Adaptation of kernel functions-based approach with ABC distributed order derivative for solutions of fuzzy fractional Volterra and Fredholm integrodifferential equations

Omar Abu Arqub [1,2*], Jagdev Singh [3], Mohammed Alhodaly [2]

[1] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan
[2] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[3] Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India

[*] Corresponding author: Omar Abu Arqub, e-mail address: o.abuarqub@bau.edu.jo

Abstract. Mathematical modeling of uncertain FIDEs is an extremely significant topic in electric circuits, signal processing, electromagnetics, and anomalous diffusion systems. Based on the RKA, a touching numerical approach is considering in this study to solve groups of FFIDEs with ABC fractional distributed order derivatives. The solution-based approach lies in generating infinite orthogonal basis from kernel functions, where uncertain condition is fulfilled. Thereafter, an orthonormal basis is erected to figurate fuzzy ABC solutions with series shape in idioms of η-cut extrapolation in Hilbert space $\mathcal{A}(\mathbb{D})$ and $\mathcal{B}(\mathbb{D})$. In this orientation, fuzzy ABC fractional integral, fuzzy ABC fractional derivative, and fuzzy ABC FIDE are utilized and comprised. The competency and accuracy of the suggested approach are indicating by employing several experiments. From theoretical viewpoints, the acquired results signalize that the utilize approach has several merits in feasibility and opportunity for treating with many fractional ABC distributed order models. In the end, highlights and future suggested research work are eluded.

Keywords: Fuzzy fractional integrodifferential equations; Atangana-Baleanu-Caputo fractional derivative; Atangana-Baleanu fractional integral; Reproducing kernel algorithm; Fuzzy inductance-resistance-capacitance circuit

Abbreviations: FFIDE: Fuzzy fractional integrodifferential; FIDE: Fractional integrodifferential; RKA: Reproducing kernel algorithm; ABC: Atangana-Baleanu-Caputo; IRCSC: Inductance-resistance-capacitance series circuit

Identifications: $\mathcal{C}^{\mathbb{Y}}(\mathbb{X})$: space of continuous maps from $\mathbb{X}$ to $\mathbb{Y}$; $\mathfrak{D}^{\mathbb{Y}}(\mathbb{X})$: space of differentiable maps from $\mathbb{X}$ to $\mathbb{Y}$; $\mathcal{L}^{\mathbb{Y}}(\mathbb{X})$: space of integrable maps from $\mathbb{X}$ to $\mathbb{Y}$; $|\mathcal{C}|^{\mathbb{Y}}(\mathbb{X})$: space of absolutely continuous maps from $\mathbb{X}$ to $\mathbb{Y}$. $\mathcal{D}_{\gamma}^{\mathbb{Y}}(\mathbb{X})$: space of γ-differentiable maps from $\mathbb{X}$ to $\mathbb{Y}$

1 Introduction

Fractional calculus with uncertainty is one of the hot topics of applied mathematics treating with the implementation and exercises of calculus connotations of arbitrary order uncertain functions that supplies a cute mechanization for clarifying the hereditary and memory labor of uncertain systems. It is straightforward popularization of fuzzy calculus that treats with integer-order. Toughly, it has become an indispensable tool mightily used for better formulating in different areas of mathematics, physics, and engineering [1-5]. Recently, many various issues of fractional operators, such as, Riemann-Liouville, Caputo-Liouville, or conformable issue have utilized and developed concurrently with the evolution of the crisp and fuzzy theories [6-10]. Anyhow, the most common weakness among those fractional qualifiers is collected in singularity, nonlocality, or limit entity. To transact with these reversals, a new construction of FFIDEs instituted on fuzzy ABC fractional derivative is used to build and formulate new concretes fuzzy mathematical concepts. This new fuzzy fractional ABC terminology seems to be liberating of singularity, nonlocality, or limit entity; because the kernel function depends on the nature exponential decay which makes FFIDEs more pragmatic in formulating various uncertain physical models [11-21].

The RKA is meshless approach and provide global continuous higher accuracy approximations by identifying Green functions representations and series solutions for various categories of integral-differential models in ordinary, fractional, or fuzzy derivatives function types. The RKA is an alternative numerical approach that is easy to use in generating infinite series representation solutions for nonlinear models emerging in applied sciences without perturbation and linearization. Anyhow, concrete overview on the reproducing kernel theory results can be viewed from [22-24] and quick overview on its applications fields including its properties and characterestics can be collected from [25-41].

Several issues of uncertain physical and engineering problems are formulated by FFIDEs and many efforts achieved to solve them numerically. Actually, scientists researchers are conscious of the fact that arbitrary distributed order derivatives treat with uncertain models more precisely from integer order ones. The current study chiefly aims to define fuzzy fractional ABC calculus and expand the purposes of the RKA in numerically solving FFIDEs of Volterra and Fredholm kinds. To justify more, those analyses appointed the discussions on the following underlying FFIDEs:

$$\begin{cases} {}^{ABC}_{\ 0}\mathfrak{D}^{\gamma}_{t} z(t) = f\big(t, z(t)\big) + \int_0^t k(t,\tau) g\big(z(\tau)\big) d\tau, \\ z(0) = z_0. \end{cases} \tag{1}$$

In this scope, we symbolized the following underlying icons: $\mathbb{D} \coloneqq [0,1]$, $t \in \mathbb{D}$, $z_0 \in \mathbb{R}_{\mathcal{F}}$, $\gamma \in (0,1]$, $\mathbb{R}$ the set real number, and $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers on $\mathbb{R}$, and ${}^{ABC}_{\ 0}\mathcal{D}^{\gamma}_{t} z(t)$ the fuzzy ABC fractional derivative of z in t over $\mathbb{D}$. Whilst,

$$\begin{cases} z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}, \\ f: \mathbb{D} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}, \\ g: \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}, \\ k: \mathbb{D} \times \mathbb{D} \to \mathbb{R}. \end{cases} \tag{2}$$

In contract of the preface; the enduring sections are synopsised sequentially as follows: Section 2: essential tools in fuzzy analysis: materials and requirements. Section 3: fuzzy AB fractional integral: definitions and fuzzy AB modifier integral operator. Section 4: fuzzy ABC fractional derivative: rules and formulation. Section 5: fuzzy ABC FIDE: structures and tools. Section 6: essential tools in the RKA: structures of kernel functions. Section 7: figuration of fuzzy ABC FIDE: formulation and series solution shape. Section 8: convergence and error frameworks: ensuring and existence of fuzzy ABC solutions. Section 9: computational algorithms: finding and ensuring of numerical solutions. Section 10: applications on fuzzy ABC FFIDEs: fuzzy ABC IRCSC and fuzzy ABC forcing term. Section 11: tables, graphs, and analysis: results and discussions. Ultimately, Section 12 utilized some highlight and future research.

2 Essential Tools in Fuzzy Analysis

The contents of this part are organized systematically as follows; review on fuzzy sets and fuzzy numbers; fuzzy metric structure; fuzzy arithmetic operations and fuzzy equivalence; fuzzy continuity; and fuzzy integral approach. More theoritcal outcomes on the fuzzy calclus and its implementation can be catched from [1-10, 42-46].

Principally, a fuzzy set $\mathfrak{A}$ in $\mathbb{R}$ can be distinguish by its membership task as $\mathfrak{A}: \mathbb{U} \subset \mathbb{R} \to \Sigma: [0,1]$. Fundamental analytic properties of fuzzy sets such as convexity, uppersemicontinuous, normal, and boundedness support can be collected in detalis from [42]. The η-cut extrapolation occupies remarkable and fundamental position in fuzzy mathematics policy. Anyhow, $\forall \eta \in \Sigma - \{0\}$, elect $[\mathfrak{A}]^{\eta} = \{s \in \mathbb{R} | \mathfrak{A}(s) \geq \eta\}$ and $[\mathfrak{A}]^{0} = \overline{\{s \in \mathbb{R} | \mathfrak{A}(s) > 0\}}$. So, $\mathfrak{A} \in \mathbb{R}_{\mathcal{F}}$ iff $[\mathfrak{A}]^{1} \neq \phi$ and $[\mathcal{U}]^{\eta}$ is convex compact in $\mathbb{R}$ [43]. Certainly, if $\mathfrak{A} \in \mathbb{R}_{\mathcal{F}}$, then $[\mathfrak{A}]^{\eta} = [\mathfrak{A}_1(\eta), \mathfrak{A}_2(\eta)]$ providing

$$\begin{cases} \mathfrak{A}_1(\eta) = \min\{s | s \in [\mathfrak{A}]^{\eta}\}, \\ \mathfrak{A}_2(\eta) = \max\{s | s \in [\mathfrak{A}]^{\eta}\}. \end{cases} \tag{3}$$

Hither, $[\mathfrak{A}]^{\eta}$ denots the η-cut extrapolation of $\mathfrak{A}$ and $\mathfrak{A}_{1\eta}$ and $\mathfrak{A}_{2\eta}$ composes to $\mathfrak{A}_1(\eta)$ and $\mathfrak{A}_2(\eta)$, simultaneously.

Definition 1 [43] A fuzzy number $\mathfrak{A}$ is a fuzzy subset in $\mathbb{R}$ with normal, convex, and upper semicontinuous with bounded support.

Theorem 1 [43] Suppose that $\mathfrak{A}_{1,2}: \Sigma \to \mathbb{R}$ satisfy the following underlying requirements:

i. $\mathfrak{A}_1$ nondecreasing bounded and $\mathfrak{A}_2$ nonincreasing bounded,
ii. $\lim_{\eta \to h^-} \mathfrak{A}_{(1,2)\eta} = \mathfrak{A}_{(1,2)h}$ and $\lim_{\eta \to 0^+} \mathfrak{A}_{(1,2)\eta} = \mathfrak{A}_{(1,2)0}$,
iii. $\mathfrak{A}_{11} \leq \mathfrak{A}_{21}$.

Then, $\mathfrak{A}: \mathbb{R} \to \Sigma$ constructed as $\mathfrak{A}(s) = \sup\{\eta | \mathfrak{A}_{1\eta} \leq s \leq \mathfrak{A}_{2\eta}\}$ belong to $\mathbb{R}_{\mathcal{F}}$ with extrapolation $[\mathfrak{A}_{1\eta}, \mathfrak{A}_{1\eta}]$. Indeed, if $\mathfrak{A}_{1,2}: \Sigma \to \mathbb{R}$ belong to $\mathbb{R}_{\mathcal{F}}$ with impersonation $[\mathfrak{A}_{1\eta}, \mathfrak{A}_{1\eta}]$, then $\mathfrak{A}_{1,2}$ are satisfies the aforesaid conditions.

Definition 2 [42] A mappnig $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ is continuous at $t^* \in \mathbb{D}$, if $\forall \epsilon > 0$ and $\forall t \in \mathbb{D}$, $\exists \rho > 0$ such that $\mathcal{D}_t(z(t), z(t^*)) < \epsilon$ whenever $|t - t^*| < \rho$, where $\mathfrak{H}_t$ is the Hausdorff space on $\mathbb{R}_{\mathcal{F}}$ and can be viewed as

$$\begin{cases} \mathfrak{H}_t: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+, \\ \mathfrak{H}_t(\mathfrak{A}, \mathfrak{B}) = \sup_{\eta \in \Sigma} \max\{|\mathfrak{A}_{1\eta} - \mathfrak{B}_{1\eta}|, |\mathfrak{A}_{2\eta} - \mathfrak{B}_{2\eta}|\}. \end{cases} \tag{4}$$

Indeed, z is continuous on $\mathbb{D}$ if its continuous $\forall t \in \mathbb{D}$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathbb{R}_{\mathcal{F}}$, if $\exists \mathfrak{C} \in \mathbb{R}_{\mathcal{F}}$ satisfies $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$, then $\mathfrak{C}$ invite as $\mathcal{H}$-difference of $(\mathfrak{A}, \mathfrak{B})$ and denoted by $\mathfrak{A} \ominus \mathcal{V}$. Hither, $\ominus$ attitudes constantly for $\mathcal{H}$-difference. Indeed, $\mathfrak{A} \ominus \mathfrak{B} \neq \mathfrak{A} + (-1)\mathfrak{B} = \mathfrak{A} - \mathfrak{B}$ and if $\mathfrak{A} \ominus \mathfrak{B}$ exists, then $[\mathfrak{A} \ominus \mathfrak{B}]^\eta = [\mathfrak{A}_{1\eta} - \mathfrak{B}_{1\eta}, \mathfrak{A}_{2\eta} - \mathfrak{B}_{2\eta}]$.

The arithmetic operations and scalar multiplication on $\mathbb{R}_{\mathcal{F}}$ are defined $\forall \eta \in \Sigma$ and $c \in \mathbb{R}$ as follows:

$$\begin{aligned} &[\mathfrak{A} + \mathfrak{B}]^\eta = [\mathfrak{A}]^\eta + [\mathfrak{B}]^\eta = [\mathfrak{A}_{1\eta} + \mathfrak{B}_{1\eta}, \mathfrak{A}_{2\eta} + \mathfrak{B}_{2\eta}], \\ &[c\mathfrak{A}]^\eta = c[\mathfrak{A}]^\eta = [\min\{c\mathfrak{A}_{1\eta}, c\mathfrak{A}_{2\eta}\}, \max\{c\mathfrak{A}_{1\eta}, c\mathfrak{A}_{2\eta}\}], \\ &[\mathfrak{A}\mathfrak{B}]^\eta = [\mathfrak{A}]^\eta[\mathfrak{B}]^\eta = [\min \Phi_1, \max \Phi_1], \\ &\left[\frac{\mathfrak{A}}{\mathfrak{B}}\right]^\eta = \frac{[\mathfrak{A}]^\eta}{[\mathfrak{B}]^\eta} = [\min \Phi_2, \max \Phi_2], \end{aligned} \tag{5}$$

whenever $\Phi_1 = \{\mathfrak{A}_{1\eta}\mathfrak{B}_{1\eta}, \mathfrak{A}_{1\eta}\mathfrak{B}_{2\eta}, \mathfrak{A}_{2\eta}\mathfrak{B}_{1\eta}, \mathfrak{A}_{2\eta}\mathfrak{B}_{2\eta}\}$ and $\Phi_1 = \left\{\frac{\mathfrak{A}_{1\eta}}{\mathfrak{B}_{1\eta}}, \frac{\mathfrak{A}_{1\eta}}{\mathfrak{B}_{2\eta}}, \frac{\mathfrak{A}_{2\eta}}{\mathfrak{B}_{1\eta}}, \frac{\mathfrak{A}_{2\eta}}{\mathfrak{B}_{2\eta}}\right\}$. As a conclusion, $\mathfrak{A} = \mathfrak{B}$ only and only if $[\mathfrak{A}]^\eta = [\mathfrak{B}]^\eta$ only and only if $\mathfrak{A}_{1\eta} = \mathfrak{B}_{1\eta}$ and $\mathfrak{A}_{2\eta} = \mathfrak{B}_{2\eta}$.

Definition 3 [42] Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $\mathcal{P} = \{t_0^*, t_1^*, \dots, t_n^*\}$ is a partition of $\mathbb{D}$, $\tau_i \in [t_{i-1}^*, t_i^*]$ with $1 \le i \le n$, $\mathcal{S}_{\mathcal{P}} = \sum_{i=1}^n z(\tau_i)(t_i^* - t_{i-1}^*)$, and $\Omega = \max_{1 \le i \le n} |t_i^* - t_{i-1}^*|$. Then the integral of $z(t)$ over $\mathbb{D}$ is defined as $\int_{\mathbb{D}} z(t)dt = \lim_{\Omega \to 0} \mathcal{S}_{\mathcal{P}}$ as long as the limit exists in the metric space $(\mathbb{R}_{\mathcal{F}}, \mathfrak{H}_t)$.

Theorem 2 [42] Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ be continuous and put $[z(t)]^\eta = [z_{1\eta}(t), z_{2\eta}(t)]$. Then $\int_{\mathbb{D}} z(t)dt$ exist, belong to $\mathbb{R}_{\mathcal{F}}$, $z_{1\eta}, z_{2\eta} \in \mathcal{L}^{\mathbb{R}}(\mathbb{D})$, and $\left[\int_{\mathbb{D}} z(t)dt\right]^\eta = \int_{\mathbb{D}} [z(t)]^\eta dt = \left[\int_{\mathbb{D}} z_{1\eta}(t)dt, \int_{\mathbb{D}} z_{2\eta}(t)dt\right]$.

3 Fuzzy AB Fractional Integral

In this passage, the definitions of fuzzy ABC fractional integral and fuzzy AB modifier integral operator are utilized. Moreover, some useful relationships and some characterizations of fuzzy ABC fractional integral are also derived and utilized specifically.

