ON THE ACCELERATION AND JERK IN MOTION ALONG A SPACE CURVE WITH QUASI-FRAME IN EUCLIDEAN 3-SPACE

A. ELSHARKAWY AND A. M. ELSHENHAB

Abstract. In this paper, we consider a particle moves on a space curve in the Euclidean 3-space and resolve its acceleration and jerk vectors according to quasi-frame. In this resolution, by appyling Siacci's theorem, we state the acceleration vector as the sum of its tangential and radial components, and obtain the jerk vector along the tangential direction and radial directions in osculating and rectifying planes. On the basis of the jerk vector formula , we give the maximum admissible speed on a space curve at all trajectory points. Furthermore, we present illustrative examples to explain how our results work.

1. Introduction

In Newtonian physics, it is well known that the force acting on a particle is concerned with its acceleration through the equation $F = m\mathbf{a}$. A particle, which moves under the influence of arbitrary forces in 3-dimensional Euclidean space, has an acceleration $\mathbf{a}$ which is obtained by the time derivative of the velocity vector, and thus by two time derivative of the position vector. For some applications, to state the acceleration vector as the sum of its tangential and normal components is practical. However, when the angular momentum of the particle is constant, to state the acceleration vector as the sum of its tangential and radial components is more practical. In 1879, Siacci in [15] stated the acceleration vector as the sum of two special oblique components in the osculating plane to the curve. After Siacci, in 1944, Whittaker in [17] dealt with Siacci's theorem and gave a geometrical proof of it in the plane. Although Siacci's formulas are very remarkable, his formulation of the theorem is inaccurate and his proof is burdensome. Therefore, In 2011, Casey in [2] presented a proof of Siacci's theorem in the space by using the Serret-Frenet frame. After that, in 2012, Küçükarslan et al. in [6] studied Siacci's theorem for curves in Finsler Manifold $\mathbf{F}^3$. In 2017, Özen et al. in [11] studied Siacci's theorem for the curves on regular surfaces in E^3 according to Darboux frame. Recently, in 2020, Özen et al. in [10] have studied Siacci's theorem for the curves in E^3 according to the modiefied orthogonal frame. In the same year, Özen has studied Siacci's theorem for the curves in Minkowski 3-space by using the Serret-Frenet frame.

On the other hand, the jerk vector $\mathbf{j}$ is the time derivative of the acceleration vector. Thus, the equality $\mathbf{j} = \frac{1}{m}\frac{dF}{dt}$ is satisfied for the particle which has a constant mass. In 1862, Resal in [12] resolved the jerk vector along the tangent, normal and binormal unit vectors of Serret-Frenet frame in Euclidean space. This concept is still an issue of interest. Recently, in 2019, Özen et al. in [9] have presented a new

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decomposition of jerk vector along the tangential direction and radial directions in osculating and rectifying planes by using the Serret- Frenet frame. In the same year, Güner in [5] has obtained the resolution of the jerk vector for the curves in E^3 according to the Bishop frame. After that, in 2020, Özen et al. in [10] have obtained the resolution of the jerk vector for the curves in E^3 according to the modiefied orthogonal frame.When a gymnast does gymnastic exercises or a stock-car racer races on track or a machinist drives a high- speed train, the acceleration changes suddenly. In these kind of situations, to estimate the lower threshold of just noticeable jerk and upper values of the jerk that can be tolerated by humans without undue discomfort is very important, see [13]. Also, in 2017, Tsirlin in [16], on the basis of the jerk vector formula, gave the maximum admissible speed on a space curve at all trajectory points.

The Serret-Frénet frame is inadequate for studying the space curves which its curvatures have discrete points zero since, in this case, the principlal normal and binormal vectors are discontinuous at points of inflections or along the straight sections of the curve. Therefore, to solve this problem, Dede et al. in [3] introduced a new adapted frame along a space curve as an alternative frame to the Serret-Frénet frame and denoted this the quasi-frame.

Motivated by these papers, we consider a particle moves on a space curve according to quasi-frame in the Euclidean 3-spac under the influence of arbitrary forces. The paper is organized as follows: In Section 2, we present some basic definitions about the Euclidean 3-spac E^3, the Serret-Frénet frame and quasi-frame, and the relation between the quasi-frame and the classical Serret-Frénet frame in the Euclidean 3-spac E^3. Furthermore, we resolve the acceleration vector $\mathbf{a}$ and the jerk vector $\mathbf{j}$ of a particle moves on a space curve according to quasi basis. In Section 3, we present alternative resolutions of acceleration and jerk vectors, and we resolve the acceleration vector $\mathbf{a}$ along the radial direction and tangential direction in osculating plane, and also try to resolve the jerk vector $\mathbf{j}$ along the tangential direction, radial direction in osculating plane and radial direction in the rectifying plane. In Section 4, we present illustrative examples to explain how our results work.

2. Preliminaries

In this section, we present some preliminaries used in our subsequent discussions. The Euclidean space is the metric space $E^3 = \left(\mathbb{R}^3, \langle , \rangle\right)$ where the metric $\langle , \rangle$ is the standard inner product given by

$$\langle X, Y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ are arbitrary vectors in E^3. Based on this metric, the *norm* of a vector $X \in E^3$ is given by $\|X\| = \sqrt{\langle X, X\rangle}$. A curve $\alpha = \alpha(s) : I \subseteq \mathbb{R} \rightarrow E^3$ is a unit speed curve if $\|\alpha'(s)\| = 1$ for all $s \in I$. In this case, s is called arc-length parameter of the curve $\alpha(s)$.

