{V}$ is defined by $\|U\|_1 := \langle U, U \rangle_{\mathcal{V}}^{1/2}$. Similarly, we define the inner product in $\mathcal{H}$ by

$$\begin{aligned}
&\langle U, \widetilde{U} \rangle_{\mathcal{H}} := \langle v, \widetilde{v} \rangle + \langle T, \widetilde{T} \rangle, \\
&\langle v, \widetilde{v} \rangle := \int_{\mho} v \cdot \widetilde{v} dxdydz,\ \langle T, \widetilde{T} \rangle = \int_{\mho} T \widetilde{T} dxdydz.
\end{aligned}$$

2.2 Operators

We define the bilinear forms $a : \mathcal{V}\times\mathcal{V}\to\mathbb{R}, a_i : \mathcal{V}_i\times\mathcal{V}_i\to\mathbb{R}, i = 1,2$, by

$$a_1(v,\widetilde{v}) = \langle v,\widetilde{v}\rangle_{\mathcal{V}_1}, a_2(T,\widetilde{T}) = \langle T,\widetilde{T}\rangle_{\mathcal{V}_2}, a(U,\widetilde{U}) = \langle U,\widetilde{U}\rangle_{\mathcal{V}}.$$

Denote by $\mathcal{V}',\mathcal{V}_i'$ the dual spaces of $\mathcal{V},\mathcal{V}_i$ for $i=1,2$, then the corresponding linear operators $A:\mathcal{V}\to\mathcal{V}', A_i : \mathcal{V}_i\to\mathcal{V}_i', i = 1,2$ can be defined as follows:

$$\langle A_1 v,\widetilde{v}\rangle = a_1(v,\widetilde{v}), \langle A_2T,\widetilde{T}\rangle = a_2(T,\widetilde{T}), \langle AU,\widetilde{U}\rangle = a(U,\widetilde{U}),$$

where $U = (v,T), \widetilde{U} = (\widetilde{v},\widetilde{T})\in\mathcal{V}$.

Now for $i = 1,2$, define $D(A_i) := \{\rho\in\mathcal{V}_i, A_i\rho\in H_i\}$. Since A_i^{-1} is a self-adjoint compact operators in $\mathcal{H}_i$, by the classic spectral theory, we can define the power A_i^s for any $s\in\mathbb{R}$. Then $D(A_i)' = D(A_i^{-1})$ is the dual space of $D(A_i)$ and $\mathcal{V}_i = D(A_i^{1/2}), \mathcal{V}_i' = D(A_i^{1/2})$. Moreover, we have the compact embedding relationship

$$D(A_i)\subset\mathcal{V}_i\subset\mathcal{H}_i\subset\mathcal{V}_i'\subset D(A_i)',\ i = 1,2.$$

And

$$\|\cdot\|_1^2 = a_i(\cdot,\cdot) = \langle A_i\cdot,\cdot\rangle = \langle A_i^{1/2}\cdot, A_i^{1/2}\cdot\rangle,\ i = 1,2.$$

Hereafter we denote by $\lambda_1 > 0$ a constant such that

$$|U|_2\le\lambda_1\|U\|_1,\ U\in\mathcal{V};\qquad \|U\|_1\le\lambda_1|AU|_2,\ U\in D(A). \tag{2.1}$$

Now we define the nonlinear operator for $U = (v,T), \widetilde{U} = (\widetilde{v},\widetilde{T}), \widehat{U} = (\widehat{v},\widehat{T})\in\mathcal{V}$,

$$b(U,\widetilde{U},\widehat{U}) = \langle B(U,\widetilde{U}),\widehat{U}\rangle = b_1(v,\widetilde{v},\widehat{v}) + b_2(v,\widetilde{T},\widehat{T}),$$

where

$$b_1(v,\widetilde{v},\widehat{v}) = \langle B_1(v,\widetilde{v}),\widehat{v}\rangle := \int_{\mho}\left[(v\cdot\nabla)\widetilde{v} - \left(\int_{-1}^{z}\nabla\cdot v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{v}\right]\cdot\widehat{v}dxdydz,$$

$$b_2(v,\widetilde{T},\widehat{T}) = \langle B_2(v,\widetilde{T}),\widehat{T}\rangle := \int_{\mho}\left[(v\cdot\nabla)\widetilde{T} - \left(\int_{-1}^{z}\nabla\cdot v(x,y,\xi,t)d\xi\right)\partial_z\widetilde{T}\right]\cdot\widehat{T}dxdydz.$$

There are some basic properties for the nonlinear operator B. The proof can be found in [31].

Lemma 2.1. *There exists a constant $c_0 > 0$ that is independent of $U,\widetilde{U},\widehat{U}$ such that*

$$b(U,\widetilde{U},\widetilde{U}) = \langle B(U,\widetilde{U}),\widetilde{U}\rangle = 0,\quad \text{for } U\in\mathcal{V},\widetilde{U}\in D(A).$$

$$|b(U,\widetilde{U},\widehat{U})|\le c_0\|U\|_1^{1/2}|AU|_2^{1/2}\|\widetilde{U}\|_1^{1/2}|A\widetilde{U}|_2^{1/2}\|\widehat{U}\|_1,\quad \text{for } U,\widetilde{U}\in D(A),\widehat{U}\in\mathcal{V}.$$

$$|b(U,\widetilde{U},\widehat{U})|\le c_0\|U\|_1^{1/2}|AU|_2^{1/2}\|\widetilde{U}\|_1\|\widehat{U}\|_1^{1/2}|\widehat{U}|_2^{1/2},\quad \text{for } U\in D(A),\widetilde{U},\widehat{U}\in\mathcal{V}.$$

$$|b(U,\widetilde{U},\widehat{U})|\le c_0\|U\|_1\|\widetilde{U}\|_1^{1/2}|A\widetilde{U}|_2^{1/2}\|\widehat{U}\|_1^{1/2}|\widehat{U}|_2^{1/2},\quad \text{for } \widetilde{U}\in D(A),U,\widehat{U}\in\mathcal{V}.$$

We also define another bilinear operator $R:\mathcal{V}\to\mathcal{V}'$ by

$$r(U,\widetilde{U}) = \langle R(U),\widetilde{U}\rangle = \langle fv^{\perp},\widetilde{v}\rangle + \langle\int_{-1}^{z}\nabla T,\widetilde{v}\rangle,\quad \text{for } U = (v,T),\widetilde{U} = (\widetilde{v},\widetilde{T})\in\mathcal{V}.$$

