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Existence and Uniqueness of Weak Solution for chemotaxis model coupled with heat equation †

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Summary

Keller-Segel chemotaxis model is described by a system of nonlinear PDE : a convection diffusion equation for the cell density coupled with a reaction-diffusion equation for chemoattractant concentration. In this work, we study the phenomenon of Keller Segel model coupled with a heat equation, because The heat has an effect the density of the cells as well as the signal of chemical concentration, since the heat is a factor affecting the spread and attraction of cells as well in relation to the signal of chemical concentration, The main objectives of this work is the study of the global existence and uniqueness and boundedness of the weak solution for the problem defined in (8) for this we use the technical of Galerkin method.

KEYWORDS:
Chemotaxis, Keller-Segel, global existence, boundedness, Galerkin method

1 | INTRODUCTION

Biological pattern forming is a subject of increasing interest in applied mathematics, both because of the potential for creating new mathematics and because of the large range of important applications it may have [19]. Chemotaxis is well understood to play a key role in the self-organization of many biological processes at the cellular level. Chemotaxis is the regulated movement of an organism in response to chemical gradients in the environment, which are also differentiated by the cells. Chemotactic aggregation is a phenomenon that occurs when chemical products are attractive (hence the term chemoattractant). Keller-Segel model cite2, which can be summarized as follows, is one of the most important partial differential structures for understanding chemotactic aggregation.:

$$\partial_t \rho = D_\rho \triangle \rho - k\nabla(\rho\nabla c), \tag{1}$$

$$\delta c_t = D_c \triangle c - \tau c + \beta\rho. \tag{2}$$

Here D_ρ is the cellular diffusion constant, k the chemotactic coefficient, β the rate of attractant production τ the rate of attractant depletion, D_c the chemical diffusion constant, ρ is the cell density, and c is the chemical density. The terms in Eq_1 include the diffusion of the cells and chemotactic and Eq_2 expresses the diffusion and production of attractant. There is a broad literature on this problem's study for both the Keller-Segel model and some simplifications [7], [15], [20], [10], [11], [12], [21], [13], [14] References looked into the biological significance of this mathematical reality [6], [1] This will be an important accomplishment in the field of mathematical

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biology. In this article, we look at the regularity problem from a different viewpoint.

$$\begin{cases} u_t - \nabla(m\nabla u) + \nabla(\zeta u\nabla c + \mu u\nabla v) = 0 \quad ,(x,t)\in\Omega\times\mathbb{R}^+, \\ \delta c_t - \triangle c + \tau c + \rho u + Kcv = 0 \quad ,(x,t)\in\Omega\times\mathbb{R}^+, \\ v_t - \alpha \triangle v = 0 \quad ,(x,t)\in\Omega\times\mathbb{R}^+. \end{cases} \tag{3}$$

Where $u = u(x,t)$ denots the density of the cells in position $x \in \mathbb{R}^d$ at time t, $c = c(x,t)$ is the concentration of chemical signal substance, $\delta \geq 0$ represents the relaxation time, the parameter ζ is the chemotactic coefficient and m, τ, μ and ρ, k are given smooth functions, the term v_t is heat distrubition over time and the term $\triangle v$ corresponds to a variation of v compared to its average and α is the heat coefficient and the terms $\nabla(\mu u\nabla v)$, Kcv are directed cell movement by a heat and chemical degradation by a heat factor (Respectively). This problem deals with the extent of the influence of heat on the attraction and graduation of cell density and concentration of the chemical solution signal.
The main objectives of this work is to study of the problem Keller-Segel coupled with a heat equation on the form: $\delta = 1$ and m, τ and ρ, k are the positive constants and $\mu = \zeta$ are the negatif constant and $\alpha = 1$. and we demonstrate the global existence and uniqueness of a weak solution for parabolic- parabolic-parabolic problem with the Dirichlet conditions and initials conditions defined as:

$$\begin{cases} P1\begin{cases} u_t - \nabla(m\nabla u) - \nabla(\zeta(u\nabla c + u\nabla v)) = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ u = 0, \quad \text{in } \Gamma, \\ u(0,x) = u_0, \quad x\in\Omega, \end{cases} \\ \\ P2\begin{cases} c_t - \triangle c + \tau c + \rho u + Kcv = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ c = 0, \quad \text{in } \Gamma, \\ c(0,x) = c_0, \quad x\in\Omega, \end{cases} \\ \\ P3\begin{cases} v_t - \triangle v = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ v = 0, \quad \text{in } \Gamma, \\ v(0,x) = v_0, \ x\in\Omega. \end{cases} \end{cases} \tag{4}$$

2 | EXISTENCE AND UNIQUENESS OF WEAK SOLUTION OF THE PROBLEM

To simplify the weak solution of the problem (4) a decomposition into three subproblems ($P1$) and ($P2$) and ($P3$) are adopted. We use the Galerkin method we can demonstrate the existence and uniqness of a weak solution of subproblems ($P1$) and ($P2$) and ($P3$) therefore we have the existence and uniqueness of a weak solution of the problem (4). The following initial-boundary conditions assumption is used to prove the proposed solution of (4)

$$u_0 \in L^2(\Omega), \tag{5}$$

$$c_0 \in L^2(\Omega), \tag{6}$$

$$v_0 \in L^2(\Omega). \tag{7}$$

2.1 | Existence and uniqueness of weak solution of the problem (P1)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem ($P1$) .