Let $\alpha(s)$ be a space curve in E^3, parameterized by arc-length s. Denote by $\{T(s), N(s), B(s)\}$ the moving Serret-Frénet frame along the unit speed curve $\alpha(s)$, where $T(s)$, $N(s)$ and $B(s)$ are the unit tangent, principal normal and binormal vectors, respectively, and they are defined as follows

$$T(s) = \alpha'(s), \ N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \ B(s) = T(s) \times N(s). \tag{2.1}$$

On the other hand, the Serret-Frénet formulas are given by

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.2}$$

where the curvature function $\kappa(s)$ and the torsion function $\tau(s)$ are defined as follows: $\kappa = \kappa(s) = \|T'(s)\|$, $\tau = \tau(s) = -\langle B'(s), N(s)\rangle$, [1].

Now, as alternative to the Serret-Frénet frame, donate by $\{T(s), N_q(s), B_q(s), \xi\}$ the quasi-frame (or simply q-frame) along a space curve $\alpha(s)$, where $T(s)$, $N_q(s)$, $B_q(s)$ and ξ are the unit tangent, the quasi-normal, the quasi-binormal and the projection vectors, respectively, and they are defined as follows:

$$T(s) = \alpha'(s), \quad N_q(s) = \frac{T \times \xi}{\|T \times \xi\|}, \quad B_q(s) = T \times N_q, \tag{2.3}$$

where ξ is the projection vector can be chosen as $\xi = (1,0,0)$ or $\xi = (0,1,0)$ or $\xi = (0,0,1)$. For simplicity, we can choose the projection vector $\xi = (1,0,0)$ in this paper. However, the q-frame is singular in all cases where T and ξ are parallel. Thus, in those cases where T and ξ are parallel the projection vector ξ can be chosen as $\xi = (0,1,0)$ or $\xi = (0,0,1)$. We can define Euclidean angle θ between the principal normal N and quasi-normal N_q vectors. Then, The relation between the quasi-frame and the classical Serret-Frénet frame is given as follows:

$$\begin{bmatrix} T(s) \\ N_q(s) \\ B_q(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}. \tag{2.4}$$

Thus, we have

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T(s) \\ N_q(s) \\ B_q(s) \end{bmatrix}. \tag{2.5}$$

By taking the derivative of (2.4), then substituting (2.2) and (2.5) into the results, we obtain the variation equations of the q-frame in the following form

$$\begin{bmatrix} T'(s) \\ N_q'(s) \\ B_q'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_q(s) \\ B_q(s) \end{bmatrix}, \tag{2.6}$$

where

$$\begin{aligned} \kappa_1(s) &= \kappa(s)\cos\theta, \qquad \kappa^2(s) = \kappa_1^2(s) + \kappa_2^2(s), \\ \kappa_2(s) &= -\kappa(s)\sin\theta, \qquad \theta = -\arctan\left(\frac{\kappa_2}{\kappa_1}\right), \\ \kappa_3(s) &= \theta'(s) + \tau(s). \end{aligned} \tag{2.7}$$

and the triple $(\kappa_1, \kappa_2, \kappa_3)$ is called the quasi-curvature functions of $\alpha(s)$, [3].

Let a particle P of mass $m > 0$ moves on a space curve according to quasi-frame in Euclidean space E^3 under the influence of arbitrary forces. Choose an arbitrary fixed origin O in the space E^3, and let $\mathbf{x}$ be the postion vector of P at time t. Let C, parametrized by the arc-length s described at time t, be the oriented curve traced out by P. Therefore, the unit tangent vector for the curve C is given as follows:

$$T = \frac{d\mathbf{x}}{ds}. \tag{2.8}$$

Then, from (2.6) and (2.8), we deduce the velocity vector $\mathbf{v}$, the acceleration vector $\mathbf{a}$ and the jerk $\mathbf{j}$ vector of P at time t with quasi-frame as follows:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{ds}{dt}T, \tag{2.9}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2}T + \kappa_1\left(\frac{ds}{dt}\right)^2 N_q + \kappa_2\left(\frac{ds}{dt}\right)^2 B_q,$$

or

$$\mathbf{a} = \frac{d^2s}{dt^2}T + \sqrt{\kappa_1^2(s) + \kappa_2^2(s)}\left(\frac{ds}{dt}\right)^2 \cos\theta N_q - \sqrt{\kappa_1^2(s) + \kappa_2^2(s)}\left(\frac{ds}{dt}\right)^2 \sin\theta B_q \tag{2.10}$$

and

$$\mathbf{j} = \frac{d\mathbf{a}}{dt} = C_T T + C_{N_q} N_q + C_{B_q} B_q, \tag{2.11}$$

where

$$C_T = \frac{d^3s}{dt^3} - \left(\kappa_1^2 + \kappa_2^2\right)\left(\frac{ds}{dt}\right)^3,$$

$$\begin{aligned} C_{N_q} &= \cos\theta\left[3\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) + \left(\frac{ds}{dt}\right)^3 \frac{d}{ds}\left(\sqrt{\kappa_1^2 + \kappa_2^2}\right)\right] \\ &\quad + \sin\theta\left[\left(\kappa_3 - \theta'\right)\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)^3\right], \end{aligned}$$

$$\begin{aligned} C_{B_q} &= -\sin\theta\left[3\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) + \left(\frac{ds}{dt}\right)^3 \frac{d}{ds}\left(\sqrt{\kappa_1^2 + \kappa_2^2}\right)\right] \\ &\quad + \cos\theta\left[\left(\kappa_3 - \theta'\right)\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)^3\right]. \end{aligned}$$

3. Alternative Resolutions of Acceleration and Jerk Vectors According to Quasi-Frame

In this section, we resolve the acceleration vector $\mathbf{a}$ along the radial direction and tangential direction in osculating plane, and also try to resolve the jerk vector $\mathbf{j}$ along the tangential direction, radial direction in osculating plane and radial direction in the rectifying plane.