Recall that for $U\in\mathcal{V},\widetilde{U}\in\mathcal{H}$, there exists $\alpha_0 > 0$ such that

$$|r(U,\widetilde{U})| = |\langle R(U),\widetilde{U}\rangle|\le\alpha_0\|U\|_1|\widetilde{U}|_2 \tag{2.2}$$

Finally, we write $Q := (0,Q_2)$ as the forcing term. We write $G(t) = (G_1(t),G_2(t)$ as the coefficients for the random term. Hereafter we set

$$\bar{\nu}_i = \min(\nu_i,\mu_i),\ i = 1,2;\ \bar{\nu} = \min(\bar{\nu}_1,\bar{\nu}_2).$$

Thus, we consider the stochastic 3D PEs of the ocean, driven by fractional Brownian motion, written in the following abstract mathematical setting

$$dU(t) = [-\nu AU(t) - B(U(t)) - R(U(t)) + Q(t)]dt + G(t)dW^H(t),\ U(0) = U_0 := (v_0,T_0). \tag{2.3}$$

2.3 Fractional Brownian motion

Set $\alpha \in (0,1)$, for a function $f:[0,T]\to\mathbb{R}$ that is regular enough, we define the Weyl fractional derivatives as follows:

$$
\begin{aligned}
D_{0+}^{\alpha} f(t) =& \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t)-f(u)}{(t-u)^{\alpha+1}}du\right),\\
D_{T-}^{\alpha} f(t) =& \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t)-f(u)}{(u-t)^{\alpha+1}}du\right),
\end{aligned}
$$

provided the singular integrals on the right hand side exist for almost all $t\in[0,T]$, and Γ stands for the Gamma function.

For $\phi \in L^1([0,T];\mathbb{R})$, we define the left and right hand side fractional Riemann-Liouville integrals of ϕ of order α for almost all $t\in(0,T)$ by

$$
\begin{aligned}
I_{0+}^{\alpha}\phi(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du,\\
I_{T-}^{\alpha}\phi(t) =& \frac{(-1)^{-\alpha}}{\Gamma(\alpha)}\int_t^T (u-t)^{\alpha-1}\phi(u)du.
\end{aligned}
$$

If $f = I_{0+}^{\alpha}\phi$, then the Weyl left-side derivative of f exists and $D_{0+}^{\alpha} f = \phi$. A similar result holds for the right-side fractional integral. See [45] for a comprehensive introduction for the theory of fractional integrals and derivatives.

We now define $W^{\alpha,1}([0,T];\mathbb{R})$ as the space of measurable functions $f:[0,T]\to\mathbb{R}$ with

$$
\|f\|_{\alpha,1} := \int_0^T \left(\frac{|f(s)|}{s^{\alpha}} + \int_0^s \frac{|f(s)-f(u)|}{(s-u)^{\alpha+1}}du\right)ds < \infty,
$$

where $\alpha\in(0,\frac{1}{2})$.

Following [47], we define the generalized Stieltjes integral $\int_0^T f dg$ by

$$
\int_0^T f dg = (-1)^{\alpha}\int_0^T D_{0+}^{\alpha} f(s) D_{T-}^{1-\alpha} g_{T-}(s)ds, \tag{2.4}
$$

where $g_{T-}(s) = g(s)-g(T)$. Under the above hypotheses, the above integral exists for all $t\in[0,T]$, and by [43], we have

$$
\int_0^t f dg = \int_0^T f 1_{(0,t)} dg.
$$

Furthermore, we have the following estimate

$$
\left|\int_0^t f dg\right| \le C_{\alpha}(g)\|f\|_{\alpha,1}, \tag{2.5}
$$

where

$$
C_{\alpha}(g) := \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\sup_{0<s<t<T}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(u)-g(s)|}{(u-s)^{2-\alpha}}du\right).
$$

On a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, for $H\in(0,1)$, we let $B_i^H = (B_i^H(t)_{t\in\mathbb{R}})$ be a sequence of independent, identically distributed continuous centered Gaussian process with covariance function

$$
R(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),\ s,t\in\mathbb{R}.
$$

This type of process is called a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H. When $H=\frac{1}{2}$, $B_i^{1/2}$ is the standard Brownian motion.

For $j \in \{1,2\}$, let H_j be linear, self-adjoint, positive trace-class operators on $\mathcal{H}_j$, that is, for a complete orthonormal basis $(e_{i,j})_{i\in\mathbb{N}_+}$ in $\mathcal{H}_j$, there exists a sequence of nonnegative values $(\lambda_{i,j})_{i\in\mathbb{N}_+}$ such that $\mathrm{tr}H_j = \sum_{i=1}^{\infty}\lambda_{i,j} < \infty$. We now introduce $\mathcal{H}_j$-valued fractional Brownian motion $W_j^H, j = 1,2$, with covariance operator H_j, and Hurst parameter H, as follows:

$$W_j^H(t) := \sum_{i=1}^{\infty}\sqrt{\lambda_{i,j}}e_{i,j}B_i^H(t). \tag{2.6}$$

For $H > \frac{1}{2}$, take a parameter $\alpha \in (1-H, \frac{1}{2})$ which will be fixed throughout the paper. Let $f \in W^{\alpha,1}([0,T];\mathbb{R})$, we define $\int_0^T f(s)dB_i^H(s)$ in the sense of (2.4) pathwise. By [43], $C_\alpha(B_i^H) < \infty$, $\mathbb{P}$-a.s. for $t \in \mathbb{R}_+, i \in \mathbb{N}_+$.

For $j = 1,2$, denote by $\mathcal{L}(\mathcal{V}_j)$ the space of linear bounded operators on $\mathcal{V}_j$, now suppose $l : \Omega \times [0,T] \to \mathcal{L}(\mathcal{V}_j)$ is an operator-valued function such that $le_{i,j} \in W^{\alpha,1}([0,T];\mathcal{V}_j)$ for any $i \in \mathbb{N}_+, \omega \in \Omega$. Now we define

$$\begin{aligned}\int_0^T \varphi(s)dW_j^H(s) :=& \sum_{i=1}^{\infty}\int_0^T l(s)H_j^{1/2}e_{i,j}dB_i^H(s)\\ =& \sum_{i=1}^{\infty}\sqrt{\lambda_{i,j}}\int_0^T l(s)e_{i,j}dB_i^H(s),\end{aligned} \tag{2.7}$$

where the convergence of the above series is understood as $\mathbb{P}$-a.s. convergence in $\mathcal{V}_j$.