Definition 4 [11] Let $\mathbb{z}: \mathbb{D} \to \mathbb{R}$, $\mathbb{z}, \mathbb{z}' \in \mathcal{L}^{\mathbb{R}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the crisp ABC fractional integral of order γ of $\mathbb{z}$ at the base node $\mathrm{t} = 0$, symbolizes by ${}_0^{AB}\mathcal{I}^\gamma \mathbb{z}(t)$, is defined as

$${}_0^{AB}\mathcal{I}^\gamma \mathbb{z}(t) = \frac{1 - \gamma}{\Lambda(\gamma)} \mathbb{z}(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t \mathbb{z}(s)(t - s)^{\gamma - 1} ds. \tag{6}$$

One must be focusing here; the normalization $\Lambda(\gamma)$ relation is selected such as $\Lambda(0) = \Lambda(1) = 1$ and in the following subordinate analysis it is composited as $\Lambda(\gamma) = 1 - \gamma + \frac{\gamma}{\Lambda(\gamma)}$. Whilst, $\Gamma(\gamma) = \int_0^\infty s^{\gamma-1} e^{-s} ds$ is the integral representation of the Gamma function.

Definition 5 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, z' \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the fuzzy AB fractional integral of z, symbolizes by ${}_0^{AB}\mathfrak{J}^\gamma z(t)$, is defined as

$${}_0^{AB}\mathfrak{J}^\gamma z(t) = \frac{1 - \gamma}{\Lambda(\gamma)} z(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t z(s)(t - s)^{\gamma - 1} ds. \tag{7}$$

Theorem 3 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, z' \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the underlying is achieved

i. ${}_0^{AB}\mathfrak{J}^\gamma z(t) \in \mathbb{R}_{\mathcal{F}}$.
ii. $[{}_0^{AB}\mathfrak{J}^\gamma z(t)]^\eta = [{}_0^{AB}\mathcal{I}^\gamma z_{1\eta}(t), {}_0^{AB}\mathcal{I}^\gamma z_{2\eta}(t)]$.

Proof. For phase (i): in view of $\frac{1-\gamma}{\Lambda(\gamma)}, \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \ge 0$ and $z_{2\eta}(s) - z_{1\eta}(s), (t - s)^{\gamma-1}, t - s \ge 0$ in the zone $\forall t, s \in \mathbb{D}$, $\forall \eta \in \Sigma$, and $\forall \gamma \in (0,1]$ with $s < t$, yields that

$$(t - s)^{\gamma - 1} z_{2\eta}(s) - (t - s)^{\gamma - 1} z_{1\eta}(s) \ge 0. \tag{8}$$

Using properties of crisp integrals, one has

$$\int_0^t (t-s)^{\gamma-1} z_{2\eta}(s)ds \geq \int_0^t (t-s)^{\gamma-1} z_{1\eta}(s)ds. \tag{9}$$

In different configuration by insert the parameters coefficients, we get

$$\frac{1-\gamma}{\Lambda(\gamma)} z_{2\eta}(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{2\eta}(s)ds \geq \frac{1-\gamma}{\Lambda(\gamma)} z_{1\eta}(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{1\eta}(s)ds. \tag{10}$$

This will summarized that

$${}^{AB}_{0}\mathcal{J}^{\gamma} z_{2\eta}(t) \geq {}^{AB}_{0}\mathcal{J}^{\gamma} z_{1\eta}(t). \tag{11}$$

While that, if one set $\Delta(t,\eta) := \left[{}^{AB}_{0}\mathcal{J}^{\gamma} z_{1\eta}(t), {}^{AB}_{0}\mathcal{J}^{\gamma} z_{2\eta}(t)\right]$ in the zone $\forall t \in \mathbb{D}$, $\forall\, \eta \in \Sigma$, and $\forall \gamma \in (0,1]$; then the set $\Delta(t,\eta)$ is compact convex subset in $\mathbb{R}$ with $\Delta(t,1) \neq \phi$, and by the result of [43]; $\Delta(t,\eta) \in \mathbb{R}_{\mathcal{F}}$. In another arrangement, one can get phase (ii) from

$$\begin{aligned}
\Delta(t,\eta) &:= \left[{}^{AB}_{0}\mathcal{J}^{\gamma} z_{1\eta}(t), {}^{AB}_{0}\mathcal{J}^{\gamma} z_{2\eta}(t)\right] \\
&= \left[\frac{1-\gamma}{\Lambda(\gamma)} z_{1\eta}(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{1\eta}(s)ds\,, \frac{1-\gamma}{\Lambda(\gamma)} z_{2\eta}(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{2\eta}(s)ds\right] \\
&= \left[\frac{1-\gamma}{\Lambda(\gamma)} z_{1\eta}(t), \frac{1-\gamma}{\Lambda(\gamma)} z_{2\eta}(t)\right] + \left[\frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{1\eta}(s)ds\,, \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z_{2\eta}(s)ds\right] \\
&= \frac{1-\gamma}{\Lambda(\gamma)} \left[z_{1\eta}(t), z_{2\eta}(t)\right] + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t \left[(t-s)^{\gamma-1} z_{1\eta}(s), (t-s)^{\gamma-1} z_{2\eta}(s)\right] ds \\
&= \frac{1-\gamma}{\Lambda(\gamma)} [z(t)]^{\eta} + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t [(t-s)^{\gamma-1} w(t)]^{\eta} ds \\
&= \frac{1-\gamma}{\Lambda(\gamma)} [z(t)]^{\eta} + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \left[\int_0^t (t-s)^{\gamma-1} z(t) ds\right]^{\eta} \\
&= \left[\frac{1-\gamma}{\Lambda(\gamma)} z(t) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} z(t) ds\right]^{\eta} \\
&= \left[{}^{AB}_{0}\mathfrak{J}^{\gamma} z(t)\right]^{\eta}. \blacksquare
\end{aligned} \tag{12}$$

Definition 6 Let $\mathbb{z}: \mathbb{D} \to \mathbb{R}$, $\mathbb{z}, \mathbb{z}' \in \mathcal{C}^{\mathbb{R}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the crisp AB modifier integral operator of $\mathbb{z}$, symbolizes by ${}^{AB}_{0}\mathcal{M}(t)$, is defined as

$$\begin{cases} {}^{AB}_{0}\mathcal{M}^{\gamma}: \mathcal{C}^{\mathbb{R}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}}(\mathbb{D}) \to \mathcal{C}^{\mathbb{R}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}}(\mathbb{D}), \\ {}^{AB}_{0}\mathcal{M}^{\gamma} \mathbb{z}(t) = \dfrac{\Lambda(\gamma)}{1-\gamma} \displaystyle\int_0^t \mathbb{z}(s) \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) ds\,. \end{cases} \tag{13}$$

Recalling that, $\mathcal{G}_{\gamma}(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\gamma+1)} t^n$ is the Mittag-Leffler function in one parameter with $\gamma > 0$ and $t \in \mathbb{R}$.

Definition 7 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, z' \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the fuzzy AB modifier integral operator of z, symbolizes by ${}^{AB}_{0}\mathfrak{M}(t)$, is defined as

$$\begin{cases} {}^{AB}_{0}\mathfrak{M}^{\gamma}: \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \to \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}), \\ {}^{AB}_{0}\mathfrak{M}^{\gamma} z(t) = \dfrac{\Lambda(\gamma)}{1-\gamma} \displaystyle\int_0^t z(s) \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) ds\,. \end{cases} \tag{14}$$

Theorem 4 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, z' \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the underlying is achieved

i. ${}^{AB}_{0}\mathfrak{M}^{\gamma} z(t) \in \mathbb{R}_{\mathcal{F}}$.
ii. $\left[{}^{AB}_{0}\mathfrak{M}^{\gamma} z(t)\right]^{\eta} = \left[{}^{AB}_{0}\mathcal{M}^{\gamma} z_{1\eta}(t), {}^{AB}_{0}\mathcal{M}^{\gamma} z_{2\eta}(t)\right]$.

Proof. Beside of $\mathcal{G}_{\gamma}(-t), \frac{\Lambda(\gamma)}{1-\gamma} \geq 0$ and in view of $z_{2\eta}(s) - z_{1\eta}(s), \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) \geq 0$ in the zone $\forall t \in \mathbb{D}$, $\forall\, \eta \in \Sigma$, and $\forall \gamma \in (0,1]$, one can sighted phase (i) as

$$\mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) z_{2\eta}(s) - \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) z_{1\eta}(s) \geq 0. \tag{15}$$

Using properties of crisp integrals, one has

$$\int_0^t \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) z_{2\eta}(s) ds \geq \int_0^t \mathcal{G}_{\gamma}\left(-\frac{\gamma}{1-\gamma}(t-s)^{\gamma}\right) z_{1\eta}(s) ds. \tag{16}$$

In different configuration by insert the parameters coefficients, one get

$$\frac{\Lambda(\gamma)}{1-\gamma}\int_0^t \mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)z_{2\eta}(s)ds \geq \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t \mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)z_{1\eta}(s)ds. \tag{17}$$

This will summarized that

$$ {}_0^{AB}\mathcal{M}^\gamma z_{2\eta}(t) \geq {}_0^{AB}\mathcal{M}^\gamma z_{1\eta}(t). \tag{18}$$

While that, if one set $\Delta(t,\eta) := \left[{}_0^{AB}\mathcal{M}^\gamma z_{1\eta}(t), {}_0^{AB}\mathcal{M}^\gamma z_{2\eta}(t)\right]$, then the set $\Delta(t,\eta)$ is compact convex subset of $\mathbb{R}$ with $\Delta(t,1) \neq \phi$, and by the result of [43]; $\Delta(t,\eta) \in \mathbb{R}_\mathcal{F}$. In another arrangement, one has for phase (ii):

$$\begin{aligned}
\Delta(t,\eta) &= \left[\frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z_{1\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds, \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z_{2\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds\right] \\
&= \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t \left[z_{1\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right), z_{2\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)\right]ds \\
&= \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t \left[z(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)\right]^\eta ds \qquad (19) \\
&= \left[\frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds\right]^\eta \\
&= \left[{}_0^{AB}\mathfrak{M}^\gamma z(t)\right]^\eta. \blacksquare
\end{aligned}$$

4 Fuzzy ABC Fractional Derivative

To describe the fundamental steps in fuzzy ABC approach; we should present the body mathematical structure of fuzzy ABC fractional derivative. After that, based on gained results equivalent statements for fuzzy ABC fractional derivative in idioms of crisp ABC fractional process are derived and utilized.

The strongly fuzzy ABC fractional differentiability allows us not to lose the possible fuzzy ABC solutions when solving FFIDEs in ABC emotion. Additionally, using strongly fuzzy tactic one can applying numerical methods strictly and easily.

Definition 8 [11] Let $\mathbb{z}: \mathbb{D} \to \mathbb{R}$, $\mathbb{z}, \mathbb{z}' \in \mathcal{L}^\mathbb{R}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the crisp ABR fractional derivative of order γ of z at $t = 0$, symbolize by ${}_0^{ABC}\mathcal{D}_t^\gamma \mathbb{z}(t)$ is defined as

$$ {}_0^{ABC}\mathcal{D}_t^\gamma \mathbb{z}(t) = \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t \mathbb{z}'(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds. \tag{20}$$

Now, in somewhat and one way or another we generalize crisp ABR fractional derivative into fuzzy ABC fractional derivative depending on the limits approach. Further, one should note that ${}_0^{ABC}\mathcal{D}_t^\gamma \mathbb{z}(t) = {}_0^{AB}\mathcal{M}^\gamma \mathbb{z}'(t)$.

Definition 9 Let $z: \mathbb{D} \to \mathbb{R}_\mathcal{F}$, $z \in \mathcal{C}^{\mathbb{R}_\mathcal{F}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_\mathcal{F}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the strongly fuzzy ABC fractional derivative of z of order γ at the base node $t = 0$, symbolize by ${}_0^{ABC}\mathfrak{D}_t^\alpha z(t)$, is defined as

$$ {}_0^{ABC}\mathfrak{D}_t^\gamma z(t) = {}_0^{AB}\mathfrak{M}^\gamma \mathfrak{D}_t z(t), \tag{21}$$

provided that ${}_0^{ABC}\mathfrak{D}_t^\gamma z(t) \in \mathbb{R}_\mathcal{F}$ and one of the following underlying requirements is achieved:

i. $\forall\omega > 0$ small enough; the $\mathcal{H}$-differences $z(t+\omega) \ominus z(t)$ and $z(t) \ominus z(t-\omega)$ exist together

$$\begin{aligned}
\mathfrak{D}_t z(t) &= \lim_{\omega\to 0^+} \frac{z(t+\omega) \ominus z(t)}{\omega} \\
&= \lim_{\omega\to 0^+} \frac{z(t) \ominus z(t-\omega)}{\omega}.
\end{aligned} \tag{22}$$

ii. $\forall\omega > 0$ small enough; the $\mathcal{H}$-differences $z(t) \ominus z(t+\omega)$ and $z(t-\omega) \ominus z(t)$ exist together

$$\begin{aligned}
\mathfrak{D}_t z(t) &= \lim_{\omega\to 0^+} \frac{z(t) \ominus z(t+\omega)}{-\omega} \\
&= \lim_{\omega\to 0^+} \frac{z(t-\omega) \ominus z(t)}{-\omega}.
\end{aligned} \tag{23}$$

In Definition 9; $\lim_{\omega\to 0^+}(\cdot)$ is considered in $(\mathbb{R}_\mathcal{F}, \mathfrak{H}_t)$ and at $\delta(\mathbb{D})$ we consider the unidirectional derivatives only. Indeed, z is differentiable on $\mathbb{D}$ as long as z is differentiable $\forall t \in \mathbb{D}$.

Definition 10 For a map $z: \mathbb{D} \to \mathbb{R}_\mathcal{F}$, $z, {}_0^{ABC}\mathcal{D}_t^\gamma z \in \mathcal{C}^{\mathbb{R}_\mathcal{F}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_\mathcal{F}}(\mathbb{D})$, and $\gamma \in (0,1]$; the following underlying is achieved:

i. z is called γ_1-fuzzy differentiable on $\mathbb{D}$ if z is differentiable in phase (i) of Definition 3 and its related derivatives symbolize as ${}^{ABC}_{0}\mathfrak{D}_t^{\gamma(1)}z(t)$ with $\mathfrak{D}_t^1 z(t)$.

ii. z is called γ_2-fuzzy differentiable on $\mathbb{D}$ if z is differentiable in phase (ii) of Definition 3 and its related derivatives symbolize as ${}^{ABC}_{0}\mathfrak{D}_t^{\gamma(2)}z(t)$ with $\mathfrak{D}_t^2 z(t)$.

It is to be noted here, in the phase of Eq. (22), we have $z_{1\eta}, z_{2\eta} \in \mathcal{D}^{\mathbb{R}}(\mathbb{D})$ together with $[\mathfrak{D}_t^1 z(t)]^\eta = \left[z'_{1\eta}(t), z'_{2\eta}(t)\right]$. Whilst in the phase of Eq. (23), we have $z_{1\eta}, z_{2\eta} \in \mathcal{D}^{\mathbb{R}}(\mathbb{D})$ together with $[\mathfrak{D}_t^2 z(t)]^\eta = \left[z'_{2\eta}(t), z'_{1\eta}(t)\right]$.