The acceleration and jerk vectors in (2.10) and (2.11) can be expressed as follows:

$$\mathbf{a} = \frac{d^2s}{dt^2}T + \sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)^2\left(\cos\theta N_q - \sin\theta B_q\right), \tag{3.1}$$

and

$$\mathbf{j} = \left[\frac{d^3s}{dt^3} - \left(\kappa_1^2 + \kappa_2^2\right)\left(\frac{ds}{dt}\right)^3\right] T$$
$$+ \left[3\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) + \left(\frac{ds}{dt}\right)^3 \frac{d}{ds}\left(\sqrt{\kappa_1^2 + \kappa_2^2}\right)\right](\cos\theta N_q - \sin\theta B_q)$$
$$+ \left[(\kappa_3 - \theta')\sqrt{\kappa_1^2 + \kappa_2^2}\left(\frac{ds}{dt}\right)^3\right](\sin\theta N_q + \cos\theta B_q). \tag{3.2}$$

We note that, since we define $\{T, N_q, B_q\}$ as a right-handed orthonormal basis, the vectors $\{T, (\cos\theta N_q - \sin\theta B_q), (\sin\theta N_q + \cos\theta B_q)\}$ comprise a right-handed orthonormal system.

Let a particle P moves on a space curve $\alpha = \alpha(s)$. Therefore, P has a postion vector according to quasi-frame. Assume that the postion vector of P be resolved as follows:

$$\mathbf{x} = \lambda_1 T - \lambda_2 (\cos\theta N_q - \sin\theta B_q) + \lambda_3 (\sin\theta N_q + \cos\theta B_q), \tag{3.3}$$

where

$$\lambda_1 = \langle \mathbf{x}, T\rangle, \quad -\lambda_2 = \langle \mathbf{x}, (\cos\theta N_q - \sin\theta B_q)\rangle, \quad \lambda_3 = \langle \mathbf{x}, (\sin\theta N_q + \cos\theta B_q)\rangle. \tag{3.4}$$

Denote by $\mathbf{r}$ and $\mathbf{r}^*$ the vectors

$$\mathbf{r} = \lambda_1 T - \lambda_2 (\cos\theta N_q - \sin\theta B_q), \quad \mathbf{r}^* = \lambda_1 T + \lambda_3 (\sin\theta N_q + \cos\theta B_q), \tag{3.5}$$

which lie in the osculating plane and rectifying plane to C at P, respectively. Then, we have

$$r^2 = \langle \mathbf{r}, \mathbf{r}\rangle = \lambda_1^2 + \lambda_2^2, \quad (r^*)^2 = \langle \mathbf{r}^*, \mathbf{r}^*\rangle = \lambda_1^2 + \lambda_3^2, \tag{3.6}$$

where r and r^* are the lengths of the vectors $\mathbf{r}$ and $\mathbf{r}^*$, respectively.

It is well known that the angular momentum vector $\mathbf{H}^O$ of P about O is given by

$$\mathbf{H}^O = \mathbf{x} \times m\mathbf{v}.$$

Thus, from (2.9) and (3.3), we obtain

$$\mathbf{H}^O = m\lambda_3 \left(\frac{ds}{dt}\right)(\cos\theta N_q - \sin\theta B_q) + m\lambda_2 \left(\frac{ds}{dt}\right)(\sin\theta N_q + \cos\theta B_q). \tag{3.7}$$

Now we try to resolve the acceleration vector $\mathbf{a}$ in (2.10) along the radial direction and tangential direction in osculating plane, and also try to resolve the jerk vector $\mathbf{j}$ in (2.11) along the tangential direction, radial direction in osculating plane and radial direction in the rectifying plane. To do so, let us express the vector $(\cos\theta N_q - \sin\theta B_q)$ in terms of $\mathbf{r}$ and T. In view of (3.5), we can conclude that this is possible if and only if $\lambda_2 \neq 0$. By making the physical assumption that the component of angular momentum along the vector $(\sin\theta N_q + \cos\theta B_q)$ never vanishes, we can ensure that λ_2 is nonzero. Secondly, let us express the vector $(\sin\theta N_q + \cos\theta B_q)$ in terms of $\mathbf{r}^*$ and T. In view of (3.5), this possible if and only if $\lambda_3 \neq 0$. By making the second physical assumption that the component of angular momentum along the vector $(\cos\theta N_q - \sin\theta B_q)$ never vanishes, we can ensure that λ_3 is nonzero. Thus, we obtain the following equations:

$$\cos\theta N_q - \sin\theta B_q = \frac{1}{\lambda_2}(-\mathbf{r} + \lambda_1 T), \quad \sin\theta N_q + \cos\theta B_q = \frac{1}{\lambda_3}(\mathbf{r}^* - \lambda_1 T). \tag{3.8}$$