2.4 Regularity of O-U processes

Now we consider the stochastic equations for $j = 1,2$, $\beta > 1$,

$$dZ_j(t) = (-A_jZ_j - \beta Z_j)dt + G_j(t)dW_j^H(t),\ Z_j(0) = 0. \tag{2.8}$$

The solution can be interpreted pathwisely in the mild sense, that is, the solution $(Z_j(t))_{t\in[0,\mathcal{T}]}$ is a $\mathcal{V}_j$-valued process whose paths are elements of the space $W^{\alpha,1}([0,\mathcal{T}];\mathcal{V}_j)$ with probability one, for $\alpha \in (1-H, \frac{1}{2})$, such that,

$$Z_j(t) = \int_0^t e^{-(t-s)(A_j+\beta)}G_j(s)dW_j^H(s). \tag{2.9}$$

We first have the following result which gives the growth rates of $Z_j(t), j = 1,2$, under certain conditions on the forcing terms.

Proposition 2.2. *For $j = 1,2$, denote by $0 < \gamma_{1,j} \le \gamma_{2,j} \le \cdots$ the eigenvalues of A_j with corresponding eigenvectors $e_{1,j}, e_{2,j}, \cdots$, assume the following condition holds*

$$\sum_{i=1}^{\infty}\lambda_{i,j}^{1/2}\gamma_{i,j}^{5/2} < \infty, \quad \text{for } j = 1,2. \tag{2.10}$$

Then for $\mathcal{T} > 0$, $(Z_1(t))_{t\in[0,\mathcal{T}]}, (Z_2(t))_{t\in[0,\mathcal{T}]}$ exist as generalized Stieltjes integrals in the sense of [47], and

$$(Z_1(t))_{t\in[0,\mathcal{T}]} \in C([0,\mathcal{T}];(H^3(\mho))^2), a.s. \text{ and } (Z_2(t))_{t\in[0,\mathcal{T}]} \in C([0,\mathcal{T}];H^3(\mho)),\ a.s..$$

Assume furtherly that the forcing terms $G_1(t), G_2(t)$ only depend on t and satisfy

$$|G_1(t)| + |G_1'(t)| \le M_1(1+t)^{-2},\ |G_2(t)| + |G_2'(t)| \le M_2(1+t)^{-2}. \tag{2.11}$$

Then for $\beta > 0$, $Z_j(t), j = 1,2$ are uniformly bounded in H^3 norms, in the sense that there exist random variables $C_1(\omega), C_2(\omega)$ taking finite values such that

$$\sup_{t\in[0,\infty)}\|Z_1(t)\|_3 \le C_1(\omega) < \infty, a.s., \qquad \sup_{t\in[0,\infty)}\|Z_2(t)\|_3 \le C_2(\omega), a.s..$$

Proof. For the proof of the continuity of Z_1 and Z_2, one can refer to Propositions 2.1 of [48]. Here we will give a short proof of the uniform boundedness of Z_1 and Z_2. By the definition in (2.9) and the property of Riemann-Stieltjes integral (see Theorem 4.2.1 in [47]), we have

$$\begin{aligned}\|Z_1(t)\|_3 =&\left\| \int_0^t e^{-(t-s)(A_1+\beta)} G_1(s) dW_1^H(s) \right\|_3\\ =&\left\| \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} \int_0^t e^{-(t-s)(\gamma_{k,1}+\beta)} e_{k,1} G_1(s) dB_k^H(s) \right\|_3\\ =&\left\| \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} e_{k,1} \Big[G_1(t) B_k^H(t) \right.\\ &\left. - \int_0^t B_k^H(s)[(\gamma_{k,1}+\beta) e^{-(t-s)(\gamma_{k,1}+\beta)} G_1(s) + e^{-(t-s)(\gamma_{k,1}+\beta)} G_1'(s)] ds \Big] \right\|_3.\end{aligned}$$

Since $|B_k^H(t)| \le t^2 + c(\omega)$ (see Lemma 2.6 of [38]), $|G_1(t)| + |G_1'(t)| \le M_1(1+t)^{-2}$, we get

$$\begin{aligned}\|Z_1(t)\|_3 \le& M_1(1+t)^{-2}(t^2+c(\omega)) \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} \gamma_{k,1}^{\frac{3}{2}}\\ &+ M_1 \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} \gamma_{k,1} \gamma_{k,1}^{\frac{3}{2}} \int_0^t (s^2+c(\omega))(1+s)^{-2} e^{-(t-s)(\gamma_{k,1}+\beta)} ds\\ \le& M_1 c \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} \gamma_{k,1}^{\frac{3}{2}} + M_1 c \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} \gamma_{k,1}^{\frac{5}{2}} \int_0^t e^{-s(\gamma_{k,1}+\beta)} ds\\ \le& C_1(\omega).\end{aligned}$$

The estimate for $Z_2(t)$ follows similarly.

□

2.5 Global well-posedness

Let $u(t) = v(t) - Z_1(t)$ and $\theta(t) = T(t) - Z_2(t)$, a stochastic process $U(t,\omega) = (v, T)$ is a strong solution to (1.1)-(1.5) on $[0, \mathcal{T}]$, if and only if (u, θ) is a strong solution to the following problem on $[0, \mathcal{T}]$:

$$\partial_t u + L_1 u + [(u+Z_1)\cdot\nabla](u+Z_1) + w(u+Z_1)\partial_z(u+Z_1)$$
$$+ f(u+Z_1)^{\perp} + \nabla p_s - \int_{-1}^{z} \nabla(\theta+Z_2)(x,y,\xi,t) d\xi = \beta Z_1; \tag{2.12}$$
$$\partial_t \theta + L_2\theta + [(u+Z_1)\cdot\nabla](\theta+Z_2) + w(u+Z_1)\partial_z(\theta+Z_2) = Q + \beta Z_2; \tag{2.13}$$
$$\int_{-1}^{0} \nabla\cdot u dz = 0; \tag{2.14}$$
$$\partial_z u|_{\Gamma_u} = \partial_z u|_{\Gamma_b} = 0,\ u\cdot\vec{n}|_{\Gamma_s} = 0,\ \partial_{\vec{n}} u \times \vec{n}|_{\Gamma_s} = 0; \tag{2.15}$$
$$(\partial_z\theta + \alpha\theta)|_{\Gamma_u} = \partial_z\theta|_{\Gamma_b} = 0,\ \partial_{\vec{n}}\theta|_{\Gamma_s} = 0; \tag{2.16}$$
$$(u(0), \theta(0)) = (v_0, T_0). \tag{2.17}$$

Then the global well-posedness of 3D SPEs driven by fractional Brownian motion follows from the result in [48].