Definition 1. We say $u \in L^2(0,T; H_0^1(\Omega)) \times H_0^1(\Omega)$ with $u_t \in L^2(0,T; H^{-1}(\Omega))$ is a weak solution of the problem ($P1$) if and only if

$$\langle u_t, \Phi\rangle + B(u, \Phi, t) = 0, \tag{8}$$

where

$$B(u,\Phi,t) = \int\limits_{\Omega} [m(\nabla u \nabla \Phi) + \zeta(u\nabla c\nabla\Phi + u\nabla v)\nabla\Phi]dx, \tag{9}$$

for all $\Phi \in H_0^1(\Omega)$,$0 \leq t \leq T$,and

$$u(0,x) = u_0 \in L^2(\Omega). \tag{10}$$

Remark 1. Note that $u \in C([0,T]; L^2(\Omega))$ as $u \in L^2(0,T; H_0^1(\Omega))$ and $u_t \in L^2(0,T; H^{-1}(\Omega))$ Then equality (10) makes sense.

Before proving the existence and uniqueness of weak solution of the problem $(P1)$, we need the following lemma:

Lemma 1. i) For all $\Phi \in H_0^1(\Omega)$ the $B(u,\Phi,t)$is continuous in $H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a constant positive C such that

$$| B(u,\Phi,t) | \leq C \parallel u \parallel_{H_0^1(\Omega)} \parallel \Phi \parallel_{H_0^1(\Omega)} . \tag{11}$$

ii)For any $u \in H_0^1(\Omega)$ Then there exists a constant positive β such that

$$\beta \parallel u \parallel_{H_0^1(\Omega)} \leq B(u,u,t), \quad \forall u \in H_0^1(\Omega). \tag{12}$$

Proof. i) We use the Cauchy-Schwarz inequality on (9) we obtain

$$\begin{aligned} | B(u,\Phi,t) | \leq & | m | \| \nabla u \|_{L^2(\Omega)} \| \nabla \Phi \|_{L^2(\Omega)} \\ & + | \zeta | \| u \|_{L^2(\Omega)} \| \nabla c \|_{L^4(\Omega)} \| \nabla \Phi \|_{L^4(\Omega)} \\ & \times | \zeta | \| u \|_{L^2(\Omega)} \| \nabla v \|_{L^4(\Omega)} \| \nabla \Phi \|_{L^4(\Omega)}, \end{aligned}$$

and we have

$$| B(u,\Phi,t) | \leq C \parallel u \parallel_{H^1(\Omega)} \parallel \Phi \parallel_{H^1(\Omega)} .$$

ii) The expression of $B(u,u,t)$ we obtain

$$B(u,u,t) = \int_\Omega (m(\nabla u)^2 + \zeta(u\nabla c\nabla u + u\nabla v\nabla u)dx,$$

and we have

$$B(u,u,t) \geq \int_\Omega (m(\nabla u)^2 dx = \grave{m} \parallel \nabla u \parallel^2_{L^2(\Omega)},$$

finally, inequality Poincares, gives $B(u,u,t) \geq \beta \parallel u \parallel^2_{H_0^1(\Omega)}$. ☐

To demonstrate the existence of weak solution of problem $(P1)$ we use the method of Galerkin we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m, \text{ its linearly independent,} \\ \text{ the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{13}$$

We are looking for $u_m = u_m(t)$ solution <<approached>> of the problem in the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{14}$$

and g_{im} to be determined by the conditions:

$$\begin{cases} \langle u_m', w_j \rangle + B(u_m, w_j, t) = 0, \\ \qquad 1 \leq j \leq m. \end{cases} \tag{15}$$

The nonlinear differential equation system is to be completed by the conditional:

$$u_m(0) = u_{0m}, \; u_{0m} = \sum_{i=1}^m \alpha_{im} w_j \to u_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty.$$

2.1.1 | Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions u_m of the approximation problems (15) and (16) converges to a weak solution of $(P1)$. For this we will need some uniform estimates.