Hence, in view of (3.6), $r \neq 0$ and $r^* \neq 0$. So, we can define the unit vectors $\mathbf{e_r}$ and $\mathbf{e_{r^*}}$ as follows:

$$\mathbf{e_r} = \frac{1}{r}\mathbf{r}, \qquad \mathbf{e_{r^*}} = \frac{1}{r^*}\mathbf{r}^*. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\cos\theta N_q - \sin\theta B_q = \frac{1}{\lambda_2}\left(-r\mathbf{e_r} + \lambda_1 T\right), \quad \sin\theta N_q + \cos\theta B_q = \frac{1}{\lambda_3}\left(r^*\mathbf{e_{r^*}} - \lambda_1 T\right). \tag{3.10}$$

Substituting (3.10) in (3.1) and (3.2), we obtain the acceleration $\mathbf{a}$ vector and jerk $\mathbf{j}$ vector as follows:

$$\begin{aligned} \mathbf{a} &= \left(\frac{d^2s}{dt^2} + \frac{\lambda_1\sqrt{\kappa_1^2+\kappa_2^2}}{\lambda_2}\left(\frac{ds}{dt}\right)^2\right)T + \left(\frac{-r\sqrt{\kappa_1^2+\kappa_2^2}}{\lambda_2}\left(\frac{ds}{dt}\right)^2\right)\mathbf{e_r} \\ &= A_tT + A_r\mathbf{e_r}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \mathbf{j} &= \Bigg[\frac{d^3s}{dt^3} - \left(\kappa_1^2+\kappa_2^2\right)\left(\frac{ds}{dt}\right)^3 + 3\sqrt{\kappa_1^2+\kappa_2^2}\frac{\lambda_1}{\lambda_2}\frac{ds}{dt}\frac{d^2s}{dt^2} \\ &\quad + \frac{\lambda_1}{\lambda_2}\left(\frac{ds}{dt}\right)^3\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right) - \frac{\lambda_1\left(\kappa_3-\theta'\right)}{\lambda_3}\sqrt{\kappa_1^2+\kappa_2^2}\left(\frac{ds}{dt}\right)^3\Bigg]T \\ &\quad + \left[-\frac{3r}{\lambda_2}\sqrt{\kappa_1^2+\kappa_2^2}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) - \frac{r}{\lambda_2}\left(\frac{ds}{dt}\right)^3\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right)\right]\mathbf{e_r} \\ &\quad + \left[\frac{r^*\left(\kappa_3-\theta'\right)}{\lambda_3}\sqrt{\kappa_1^2+\kappa_2^2}\left(\frac{ds}{dt}\right)^3\right]\mathbf{e_{r^*}}. \\ &= J_tT + J_r\mathbf{e_r} + J_{r^*}\mathbf{e_{r^*}}. \end{aligned} \tag{3.12}$$

Here, A_t and A_r are tangential and radial Siacci components of the acceleration, while J_t, J_r and J_{r^*} are tangential and radial components of the jerk. By taking into consideration the above conclusion about the acceleration and jerk vectors of the particle P, we can state the following theorems:

Theorem 1. *(Siacci's Theorem According to Quasi-Frame). Let P be a particle whose mass is m and which moves along an analytic space curve $\alpha = \alpha(s)$ with respect to quasi-frame. Assume that the component of its angular momentum which is along the vector $(\sin\theta N_q + \cos\theta B_q)$ never takes the value zero. In this case, the acceleration vector $\mathbf{a}$ of P can be expressed as in (3.11). A_t lies along the tangent line of α, while A_r is directed from the particle P towards the foot of the perpendicular that is from the origin to osculating plane to α at P.*

Theorem 2. *Let P be a particle whose mass is m and which moves along an analytic space curve $\alpha = \alpha(s)$ with respect to quasi-frame. Assume that each of the components of its angular momentum never takes vanishes. In this case, the jerk $\mathbf{j}$ of P can be expressed as in (3.12). The component J_t lies along the tangent line of α, while the component J_r lies along the line that passes through the particle P towards the foot of the perpendicular that is from the origin to osculating plane to α at P, and the component J_{r^*} lies along the line that passes through the particle P towards the foot of the perpendicular that is from the origin to rectifying plane to α at P.*

Remark 1. *We note that if $\kappa_3 = 0$, then the quasi-frame $\{T(s), N_q(s), B_q(s)\}$ becomes the Bishop frame. In this case, Theorem 2 reduces to Theorem 1 in [5].*

Corollary 1. *In Euclidean 3-space, Let the particle P move along a curve with respect to quasi-frame and lie in the osculating plane which deos not contain the origin of space. Assume that the component of its angular momentum vector along the normal vector of this plane never vanishes. In that case, the jerk vector reduces to*

$$\begin{aligned}\mathbf{j} \;=\;& \left[\frac{d^3s}{dt^3} - \left(\kappa_1^2+\kappa_2^2\right)\left(\frac{ds}{dt}\right)^3 + 3\sqrt{\kappa_1^2+\kappa_2^2}\frac{\lambda_1}{\lambda_2}\frac{ds}{dt}\frac{d^2s}{dt^2}\right.\\ &\left. + \frac{\lambda_1}{\lambda_2}\left(\frac{ds}{dt}\right)^3\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right)\right]T\\ &+\left[-\frac{3r}{\lambda_2}\sqrt{\kappa_1^2+\kappa_2^2}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right) - \frac{r}{\lambda_2}\left(\frac{ds}{dt}\right)^3\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right)\right]\mathbf{e_r}.\end{aligned}$$