Theorem 2.3. *Let $Q_2 \in L^2(\mho)$, $v_0 \in \mathcal{V}_1, T_0 \in \mathcal{V}_2, \mathcal{T} > 0$. Assume the condition (2.10) holds, then there exists a unique strong solution (v,T) of the system (1.1)-(1.5), or equivalently, (u,θ) of the system (2.12)-(2.17) on the interval $[0,\mathcal{T}]$ which is Lipschitz continuous with respect to the initial data and the noises in $\mathcal{V}$ and $C([0,\mathcal{T}];\mathcal{V})$, respectively.*

3 Exponential stability of solutions

3.1 Steady state solution

A stationary solution to (2.3) is $U^* = (v^*, T^*)$ satisfying

$$L_1 v^* + (v^* \cdot \nabla)v^* - \left(\int_{-1}^{z} \nabla \cdot v^*(x, y, \xi, t) d\xi\right) \partial_z v^* + f(v^*)^{\perp} + \nabla p_s(x, y, z, t) - \int_{-1}^{z} \nabla T^*(x, y, \xi, t) d\xi = 0,$$
$$L_2 T^* + (v^* \cdot \nabla)T^* - \left(\int_{-1}^{z} \nabla \cdot v^*(x, y, \xi, t) d\xi\right) \partial_z T^* = Q_2,$$

with boundary conditions (1.3)-(1.4) held. In abstract setting, the above equation can be rewritten as

$$\nu A U^* + B(U^*) + R(U^*) = Q. \tag{3.1}$$

The existence of steady state solutions follows from the result in [39].

Theorem 3.1. *Suppose $Q_2 \in L^2(\mho)$ and $\bar{\nu} > 0$ is large enough, then (3.1), together with boundary conditions (1.3)-(1.4), has a unique solution $U^* = (v^*, T^*)$. Moreover, we have the following estimate*

$$|AU^*|_2^2 = |A_1 v^*|_2^2 + |A_2 T^*|_2^2 \leq K, \tag{3.2}$$

where K is a constant that depends on Q, ν.

3.2 Exponential stability of steady state solutions

In this section, we discuss the stability of steady state solutions. In the following, we first give the definition.

Definition 3.2. *We say the solution $U(t)$ to (2.3) converges to $U^* \in \mathcal{H}$ almost surely exponentially if there exists $\gamma > 0$ such that*

$$\lim_{t \to \infty} \frac{1}{t} \log |U(t) - U^*|_2 \leq -\gamma. \tag{3.3}$$

*We say that U^* is **almost surely exponentially stable** if any solution to (2.3) converges to U^* almost surely exponentially with the same $\gamma > 0$.*

The following Lemma is called uniform Gronwall lemma which will be used repeatedly in the proof of Proposition 3.4. One can refer to Foias and Prodi [18] and Temam [46] for a proof of Lemma 3.3.

Lemma 3.3. *Let f, g and h be three non-negative locally integrable functions on (t_0, ∞) such that*

$$\frac{df}{dt} \leq gf + h, \quad \forall\, t \geq t_0,$$

and

$$\int_t^{t+r} f(s)ds \leq a_1, \quad \int_t^{t+r} g(s)ds \leq a_2, \quad \int_t^{t+r} h(s)ds \leq a_3, \quad \forall\, t \geq t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$f(t+r) \leq (\frac{a_1}{r} + a_3)e^{a_2}, \quad \forall\, t \geq t_0.$$

Proposition 3.4. *For any fixed ω, there exists a random variable $C(\omega)$ taking values in $\mathbb{R}^+ := (0, \infty)$ such that*

$$\sup_{t \in [0,\infty)} \|U(t)\|_1 \leq C(\omega) < \infty, a.s.. \tag{3.4}$$

Proof. According to (5.106) in [48], there exists positive constants C and γ_1 such that

$$\begin{aligned}
&|u(t)|_2^2+|\theta(t)|_2^2\\
\leq&(|u(0)|_2^2+|\theta(0)|_2^2)\exp\left\{\int_0^t -\gamma_1+C(\|Z_1\|_2^2+\|Z_1\|_2^4+\|Z_2\|_3^2)ds\right\}\\
&+\int_0^t \exp\left\{\int_s^t -\gamma_1+C(\|Z_1\|_2^2+\|Z_1\|_2^4+\|Z_2\|_3^2)dx\right\}(|Q|_2^2+\|Z_1\|_1^2)ds. \qquad (3.5)
\end{aligned}$$

By [38], we know that $(C_0(\mathbb{R},\mathcal{V}),\mathcal{B}(C_0(\mathbb{R},\mathcal{V})),\mathbb{P},\vartheta)$ is an ergodic metric dynamical system, it is shown in [48] that $Z_j(t)$ is adapted with respect to $\mathcal{F}_t:=\sigma(W_j^H(s),j=1,2,s\leq t)$. Therefore, applying the properties of ergodic metric dynamical system, there exists a β big enough such that

$$\lim_{t\to\infty}\frac{C\int_0^t(\|Z_1\|_2^2+\|Z_1\|_2^4+\|Z_2\|_3^2)ds}{t}=C\mathbb{E}[\|Z_1(0)\|_1^2+\|Z_1(0)\|_2^4+\|Z_2(0)\|_3^2]<\frac{\gamma_1}{2}.$$