Theorem 5 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, {}^{ABC}_{0}\mathcal{D}_t^{\alpha} z \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the following underlying is achieved:

i. If z is $\gamma(1)$-fuzzy differentiable on $\mathbb{D}$, then $z_{1\eta}, z_{2\eta} \in \mathcal{D}_{\gamma}^{\mathbb{R}}(\mathbb{D})$ jointly with

$$\left[{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1} z(t)\right]^\eta = \left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2\eta}(t)\right]. \tag{24}$$

ii. If z is $\gamma(2)$-fuzzy differentiable on $\mathbb{D}$, then $z_{1\eta}, z_{2\eta} \in \mathcal{D}_{\gamma}^{\mathbb{R}}(\mathbb{D})$ jointly with

$$\left[{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_2} z(t)\right]^\eta = \left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t)\right]. \tag{25}$$

Proof. Beside of $\mathcal{G}_\gamma(-t), \frac{\Lambda(\gamma)}{1-\gamma} \geq 0$ in the zone $\forall t \in \mathbb{D}$, $\forall\, \eta \in \Sigma$, and $\forall \gamma \in (0,1]$, one can sighted phase (i) as

$$\begin{aligned}
\left[{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1} z(t)\right]^\eta &= \left[{}^{AB}_{0}\mathfrak{M}^{\gamma}\mathfrak{D}_t^1 z(t)\right]^\eta \\
&= \left[{}^{AB}_{0}\mathcal{M}^{\gamma} z'_{1\eta}(t), {}^{AB}_{0}\mathcal{M}^{\gamma} z'_{2\eta}(t)\right] \\
&= \left[\frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z'_{1\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds, \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z'_{2\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds\right] \\
&= \left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2\eta}(t)\right].
\end{aligned} \tag{26}$$

Likewise, one can sighted phase (ii) as

$$\begin{aligned}
\left[{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_2} z(t)\right]^\eta &= \left[{}^{AB}_{0}\mathfrak{M}^{\gamma}\mathfrak{D}_t^2 z(t)\right]^\eta \\
&= \left[{}^{AB}_{0}\mathcal{M}^{\gamma} z'_{2\eta}(t), {}^{AB}_{0}\mathcal{M}^{\gamma} z'_{1\eta}(t)\right] \\
&= \left[\frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z'_{2\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds, \frac{\Lambda(\gamma)}{1-\gamma}\int_0^t z'_{1\eta}(s)\mathcal{G}_\gamma\left(-\frac{\gamma}{1-\gamma}(t-s)^\gamma\right)ds\right] \\
&= \left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t)\right]. \blacksquare
\end{aligned} \tag{27}$$

In the crisp case when $\mathbb{z}: \mathbb{D} \to \mathbb{R}$, $\mathbb{z}, \mathbb{z}' \in \mathcal{L}^{\mathbb{R}}(\mathbb{D})$, and $\gamma \in (0,1]$ it is easy to view that ${}^{AB}_{0}\mathcal{J}^{\gamma}\left({}^{ABC}_{0}\mathcal{D}_t^{\gamma}\mathbb{z}(t)\right) = {}^{AB}_{0}\mathcal{J}^{\gamma}\left({}^{AB}_{0}\mathcal{M}^{\gamma}\mathbb{z}'(t)\right) = \mathbb{z}(t) - \mathbb{z}(0)$. Hither, the Newton-Leibniz fuzzy formulas for integral-derivative are required.

Theorem 6 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, {}^{ABC}_{0}\mathcal{D}_t^{\alpha} z \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Then the following underlying is achieved:

i. If z is γ_1-fuzzy ABC fractional derivative, then

$${}^{AB}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1} z(t)\right) = z(t) \ominus z(0). \tag{28}$$

ii. If z is γ_2-fuzzy ABC fractional derivative, then

$${}^{AB}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}_t^{\gamma_2} z(t)\right) = (-1)z(0) \ominus (-1)z(t). \tag{29}$$

Proof. In the case of z is γ_1-fuzzy ABC fractional derivative and in the zone $\forall t \in \mathbb{D}$, $\forall\, \eta \in \Sigma$, and $\forall \gamma \in (0,1]$, one can sighted phase (i) as

$$\begin{aligned}
\left[{}^{AB}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1} z(t)\right)\right]^\eta &= \left[{}^{ABC}_{0}\mathcal{J}^{\gamma}\left({}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t)\right), {}^{ABC}_{0}\mathcal{J}^{\gamma}\left({}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2\eta}(t)\right)\right] \\
&= \left[z_{1\eta}(t) - z_{1\eta}(0), z_{2\eta}(t) - z_{2\eta}(0)\right] \\
&= \left[z_{1\eta}(t), z_{2\eta}(t)\right] \ominus \left[z_{1\eta}(0), z_{2\eta}(0)\right] \\
&= [z(t)]^\eta \ominus [z(0)]^\eta.
\end{aligned} \tag{30}$$

Synonymously, if z is γ_2-fuzzy ABC fractional derivative, then phase (ii) can be sighted as

$$\begin{aligned}
\left[{}^{ABC}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}_t^{\gamma_2} z(t)\right)\right]^\eta &= \left[{}^{ABC}_{0}\mathcal{J}^{\gamma}\left({}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{2r}(t)\right), {}^{ABC}_{0}\mathcal{J}^{\gamma}\left({}^{ABC}_{0}\mathcal{D}_t^{\gamma} z_{1\eta}(t)\right)\right] \\
&= \left[z_{2\eta}(t) - z_{2\eta}(0), z_{1r}(t) - z_{1\eta}(0)\right]
\end{aligned} \tag{31}$$

$$= [-z_{2\eta}(0), -z_{1\eta}(0)] \ominus [-z_{2\eta}(t), -z_{1\eta}(t)]$$
$$= -[z_{1\eta}(0), z_{2\eta}(0)] \ominus (-[z_{1\eta}(t), z_{2\eta}(t)])$$
$$= -[z(0)]^\eta \ominus (-[z(t)]^\eta).$$

By neglecting η-values -since it is arbitrary- we completely be sighted the two phases. Note here that, phase (ii) can be expressed likewise $z(t) = z(0) \ominus (-1)\, {}^{AB}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}^{\gamma_2}_{t} z(t)\right)$ or $z(0) = z(t) - {}^{AB}_{0}\mathfrak{J}^{\gamma}\left({}^{ABC}_{0}\mathfrak{D}^{\gamma_2}_{t} z(t)\right)$. ■

5 Fuzzy ABC FIDEs

The formulation of FFIDEs by employed fuzzy characterization theorem and strongly fuzzy ABC fractional derivative is very important task in utilizing the numerical methods in the fuzzy emotion. Next, characterization based and theoretical results related to γ_1- and γ_2-fuzzy ABC fractional solutions are derivable and inferred.

Let us primarily theorizing the main fuzzy structure of our ABC FFIDE of the form

$$\begin{cases} {}^{ABC}_{0}\mathfrak{D}^{\gamma}_{t} z(t) = f(t, z(t)) + \int_0^t k(t,\tau) g(z(\tau)) d\tau, \\ z(0) = z_0. \end{cases} \tag{32}$$

Mainly, the η-cut extrapolation of $\left(z(t), {}^{ABC}_{0}\mathcal{D}^{\gamma}_{t} z(t), f(t, z(t)), g(z(\tau))\right)$ should be acquired, whilst the most important idioms are

$$\begin{cases} [f(t, z(t))]^{\gamma} = \left[f_{1r}\left(t, z_{1\eta}(t), z_{2\eta}(t)\right), f_{2r}\left(t, z_{1\eta}(t), z_{2\eta}(t)\right)\right], \\ [g(z(\tau))]^{\gamma} = \left[g_{1r}\left(z_{1\eta}(\tau), z_{2\eta}(\tau)\right), g_{2r}\left(z_{1\eta}(\tau), z_{2\eta}(\tau)\right)\right]. \end{cases} \tag{33}$$

To deal with ABC FFIVP in realistic approach, one can find the underlying coupled crisp systems of FIDEs relate to Eq. (32) in the ABC approach performed by stratifying γ_1- or γ_2-fuzzy ABC fractional derivative, simultaneously, and taking the η-cut extrapolation for both its sides as follows:

- System of γ_1-crisp ABC FIDE:

$$\begin{cases} {}^{ABC}_{0}\mathfrak{D}^{\gamma_1}_{t} z(t) = f(t, x(t)) + \int_0^t k(t,\tau) g(z(\tau)) d\tau, \\ z(0) = z_0. \end{cases} \tag{34}$$

- System of $\alpha(2)$-crisp ABC FDE:

$$\begin{cases} {}^{ABC}_{0}\mathfrak{D}^{\gamma_2}_{t} z(t) = f(t, x(t)) + \int_0^t k(t,\tau) g(z(\tau)) d\tau, \\ z(0) = z_0. \end{cases} \tag{35}$$

Definition 11 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z, {}^{ABC}_{0}\mathfrak{D}^{\alpha}_{t} z \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, and $\gamma \in (0,1]$. Assume that z be such that ${}^{ABC}_{0}\mathfrak{D}^{\gamma(1)}_{t} z(t)$ or ${}^{ABC}_{0}\mathfrak{D}^{\gamma(1)}_{t} z(t)$ exists. Then the following underlying is achieved:

i. If $z(t)$ and ${}^{ABC}_{0}\mathfrak{D}^{\gamma(1)}_{t} z(t)$ satisfy Eq. (34), then $z(t)$ is said a γ_1-fuzzy ABC solution of Eq. (32).
ii. If $z(t)$ and ${}^{ABC}_{0}\mathfrak{D}^{\gamma(1)}_{t} z(t)$ satisfy Eq. (35), then $z(t)$ is said a γ_2-fuzzy ABC solution of Eq. (32).

Theorem 7 Let $z: \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, $z \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}) \cap \mathcal{L}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D})$, $f \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D} \times \mathbb{R}_{\mathcal{F}})$, $g \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{R}_{\mathcal{F}})$, $k \in \mathcal{C}^{\mathbb{R}}(\mathbb{D} \times \mathbb{D})$, and $\gamma \in (0,1]$. Then fuzzy ABC FFIDE of Eq. (32) is equivalent to one of the underlying fuzzy ABC fractional integral equations:

i. If z is γ_1-fuzzy ABC fractional derivative, then

$$\begin{aligned} z(t) = z_0 &+ \frac{1-\gamma}{\Lambda(\gamma)} f(t, z(t)) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t f(s, z(s))(t-s)^{\gamma-1} ds + \frac{1-\gamma}{\Lambda(\gamma)} \int_0^t k(t,\tau) g(z(\tau)) d\tau \\ &+ \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t \left(\int_0^s k(s,\tau) g(z(\tau))(t-s)^{\gamma-1} d\tau \right) ds \end{aligned} \tag{36}$$

ii. If z is γ_2-fuzzy ABC fractional derivative, then

$$\begin{aligned} z(t) = z_0 \ominus (-1) &\left(\frac{1-\gamma}{\Lambda(\gamma)} f(t, z(t)) + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t f(s, z(s))(t-s)^{\gamma-1} ds \right. \\ &\left. + \frac{1-\gamma}{\Lambda(\gamma)} \int_0^t k(t,\tau) g(z(\tau)) d\tau + \frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)} \int_0^t \left(\int_0^s k(s,\tau) g(z(\tau))(t-s)^{\gamma-1} d\tau \right) ds \right). \end{aligned} \tag{37}$$

Proof. Because $f \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D} \times \mathbb{R}_{\mathcal{F}})$, $g \in \mathcal{C}^{\mathbb{R}_{\mathcal{F}}}(\mathbb{R}_{\mathcal{F}})$, and $k \in \mathcal{C}^{\mathbb{R}}(\mathbb{D} \times \mathbb{D})$, then they are integrable. For phase (i), one can implement fuzzy integration and fuzzy inversion formula in Eq. (28) to gain the following:

$$z(t)=z(0)+\frac{1-\gamma}{\Lambda(\gamma)}f\big(t,z(t)\big)+\frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)}\int_0^t f\big(s,z(s)\big)(t-s)^{\gamma-1}ds+\frac{1-\gamma}{\Lambda(\gamma)}\int_0^t k(t,\tau)g\big(z(\tau)\big)d\tau$$
$$+\frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)}\int_0^t\left(\int_0^s k(s,\tau)g\big(z(\tau)\big)(t-s)^{\gamma-1}d\tau\right)ds. \tag{38}$$

For phase (ii), one can implement fuzzy integration and fuzzy inversion formula in Eq. (29) to get the following:

$$z(0)=z(t)\ominus(-1)\left(\frac{1-\gamma}{\Lambda(\gamma)}f\big(t,z(t)\big)+\frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)}\int_0^t f\big(s,z(s)\big)(t-s)^{\gamma-1}ds\right.$$
$$\left.+\frac{1-\gamma}{\Lambda(\gamma)}\int_0^t k(t,\tau)g\big(z(\tau)\big)d\tau+\frac{\gamma}{\Lambda(\gamma)\Gamma(\gamma)}\int_0^t\left(\int_0^s k(s,\tau)g\big(z(\tau)\big)(t-s)^{\gamma-1}d\tau\right)ds\right). \blacksquare \tag{39}$$

Theorem 8 Set $h\big(t,\tau,z(\tau)\big)=k(t,\tau)g\big(z(\tau)\big)$. Assume that $f\in C^{\mathbb{R}_{\mathcal{F}}}(\mathbb{D}\times\mathbb{R}_{\mathcal{F}})$, $g\in C^{\mathbb{R}_{\mathcal{F}}}(\mathbb{R}_{\mathcal{F}})$, and $k\in C^{\mathbb{R}}(\mathbb{D}\times\mathbb{D})$. If $\exists\mathcal{K}_1,\mathcal{K}_2>0$ in the zone $\forall t_{1,2},\tau_{1,2}\in\mathbb{D}$, $\forall\eta\in\Sigma$, and $z(t),w(t)\in\mathbb{R}_{\mathcal{F}}$ such that

$$\left|f_{(1,2)\eta}\left(t_1,z_{1\eta}(t_1),z_{2\eta}(t_1)\right)-f_{(1,2)\eta}\left(t_2,w_{1\eta}(t_2),w_{2\eta}(t_2)\right)\right|$$
$$\le\mathcal{K}_1\max\{|t_2-t_1|,|z_{1\eta}(t_1)-w_{1\eta}(t_2)|,|z_{2\eta}(t_1)-w_{2\eta}(t_2)|\},$$
$$\left|g_{(1,2)\eta}\left(t_1,\tau_1,z_{1\eta}(\tau_1),z_{2\eta}(\tau_1)\right)-g_{(1,2)\eta}\left(t_2,\tau_2,w_{1\eta}(\tau_2),w_{2\eta}(\tau_2)\right)\right|$$
$$\le\mathcal{K}_2\max\{|t_1-t_2|,|\tau_1-\tau_2|,|z_{1\eta}(\tau_1)-w_{1\eta}(\tau_2)|,|z_{2\eta}(\tau_1)-w_{2\eta}(\tau_2)|\}. \tag{40}$$

Then, the following underlying is achieved:

i. For γ_1-fuzzy ABC fractional derivative; the ABC FFIDE of Eq. (32) and the system of γ_1-crisp ABC FDIE of Eq. (34) are equivalent.
ii. For γ_1-fuzzy ABC fractional derivative; the ABC FDE of Eq. (32) and the system of γ_1-crisp ABC FDE of Eq. (35) are equivalent.

Proof. Similar to proof of Theorem 8 in [6], taking into account the modification in the integral sign of Eq. (32). ■

6 Essential Tools in the RKA

With the growth of technique and science, many phenomena cannot be well utilized by fuzzy integrodifferential problems. For example, various uncertain physical processes own memory and hereditary ownerships and cannot be well drawn unless if one used FFIDEs. Anyhow, when we have faced these challenges; we must build up new excellent numerical tool. Hither, the RKA is presented hither as a version solver for ABC FFIDEs.

Firstly, we set the following subordinate important determinants: $[z(t)]_\eta=\big(z_{1\eta}(t),z_{2\eta}(t)\big)$, $[w(t)]_\eta=\big(w_{1\eta}(t),w_{2\eta}(t)\big)$, While remembering that $[z(t)]^\eta=\big[z_{1\eta}(t),z_{2\eta}(t)\big]$ and $[w(t)]^\eta=\big[w_{1\eta}(t),w_{2\eta}(t)\big]$ as formerly discussed. A mapping $\mathbb{P}\in C^{\mathbb{R}}(\mathbb{D}\times\mathbb{D})$ is a reproducing kernel of $\mathbb{K}\neq\emptyset$ if the following underlying requirements are satisfied, wheresoever $\mathbb{H}$ is a Hilbert space of functions defined on $\mathbb{D}$:

i. $\forall t\in\mathbb{D}$: $\mathbb{P}(\cdot,t)\in\mathbb{H}$,
ii. $\forall\psi\in\mathbb{H}$ and $\forall t\in\mathbb{D}$: $\langle\psi(\cdot),\mathbb{P}(\cdot,t)\rangle_{\mathbb{H}}=\psi(t)$.