Proof. The proof can be completed directly by considering $\kappa_3 - \theta' = \tau$ in Theorem 2, and Putting $\tau = 0$ for the planar case. □

Corollary 2. *In Euclidean 3-space, Let the particle P move along a curve with a uniform motion with a speed V, a velocity vector $\mathbf{v}$, an acceleration vector $\mathbf{a}$ and a jerk $\mathbf{j}$ vector at time t with respect to quasi-frame such that the jerk satisfy the condition $\|\mathbf{j}\| \leq j_{\max}$, then the maximum speed V admissible on the curve at all trajectory points must satisfy*

$$V \leq \frac{\sqrt[3]{j_{\max}}}{\sqrt[6]{\max\Phi(s)}},$$

where

$$\Phi(s) = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \frac{2\lambda_1}{r}\Phi_1\Phi_2 + \frac{2\lambda_1}{r^*}\Phi_1\Phi_3 + \frac{2\lambda_1^2}{rr^*}\Phi_2\Phi_3,$$

and

$$\Phi_1(s) = \left[\frac{\lambda_1}{\lambda_2}\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right) - \left(\kappa_1^2+\kappa_2^2\right) - \frac{\lambda_1(\kappa_3-\theta')}{\lambda_3}\sqrt{\kappa_1^2+\kappa_2^2}\right],$$

$$\Phi_2(s) = -\frac{r}{\lambda_2}\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right),$$

$$\Phi_3(s) = \frac{r^*(\kappa_3-\theta')}{\lambda_3}\sqrt{\kappa_1^2+\kappa_2^2}.$$

Proof. In the case of uniform motion, Let the particle P move along a curve with a uniform motion with $ds/dt = V$, $d^2s/dt^2 = 0$ and $d^3s/dt^3 = 0$. Thus, from Theorem 2, we get

$$\begin{aligned}J_t \;=\;& \left[\frac{\lambda_1}{\lambda_2}\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right) - \left(\kappa_1^2+\kappa_2^2\right) - \frac{\lambda_1(\kappa_3-\theta')}{\lambda_3}\sqrt{\kappa_1^2+\kappa_2^2}\right]V^3\\ =\;& \Phi_1(s)V^3,\end{aligned}$$

$$J_r = -\left[\frac{r}{\lambda_2}\frac{d}{ds}\left(\sqrt{\kappa_1^2+\kappa_2^2}\right)\right]V^3 = \Phi_2(s)V^3,$$

and

$$J_{r^*} = \left[\frac{r^* (\kappa_3 - \theta')}{\lambda_3} \sqrt{\kappa_1^2 + \kappa_2^2} \right] V^3 = \Phi_3 (s) V^3.$$

Then

$$\begin{aligned} \|\mathbf{j}\| &= V^3 \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \frac{2\lambda_1}{r} \Phi_1 \Phi_2 + \frac{2\lambda_1}{r^*} \Phi_1 \Phi_3 + \frac{2\lambda_1^2}{rr^*} \Phi_2 \Phi_3} \\ &= V^3 \sqrt{\Phi}. \end{aligned}$$

Which implies that

$$V \leq \frac{\sqrt[3]{j_{\max}}}{\sqrt[6]{\max \Phi (s)}}.$$

The proof is complete. □

4. Applications

In this section, we give illustrative examples to calculate the components of acceleration and jerk vectors with respect to quasi-frame by applying (2.10) and (2.11), and Theorem 1, Theorem 2 and Corollary 1. Furthermore, we calculate the maximum admissible speed on a space curve at all trajectory points by applying Corollary 2.

Example 1. *Suppose that a particle P travels a helical curve over clothoid (Cornu spiral or Euler spiral) in E^3, and in Cartesian coordinates, the postion vector of P is given as follows:*

$$\mathbf{x} = \left(\frac{1}{\sqrt{2}} \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \frac{1}{\sqrt{2}} \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du, \frac{t}{\sqrt{2}} \right), \tag{4.1}$$

where $\int_0^t \cos\left(\frac{\pi u^2}{2}\right) du$ and $\int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$ are called Fresnel integrals. Recently, this curve has many applications in the real life, for example, the highway, railway route design or roller coasters, etc. The velocity vector, acceleration vector and jerk vector of can be calculated as follows:

$$\mathbf{v} = \left(\frac{1}{\sqrt{2}} \cos\left(\frac{\pi t^2}{2}\right), \frac{1}{\sqrt{2}} \sin\left(\frac{\pi t^2}{2}\right), \frac{1}{\sqrt{2}} \right), \tag{4.2}$$

$$a = \left(\frac{-\pi t}{\sqrt{2}} \sin\left(\frac{\pi t^2}{2}\right), \frac{\pi t}{\sqrt{2}} \cos\left(\frac{\pi t^2}{2}\right), 0 \right),$$

$$\mathbf{j} = \left(\frac{-\pi^2 t^2}{\sqrt{2}} \cos\left(\frac{\pi t^2}{2}\right) - \frac{\pi}{\sqrt{2}} \sin\left(\frac{\pi t^2}{2}\right), \frac{-\pi^2 t^2}{\sqrt{2}} \sin\left(\frac{\pi t^2}{2}\right) + \frac{\pi}{\sqrt{2}} \cos\left(\frac{\pi t^2}{2}\right), 0 \right).$$

From (4.2), we can write the following equalities:

$$dx = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi t^2}{2}\right) dt, \quad dy = \frac{1}{\sqrt{2}} \sin\left(\frac{\pi t^2}{2}\right) dt, \quad dz = \frac{1}{\sqrt{2}} dt.$$