Hence, for any fixed path ω, there exists $T(\omega)$ big enough such that when $t>T(\omega)$,

$$\int_0^t(\|Z_1\|_2^2+\|Z_1\|_2^4+\|Z_2\|_3^2)ds\leq\frac{\gamma_1}{2}t.$$

Now with (3.5), for $t>T(\omega)$, we have

$$\begin{aligned}
|U(t)|_2^2=&|u(t)|_2^2+|\theta(t)|_2^2+|Z_1|_2^2+|Z_2|_2^2\\
&\leq(|u(0)|_2^2+|\theta(0)|_2^2)e^{-\gamma_1 t/2}+\frac{1}{\gamma_1}e^{-\gamma_1 t/2}\int_0^t(|Q|_2^2+\|Z_1\|_1^2)ds\\
&\quad+|Z_1|_2^2+|Z_2|_2^2. \qquad (3.6)
\end{aligned}$$

By Proposition 2.2, there exist $C_1(\omega),C_2(\omega)$ such that

$$\sup_{t\in[0,\infty)}\|Z_1(t)\|_1\leq C_1(\omega)<\infty \text{ and } \sup_{t\in[0,\infty)}\|Z_2(t)\|_1\leq C_2(\omega)<\infty.$$

Thus, back to (3.6), by the continuity of $|U(t)|_2$ with respect to time t, there exists $C(\omega)$,

$$\sup_{t\geq 0}|U(t)|_2^2\leq C(\omega). \qquad (3.7)$$

In view of (5.105) in [48], (3.7), Proposition 2.2 and Lemma 3.3, we have

$$\int_t^{t+1}\|U(s)\|_1^2ds<C(\omega). \qquad (3.8)$$

By the formula above (5.113) in [48], (3.7), (3.8), Proposition 2.2 and Lemma 3.3, we obtain

$$\sup_{t\in[0,\infty)}|\theta(t)|_4^2<C(\omega). \qquad (3.9)$$

Recall that $\bar{u}(x,y)=\int_{-1}^0 u(x,y,z)dz$ and $\tilde{u}=u-\bar{u}$. By (5.122) in [48], (3.7)-(3.9), Proposition 2.2 and Lemma 3.3, we obtain

$$\sup_{t\in[0,\infty)}|\tilde{u}(t)|_4^2<C(\omega). \qquad (3.10)$$

By virtue of (5.121) in [48], (3.7)-(3.10), Proposition 2.2 and Lemma 3.3, we obtain

$$\int_t^{t+1}\int_{\mho}\Big(|\nabla(|\tilde{u}(s)|^2)|^2+|\partial_z(|\tilde{u}(s)|^2)|^2ds+\int_t^{t+1}\int_{\mho}|\tilde{u}|^2(|\nabla\tilde{u}(s)|^2+|\partial_z\tilde{u}(s)|^2)ds<C(\omega). \qquad (3.11)$$

By (5.127) in [48], (3.7)-(3.11), Proposition 2.2 and Lemma 3.3, we obtain

$$\int_t^{t+1} |\Delta \bar{u}|_2^2 ds < C(\omega) \tag{3.12}$$

and

$$\sup_{t\in[0,\infty)} |\nabla \bar{u}(t)|_2^2 ds < C(\omega). \tag{3.13}$$

By (5.137) in [48], (3.7)-(3.13), Proposition 2.2 and Lemma 3.3, we obtain

$$\int_t^{t+1} (|\nabla u_z|_2^2 + |u_{zz}|_2^2) ds < C(\omega) \tag{3.14}$$

and

$$\sup_{t\in[0,\infty)} |u_z(t)|_2^2 ds < C(\omega). \tag{3.15}$$

By (5.141) in [48], (3.7)-(3.15), Proposition 2.2 and Lemma 3.3, we obtain

$$\int_t^{t+1} |\Delta u(s)|_2^2 ds < C(\omega) \tag{3.16}$$

and

$$\sup_{t\in[0,\infty)} |\nabla u(t)|_2^2 < C(\omega). \tag{3.17}$$

By (5.143) in [48], (3.7)-(3.17), Proposition 2.2 and Lemma 3.3, we obtain

$$\sup_{t\in[0,\infty)} (|\nabla \theta(t)|_2^2 + |\theta_z|_2^2 + \alpha|\theta(z=0)|_2^2) < C(\omega). \tag{3.18}$$

Finally, the result follows from (3.15), (3.17), (3.18). □

In the following, we give the main result of this paper.

Theorem 3.5. *Suppose $Q_2 \in L^2(\mho)$. We assume the conditions for G_1, G_2 stronger than (2.11), that is, there exist $M_1, M_2, \rho_1, \rho_2 > 0$ such that*

$$|G_1(t)| + |G_1'(t)| \leq M_1 e^{-\rho_1 t}, \ |G_2(t)| + |G_2'(t)| \leq M_2 e^{-\rho_2 t}. \tag{3.19}$$

Let $U^* \in D(A)$ be the unique solution to (3.1) with boundary and initial conditions (1.3)-(1.5), and let ν be large enough such that

$$\min\{\nu_1, \mu_1, \nu_2, \mu_2\} > c_0\lambda_1 |AU^*|_2^2 + \alpha_0\lambda_1,$$

the solution U to (2.3) converges to the stationary solution U^* almost surely exponentially. More precisely, there exists $0 < \lambda < \rho_1 \wedge \rho_2 \wedge 2\lambda_1^{-2}(\bar{\nu} - c_0\lambda_1|AU^*|_2^2 - \alpha_0\lambda_1)$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log |U(t) - U^*|_2 \leq -\lambda/2. \tag{3.20}$$

Proof. Firstly, we have

$$\begin{aligned}
\frac{d}{dt}[e^{\lambda t}|U(t) - U^*|_2^2] =& \lambda e^{\lambda t}|U(t) - U^*|_2^2 + 2e^{\lambda t}\langle \frac{d}{dt}U(t), U(t) - U^*\rangle \\
=& \lambda e^{\lambda t}|U(t) - U^*|_2^2 - 2e^{\lambda t}\nu\langle AU(t), U(t) - U^*\rangle \\
&- 2e^{\lambda t}\langle B(U(t)) + R(U(t)), U(t) - U^*\rangle + 2e^{\lambda t}\langle Q(t), U(t) - U^*\rangle \\
&+ 2e^{\lambda t}\langle G(t) d\dot{W}^H(t), U(t) - U^*\rangle.
\end{aligned}$$