Definition 12 [6] The space $\mathcal{A}(\mathbb{D})$ is fully construct as

$$\begin{cases}\mathcal{A}(\mathbb{D})=\{[z(t)]_\eta^T: z_{(1,2)\eta}\in|\mathcal{C}|^{\mathbb{R}}(\mathbb{D}), z''_{(1,2)\eta}\in L^2(\mathbb{D}), \text{and } z_{(1,2)\eta}(0)=0\},\\ \langle[z(t)]_\eta^T,[w(t)]_\eta^T\rangle_{\mathcal{A}}=\displaystyle\sum_{i=1}^2\left(z_{i\eta}(0)w_{i\eta}(0)+z'_{i\eta}(0)w'_{i\eta}(0)+\int_{\mathbb{D}}z''_{i\eta}(t)w''_{i\eta}(t)dt\right),\\ \left\|[z]_\eta^T\right\|_{\mathcal{A}}=\sqrt{\langle[z(t)]_\eta,[z(t)]_\eta\rangle_{\mathcal{A}}}.\end{cases} \tag{41}$$

Definition 13 [6] The space $\mathcal{B}(\mathbb{D})$ is fully construct as $[z(t)]_\eta^T$

$$\begin{cases}\mathcal{B}(\mathbb{D})=\{[z(t)]_\eta: z_{(1,2)\eta}\in|\mathcal{C}|^{\mathbb{R}}(\mathcal{T}), z'_{(1,2)\eta}\in L^2(\mathbb{D})\},\\ \langle[z(t)]_\eta^T,[w(t)]_\eta^T\rangle_{\mathcal{B}}=\displaystyle\sum_{i=1}^2 z_{i\eta}(0)w_{i\eta}(0)+\int_{\mathbb{D}}z'_{i\eta}(t)w'_{i\eta}(t)dt,\\ \left\|[z]_\eta^T\right\|_{\mathbb{V}}=\sqrt{\langle[z(t)]_\eta,[z(t)]_\eta\rangle_{\mathcal{B}}}.\end{cases} \tag{42}$$

Theorem 9 [6] The space $\mathcal{A}(\mathbb{D})$ is a complete reproducing kernel with $\bar{\mathcal{P}}_t(s)=\big(\mathcal{P}_t(s),\mathcal{P}_t(s)\big)$ and

$$\mathcal{P}_t(s) = \frac{1}{6}\begin{cases} s(-s^2 + 6t + 3ts), s \le t, \\ t(-t^2 + 6s + 3ts), s > t. \end{cases} \tag{43}$$

Theorem 10 [6] The space $\mathcal{B}(\mathbb{D})$ is a complete reproducing kernel with $\bar{\mathcal{Q}}_t(s) = \big(\mathcal{Q}_t(s), \mathcal{Q}_t(s)\big)$ and

$$\mathcal{Q}_t(s) = \begin{cases} 1 + t, & s \le t, \\ 1 + s, & s > t, \end{cases} \tag{44}$$

When the RKA is used, we must partitioned the interval $\mathbb{D}$ into uniform pieces of t_i. This will be gained the set $\{t_i\}_{i=1}^{\infty}$ which be dense in $\mathbb{D}$. Anyhow, we seek to cover the set $\mathbb{D}$ as well as the approximation procedure should finish up in finite phases.

Theorem 11 The image set of $\mathcal{P} \in \mathcal{C}^{\mathbb{R}}(\mathbb{D} \times \mathbb{D})$ and $\mathcal{Q} \in \mathcal{C}^{\mathbb{R}}(\mathbb{D} \times \mathbb{D})$ are [0,1.5] and [1,2], simultaneously.

Proof. Using benefit continuity of $\mathcal{P}$ and $\mathcal{Q}$ together with replacing of variables s and t by their corresponding intervals images, one has

$$\begin{aligned} \mathcal{P}_{\mathbb{D}}(\mathbb{D}) &= \frac{1}{6}[0,1](-[0,1]^2 + 6[0,1] + 3[0,1][0,1]) \\ &= \frac{1}{6}[0,1]([-1,0] + [0,6] + [0,3]) \\ &= \frac{1}{6}[0,1][-1,9] \\ &= \frac{1}{6}[0,9] \\ &= [0,1.5]. \end{aligned} \tag{45}$$

Similarly, one has $\mathcal{Q}_{\mathbb{D}}(\mathbb{D}) = [1,2]$. ■

Theorem 12 In $\mathcal{A}(\mathbb{D})$; the set $\left\{\bar{\mathcal{P}}_{t_i}(s)\right\}_{i=1}^{\infty}$ is linearly independent.

Proof. If $\{\rho_i\}_{i=1}^{h}$ is picked as $\sum_{i=1}^{h} \rho_i \bar{\mathcal{P}}_{t_i}(s) = 0$ and choosing $r_k(s) \in \mathcal{A}(\mathbb{D})$ with $r_k(s_l) = \delta_{l,k}$ at $l = 1,2,\cdots,h$, then

$$\begin{aligned} 0 &= \left\langle r_k(s), \sum_{i=1}^{h} \sigma_i \bar{\mathcal{P}}_{t_i}(s) \right\rangle_{\mathcal{A}} \\ &= \sum_{i=1}^{h} \rho_i \left\langle r_k(s), \bar{\mathcal{P}}_{t_i}(s) \right\rangle_{\mathcal{A}} \\ &= \sum_{i=1}^{h} \rho_i r_k(s_i) \\ &= \rho_i, \end{aligned} \tag{46}$$

where $k = 1,2,\cdots,h$. In fact, this validate that $\left\{\bar{\mathcal{P}}_{t_i}(s)\right\}_{i=1}^{h}$ is linearly independent $\forall h \ge 1$. ■

7 Figuration of Fuzzy ABC FIDEs

The fuzzy ABC FIDE problem formalism, homogenized fuzzy initial condition, constructed of fractional ABC differential operator, orthogonal function system, completeness, exemplification of fuzzy ABC analytic solution in the adequate Hilpert spaces $\mathcal{A}(\mathbb{D})$ and $\mathcal{B}(\mathbb{D})$ are the main significations of the following subordinate part.

Prior to harvest more, we assume without the loss of commonality that $k(t,\tau) \ge 0$ on $0 \le \tau \le a$ and $k(t,\tau) \le 0$ on $a \le \tau \le t$ in ABC FFIDE of Eq. (32). So, the main part of Eq. (32) can be elicited to the following:

$$ {}^{ABC}_{0}\mathcal{D}_t^{\gamma} z(t) = f\big(t, z(t)\big) + \int_0^a k(t,\tau) g\big(z(\tau)\big) d\tau + \int_a^t k(t,\tau) g\big(z(\tau)\big) d\tau. \tag{47}$$

In the following execution, we just theorize γ_1-fuzzy ABC solution exclusively (similar execution can be utilized for γ_2-fuzzy ABC solution). Before continuance more, we must homogenized fuzzy initial condition in Eq. (14) to be in $\mathcal{A}(\mathbb{D})$ as the surrogate $z(t): \rightarrow z(t) \ominus z_0$. Eventually, we will still use $z(t)$ as

$$\begin{cases} {}^{ABC}_{0}\mathcal{D}_t^{\gamma_1} z(t) = f\big(t, z(t)\big) + \int_0^a k(t,\tau) g\big(z(\tau)\big) d\tau + \int_a^t k(t,\tau) g\big(z(\tau)\big) d\tau, \\ z(0) = 0. \end{cases} \tag{48}$$

For more theorized simplification, we put and denote for the following tantamount

$$\begin{cases} \mathcal{U}z(t) := \int_0^a \mathscr{k}(t,\tau)\mathscr{g}\big(z(\tau)\big)d\tau, \\ \mathcal{V}z(t) := \int_a^t \mathscr{k}(t,\tau)\mathscr{g}\big(z(\tau)\big)d\tau, \\ \mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big) \coloneqq \mathscr{f}\big(t, z(t)\big) + \mathcal{U}z(t) + \mathcal{V}z(t). \end{cases} \tag{49}$$

Thus, amalgamate theorized form of Eqs. (48) and (49), one can get new figuration as

$$\begin{cases} {}^{ABC}_{\;\;0}\mathcal{D}_t^{\gamma_1} z(t) = \mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big), \\ z(0) = 0. \end{cases} \tag{50}$$

To processed more, locate the following underlying fractional ABC differential operator:

$$\begin{cases} \mathcal{F}: \mathcal{A}(\mathbb{D}) \to \mathcal{B}(\mathbb{D}), \\ \mathcal{F}z(t) = {}^{ABC}_{\;\;0}\mathfrak{D}_t^{\gamma_1} z(t). \end{cases} \tag{51}$$

Subsequently, Eq. (48) can be turn into the following tantamount form:

$$\begin{cases} \mathcal{F}z(t) = \mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big), \\ z(0) = 0. \end{cases} \tag{52}$$

Hither, we will designate $[\mathcal{F}z(t)]_\eta = \left[{}^{ABC}_{\;\;0}\mathfrak{D}_t^{\gamma_1} z(t)\right]_\eta$ which intends that $\mathcal{F}_1 z_{1\eta}(t) = {}^{ABC}_{\;\;0}\mathcal{D}_t^{\gamma_1} z_{1r}(t)$ and $\mathcal{F}_2 z_{2r}(t) = {}^{ABC}_{\;\;0}\mathcal{D}_t^{\gamma_1} z_{2r}(t)$. Next, we will assign system of orthogonal functions using the following subordinate junctures: put $\mathfrak{T}_{ij}(t) = \mathcal{Q}_{t_i}(t)\mathbb{e}_j$ and $\Pi_{ij}(t) = \Pi^*\mathfrak{T}_{ij}(t)$ at $i = 1,2,3,\ldots$ and $j = 1,2$, where $\mathbb{e}_1 = (1,0)^T$ and $\mathbb{e}_2 = (0,1)^T$. Hither, $\mathcal{F}^* = \operatorname{diag}(\mathcal{F}_1^*, \mathcal{F}_2^*)$ and $\{t_i\}_{i=1}^\infty$ is dense on $\mathbb{D}$. Notice here that, the system of orthonormal functions $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $\mathcal{A}(\mathbb{D})$ can be formulated as follows:

$$\overline{\Pi}_{ij}(t) = \sum_{l=1}^{i}\sum_{k=1}^{j} \mathscr{p}_{lk}^{ij}\,\Pi_{lk}(t), i = 1,2,3,\ldots, j = 1,2, \tag{53}$$

where $\mathscr{p}_{lk}^{ij}$ are orthogonalization coefficients of $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Theorem 13 The orthonormal system $\{\Pi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is complete with $\Pi_{ij}(t) = \mathcal{F}_s\mathcal{P}_t(s)|_{s=t_i}$.

Proof. If $\langle [z(t)]_\eta^T, \Pi_{ij}(t)\rangle_{\mathcal{A}} = 0$ at $i = 1,2,\ldots, j = 1,2$, then

$$\begin{aligned} \langle [z(t)]_\eta^T, \Pi_{ij}(t)\rangle_{\mathcal{A}} &= \langle [z(t)]_\eta^T, \Pi^*\mathfrak{T}_{ij}(t)\rangle_{\mathcal{A}} \\ &= \langle \Pi[z(t)]_\eta^T, \mathfrak{T}_{ij}(t)\rangle_{\mathcal{B}} \\ &= \Pi(t_i) = 0. \end{aligned} \tag{54}$$

But, $[z(t)]_r^T = \sum_{j=1}^2 z_{jr}(t)\mathbb{e}_j = \sum_{j=1}^2 \langle [z(\cdot)]_\eta^T, \mathcal{P}_t(\cdot)\mathbb{e}_j\rangle_{\mathcal{A}}\mathbb{e}_j$ and $\mathcal{F}[z(t)]_r^T = \sum_{j=1}^2 \langle \mathcal{F}[z(t)]_\eta^T, \mathfrak{T}_{ij}(t)\rangle_{\mathcal{A}}\mathbb{e}_j = 0$. By the density of $\{t_i\}_{i=1}^\infty$ on $\mathbb{D}$, we gained $\mathcal{F}[z(t)]_\eta^T = 0$. Through the existence of $\mathcal{F}^{-1}$, produces that $[z(t)]_\eta^T = 0$. Posteriorly, $\{\Pi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is a complete on $\mathcal{A}(\mathbb{D})$. Againe, visibly one has

$$\begin{aligned} \Pi_{ij}(t) &= \mathcal{F}^*\mathfrak{T}_{ij}(t) \\ &= \langle \mathcal{F}^*\mathfrak{T}_{ij}(s), \mathcal{P}_t(s)\rangle_{\mathcal{A}} \\ &= \langle \mathfrak{T}_{ij}(s), \mathcal{F}_s\mathcal{P}_t(s)\rangle_{\mathcal{B}} \\ &= \mathcal{F}_s\mathcal{P}_t(s)|_{s=t_i}. \blacksquare \end{aligned} \tag{55}$$

The readers should distinguish in between $\left[\mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big)\right]^\eta$ and $\left[\mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big)\right]_\eta$, which will be establish and employ in following analysis. Anyhow, one can notice that

$$\begin{aligned} &\left[\mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big)\right]_\eta \\ &\qquad = \Big(\mathfrak{P}_{1\eta}\big(t, [z(t)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\big), \mathfrak{P}_{2\eta}\big(t, [z(t)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\big)\Big), \\ &\left[\mathfrak{P}\big(t, z(t), \mathcal{U}z(t), \mathcal{V}z(t)\big)\right]^\eta \\ &\qquad = \left[\mathfrak{P}_{1\eta}\big(t, [z(t)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\big), \mathfrak{P}_{2\eta}\big(t, [z(t)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\big)\right]. \end{aligned} \tag{56}$$

Theorem 14 Let $\mathscr{p}_{lk}^{ij}$ are orthogonalization coefficients for $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$. Then the analytic solution of Eq. (52) fulfill well

$$[z(t)]_\eta^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \wp_{lk}^{ij}\mathfrak{P}_{k\eta}\left(t_l, [z(t_l)]_\eta^T, [\mathcal{U}z(t_l)]_\eta^T, [\mathcal{V}z(t_l)]_\eta^T\right)\overline{\Pi}_{ij}(t). \tag{57}$$

Proof. Because $\left\langle [z(t)]_\eta^T, \mathfrak{T}_{ij}(t)\right\rangle_{\mathcal{A}} = z_{j\eta}(t_i)$ and $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}}\overline{\Pi}_{ij}(t)$ is the Fourier series of $\left\{\overline{\Pi}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$. Then the mentioned series is convergent in the feeling of $\|\cdot\|_{\mathcal{A}}$. From here,

$$\begin{aligned}
[z(t)]_\eta^T &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}}\overline{\Pi}_{ij}(t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\Pi_{lk}(t)\right\rangle_{\mathcal{A}}\overline{\Pi}_{ij}(t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\left\langle [z(t)]_\eta^T, \mathcal{F}^*\mathfrak{T}_{lk}(t)\right\rangle_{\mathcal{A}}\overline{\Pi}_{ij}(t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\left\langle \mathcal{F}[z(t)]_\eta^T, \mathfrak{T}_{lk}(t)\right\rangle_{\mathcal{B}}\overline{\Pi}_{ij}(t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\left\langle \mathfrak{P}_{k\eta}\left(t, [z(t)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\right), \mathfrak{T}_{lk}(t)\right\rangle_{\mathcal{B}}\overline{\Pi}_{ij}(t) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\mathfrak{P}_{k\eta}\left(t_l, [z(t_l)]_\eta^T, [\mathcal{U}z(t)]_\eta^T, [\mathcal{V}z(t)]_\eta^T\right)\overline{\Pi}_{ij}(t)\,. \blacksquare
\end{aligned} \tag{58}$$

8 Convergence and Error Frameworks

In order to more control on fuzzy ABC numerical solution obtained from the RKA and to know more results on the behavior solution and its limits; we ought studing its convergence and error frameworks. These concepts will discuss and derive in details in the following part.

Principally, for the numerical calculations on software package used, we sholud amputated the expresion in Eq. (57) to engendering the n- idiom crisp ABC numerical solution of $[z(t)]_\eta^T$ as

$$[z^n(t)]_\eta^T = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \wp_{lk}^{ij}\mathfrak{P}_{k\eta}\left(t_l, [z(t_l)]_\eta^T, [\mathcal{U}z(t_l)]_\eta^T, [\mathcal{V}z(t_l)]_\eta^T\right)\overline{\Pi}_{ij}(t). \tag{59}$$

Hither, we assume the following subordinate: $\left\|[z^{n-1}]_\eta^T\right\|_{\mathcal{A}}$ is bounded as $n \to \infty$, $\{t_i\}_{i=1}^{\infty}$ is dense on $\mathbb{D}$, and solution of Eq. (52) unique and exists in $\mathcal{A}(\mathbb{D})$. Now, we will show the convergence of $[z(t)]_\eta^T$ in $\mathbb{D}$ over $\mathcal{A}(\mathbb{D})$.