Using $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, we obtain

$$\frac{ds}{dt} = 1, \quad \frac{d^2 s}{dt^2} = 0, \quad \frac{d^3 s}{dt^3} = 0.$$

We see that the oriented curve traced out by the particle P can be parameterized by the arc-length $s = s(t) = t$ as follows:

$$\delta(s) = \left(\frac{1}{\sqrt{2}} \int_0^s \cos\left(\frac{\pi u^2}{2}\right) du, \frac{1}{\sqrt{2}} \int_0^s \sin\left(\frac{\pi u^2}{2}\right) du, \frac{s}{\sqrt{2}} \right). \tag{4.3}$$

Then, from (2.1), we can obtain the Serret-Frénet frame as follows:

$$\begin{aligned} T &= \left(\frac{1}{\sqrt{2}} \cos\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}} \sin\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}} \right), \\ N &= \left(\frac{-s}{|s|} \sin\left(\frac{\pi s^2}{2}\right), \frac{s}{|s|} \cos\left(\frac{\pi s^2}{2}\right), 0 \right), \\ B &= \left(\frac{-s}{\sqrt{2}\,|s|} \cos\left(\frac{\pi s^2}{2}\right), \frac{-s}{\sqrt{2}\,|s|} \sin\left(\frac{\pi s^2}{2}\right), \frac{s}{\sqrt{2}\,|s|} \right), \end{aligned}$$

and the curvature and the torsion as

$$\kappa = \frac{\pi\,|s|}{\sqrt{2}}, \quad \tau = \frac{\pi s}{\sqrt{2}}.$$

Thus, we note that the Serret-Frénet frame is inadequate for studying the space curves which its curvatures have discrete points zero since as we shown the principlal normal and binormal vectors are discontinuous at $s = 0$, and the curvature is not differentiable as well. Therefore, to solve this problem, we use the quasi-frame as an alternative frame to the Serret-Frénet frame. From (2.7), we obtain

$$\begin{aligned} \kappa_1 &= \frac{\pi\,|s|}{\sqrt{2}} \cos\theta, \\ \kappa_2 &= -\frac{\pi\,|s|}{\sqrt{2}} \sin\theta, \\ \kappa_3 &= \theta'(s) + \frac{\pi s}{\sqrt{2}}, \quad \theta = -\arctan\left(\frac{\kappa_2}{\kappa_1}\right). \end{aligned}$$

If we consider (2.3), we get the following quasi-frame:

$$\begin{aligned} T &= \left(\frac{1}{\sqrt{2}} \cos\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}} \sin\left(\frac{\pi s^2}{2}\right), \frac{1}{\sqrt{2}} \right), \\ N_q &= \left(\frac{1}{\sqrt{2}} \sin\left(\frac{\pi s^2}{2}\right), \frac{-1}{\sqrt{2}} \cos\left(\frac{\pi s^2}{2}\right), 0 \right), \\ B_q &= \left(\frac{1}{2} \cos\left(\frac{\pi s^2}{2}\right), \frac{1}{2} \sin\left(\frac{\pi s^2}{2}\right), \frac{-1}{2} \right). \end{aligned}$$

By considering (2.10) and (2.11), we get

$$\mathbf{a} = \frac{\pi\,|s|}{\sqrt{2}} \cos\theta N_q - \frac{\pi\,|s|}{\sqrt{2}} \sin\theta B_q,$$

and

$$\begin{aligned} C_T &= -\frac{\pi^2 s^2}{2}, \\ C_{N_q} &= \left(\frac{\pi s}{\sqrt{2}\,|s|} \cos\theta + \frac{\pi^2 s\,|s|}{2} \sin\theta \right), \quad \textit{for } s \neq 0, \\ C_{B_q} &= \left(-\frac{\pi s}{\sqrt{2}\,|s|} \sin\theta + \frac{\pi^2 s\,|s|}{2} \cos\theta \right), \quad \textit{for } s \neq 0, \end{aligned}$$

where θ is the Euclidean angle between the principal normal N and quasi-normal N_q vectors. Then

$$\mathbf{j} = C_T T + C_{N_q} N_q + C_{B_q} B_q.$$

Example 2. *Assume that a particle P moves a long a right-handed circular helix which lies on a clylinder of radius a and that the angular frequency ω of P is not time dependent. In that case, in Cartesian coordinates, the postion vector of P is given as follows:*

$$\mathbf{x} = (a\cos(\omega t), a\sin(\omega t), bt), \tag{4.4}$$

where t is the time and a, b are postive constants. Let the helix axis be the z-axis, and φ be the helix angle satisfying $\tan\varphi = \frac{a\omega}{b}$. The velocity vector, acceleration vector and jerk vector of can be calculated as follows:

$$\mathbf{v} = (-a\omega\sin(\omega t), a\omega\cos(\omega t), b),$$
$$\mathbf{a} = \left(-a\omega^2\cos(\omega t), -a\omega^2\sin(\omega t), 0\right),$$
$$\mathbf{j} = \left(a\omega^3\sin(\omega t), -a\omega^3\cos(\omega t), 0\right).$$

From (4.4), we can write the following equalities:

$$dx = -a\omega\sin(\omega t)\,dt, \quad dy = a\omega\cos(\omega t), \quad dz = bdt.$$

Using $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, the speed υ of the particle P, and its first and second derivatives can be given by

$$\upsilon = \frac{ds}{dt} = \sqrt{a^2\omega^2 + b^2}, \quad \frac{d^2s}{dt^2} = 0, \quad \frac{d^3s}{dt^3} = 0.$$