By (3.1), U^* satisfies

$$e^{\lambda t}\langle \nu AU^* + B(U^*) + R(U^*), U(t) - U^*\rangle = e^{\lambda t}\langle Q, U(t) - U^*\rangle,$$

we obtain that

$$\begin{aligned} e^{\lambda t}|U(t) - U^*|_2^2 =& |U(0) - U^*|_2^2 + \int_0^t \lambda e^{\lambda s}|U(s) - U^*|_2^2 ds - 2\int_0^t e^{\lambda s}\nu\langle A(U(s) - U^*), U(s) - U^*\rangle ds \\ & - 2\int_0^t e^{\lambda s}\langle B(U(s)) - B(U^*), U(s) - U^*\rangle ds \\ & - 2\int_0^t e^{\lambda s}\langle R(U(s)) - R(U^*), U(s) - U^*\rangle ds \\ & + 2\int_0^t e^{\lambda s}\langle G(s)dW^H(s), U(s) - U^*\rangle \\ =& I_1 + \cdots + I_6. \end{aligned} \tag{3.21}$$

First by (2.1), we have

$$I_2 \le \int_0^t \lambda\lambda_1^2 e^{\lambda s}\|U(s) - U^*\|_1^2 ds,\ \ I_3 \le -2\bar{\nu}\int_0^t e^{\lambda s}\|U(s) - U^*\|_1^2 ds.$$

For the bilinear operator B, by (2.1), Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned} I_4 =& -2\int_0^t e^{\lambda s}\langle B(U(s), U(s) - U^*), U(s) - U^*\rangle ds - 2\int_0^t e^{\lambda s}\langle B(U(s) - U^*, U^*), U(s) - U^*\rangle ds \\ =& -2\int_0^t e^{\lambda s}\langle B(U(s) - U^*, U^*), U(s) - U^*\rangle ds \\ \le& 2c_0\lambda_1|AU^*|_2^2\int_0^t e^{\lambda s}\|U(s) - U^*\|_1^2 ds \end{aligned}$$

By (2.1) and (2.2),

$$I_5 \le 2\alpha_0\int_0^t e^{\lambda s}\|U(s) - U^*\|_1|U(s) - U^*|_2 ds \le 2\alpha_0\lambda_1\int_0^t e^{\lambda s}\|U(s) - U^*\|_1^2 ds.$$

By the assumption, we see that for $\bar{\nu} > 0$ large enough such that $\bar{\nu} > c_0\lambda_1|AU^*|_2^2 + \alpha_0\lambda_1$, then for small λ so that

$$\lambda < 2\lambda_1^{-2}(\bar{\nu} - c_0\lambda_1|AU^*|_2^2 - \alpha_0\lambda_1), \tag{3.22}$$

we have

$$I_2 + I_3 + I_4 + I_5 < 0.$$

By the definition of fractional Brownian motions in (2.6), we get that

$$\begin{aligned} I_6 =& 2\int_0^t e^{\lambda s}\langle G_1(s)dW_1^H(s), v(s) - v^*\rangle + 2\int_0^t e^{\lambda s}\langle G_2(s)dW_1^H(s), T(s) - T^*\rangle \\ =& 2\int_0^t e^{\lambda s}\langle\sum_{k=1}^\infty \sqrt{\lambda_{k,1}}e_{k,1}G_1(s)dB_k^H(s), v(s) - v^*\rangle + 2\int_0^t e^{\lambda s}\langle\sum_{k=1}^\infty \sqrt{\lambda_{k,2}}e_{k,2}G_2(s)dB_k^H(s), T(s) - T^*\rangle \\ =& 2\sum_{k=1}^\infty \sqrt{\lambda_{k,1}}\int_0^t \langle v(s) - v^*, e_{k,1}\rangle e^{\lambda s}G_1(s)dB_k^H(s) + 2\sum_{k=1}^\infty \sqrt{\lambda_{k,2}}\int_0^t \langle T(s) - T^*, e_{k,2}\rangle e^{\lambda s}G_2(s)dB_k^H(s). \end{aligned}$$

Now applying the inequality (2.5), and by condition (2.11),

$$
\begin{aligned}
&\Big|\int_0^t e^{\lambda s}\langle G(s)dW^H(s), U(s)-U^*\rangle\Big|\\
\le&\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\Big|\sum_{j=1}^{[t]+1}\int_{j-1}^{j}\langle v(s)-v^*, e_{k,1}\rangle e^{\lambda s}G_1(s)dB_k^H(s)\Big|\\
&+\sum_{k=1}^{\infty}\sqrt{\lambda_{k,2}}\Big|\sum_{j=1}^{[t]+1}\int_{j-1}^{j}\langle T(s)-T^*, e_{k,2}\rangle e^{\lambda s}G_2(s)dB_k^H(s)\Big|\\
\le&\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\frac{|\langle v(s)-v^*, e_{k,1}\rangle|e^{-(\rho_1-\lambda)s}}{[s-(j-1)]^\alpha}ds\\
&+\sum_{k=1}^{\infty}\sqrt{\lambda_{k,2}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\frac{|\langle T(s)-T^*, e_{k,2}\rangle|e^{-(\rho_2-\lambda)s}}{[s-(j-1)]^\alpha}ds\\
&+\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v^*, e_{k,1}\rangle e^{\lambda s}G_1(s)-\langle v(x)-v^*, e_{k,1}\rangle e^{\lambda x}G_1(x)|}{(s-x)^{1+\alpha}}dxds\\
&+\sum_{k=1}^{\infty}\sqrt{\lambda_{k,2}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle T(s)-T^*, e_{k,2}\rangle e^{\lambda s}G_2(s)-\langle T(x)-T^*, e_{k,2}\rangle e^{\lambda x}G_2(x)|}{(s-x)^{1+\alpha}}dxds\\
=:&J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.23}
$$

Firstly, we have

$$
\begin{aligned}
J_1\le&\frac{1}{2}\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\Big[\int_{j-1}^{j}|v(s)-v^*|_2^2e^{-(\rho_1-\lambda)s}ds+\int_{j-1}^{j}\frac{e^{-(\rho_1-\lambda)s}}{[s-(j-1)]^{2\alpha}}ds\Big]\\
\le&\frac{1}{2}\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\Big[\int_{j-1}^{j}|v(s)-v^*|_2^2e^{-(\rho_1-\lambda)s}ds+e^{-(\rho_1-\lambda)(j-1)}\frac{1}{1-2\alpha}\Big]\\
\le&\frac{1}{2}\sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\Big[\int_{j-1}^{j}|v(s)-v^*|_2^2e^{-(\rho_1-\lambda)s}ds+\frac{e^{-(\rho_1-\lambda)(j-1)}}{1-2\alpha}\Big].
\end{aligned}
$$