Theorem 15 Let $\left[\not{f}\left(t, [z(t)]_\eta^T\right)\right]_\eta \in \mathcal{C}^{\mathbb{R}}(\mathbb{D}\times\mathbb{R}\times\mathbb{R})$ and $\left[\mathscr{g}\left([z(t)]_\eta^T\right)\right]_\eta \in \mathcal{C}^{\mathbb{R}}(\mathbb{R}\times\mathbb{R})$. If $\left\|[z^{n-1}]_\eta^T - [z]_\eta^T\right\|_{\mathcal{A}} \to 0$ and $t_n \to s$ as $n \to \infty$, then as $n \to \infty$, one can get

$$\left[\mathfrak{P}\left(t_n, [z^{n-1}(t_n)]_\eta^T, [\mathcal{U}z^{n-1}(t_n)]_\eta^T, [\mathcal{V}z^{n-1}(t_n)]_\eta^T\right)\right]_\eta \to \left[\mathfrak{P}\left(s, [z^{n-1}(s)]_\eta^T, [\mathcal{U}z^{n-1}(s)]_\eta^T, [\mathcal{V}z^{n-1}(s)]_\eta^T\right)\right]_\eta. \tag{60}$$

Proof. Firstly, we have to start by proving that $[z^{n-1}(t_n)]_\eta^T \to [z(s)]_\eta^T$. Since,

$$\begin{aligned}
\left|[z^{n-1}(t_n)]_\eta^T - [z(s)]_\eta^T\right| &= \left|[z^{n-1}(t_n)]_\eta^T - [z^{n-1}(s)]_\eta^T + [z^{n-1}(s)]_\eta^T - [z(s)]_\eta^T\right| \\
&\le \left|[z^{n-1}(t_n)]_\eta^T - [z^{n-1}(s)]_\eta^T\right| + \left|[z^{n-1}(s)]_\eta^T - [z(s)]_\eta^T\right| \\
&\le \left|\left([z^{n-1}(\tau)]_\eta^T\right)'\right||t_n - s| + \left|[z^{n-1}(s)]_\eta^T - [z(s)]_\eta^T\right|,
\end{aligned} \tag{61}$$

where $\tau \in (\min\{t_n, s\}, \max\{t_n, s\})$. So, $\left|[z^{n-1}(t_n)]_\eta^T - [z(s)]_\eta^T\right| \to 0$ as $n \to \infty$. By the fact that $\left[\mathscr{g}\left([z(t)]_\eta^T\right)\right]_\eta \in \mathcal{C}^{\mathbb{R}}(\mathbb{R}\times\mathbb{R})$ one has $[\mathcal{U}z(t_l)]_\eta^T, [\mathcal{V}z(t_l)]_\eta^T \in \mathcal{C}^{\mathbb{R}}(\mathbb{R}\times\mathbb{R})$. Using too $\left[\not{f}\left(t, [z(t)]_\eta^T\right)\right]_\eta \in \mathcal{C}^{\mathbb{R}}(\mathbb{D}\times\mathbb{R}\times\mathbb{R})$, one get the demanded requirements. $\blacksquare$

Next, we stand for $\mathfrak{R}_{(n,j)\eta} = \sum_{l=1}^{n}\sum_{k=1}^{j}\wp_{lk}^{ij}\mathfrak{P}_{k\eta}\left(t_l, [z(t_l)]_\eta^T, [\mathcal{U}z(t_l)]_\eta^T, [\mathcal{V}z(t_l)]_\eta^T\right)$. De facto, this authorize one to put $[z^n(t)]_\eta^T$ as

$$[z^n(t)]_\eta^T = \sum_{i=1}^{n}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}\, \overline{\Pi}_{ij}(t). \tag{62}$$

Theorem 16 In the monotonous recipe of Eqs. (62), one has $[z^n(t)]_\eta^T \to [z(t)]_\eta^T$ as $n \to \infty$.

Proof. From Eq. (62), one infer that $[z^{n+1}(t)]_\eta^T = [z^n(t)]_\eta^T + \sum_{j=1}^{2} \mathfrak{R}_{(n+1,j)\eta}\, \overline{\Pi}_{(n+1)j}(t)$. By the orthogonality of $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$, one win

$$\begin{aligned}
\left\|[z^{n+1}]_\eta^T\right\|_{\mathcal{A}}^2 &= \left\|[z^{n}]_\eta^T\right\|_{\mathcal{A}}^2 + \sum_{j=1}^{2} \mathfrak{R}_{(n+1,j)\eta}^2 \\
&= \left\|[z^{n-1}]_\eta^T\right\|_{\mathcal{A}}^2 + \sum_{j=1}^{2} \mathfrak{R}_{(n,j)\eta}^2 + \sum_{j=1}^{2} \mathfrak{R}_{(n+1,j)\eta}^2 \\
&= \cdots \\
&= \left\|[z^{0}]_\eta^T\right\|_{\mathcal{A}}^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}^2.
\end{aligned} \tag{63}$$

Thereafter, $\left\|[z^{n+1}]_\eta^T\right\|_{\mathcal{A}} \ge \left\|[z^{n}]_\eta^T\right\|_{\mathcal{A}}$ and $\exists \mathbb{c} \in \mathbb{R}$ with $\sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}^2 = \mathbb{c}$, which reap that $\left\{\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}^2\right\}_{i=1}^{\infty} \in l^2$. Over and above,

$$[z^m(t)]_\eta^T - [z^{m-1}(t)]_\eta^T \perp [z^{m-1}(t)]_\eta^T - [z^{m-2}(t)]_\eta^T \perp \cdots \perp [z^{n+1}(t)]_\eta^T - [z^{n}(t)]_\eta^T, \tag{64}$$

it pull off for $m > n$ that

$$\begin{aligned}
\left\|[z^m]_\eta^T - [z^n]_\eta^T\right\|_{\mathcal{A}}^2 &= \left\|[z^m]_\eta^T - [z^{m-1}]_\eta^T + [z^{m-1}]_\eta^T - \cdots + [z^{n+1}]_\eta^T - [z^n]_\eta^T\right\|_{\mathcal{A}}^2 \\
&= \left\|[z^m]_\eta^T - [z^{m-1}]_\eta^T\right\|_{\mathcal{A}}^2 + \left\|[z^{m-1}]_\eta^T - [z^{m-2}]_\eta^T\right\|_{\mathcal{A}}^2 + \ldots + \left\|[z^{n+1}]_\eta^T - [z^n]_\eta^T\right\|_{\mathcal{A}}^2.
\end{aligned} \tag{65}$$

While, $\left\|[z^m]_\eta^T - [z^{m-1}]_\eta^T\right\|_{\mathcal{A}}^2 = \sum_{j=1}^{2} \mathfrak{R}_{(m,j)\eta}^2$. So, as $n, m \to \infty$, we get $\left\|[z^m]_\eta^T - [z^n]_\eta^T\right\|_{\mathcal{A}}^2 = \sum_{l=n+1}^{m}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}^2 \to 0$. As the completeness, $\exists [z^n(t)]_\eta^T \in \mathcal{A}(\mathbb{D})$ with $[z^n(t)]_\eta^T \to [z(t)]_\eta^T$ as $n \to \infty$ in the feeling of $\|\cdot\|_{\mathcal{A}}$. ■

Theorem 17 In the monotonous recipe of Eqs. (62), one has $[z(t)]_\eta^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}\, \overline{\Pi}_{ij}(t)$ as $n \to \infty$.

Proof. By taking $\lim\limits_{n\to\infty}(\cdot)$ on two sides of Eq. (62), one has $[z(t)]_\eta^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \overline{\Pi}_{ij}(t)$. While $\mathcal{F}[z(t)]_\eta^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta}\, \mathcal{F}\overline{\Pi}_{ij}(t)$, So

$$\begin{aligned}
\mathcal{F}_k[z(t_l)]_\eta^T &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \left\langle \mathcal{F}\overline{\Pi}_{ij}(t), \mathfrak{T}_{lk}(t)\right\rangle_{\mathcal{A}} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \left\langle \overline{\Pi}_{ij}(t), \mathcal{F}^*\mathfrak{T}_{lk}(t)\right\rangle_{\mathcal{A}} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \left\langle \overline{\Pi}_{ij}(t), \Pi_{lk}(t)\right\rangle_{\mathcal{A}}.
\end{aligned} \tag{66}$$

$$\begin{aligned}
\sum_{l'=1}^{l}\sum_{k'=1}^{k} \wp_{l'k'}^{lk} \mathcal{F}_{k'}[z(t)]_\eta^T(t_{l'}) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \left\langle \overline{\Pi}_{ij}(t), \sum_{l'=1}^{l}\sum_{k'=1}^{k} \wp_{l'k'}^{lk} \Pi_{l'k'}(t)\right\rangle_{\mathcal{A}} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathfrak{R}_{(i,j)\eta} \left\langle \overline{\Pi}_{ij}(t), \overline{\Pi}_{l'k'}(t)\right\rangle_{\mathcal{A}} \\
&= \mathfrak{R}_{(l,k)\eta}.
\end{aligned} \tag{67}$$

Vindicatory, if $l = 1$, then $\mathcal{F}_j[z(t_1)]_\eta^T = \mathfrak{P}_{j\eta}\left(t_1, [z^0(t_1)]_\eta^T, [\mathcal{U}z^0(t_1)]_\eta^T, [\mathcal{V}z^0(t_1)]_\eta^T\right)$. So, one can get $\mathcal{F}[z(t_1)]_\eta^T = \left[\mathfrak{P}\left(t_1, z^0(t_1), [\mathcal{U}z^0(t_1)]_\eta^T, [\mathcal{V}z^0(t_1)]_\eta^T\right)\right]_\eta$. If $l = 2$, then $\mathcal{F}_j[z(t_2)]_\eta^T = \mathfrak{P}_{j\eta}\left(t_2, [z^1(t_2)]_\eta^T, [\mathcal{U}z^1(t_2)]_\eta^T, [\mathcal{V}z^1(t_2)]_\eta^T\right)$. So, one can get $\mathcal{F}[z(t_2)]_\eta^T = \left[\mathfrak{P}\left(t_2, [z^1(t_2)]_\eta^T, [\mathcal{U}z^1(t_2)]_\eta^T, [\mathcal{V}z^1(t_2)]_\eta^T\right)\right]_\eta$. In like trajectory, the shape form can be formulated as $\mathcal{F}[z(t_n)]_\eta^T = \left[\mathfrak{P}\left(t_n, [z^{n-1}(t_n)]_\eta^T, [\mathcal{U}z^{n-1}(t_n)]_\eta^T, [\mathcal{V}z^{n-1}(t_n)]_\eta^T\right)\right]_\eta$. Through the density, $\forall s \in \mathbb{D}$; $\exists \left\{t_{n_q}\right\}_{q=1}^{\infty}$ such that $t_{n_q} \to s$ as $q \to \infty$ or $\mathcal{F}\left[z\left(t_{n_q}\right)\right]_\eta^T = \left[\mathfrak{P}\left(t_{n_q}, \left[z\left(t_{n_q}\right)\right]_\eta^T, \left[\mathcal{U}z^{n_q-1}\left(t_{n_q}\right)\right]_\eta^T, \left[\mathcal{V}z^{n_q-1}\left(t_{n_q}\right)\right]_\eta^T\right)\right]_\eta$. Let $j \to \infty$, by Theorem 14, one has $\mathcal{F}[z(s)]_\eta^T = \left[\mathfrak{P}\left(s, [z(s)]_\eta^T, [\mathcal{U}z(s)]_\eta^T, [\mathcal{V}z(s)]_\eta^T\right)\right]_\eta$. Likewise, since $\overline{\Pi}_{ij}(t) \in \mathcal{A}(\mathcal{T})$, then $[z(t)]_\eta^T$ fulfill Eq. (52). Ultimately, the uniqueness solution of Eq. (52) harness the desired score. ■

Next, we will debate the attitude of errors for large n. In the extreme, we will denote $\mathfrak{R}_n = \left\|[z]_\eta^T - [z^n]_\eta^T\right\|_{\mathcal{A}}$ on $\mathcal{T}$ provided that $[z(t)]_\eta^T$ and $[z^n(t)]_\eta^T$ are taken away from Eqs. (57) and (59), simultaneously.

Theorem 18 $\{\mathfrak{E}_n\}_{n=1}^{\infty}$ is monotone decreasing in $\mathcal{A}(\mathbb{D})$ with $\mathfrak{E}_n \to 0$ as $n \to \infty$.

Proof. Its visible that

$$\begin{aligned}
\mathfrak{E}_n^2 &= \left\|\sum_{i=n+1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}} \overline{\Pi}_{ij}(t)\right\|_{\mathcal{A}}^2 \\
&= \sum_{i=n+1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}}^2 \\
&\le \sum_{i=n}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}}^2 \\
&= \left\|\sum_{i=n}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}} \overline{\Pi}_{ij}(t)\right\|_{\mathcal{A}}^2 \\
&= \mathfrak{E}_{n-1}^2.
\end{aligned} \tag{68}$$

Thus, $\{\mathfrak{E}_n\}_{n=1}^{\infty}$ is monotone decreasing in the feeling of $\|\cdot\|_{\mathcal{A}}$. From Theorem 13 and convergent fact on $\sum_{i=1}^{\infty}\sum_{j=1}^{2}\left\langle [z(t)]_\eta^T, \overline{\Pi}_{ij}(t)\right\rangle_{\mathcal{A}} \overline{\Pi}_{ij}(t)$ one harvest that $\mathfrak{E}_n^2 \to 0$ as $n \to \infty$. ■

9 Computational Algorithms

Hither three computational algorithms are given in order to laying the groundwork for the solutions methods. Anyhow, the first one is how to check the validity of fuzzy ABC analytical solutions; the second one is to generate systems of orthonormal functions; whiles the third one is how to apply the utlized RKA steps.

The inputs requirements for the three algorithms are as pursues, simultaneously: $[z^n(t)]_\eta^T$, $[z(t)]_\eta^T$, $\Pi_{lk}(t)$, $\overline{\Pi}_{ij}(t)$, $\gamma_{1,2}$-fuzzy ABC fractional derivatives, truth interval Σ, truth values η, order γ of ABC fractional derivative, n collocation points, and the domain $\mathbb{D}$.

Algorithm 1 Finding and ensuring the validity of the γ_1- or a γ_2-fuzzy ABC analytic solutions of FFIDE of Eq. (52):

Phase I. If $z(t)$ is γ_1-fuzzy ABC fractional differentiable on $\mathbb{D}$, then applying the underlying:

- **i.** Solve γ_1-crisp ABC FIDE system to the references $\left[z_{1\eta}(t), z_{2\eta}(t)\right]$,
- **ii.** Assure $\left[z_{1\eta}(t), z_{2\eta}(t)\right]$ and $\left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma_1} z_{1\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma_1} z_{2\eta}(t)\right]$ are righteous sets,
- **iii.** Forecast γ_1-fuzzy ABC solution of $z(t)$ as $[z(t)]^\eta = \left[z_{1\eta}(t), z_{2\eta}(t)\right]$.

Phase II. If $z(t)$ is γ_2-fuzzy ABC fractional differentiable on $\mathbb{D}$, then applying the underlying:

- **i.** Solve the γ_2-crisp ABC FDE system to the references $\left[z_{1\eta}(t), z_{2\eta}(t)\right]$,
- **ii.** Assure $[z_{1r}(t), z_{2r}(t)]$ and $\left[{}^{ABC}_{0}\mathcal{D}_t^{\gamma_2} z_{2\eta}(t), {}^{ABC}_{0}\mathcal{D}_t^{\gamma_2} z_{1\eta}(t)\right]$ are righteous sets,
- **iii.** Forecast a γ_2-fuzzy ABC solution of $z(t)$ as $[z(t)]^\eta = \left[z_{1\eta}(t), z_{2\eta}(t)\right]$.

Algorithm 2 Applying the Gram-Schmidt process to finding $\wp_{lk}^{ij}$ and $\left\{\overline{\Pi}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$ in Eq. (53):

Phase 1: At $l = 1,2,\ldots$, $k = 1,2,\ldots,l$, $i = 1,2,3,\ldots$, and $j = 1,2$ applying the underlying:

$$\wp_{lk}^{ij}=\begin{cases}\dfrac{1}{\|\Pi_{11}\|_{\mathcal{A}}}, l=k=1,\\ \dfrac{1}{\sqrt{\|\Pi_{lk}\|_{\mathcal{A}}^2-\sum_{p=1}^{l-1}\langle\Pi_{lk}(t),\overline{\Pi}_{ij}(t)\rangle_{\mathcal{A}}^2}}, l=k\neq 1,\\ -\dfrac{\sum_{p=k}^{l-1}\langle\Pi_{lk}(t),\overline{\Pi}_{ij}(t)\rangle_{\mathcal{A}}z_{pk}^{ij}}{\sqrt{\|\Pi_{lk}\|_{\mathcal{A}}^2-\sum_{p=1}^{l-1}\langle\Pi_{lk}(t),\overline{\Pi}_{ij}(t)\rangle_{\mathcal{A}}^2}}, l>k,\end{cases}\tag{69}$$

Output: The orthogonalization coefficients $\wp_{lk}^{ij}$.