We see that the oriented curve traced out by the particle P can be parameterized by the arc-length $s = s(t) = \upsilon t$ as follows:

$$\alpha(s) = \left(a\cos\left(\frac{\omega s}{\upsilon}\right), a\sin\left(\frac{\omega s}{\upsilon}\right), \frac{bs}{\upsilon}\right). \tag{4.5}$$

Then, from (2.1) and (4.5), we can obtain the Serret-Frénet frame as follows:

$$T = \left(-\sin\varphi\sin\left(\frac{\omega s}{\upsilon}\right), \sin\varphi\cos\left(\frac{\omega s}{\upsilon}\right), \cos\varphi\right),$$
$$N = \left(-\cos\left(\frac{\omega s}{\upsilon}\right), -\sin\left(\frac{\omega s}{\upsilon}\right), 0\right),$$
$$B = \left(\cos\varphi\sin\left(\frac{\omega s}{\upsilon}\right), -\cos\varphi\cos\left(\frac{\omega s}{\upsilon}\right), \sin\varphi\right).$$

On the other hand, we can get the curvature and the torsion as

$$\kappa = \frac{a\omega^2}{\upsilon^2}, \quad \tau = \frac{b\omega}{\upsilon^2}.$$

From (2.7), we obtain

$$\kappa_1 = \frac{a\omega^2}{\upsilon^2}\cos\theta,$$
$$\kappa_2 = -\frac{a\omega^2}{\upsilon^2}\sin\theta,$$
$$\kappa_3 = \theta'(s) + \frac{b\omega}{\upsilon^2}, \quad \theta = -\arctan\left(\frac{\kappa_2}{\kappa_1}\right).$$

If we consider the relation (2.4) between the quasi-frame and the classical Serret-Frénet frame, we get the following quasi-frame:

$$T = \left(-\sin\varphi \sin\left(\frac{\omega s}{v}\right), \sin\varphi\cos\left(\frac{\omega s}{v}\right), \cos\varphi\right),$$

$$\begin{aligned} N_q &= \Big(-\cos\theta\cos\left(\frac{\omega s}{v}\right) + \sin\theta\cos\varphi\sin\left(\frac{\omega s}{v}\right), \\ &\qquad -\cos\theta\sin\left(\frac{\omega s}{v}\right) - \sin\theta\cos\varphi\cos\left(\frac{\omega s}{v}\right), \\ &\qquad \sin\theta\sin\varphi\Big), \\ B_q &= \Big(\sin\theta\cos\left(\frac{\omega s}{v}\right) + \cos\theta\cos\varphi\sin\left(\frac{\omega s}{v}\right), \\ &\qquad \sin\theta\sin\left(\frac{\omega s}{v}\right) - \cos\theta\cos\varphi\cos\left(\frac{\omega s}{v}\right), \\ &\qquad \cos\theta\sin\varphi\Big). \end{aligned}$$

By considering (3.4) and (4.5), we get

$$\lambda_1 = \frac{bs}{v}\cos\varphi, \quad \lambda_2 = a, \quad \lambda_3 = \frac{bs}{v}\sin\varphi. \tag{4.6}$$

Also, from (3.3) and (4.5), we have

$$\begin{aligned} \alpha(s) &= \left(\frac{bs}{v}\cos\varphi\right)T - a\left(\cos\theta N_q - \sin\theta B_q\right) + \left(\frac{bs}{v}\sin\varphi\right)\left(\sin\theta N_q + \cos\theta B_q\right) \\ &= \left(\frac{bs}{v}\cos\varphi\right)T - \left(a\cos\theta - \frac{bs}{v}\sin\varphi\sin\theta\right)N_q + \left(a\sin\theta + \frac{bs}{v}\sin\varphi\cos\theta\right)B_q. \end{aligned}$$

On the other hand, from (3.6) and (4.6), we obtain

$$r = \sqrt{\left(\frac{bs}{v}\right)^2\cos^2\varphi + a^2}, \quad r^* = \frac{bs}{v}.$$

Therefore, by applying Theorem 1 and Theorem 2, we get the components of the acceleration and jerk vectors as follows:

$$A_t = \frac{\omega^2 b^2 s}{a^2\omega^2 + b^2}, \qquad A_r = -\omega^2\sqrt{\frac{b^4 s^2}{\left(a^2\omega^2+b^2\right)^2} + a^2},$$

and

$$J_t = -\omega^2\sqrt{a^2\omega^2+b^2}, \qquad J_r = 0, \qquad J_{r^*} = b\omega^2.$$

On the other hand, by applying Corollary 2, if the jerk satisfy the condition $\|\mathbf{j}\| \leq j_{\max}$, we can calculate the maximum admissible speed on a circular helix at all trajectory points as follows:

$$\Phi_1(s) = \frac{-\omega^2}{a^2\omega^2+b^2}, \quad \Phi_2(s) = 0, \quad \Phi_3(s) = \frac{b\omega^2}{\left(a^2\omega^2+b^2\right)^{3/2}}.$$