With similar discussion, we get

$$
J_2\le\frac{1}{2}\sum_{k=1}^{\infty}\sqrt{\lambda_{k,2}}\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\Big[\int_{j-1}^{j}|T(s)-T^*|_2^2e^{-(\rho_2-\lambda)s}ds+\frac{e^{-(\rho_2-\lambda)(j-1)}}{1-2\alpha}\Big].
$$

Thus, by Proposition 3.4, for any ω,

$$
\begin{aligned}
J_1+J_2\le&\frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}+\sqrt{\lambda_{k,2}})\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\Big[\int_{j-1}^{j}|U(s)-U^*|_2^2e^{-(\rho_1\wedge\rho_2-\lambda)s}ds+\frac{e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}}{1-2\alpha}\Big]\\
\le&\frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}+\sqrt{\lambda_{k,2}})\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}[|U^*|_2^2+C(\omega)]e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}ds\\
&+\frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}+\sqrt{\lambda_{k,2}})\sum_{j=1}^{[t]+1}C_\alpha(B_k^H)\Big|_{j-1}^{j}\frac{e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}}{1-2\alpha}\\
=:&K_1+K_2.
\end{aligned}\tag{3.24}
$$

By Lemma 7.4 and Lemma 7.5 in [43], for any $0 < \varepsilon < H$,

$$
\begin{aligned}
K_1 + K_2 \le & \frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}} + \sqrt{\lambda_{k,2}}) \sum_{j=1}^{[t]+1} C_\alpha \eta_{\varepsilon,j,j-1}\left(1 + \frac{1}{H-\varepsilon-1+\alpha}\right)[|U^*|_2^2 + C(\omega)]e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} \\
& + \frac{1}{2}\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}} + \sqrt{\lambda_{k,2}}) \sum_{j=1}^{[t]+1} C_\alpha \eta_{\varepsilon,j,j-1}\left(1 + \frac{1}{H-\varepsilon-1+\alpha}\right)\frac{e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}}{1-2\alpha},
\end{aligned}
$$

where $\eta_{\varepsilon,j,j-1}$ is a positive random variable such that $\mathbb{E}|\eta_{\varepsilon,j,j-1}|^p < C_{\varepsilon,p}$ that does not depend on j, for any $p \ge 1$. Hence, for $\lambda < \rho_1 \wedge \rho_2$,

$$
\begin{aligned}
& \mathbb{E}\sum_{j=1}^{[t]+1} C_\alpha \eta_{\varepsilon,j,j-1}\left(1 + \frac{1}{H-\varepsilon-1+\alpha}\right)e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} \\
\le & \sum_{j=1}^{\infty} C_\alpha C_{\varepsilon,1} e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} < \infty,
\end{aligned}
$$

so we get $\sup\limits_{t\in[0,\infty)}[K_1 + K_2] < \infty$, and back to (3.24), we get $\sup\limits_{t\in[0,\infty)}[J_1 + J_2] < \infty$.

Now for J_3, first by triangle inequality, and the assumption for G_1, we get

$$
\begin{aligned}
J_3 \le & \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v(x), e_{k,1}\rangle e^{\lambda x}G_1(x)|}{(s-x)^{1+\alpha}}dxds \\
& + \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v^*, e_{k,1}\rangle[e^{\lambda s}G_1(s) - e^{\lambda x}G_1(x)]|}{(s-x)^{1+\alpha}}dxds \\
\le & \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v(x), e_{k,1}\rangle|e^{-(\rho_1-\lambda)x}}{(s-x)^{1+\alpha}}dxds \\
& + \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v^*, e_{k,1}\rangle|e^{-(\rho_1-\lambda)(j-1)}(s-x)}{(s-x)^{1+\alpha}}dxds \\
\le & \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v(x), e_{k,1}\rangle|e^{-(\rho_1-\lambda)x}}{(s-x)^{1+\alpha}}dxds \\
& + \sum_{k=1}^{\infty}\sqrt{\lambda_{k,1}}\sum_{j=1}^{[t]+1} e^{-(\rho_1-\lambda)(j-1)}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|\langle v(s)-v^*, e_{k,1}\rangle|}{(s-x)^{\alpha}}dxds,
\end{aligned}
$$

there is similar estimate for J_4, hence, altogether, by Proposition 3.4, one obtain that

$$
\begin{aligned}
J_3 + J_4 \le & \sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}} + \sqrt{\lambda_{k,2}})\sum_{j=1}^{[t]+1} C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|U(s)-U(x)|_2 e^{-(\rho_1\wedge\rho_2-\lambda)x}}{(s-x)^{1+\alpha}}dxds \\
& + \sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}} + \sqrt{\lambda_{k,2}})\sum_{j=1}^{[t]+1} e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{|U(s)-U^*|_2}{(s-x)^{\alpha}}dxds \\
=: & N_1 + N_2. \qquad (3.25)
\end{aligned}
$$

Firstly, by Proposition 3.4,

$$
\begin{aligned}
N_2 \leq & \sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}+\sqrt{\lambda_{k,2}}) \sum_{j=1}^{[t]+1} e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} C_\alpha(B_k^H)\Big|_{j-1}^{j} \int_{j-1}^{j} (C(\omega)+|U^*|_2) \frac{(s-j+1)^{1-\alpha}}{1-\alpha} ds \\
\leq & \sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}+\sqrt{\lambda_{k,2}}) \sum_{j=1}^{\infty} \frac{(C(\omega)+|U^*|_2) e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}}{(2-\alpha)(1-\alpha)} C_\alpha(B_k^H)\Big|_{j-1}^{j}.
\end{aligned}
$$

With the same discussion as before, for $\lambda < \rho_1 \wedge \rho_2$, we get

$$
\mathbb{E} \sum_{j=1}^{\infty} e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} C_\alpha(B_k^H)\Big|_{j-1}^{j} < \infty,
$$

so $\sup_{t\in[0,\infty)} N_2 < \infty$, a.s..