Phase 2: At $i=1,2,3,\ldots$ and $j=1,2$ put

$$\overline{\Pi}_{ij}(t)=\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\,\Pi_{lk}(t),\tag{70}$$

Output: systems of orthonormal functions $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Algorithm 3 Finding γ_1-fuzzy ABC numerical solution using the RKA for FFIDE of Eq. (52):

Phase I: Fixed t,s in $\mathbb{D}$ and do the underlying:

- **i.** Put $t_i=\frac{1}{n}i$ at $i=0,1,\ldots,n$,
- **ii.** Put $\eta_l=\frac{1}{m}l$ at $l=0,1,\ldots,m$,
- **iii.** Put $\Pi_{ij}(t)=\mathcal{F}_s\mathcal{P}_t(s)|_{s=t_i}$ at $i=1,2,\ldots,n$ and $j=1,2$,

Output: The orthogonal function system $\{\overline{\Pi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Phase II: At $l=1,2,\ldots$ and $k=1,2,\ldots,l$ do Algorithm 2,

Output: The orthogonalization coefficients $\wp_{lk}^{ij}$.

Phase III: Set $\overline{\Pi}_{ij}(t)=\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\,\Pi_{lk}(t)$ at $i=1,2,\ldots,n$ and $j=1,2$,

Output: The orthonormal function system $\overline{\Pi}_{ij}(t)$.

Phase IV: Put $[z^0(t_1)]_\eta^T=0$ and at $i=1,2,\ldots,n$ do the underlying:

- **i.** Put $\left[z^i(t_i)\right]_\eta^T=\left[z^{i-1}(t_i)\right]_\eta^T$,
- **ii.** Put $\Re_{(i,j)\eta}=\sum_{l=1}^{i}\sum_{k=1}^{j}\wp_{lk}^{ij}\mathfrak{P}_{k\eta}\left(t_l,[z(t_l)]_\eta^T,[\mathcal{U}z(t_l)]_\eta^T,[\mathcal{V}z(t_l)]_\eta^T\right)$,
- **iii.** Put $\left[z^i(t)\right]_\eta^T=\sum_{k=1}^{i}\sum_{j=1}^{2}\Re_{(i,j)\eta}\,\overline{\Pi}_{ij}(t)$,

Output: The n- idiom numerical reckoning $[z^n(t)]_\eta^T$ of $[z(t)]_\eta^T$.

---

10 Applications on fuzzy ABC FFIDEs

To navigate more in the utilized fuzzy analyses, we must add some applications to show the strength of the presented study and the strength of the presented numerical method. Anyhow, two ABC FFIDEs are discussed and utilized for the first time in this section by displaying a number of tables, figures, and analyzes.

To discuss our utilized outcomes in shape of realistic fuzzy models; duo applications are debated here. The first one relates to fuzzy ABC IRCSC, whilist, the last focuses on fuzzy ABC forcing idiom effects.

Application 1 Look firstly for the underlying fuzzy ABC IRCSC of Volterra type:

$$\begin{cases}{}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma}z(t)=f\big(t,z(t)\big)+\displaystyle\int_0^t k(t,\tau)g\big(z(\tau)\big)d\tau,\\ z(0)=z_0.\end{cases}\tag{71}$$

Indeed, $z:\mathbb{D}\to\mathbb{R}_{\mathcal{F}}$, $f:\mathbb{D}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$, $g:\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$, $k(t,\tau):\mathbb{D}\times\mathbb{D}\to\mathbb{R}$, and $\gamma\in(0,1]$ with

$$\begin{cases}f\big(t,z(t)\big)=-RL^{-1}z(t)+v(t),\\ g\big(z(\tau)\big)=z(\tau),\\ k(t,\tau)=-LC^{-1}.\end{cases}\tag{72}$$

$$z_0(s)=\begin{cases}25s-24, s\in[0.96,1],\\ -100s+101, s\in[1,1.01],\\ 0, s\in\mathbb{R}-[0.96,1.01].\end{cases}\tag{73}$$

Hither, the following underlying crisp parametes values are fixed: $R=1$ (Ohm), $L=1$ (Henry), $C=1$ (Farad). Whilst, the voltage function is fixed as $\mathcal{v}(t)=\sin(t)$.

In fact, the underlying subsequent coupled crisp systems of ABC FIDEs in idiom of η-cut extrapolation that are linked to γ_1- and γ_2-fuzzy ABC FFIDE of Eqs. (71), (72), and (73) can be appearing, simultaneously, as

Status 1. The system of γ_1-crisp ABC FIDE corresponding to γ_1-fuzzy ABC fractional derivative is

$$\begin{cases}{}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma_1} z_{1\eta}(t)=-z_{2\eta}(t)+\sin(t)-\int_0^t z_{2\eta}(\tau)d\tau,\\ {}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma_1} z_{2\eta}(t)=-z_{1\eta}(t)+\sin(t)-\int_0^t z_{1\eta}(\tau)d\tau,\\ z_{1\eta}(0)=0.96+0.04\eta,\\ z_{2\eta}(0)=1.01-0.01\eta.\end{cases}\tag{74}$$

The analytic fuzzy solutions of Eq. (74) when $\gamma=1$ is

$$\begin{cases}z_{1\eta}(t)=\mathfrak{a}(\eta)e^{0.5(1-\sqrt{5})t}+\mathfrak{b}(\eta)e^{0.5(1+1\sqrt{5})t}+e^{-0.5t}\left(\mathfrak{c}(\eta)\cos\left(0.5\sqrt{3}t\right)+\mathfrak{d}(\eta)\sin\left(0.5\sqrt{3}t\right)\right)+\sin(t),\\ z_{2\eta}(t)=-\mathfrak{a}(\eta)e^{0.5(1-\sqrt{5})t}-\mathfrak{b}(\eta)e^{0.5(1+\sqrt{5})t}+e^{-0.5t}\left(\mathfrak{c}(\eta)\cos\left(0.5\sqrt{3}t\right)+\mathfrak{d}(\eta)\sin\left(0.5\sqrt{3}t\right)\right)+\sin(t).\end{cases}\tag{75}$$

Hither, the parametres functions values of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$ in term of η are given, simultaneously, as follows:

$$\begin{aligned}&\mathfrak{a}(\eta)=0.0025\left(\sqrt{5}-5\right)(1-\eta),\\ &\mathfrak{b}(\eta)=0.0025\left(\sqrt{5}+5\right)(\eta-1),\\ &\mathfrak{c}(\eta)=0.005(3\eta+197),\\ &\mathfrak{d}(\eta)=-0.005\sqrt{3}(\eta+199).\end{aligned}\tag{76}$$

Status 2. The system of γ_2-crisp ABC FIDE corresponding to γ_2-fuzzy ABC fractional derivative is

$$\begin{cases}{}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma_1} z_{1\eta}(t)=-z_{1\eta}(t)-\int_0^t z_{1\eta}(\tau)+\sin(t),\\ {}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma_1} z_{2\eta}(t)=-z_{2\eta}(t)-\int_0^t z_{2\eta}(\tau)+\sin(t),\\ z_{1\eta}(0)=0.96+0.04\eta,\\ z_{2\eta}(0)=1.01-0.01\eta.\end{cases}\tag{77}$$

The analytic fuzzy solutions of Eq. (77) when $\gamma=1$ is

$$\begin{cases}z_{1\eta}(t)=(0.96+0.04\eta)e^{-0.5t}\cos\left(0.5\sqrt{3}t\right)-\frac{\sqrt{3}}{3}(2.96+0.04\eta)e^{-0.5t}\sin\left(0.5\sqrt{3}t\right)+\sin(t),\\ z_{2\eta}(t)=(1.01-0.01\eta)e^{-0.5t}\cos\left(0.5\sqrt{3}t\right)-\frac{\sqrt{3}}{3}(3.01-0.01\eta)e^{-0.5t}\sin\left(0.5\sqrt{3}t\right)+\sin(t).\end{cases}\tag{78}$$

In the fuzzy tactic and in idioms of z_0 one can collect and represent the expression in Eq. (78) as

$$z(t)=z_0\odot e^{-0.5t}\cos\left(0.5\sqrt{3}t\right)-\frac{\sqrt{3}}{3}(2+z_0)\odot e^{-0.5t}\sin\left(0.5\sqrt{3}t\right)+\sin(t).\tag{79}$$

Application 2 Now, look for the underlying fuzzy ABC FFIVP with fuzzy forcing idiom term of Fredholm type:

$$\begin{cases}{}^{ABC}_{\ 0}\mathfrak{D}_t^{\gamma} z(t)=\mathcal{f}\left(t,z(t)\right)+\int_0^1 \mathcal{k}(t,\tau)\mathcal{g}\left(z(\tau)\right)d\tau,\\ z(0)=z_0.\end{cases}\tag{80}$$

Indeed, $z:\mathbb{D}\to\mathbb{R}_{\mathcal{F}}$, $\mathcal{f}:\mathbb{D}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$, $\mathcal{g}:\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$, $\mathcal{k}(t,\tau):\mathbb{D}\times\mathbb{D}\to\mathbb{R}$, and $\gamma\in(0,1]$ with

$$\begin{cases}\mathcal{f}\left(t,z(t)\right)=\frac{13}{48}(3\rho+8)t,\\ \mathcal{g}\left(z(\tau)\right)=z(\tau),\\ \mathcal{k}(t,\tau)=t(1-2\tau).\end{cases}\tag{81}$$

$$z_0(s)=\begin{cases}\frac{1}{3}(s-1), s\in[1,4],\\ 5-s, s\in[4,5],\\ 0, s\in\mathbb{R}-[1,5].\end{cases} \tag{82}$$

$$\rho(s)=\max(0,1-|s|), s\in\mathbb{R}. \tag{83}$$

The reader should pay attention to that the crisp kernel $\mathcal{k}(t,\tau)$ is nonnegative on $0\le\tau\le 0.5$ and nonpositive on $0.5\le\tau\le 1$ in any case effect of variable t in $\mathbb{D}$. So, if $[\mathcal{V}z(t)]^\eta=\left[\left(\mathcal{V}z(t)\right)_{1\eta},\left(\mathcal{V}z(t)\right)_{2\eta}\right]$, then

$$\begin{aligned}\left(\mathcal{V}z(t)\right)_{1\eta}&=\int_0^{0.5}t(1-2\tau)z_{1\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{2\eta}(\tau)d\tau,\\ \left(\mathcal{V}z(t)\right)_{2\eta}&=\int_0^{0.5}t(1-2\tau)z_{2\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{1\eta}(\tau)d\tau.\end{aligned} \tag{84}$$

In fact, the underlying subsequent coupled crisp systems of ABC FIDEs in idiom of η-cut extrapolation that are linked to γ_1- and γ_2-fuzzy ABC FFIDE of Eqs. (80), (81), (82), and (83) can be appearing, simultaneously, as

Status 1. The system of γ_1-crisp ABC FIDE corresponding to γ_1-fuzzy ABC fractional derivative is

$$\begin{cases}{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1}z_{1\eta}(t)=\frac{13}{48}(3\eta+5)t+\int_0^{0.5}t(1-2\tau)z_{1\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{2\eta}(\tau)d\tau,\\ {}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1}z_{2\eta}(t)=\frac{13}{48}(11-3\eta)t+\int_0^{0.5}t(1-2\tau)z_{2\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{1\eta}(\tau)d\tau,\\ z_{1\eta}(0)=3\eta+1,\\ z_{2\eta}(0)=5-\eta.\end{cases} \tag{85}$$

The analytic fuzzy solutions of Eq. (85) when $\gamma=1$ is

$$\begin{cases}z_{1\eta}(t)=\eta t^2+1+3\eta,\\ z_{2\eta}(t)=(2-\eta)t^2+5-\eta.\end{cases} \tag{86}$$

In the fuzzy tactic and in idioms of ρ and z_0, one can collect and represent the expression in Eq. (86) as

$$z(t)=(\rho+1)\odot t^2+z_0. \tag{87}$$

Status 2. The system of γ_2-crisp ABC FIDE corresponding to γ_2-fuzzy ABC fractional derivative is

$$\begin{cases}{}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1}z_{1\eta}(t)=\frac{13}{48}(11-3r)t+\int_0^{0.5}t(1-2\tau)z_{2\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{1\eta}(\tau)d\tau,\\ {}^{ABC}_{0}\mathfrak{D}_t^{\gamma_1}z_{2\eta}(t)=\frac{13}{48}(3r+5)t+\int_0^{0.5}t(1-2\tau)z_{1\eta}(\tau)d\tau+\int_{0.5}^1 t(1-2\tau)z_{2\eta}(\tau)d\tau,\\ z_{1\eta}(0)=3\eta+1,\\ z_{2\eta}(0)=5-\eta.\end{cases} \tag{88}$$

The analytic fuzzy solutions of Eq. (88) when $\gamma=1$ is

$$\begin{cases}z_{1\eta}(t)=\frac{1}{35}(64-29\eta)t^2+1+3\eta,\\ z_{2\eta}(t)=\frac{1}{35}(6+29\eta)t^2+5-\eta.\end{cases} \tag{89}$$

In the fuzzy tactic and in idioms of z_0 one can collect and represent the expression in Eq. (89) as

$$z(t)=\ominus\frac{1}{35}\sigma\odot t^2+z_0, \tag{90}$$

where $\sigma\in\mathbb{R}_{\mathcal{F}}$ such that its η-cut extrapolation is $[\sigma]^\eta=[-64+29\eta,-6-29\eta]$.

11 Tables, Graphs, and Analysis

In the previous two fuzzy applications, the readers should memorandum that the non-attendance of fuzzy ABC analytic solutions for varuous $\gamma\in(0,1]$ does not have effect on the obtained numerical results; because we have ploted fuzzy ABC numerical solutions at different values of γ which have been guaranteed from the prior convergence therorems.

Anyhow, by catching the following underlying inputs: $t_i = \frac{i}{n}$ at $i = 0,1,\dots,n = 100$ on $\mathbb{D}$ and $\eta_l = \frac{1}{m} l$ at $\eta = 0,1,\dots,5$ on Σ in approximation $[z^n(t_i)]^T_{\eta_l}$ with using of Algorithms 1, 2, and 3 in all computations effects over $\gamma \in (0,1]$ and $t \in \mathbb{D}$; set of numerical data are listed in form of figures and tables.

Primarily, the geometric dynamical behaviors over the heritage and memory characteristics are seeked. In the running individual figures; geometrical attributives have been gained and exhibited for both presented applications over $\gamma \in (0,1]$, $t \in \mathbb{D}$, and $\eta \in \Sigma$. Anyhow, Figures 1 and 2 related to Application 1 and ploted the fuzzy ABC numerical solution in phases of γ_1- and γ_2-fuzzy ABC fractional derivative, simultaneously. Whilst, Figures 3 and 4 related to Application 2 and ploted the fuzzy ABC numerical solution in phases of γ_1- and γ_2-fuzzy ABC fractional derivative, simultaneously.