Then

$$\Phi(s) = \frac{a^2\omega^6}{\left(a^2\omega^2+b^2\right)^3}.$$

Which implies that

$$\|\mathbf{j}\| = a\omega^3,$$

and

$$V \leq \frac{\sqrt[3]{j_{\max}}}{\sqrt[6]{\max \Phi(s)}} = \frac{\omega \sqrt[3]{a j_{\max}}}{\sqrt{a^2\omega^2 + b^2}},$$

then

$$V_{\max} = \frac{\omega^2 \sqrt[3]{a^2}}{\sqrt{a^2\omega^2 + b^2}}.$$

Example 3. *Assume that a particle P moves a long the logarithmic spiral curve. In Cartesian coordinates, the postion vector of P is given as follows:*

$$\mathbf{x} = \left(e^{\omega t}\cos(\omega t), 0, e^{\omega t}\sin(\omega t)\right), \tag{4.7}$$

where t is the time and ω the angular frequency. The velocity vector, acceleration vector and jerk vector of can be calculated as follows:

$$\mathbf{v} = \omega\left(e^{\omega t}\cos(\omega t) - e^{\omega t}\sin(\omega t), 0, e^{\omega t}\sin(\omega t) + e^{\omega t}\cos(\omega t)\right),$$

$$\mathbf{a} = 2\omega^2\left(-e^{\omega t}\sin(\omega t), 0, e^{\omega t}\cos(\omega t)\right),$$

$$\mathbf{j} = 2\omega^3\left(-e^{\omega t}\sin(\omega t) - e^{\omega t}\cos(\omega t), 0, e^{\omega t}\cos(\omega t) - e^{\omega t}\sin(\omega t)\right).$$

From (4.7), we can write the following equalities:

$$dx = \omega e^{\omega t}(\cos(\omega t) - \sin(\omega t))\,dt, \quad dz = \omega e^{\omega t}(\sin(\omega t) + \cos(\omega t))\,dt.$$

Using $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, we obtain

$$\frac{ds}{dt} = \sqrt{2}\omega e^{\omega t}, \quad \frac{d^2s}{dt^2} = \sqrt{2}\omega^2 e^{\omega t}, \quad \frac{d^3s}{dt^3} = \sqrt{2}\omega^3 e^{\omega t}.$$

We see that the oriented curve traced out by the particle P can be parameterized by the arc-length $s = s(t) = \sqrt{2}\left(e^{\omega t} - 1\right)$ as follows:

$$\alpha^*(s) = \frac{s+\sqrt{2}}{\sqrt{2}}\left(\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right), 0, \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right). \tag{4.8}$$

Then, from (2.1) and (4.8), we can obtain the Serret-Frénet frame as follows:

$$T = \frac{1}{\sqrt{2}}\left(\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right), 0, \cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) + \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right),$$

$$N = \frac{1}{\sqrt{2}}\left(-\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right), 0, \cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right),$$

$$B = (0, -1, 0).$$

On the other hand, we can get the curvature and the torsion as

$$\kappa = \frac{1}{s+\sqrt{2}}, \quad \tau = 0.$$

From (2.7), we obtain

$$\kappa_1 = \frac{1}{s+\sqrt{2}}\cos\theta,$$

$$\kappa_2 = -\frac{1}{s+\sqrt{2}}\sin\theta,$$

$$\kappa_3 = \theta'(s), \quad \theta = -\arctan\left(\frac{\kappa_2}{\kappa_1}\right).$$

If we consider the relation (2.4) between the quasi-frame and the classical Serret-Frénet frame, we get the following quasi-frame:

$$T = \frac{1}{\sqrt{2}}\left(\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right), 0, \cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) + \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right),$$

$$\begin{aligned} N_q &= \left(\frac{1}{\sqrt{2}}\cos\theta\left(-\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right), -\sin\theta,\right. \\ &\qquad \left.\frac{1}{\sqrt{2}}\cos\theta\left(\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)\right), \end{aligned}$$

$$\begin{aligned} B_q &= \left(\frac{-1}{\sqrt{2}}\sin\theta\left(-\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right), -\cos\theta,\right. \\ &\qquad \left.\frac{-1}{\sqrt{2}}\sin\theta\left(\cos\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right) - \sin\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)\right), \end{aligned}$$

By considering (3.4) and (4.8), we get

$$\lambda_1 = \frac{s+\sqrt{2}}{2}, \quad \lambda_2 = \frac{s+\sqrt{2}}{2}, \quad \lambda_3 = 0. \tag{4.9}$$

Also, from (3.3) and (4.8), we have

$$\alpha^*(s) = \left(\frac{s+\sqrt{2}}{2}\right)T - \left(\frac{s+\sqrt{2}}{2}\right)\cos\theta N_q + \left(\frac{s+\sqrt{2}}{2}\right)\sin\theta B_q.$$

vc v On the other hand, from (3.6) and (4.9), we obtain

$$r = \frac{s+\sqrt{2}}{\sqrt{2}} = e^{\omega t}, \quad r^* = \frac{s+\sqrt{2}}{2} = \frac{1}{\sqrt{2}}e^{\omega t}.$$

Therefore, by applying Theorem 1 and Corollary 1 and , we get the components of the acceleration and jerk vectors as follows:

$$A_t = 2\sqrt{2}\omega^2 e^{\omega t}, \qquad A_r = -2\omega^2 e^{\omega t},$$

and

$$J_t = 4\sqrt{2}\omega^3 e^{\omega t}, \qquad J_r = \frac{1}{2}\omega^3 e^{\omega t}.$$

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D̈epartment of Mathematics, Faculty of science, Tanta University, Egypt
E-mail address:

D̈epartment of Mathematics, Faculty of science, Mansura University, Egypt
E-mail address: ähmedelshenhab@mans.edu.eg