Now we will estimate N_1. Applying Theorem 4.2.1 in [47], with $|B_k^H(t)| \leq t^2 + c(\omega)$(see Lemma 2.6 of [38]), one can obtain

$$
\begin{aligned}
& \Big|\langle \int_x^s G_1 \dot{W}_1^H, e_{k,1}\rangle\Big| \\
& = \Big|\langle \sum_{j=1}^{\infty} \sqrt{\lambda_{j,1}} e_{j,1} \int_x^s G_1(r) dB_j^H(r), e_{k,1}\rangle\Big| = \sqrt{\lambda_{k,1}}\Big|\int_x^s G_1(r) dB_k^H(r)\Big| \\
& = \sqrt{\lambda_{k,1}}\Big|G_1(s)B_k^H(s) - G_1(x)B_k^H(x) - \int_x^s G_1'(r) B_k^H(r) dr\Big| \\
& \leq \sqrt{\lambda_{k,1}}[|G_1(s)B_k^H(s) - G_1(x)B_k^H(s)| + |G_1(x)B_k^H(s) - G_1(x)B_k^H(x)| + M_1(s^2+c(\omega))(s-x)] \\
& \leq \sqrt{\lambda_{k,1}} M_1[2(s-x)(s^2+c(\omega)) + (s-x)^{H-\varepsilon}\eta_{\varepsilon,x,s}].
\end{aligned}
$$

Recall that $A_j e_{j,k} = \gamma_{j,k} e_{j,k}$ for $j=1,2$, then for $\varepsilon < H-\alpha$ small enough,

$$
\begin{aligned}
& |\langle v(s)-v(x), e_{k,1}\rangle| = \Big|\int_x^s \langle L_1 v, e_{k,1}\rangle + \int_x^s \langle v\cdot\nabla v, e_{k,1}\rangle + \int_x^s \langle w\partial_z v, e_{k,1}\rangle + \int_x^s \langle G_1\dot{W}_1^H, e_{k,1}\rangle\Big| \\
\leq & \gamma_{k,1}(|v|_2 + |v|_2\|v\|_1 + \|v\|_1^2)(s-x) + \sqrt{\lambda_{k,1}} M_1[2(s-x)(s^2+c(\omega)) + (s-x)^{H-\varepsilon}\eta_{\varepsilon,x,s}] \\
\leq & C\gamma_{k,1}(s-x) + C\sqrt{\lambda_{k,1}}[(s-x)(s^2+c(\omega)) + (s-x)^{H-\varepsilon}\eta_{\varepsilon,x,s}].
\end{aligned}
$$

Similarly, we have

$$
|\langle T(s)-T(x), e_{k,2}\rangle| \leq C\gamma_{k,2}(s-x) + C\sqrt{\lambda_{k,2}}[(s-x)(s^2+c(\omega)) + (s-x)^{H-\varepsilon}\eta_{\varepsilon,x,s}].
$$

Hence, by the above estimates and Proposition 3.4,

$$
\begin{aligned}
N_1 \leq & C\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}\gamma_{k,1}+\sqrt{\lambda_{k,2}}\gamma_{k,2})\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{e^{-(\rho_1\wedge\rho_2-\lambda)x}}{(s-x)^\alpha}dxds \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}\int_{j-1}^{j}\int_{j-1}^{s}\frac{e^{-(\rho_1\wedge\rho_2-\lambda)x}(s^2+c(\omega))}{(s-x)^\alpha}dxds \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}\eta_{\varepsilon,j,j-1}\int_{j-1}^{j}\int_{j-1}^{s}e^{-(\rho_1\wedge\rho_2-\lambda)x}(s-x)^{H-\varepsilon-1-\alpha}dxds \\
\leq & C\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}\gamma_{k,1}+\sqrt{\lambda_{k,2}}\gamma_{k,2})\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}\frac{e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}}{(1-\alpha)(2-\alpha)} \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}e^{-\frac{1}{2}(\rho_1\wedge\rho_2-\lambda)(j-1)} \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\frac{1}{(1+H-\varepsilon-\alpha)(H-\varepsilon-\alpha)}\sum_{j=1}^{\infty}C_\alpha(B_k^H)\Big|_{j-1}^{j}e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}\eta_{\varepsilon,j,j-1},
\end{aligned}
$$

where we used the boundedness $\sup_{\{(x,s)\in[0,\infty)\times[0,\infty)\cap\{|x-s|\leq 1\}\}} e^{-\frac{1}{2}(\rho_1\wedge\rho_2-\lambda)x}(s^2+c(\omega)) < \infty$ in the last inequality. Again by Lemma 7.4 and Lemma 7.5 in [43], for $0<\varepsilon<H$,

$$
\begin{aligned}
N_1 \leq & C\sum_{k=1}^{\infty}(\sqrt{\lambda_{k,1}}\gamma_{k,1}+\sqrt{\lambda_{k,2}}\gamma_{k,2})\sum_{j=1}^{\infty}\eta_{\varepsilon,j,j-1}e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)} \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\sum_{j=1}^{\infty}\eta_{\varepsilon,j,j-1}e^{-\frac{1}{2}(\rho_1\wedge\rho_2-\lambda)(j-1)} \\
&+C\sum_{k=1}^{\infty}(\lambda_{k,1}+\lambda_{k,2})\sum_{j=1}^{\infty}e^{-(\rho_1\wedge\rho_2-\lambda)(j-1)}\eta_{\varepsilon,j,j-1}^2,
\end{aligned}
$$

where $\eta_{\varepsilon,j,j-1}$ is a positive random variable such that $\mathbb{E}|\eta_{\varepsilon,j,j-1}|^p < C_{\varepsilon,p}$ that does not depend on j, for any $p\geq 1$. Hence, there exists $C_2(\omega)$ such that

$$
\mathbb{E}[\sup_{t\in[0,\infty)} N_1] < C,
$$

which implies

$$
\sup_{t\in[0,\infty)} N_1 < C(\omega).
$$

Back to (3.21), we get that for any fixed ω, there exists $C_1(\omega)$, and $\lambda<\rho_1\wedge\rho_2$ such that

$$
e^{\lambda t}|U(t)-U^*|_2^2 \leq C_1(\omega),
$$

combining with (3.22), this yields that for $\lambda<\rho_1\wedge\rho_2\wedge 2\lambda_1^{-2}(\bar{\nu}-c_0\lambda_1|AU^*|_2^2-\alpha_0\lambda_1)$,

$$
\lim_{t\to\infty}\frac{1}{t}\log|U(t)-U^*|_2^2 \leq -\lambda. \tag{3.26}
$$

□

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