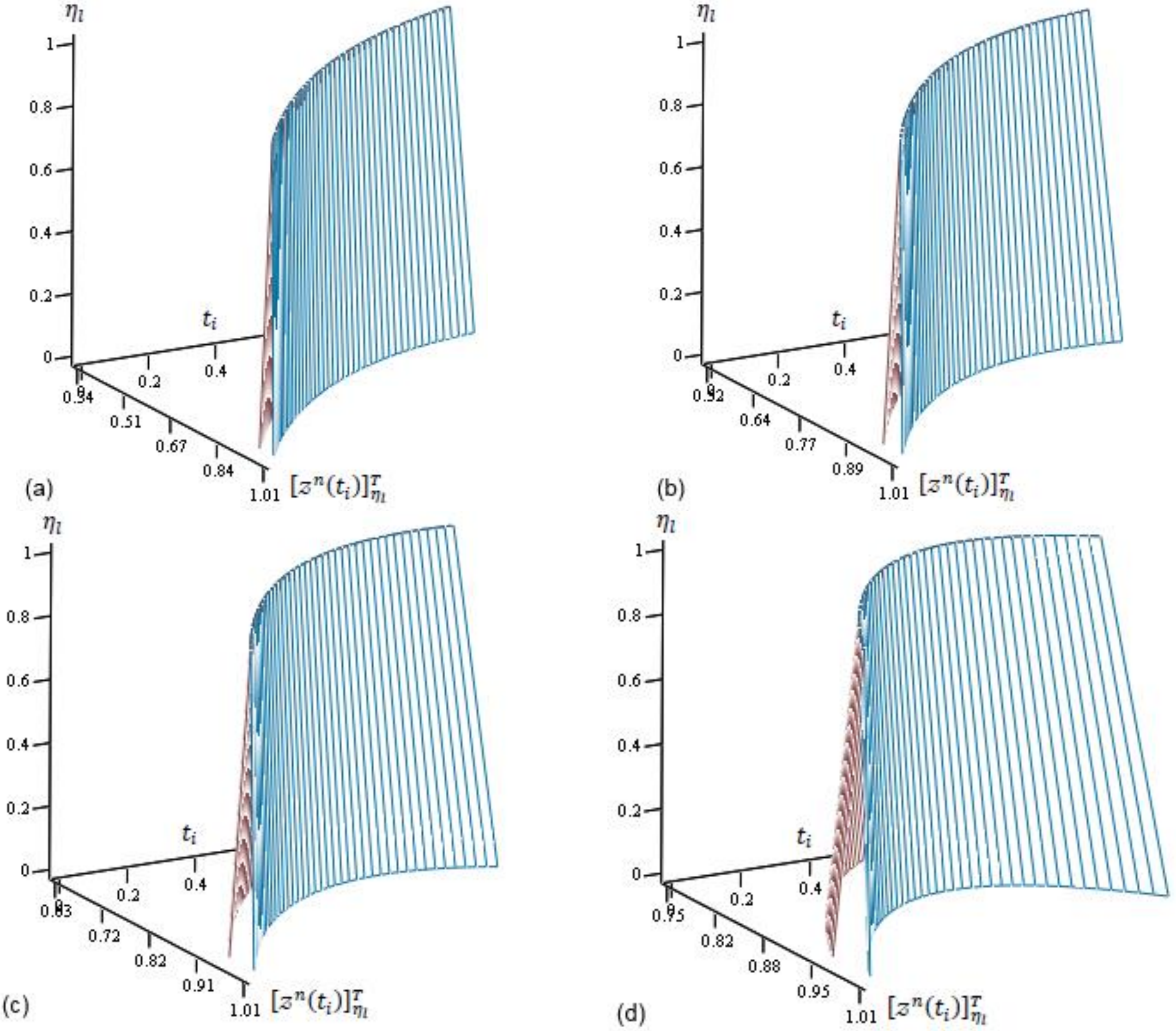


Figure 1: Computation of γ_1-fuzzy ABC solution of Application 1 obtained from the RKA: (a) $\gamma = 1$, (b) $\gamma = 0.9$, (c) $\gamma = 0.8$, and (d) $\gamma = 0.7$ wheresoever $z_{1\eta}(t)$: brown offshoot and $z_{2\eta}(t)$: blue offshoot.

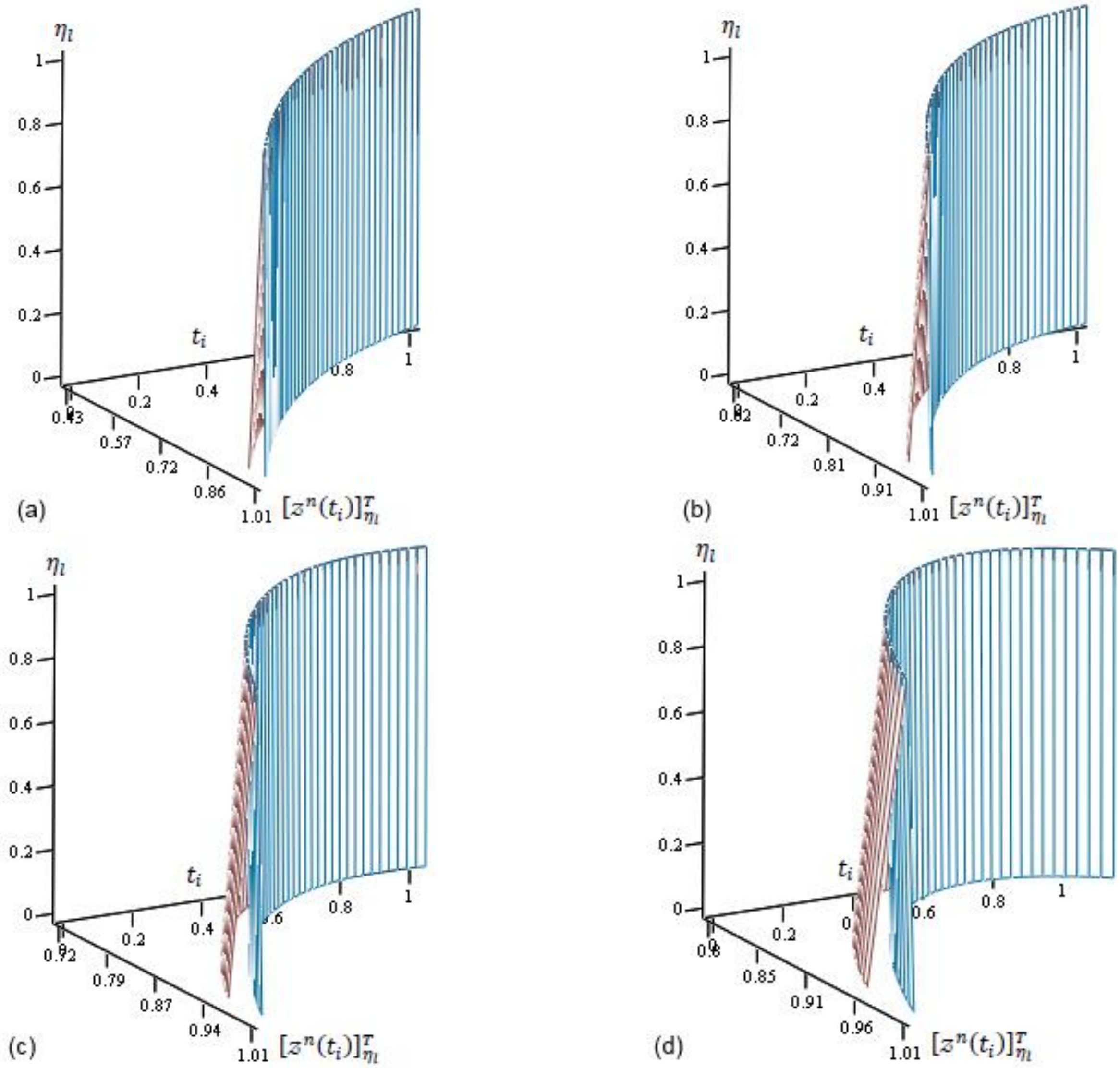
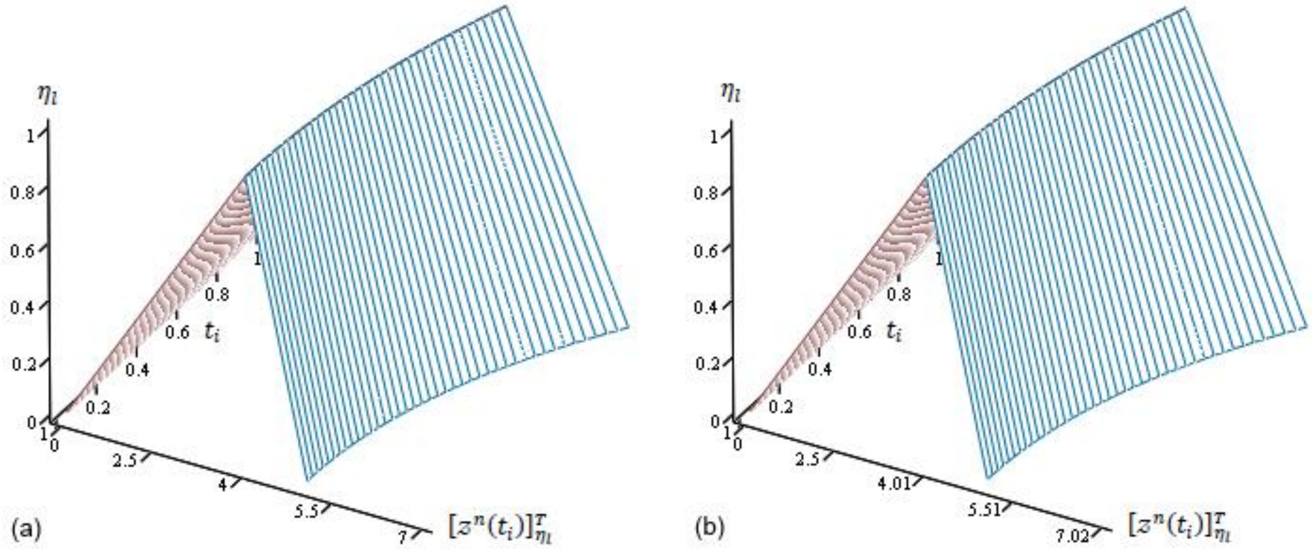


Figure 2: Computation of γ_2-fuzzy ABC solution of Application 1 obtained from the RKA: (a) $\gamma = 1$, (b) $\gamma = 0.9$, (c) $\gamma = 0.8$, and (d) $\gamma = 0.7$ wheresoever $z_{1\eta}(t)$: brown offshoot and $z_{2\eta}(t)$: blue offshoot.



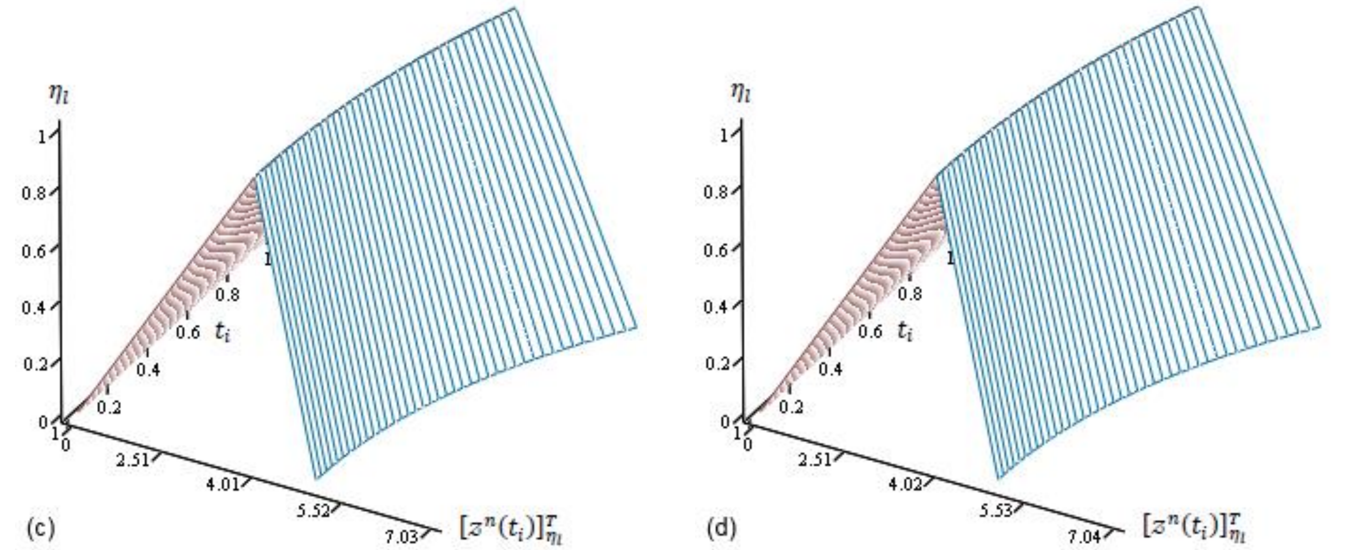


Figure 3: Computation of γ_1-fuzzy ABC solution of Application 2 obtained from the RKA: (a) $\gamma = 1$, (b) $\gamma = 0.9$, (c) $\gamma = 0.8$, and (d) $\gamma = 0.7$ wheresoever $z_{1\eta}(t)$: brown offshoot and $z_{2\eta}(t)$: blue offshoot.

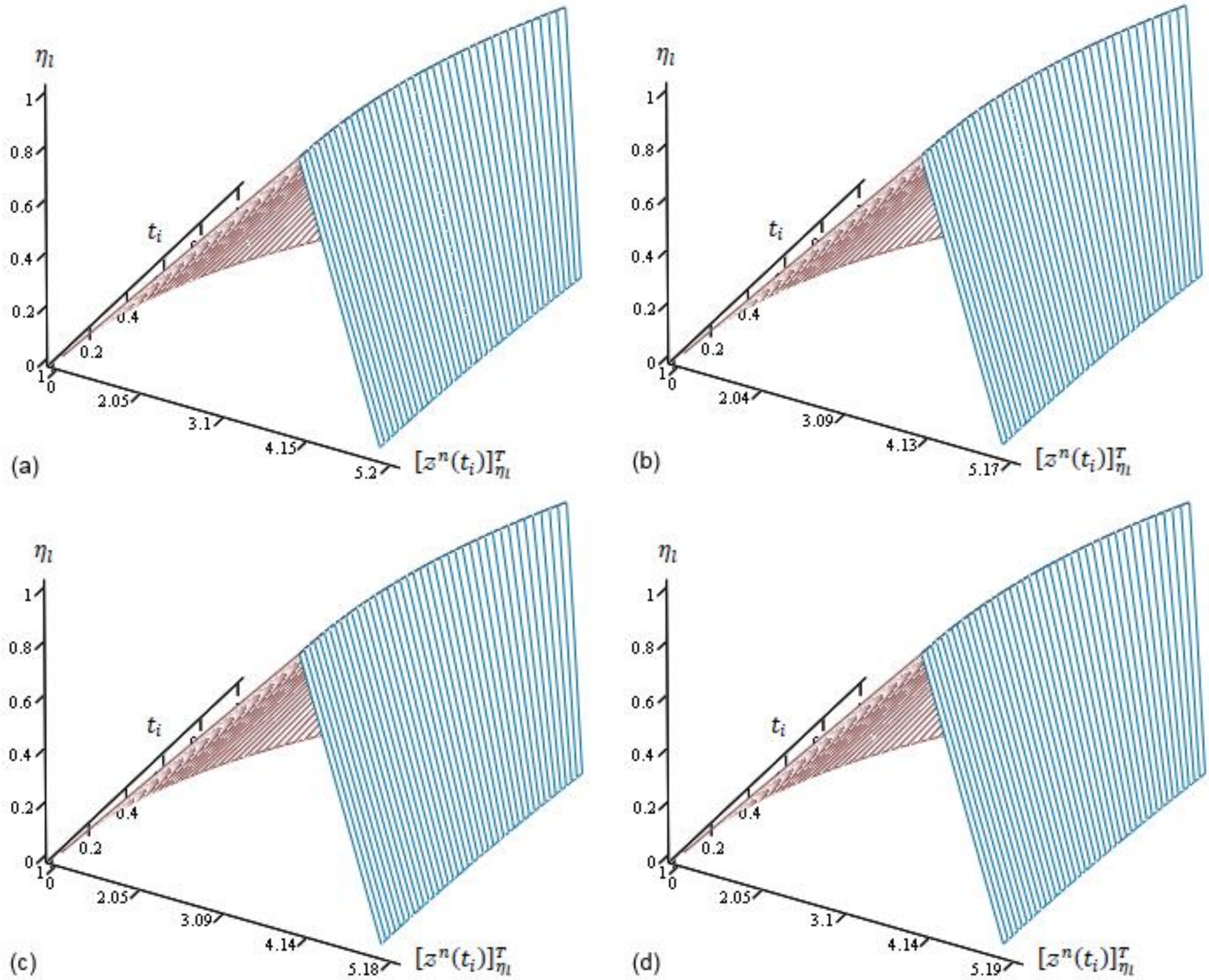


Figure 4: Computation of γ_2-fuzzy ABC solution of Application obtained from the RKA: (a) $\gamma = 1$, (b) $\gamma = 0.9$, (c) $\gamma = 0.8$, and (d) $\gamma = 0.7$ wheresoever $z_{1\eta}(t)$: brown offshoot and $z_{2\eta}(t)$: blue offshoot.

One can be observed from the figures the underlying notes: all plots almost match and are analogous in their behaviors; the plots are in pretty contract with each other, essentially when theorizing the fuzzy ABC traditional derivative of $\gamma = 1$; and the ABC FFDIE have powerful belongings on the model profiles.

Anew, in the running tables; several numerical effectiveness have been gained and exhibited for both presented applications. Anyhow, Tables 1 and 2 related to Application 1 and utilized the absolute errors for approximating the fuzzy ABC solutions in phases of $\gamma = 1$ for both γ_1-and γ_2-fuzzy ABC fractional derivative, simultaneously. Whilst, Tables 3 and 4 related to Application 2 and utilized the absolute errors for approximating the fuzzy ABC solutions in phases of $\gamma = 1$ for both γ_1- and γ_2-fuzzy ABC fractional derivative, simultaneously.

Table 1. Numerical outcomes in form of absolute errors in Applications 1 in phase of γ_1-fuzzy ABC fractional derivative using RKA when $\gamma = 1$.

	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{1\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	1.05492×10^{-5}	1.03730×10^{-5}	1.01978×10^{-5}	1.00221×10^{-5}	9.84651×10^{-5}	9.67082×10^{-5}
	0.4	2.22926×10^{-5}	2.19961×10^{-5}	2.17009×10^{-5}	2.14050×10^{-5}	2.11091×10^{-5}	2.08132×10^{-5}
	0.6	1.87915×10^{-5}	1.83081×10^{-5}	1.78246×10^{-5}	1.73411×10^{-5}	1.68576×10^{-5}	1.63742×10^{-5}
	0.8	1.64698×10^{-5}	1.57102×10^{-5}	1.49544×10^{-5}	1.41967×10^{-5}	1.34391×10^{-5}	1.26814×10^{-5}
	1.0	1.48183×10^{-5}	1.36573×10^{-5}	1.24964×10^{-5}	1.13354×10^{-5}	1.01745×10^{-5}	9.01356×10^{-5}
	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{2\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	8.64362×10^{-5}	8.84906×10^{-5}	9.05450×10^{-5}	9.25994×10^{-5}	9.46538×10^{-5}	9.67082×10^{-5}
	0.4	1.87738×10^{-5}	1.91817×10^{-5}	1.95896×10^{-5}	1.99975×10^{-5}	2.04053×10^{-5}	2.08132×10^{-5}
	0.6	1.34444×10^{-5}	1.40304×10^{-5}	1.46163×10^{-5}	1.52023×10^{-5}	1.57882×10^{-5}	1.63742×10^{-5}
	0.8	8.42369×10^{-5}	9.27524×10^{-5}	1.01267×10^{-5}	1.09783×10^{-5}	1.18298×10^{-5}	1.26814×10^{-5}
	1.0	2.80060×10^{-5}	4.04319×10^{-5}	5.28578×10^{-5}	6.52837×10^{-5}	7.77097×10^{-5}	9.01356×10^{-5}

Table 2. Numerical outcomes in form of absolute errors in Applications 1 in phase of γ_2-fuzzy ABC fractional derivative using RKA when $\gamma = 1$.

	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{1\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	9.47243×10^{-5}	$9.51211 \times 10^{-}$	9.55179×10^{-5}	9.59147×10^{-5}	9.63115×10^{-5}	9.67082×10^{-5}
	0.4	2.00667×10^{-5}	2.02160×10^{-5}	2.03653×10^{-5}	2.05146×10^{-5}	2.06639×10^{-5}	2.08132×10^{-5}
	0.6	1.56910×10^{-5}	1.58276×10^{-5}	1.59643×10^{-5}	1.61009×10^{-5}	1.62375×10^{-5}	1.63742×10^{-5}
	0.8	1.20556×10^{-5}	1.21808×10^{-5}	1.23059×10^{-5}	1.24311×10^{-5}	1.25562×10^{-5}	1.26814×10^{-5}
	1.0	8.46932×10^{-5}	8.57810×10^{-5}	8.68702×10^{-5}	8.79586×10^{-5}	8.90471×10^{-5}	9.01356×10^{-5}
	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{2\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	9.72042×10^{-5}	9.71057×10^{-5}	9.70058×10^{-5}	9.69066×10^{-5}	9.68074×10^{-5}	9.67082×10^{-5}
	0.4	2.09998×10^{-5}	2.09621×10^{-5}	2.09252×10^{-5}	2.08879×10^{-5}	2.08505×10^{-5}	2.08132×10^{-5}
	0.6	1.65450×10^{-5}	1.65102×10^{-5}	1.64766×10^{-5}	1.64425×10^{-5}	1.64083×10^{-5}	1.63742×10^{-5}
	0.8	1.28378×10^{-5}	1.28060×10^{-5}	1.27752×10^{-5}	1.27440×10^{-5}	1.27127×10^{-5}	1.26814×10^{-5}
	1.0	9.14961×10^{-5}	9.12241×10^{-5}	9.09519×10^{-5}	9.06798×10^{-5}	9.04077×10^{-5}	9.01356×10^{-5}

Table 3. Numerical outcomes in form of absolute errors in Applications 2 in phase of γ_1-fuzzy ABC fractional derivative using RKA when $\gamma = 1$.

	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{1\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	1.29998×10^{-7}	9.20931×10^{-6}	1.84186×10^{-6}	2.76279×10^{-6}	3.68372×10^{-6}	4.60465×10^{-6}
	0.4	1.97523×10^{-7}	1.15425×10^{-6}	2.30849×10^{-6}	3.46274×10^{-6}	4.61699×10^{-6}	5.77124×10^{-6}
	0.6	1.99064×10^{-7}	1.73458×10^{-7}	3.46916×10^{-7}	5.20374×10^{-7}	6.93832×10^{-7}	8.67290×10^{-7}
	0.8	2.61310×10^{-7}	3.08496×10^{-7}	6.16991×10^{-7}	9.25487×10^{-7}	1.23398×10^{-6}	1.54248×10^{-6}
	1.0	1.91299×10^{-7}	4.82025×10^{-7}	9.64049×10^{-7}	1.44607×10^{-6}	1.92809×10^{-6}	2.41012×10^{-6}
	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{2\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	9.20931×10^{-7}	8.28838×10^{-6}	7.36744×10^{-6}	6.44651×10^{-6}	5.52558×10^{-6}	4.60465×10^{-6}
	0.4	1.15425×10^{-7}	1.03882×10^{-6}	9.23398×10^{-6}	8.07970×10^{-6}	6.92549×10^{-6}	5.77124×10^{-6}
	0.6	1.73458×10^{-6}	1.56112×10^{-6}	1.38766×10^{-6}	1.21421×10^{-6}	1.04074×10^{-6}	8.67290×10^{-7}
	0.8	3.08496×10^{-6}	2.77646×10^{-6}	2.46797×10^{-6}	2.15947×10^{-6}	1.85098×10^{-6}	1.54248×10^{-6}

	1.0	4.82025×10^{-6}	4.33822×10^{-6}	3.85619×10^{-6}	3.37417×10^{-6}	2.89215×10^{-6}	2.41012×10^{-6}

Table 4 Numerical outcomes in form of absolute errors in Applications 2 in phase of γ_2-fuzzy ABC fractional derivative using RKA when $\gamma = 1$.

	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{1\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	8.20446×10^{-6}	1.57729×10^{-6}	2.33412×10^{-6}	3.09097×10^{-6}	3.84781×10^{-6}	4.60465×10^{-6}
	0.4	1.84575×10^{-6}	2.63084×10^{-6}	3.41594×10^{-6}	4.20104×10^{-6}	4.98614×10^{-6}	5.77124×10^{-6}
	0.6	2.09278×10^{-6}	1.84768×10^{-6}	1.60258×10^{-6}	1.35749×10^{-6}	1.11239×10^{-6}	8.67290×10^{-7}
	0.8	3.72062×10^{-6}	3.28499×10^{-6}	2.84936×10^{-6}	2.41374×10^{-6}	1.97810×10^{-6}	1.54248×10^{-6}
	1.0	5.81347×10^{-6}	5.13280×10^{-6}	4.45213×10^{-6}	3.77146×10^{-6}	3.09079×10^{-6}	2.41012×10^{-6}
	t_i	$\eta_0 = 0$	$\eta_1 = 0.2$	$\eta_1 = 0.4$	$\eta_2 = 0.6$	$\eta_3 = 0.8$	$\eta_4 = 1$
$z_{2\eta_l}(t_i)$	0	0	0	0	0	0	0
	0.2	8.38886×10^{-6}	7.63201×10^{-6}	6.87518×10^{-6}	6.11833×10^{-6}	5.36149×10^{-6}	4.60465×10^{-6}
	0.4	9.69672×10^{-6}	8.91163×10^{-6}	8.12653×10^{-6}	7.34143×10^{-6}	6.55633×10^{-6}	5.77124×10^{-6}
	0.6	3.58209×10^{-7}	1.13109×10^{-7}	1.31990×10^{-7}	3.77090×10^{-7}	6.22190×10^{-7}	8.67290×10^{-7}
	0.8	6.35663×10^{-7}	2.00035×10^{-7}	2.35594×10^{-7}	6.71222×10^{-7}	1.10685×10^{-6}	1.54248×10^{-6}
	1.0	9.93224×10^{-7}	3.12555×10^{-7}	3.68115×10^{-7}	1.04878×10^{-6}	1.72945×10^{-6}	2.41012×10^{-6}

One can be observed from the tables the underlying notes: all tables almost match and are analogous in the same errors order; the tables are in pretty contract with each other, essentially when theorizing the ABC crisp derivative of $\eta = 1$.

12 Highlight and Future Research

In this analysis, fuzzy ABC fractional integral, fuzzy ABC fractional derivative, fuzzy ABC FFIDE, and fuzzy ABC solutions are discussed and utilized for the first time. Indeed, equivalent intervals approach is presented to figuration fuzzy ABC fractional derivative, also, characterization theorem are likewise discussed for the first time as well. The RKA is offered in details as a novel version solver for such fuzzy ABC FFIDEs. Over and above, a computational algorithm concerned to characterizing fuzzy ABC solutions is given. In this direction, two applications on fuzzy ABC FFIDEs are fitted to confirm the approaching theoretical analysis in the fuzzy ABC calculus. Hereafter, those proposed extended can be used efficacious as a substitution planner in formulation various kinds of uncertain differential and integral problems under investigation in engineering and applied sciences. Our future research, will be analyze and arrangement fuzzy ABC fractional boundary value problems.

References

[1] T. Allahviranloo. S. Salahshour, S. Abbasbandy, Explicit solutions of fractional differential equations with uncertainty, Soft Comput 16 (2012) 297-302.

[2] T. Allahviranloo, L. Avazpour, M.J. Ebadi, D. Baleanu, S. Salahshour, Fuzzy fractional Ostrowski inequality with Caputo differentiability, Journal of Inequalities and Applications 2013 (2013) 50.

[3] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, P. Agarwal, On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem, Entropy 17 (2015) 885-902.

[4] A. Ahmadian, M. Suleiman, S. Salahshour, D. Baleanu, A Jacobi operational matrix for solving a fuzzy linear fractional differential equation, Advances in Difference Equations 2013 (2013)104.

[5] E. Khodadadi, E. Çelik, The variational iteration method for fuzzy fractional differential equations with uncertainty, Fixed Point Theory and Applications 2013 (2013)13.

[6] O. Abu Arqub, M. Al-Smadi, Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions, Soft Computing (2020) 2020 1-22.

[7] S. Salahshour, T. Allahviranloo, S. Abbasbandy, Solving fuzzy fractional differential equations by fuzzy Laplace transforms, Communications in Nonlinear Science and Numerical Simulation 17 (2012) 1372-1381.

[8] O. Abu Arqub, M. Al-Smadi, S. Momani, T. Hayat, Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method, Soft Computing 20 (2016) 3283-3302.

[9] O. Abu Arqub, M. Al-Smadi, S. Momani, T. Hayat, Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems, Soft Computing 21 (2017) 7191-7206.

[10] O. Abu Arqub, Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra

integrodifferential equations, Neural Computing & Applications 28 (2017) 1591-1610.
[11] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model, Thermal Science 20 (2016) 763-769.
[12] A. Atangana, J.J. Nieto, Numerical solution for the model of RLC circuit via the fractional derivative without singular kernel, Advances in Mechanical Engineering 7(2015) 1-7.
[13] A. Atangana, J.F. Gómez-Aguilar, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, The European Physical Journal Plus 133 (2018) 1-22.
[14] A. Atangana, J.F. Gómez-Aguilar, Fractional derivatives with no-index law property: Application to chaos and statistics, Chaos Solitons & Fractals 114 (2018) 516-535.
[15] O Abu Arqub, M Al-Smadi, Atangana–Baleanu fractional approach to the solutions of Bagley–Torvik and Painlevé equations in Hilbert space, Chaos, Solitons & Fractals 117 (2018) 161-167.
[16] O Abu Arqub, B Maayah, Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana–Baleanu fractional operator, Chaos, Solitons & Fractals 117 (2018) 117-124.
[17] O. Abu Arqub, B. Maayah, Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC–Fractional Volterra integro-differential equations, Chaos, Solitons & Fractals 126 (2019) 394-402.
[18] O. Abu Arqub, B. Maayah, Modulation of reproducing kernel Hilbert space method for numerical solutions of Riccati and Bernoulli equations in the Atangana-Baleanu fractional sense, Chaos, Solitons & Fractals 125 (2019) 163-170.
[19] A. Atangana. On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation. Applied Mathematics and Computation 273 (2016) 948-956.
[20] A. Atangana, I. Koca, On the new fractional derivative and application to Nonlinear Baggs and Freedman model, Journal of Nonlinear Science and Applications 9 (2016) 2467-2480.
[21] T. Abdeljawad, Fractional difference operators with discrete generalized Mittag-Leffler kernels, Chaos, Solitons and Fractals 126 (2019) 315-324.
[22] Cui M, Lin Y (2009) Nonlinear numerical analysis in the reproducing kernel space, Nova Science, USA.
[23] Berlinet A, Agnan CT (2004) Reproducing kernel Hilbert space in probability and statistics, Kluwer Academic Publishers, USA.
[24] Daniel A, Reproducing kernel spaces and applications, Springer, Basel, Switzerland, 2003.
[25] O. Abu Arqub, Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions, Computers & Mathematics with Applications 73 (2017) 1243-1261.
[26] O. Abu Arqub, Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm, International Journal of Numerical Methods for Heat & Fluid Flow 28 (2018) 828-856.
[27] O. Abu Arqub, The reproducing kernel algorithm for handling differential algebraic systems of ordinary differential equations, Mathematical Methods in the Applied Sciences 39 (2016) 4549-4562.
[28] O. Abu Arqub, Approximate solutions of DASs with nonclassical boundary conditions using novel reproducing kernel algorithm, Fundamenta Informaticae 146 (2016) 231-254.
[29] O. Abu Arqub, Solutions of time-fractional Tricomi and Keldysh equations of Dirichlet functions types in Hilbert space, Numerical Methods for Partial Differential Equations 34 (2018) 1759-1780.
[30] O. Abu Arqub, Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm, Calcolo 55 (2018) 1-28.
[31] O. Abu Arqub, Z. Odibat, M. Al-Smadi, Numerical solutions of time-fractional partial integrodifferential equations of Robin functions types in Hilbert space with error bounds and error estimates, Nonlinear Dynamics 94 (2018), 1819-1834.
[32] O. Abu Arqub, Numerical Algorithm for the Solutions of Fractional Order Systems of Dirichlet Function Types with Comparative Analysis, Fundamenta Informaticae 166 (2019) 111–137.
[33] O. Abu Arqub, M. Al-Smadi, An adaptive numerical approach for the solutions of fractional advection-diffusion and dispersion equations in singular case under Riesz's derivative operator, Physica A: Statistical Mechanics and its Applications 540 (2020) 123257.
[34] O. Abu Arqub, N. Shawagfeh, Application of reproducing kernel algorithm for solving Dirichlet time-fractional

diffusion-Gordon types equations in porous media, Journal of Porous Media 22 (2019) 411-434.
[35] Abu Arqub O, Al-Smadi M (2018) Numerical algorithm for solving time-fractional partial integrodifferential equations subject to initial and Dirichlet boundary conditions. Numerical Methods for Partial Differential Equations 34:1577-1597.
[36] Abu Arqub O, Shawagfeh N (2019) Solving optimal control problems of Fredholm constraint optimality via the reproducing kernel Hilbert space method with error estimates and convergence analysis. Mathematical Methods in the Applied Sciences 2019:1-18. DOI: 10.1002/mma.5530.
[37] Jiang W, Chen Z (2014) A collocation method based on reproducing kernel for a modified anomalous subdiffusion equation. Numerical Methods for Partial Differential Equations 30:289-300.
[38] Geng FZ, Qian SP, Li S (2014) A numerical method for singularly perturbed turning point problems with an interior layer. Journal of Computational and Applied Mathematics 255:97-105.
[39] Lin Y, Cui M, Yang L (2006) Representation of the exact solution for a kind of nonlinear partial differential equations. Applied Mathematics Letters 19:808-813.
[40] Zhoua Y, Cui M, Lin Y (2009) Numerical algorithm for parabolic problems with non-classical conditions. Journal of Computational and Applied Mathematics 230:770-780.
[41] Akgül A (2018) A novel method for a fractional derivative with non-local and non-singular kernel. Chaos, Solitons and Fractals 114:478-482
[42] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301-317.
[43] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31-43.
[44] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581-599.
[45] Y. Chalco-Cano, H. Román-Flores, On new solutions of fuzzy differential equations, Chaos, Solitons and Fractals 38 (2008) 112-119.
[46] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319-330.