Theorem 1. (Energy estimates.) There exists a constant C depending only on Ω, T and c, such that

$$\begin{aligned} &\max_{0\le t\le T} \parallel u_m \parallel_{L^2(\Omega)} + \parallel u_m \parallel_{L^2(0,T;H_0^1(\Omega))} \\ &+ \parallel u_m' \parallel_{L^2(0,T;H^{-1}(\Omega))} \le C \parallel u_0 \parallel_{L^2(\Omega)} . \end{aligned} \tag{16}$$

Proof. In order to prove the estimation (16) we will estime each terms in the left side of (14) one by one as follows:

1. Multiplying equation (15) by $g_{jm}(t)$ and summing for $k = 1...m$, and then recalling (14) we find

$$\langle u_m^{'}, u_m\rangle + B(u_m, u_m, t) = 0, \tag{17}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\parallel u_m \parallel^2_{L^2(\Omega)}] + B(u_m, u_m, t) = 0, \tag{18}$$

From Lemma (1) there exists constant $\beta > 0$ such that

$$\beta \parallel u_m \parallel^2_{H_0^1(\Omega)} \le B(u_m, u_m, t), \forall 0 \le t \le T, \tag{19}$$

and we have

$$\frac{d}{dt}(\parallel u_m \parallel^2_{L^2(\Omega)}) + \beta \parallel u_m \parallel^2_{H_0^1(\Omega)} \le 0, \tag{20}$$

this implies that

$$\parallel u_m \parallel^2_{L^2(\Omega)} \le \parallel u_m(0) \parallel^2_{L^2(\Omega)} \le \parallel u_0 \parallel^2_{L^2(\Omega)}, \tag{21}$$

so we have

$$\max_{0\le t\le T} \parallel u_m \parallel_{L^2(\Omega)} \le \parallel u_0 \parallel_{L^2(\Omega)} . \tag{22}$$

2. Integrate inequality (20) from 0 to T and we employ the inequality (22) to find

$$\parallel u_m \parallel^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \parallel u_m \parallel^2_{H_0^1(\Omega)} dt. \tag{23}$$

3. Fix any $v \in H_0^1(\Omega)$, with $\parallel v \parallel^2_{H_0^1(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$, $(k = 1, ..., m)$, we use (15) we deduce for all $0 \le t \le T$ that

$$(u_m^{'}, v^1) + B(u_m, v^1, t) = 0,$$

then (14) implies

$$\langle u_m^{'}, v\rangle = (u_m^{'}, v) = (u_m^{'}, v^1) = -B(u_m, v^1, t),$$

consequently

$$| \langle u_m^{'}, v\rangle | \le C \parallel u_m \parallel^2_{H_0^1(\Omega)},$$

since

$$\parallel v^1 \parallel^2_{H_0^1(\Omega)} \le \parallel v \parallel^2_{H_0^1(\Omega)} \le 1,$$

we have

$$\parallel u_m^{'} \parallel_{H^{-1}(\Omega)} \le C \parallel u_m \parallel_{H_0^1(\Omega)}$$

and therefore

$$\parallel u_m^{'} \parallel^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \parallel u_m^{'} \parallel^2_{H^{-1}(\Omega)} dt \le C \int_0^T \parallel u_m \parallel^2_{H_0^1(\Omega)} dt \le C \parallel u_0 \parallel^2_{L^2(\Omega)} .$$

□

2.1.2 | Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P1)$.

Theorem 2. (Existence of weak solution.) Under hypothesis (5), there exists a weak solution of problem $(P1)$.

Proof. According to the energy estimates (16), we see that the sequence $\{u_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ Consequently there exists a subsequence which is also noted by $\{u_m\}_{m=1}^{\infty}$ and a function $u \in L^2(0,T;H_0^1(\Omega))$, with $u^{'} \in L^2(0,T;H^{-1}(\Omega))$, such that

$$u_m \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{24}$$
$$u_m^{'} \to u^{'} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{25}$$

2. Next fix an integer N and choose a function $v \in C^1(0,T;H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} g^{(k)}(t) w_k, \tag{26}$$

where $\left\{g^{(k)}\right\}_{k=1}^{N}$ are given smooth functions, we choose $m \geq N$ and multipling equation (15) by $g^{(k)}(t)\ \forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle u_m^{'}, v\rangle + B(u_m, v, t)dt = 0, \tag{27}$$

we recall (24) and to find upon passing to weak limits that

$$\int_0^t \langle u^{'}, v\rangle + B(u, v, t)dt = 0, \ \ \forall v \in L^2(0,T;H_0^1(\Omega)), \tag{28}$$

as functions of the form (26) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle u', v\rangle + B(u, v, t) = 0, \ \ \forall v \in H_0^1(\Omega) \text{ and } \forall t \in [0,T], \tag{29}$$

and from Remark (1) we have $u \in C(0,T;L^2(\Omega))$.

3. In order to prove $u(0) = u_0$, we first note from (10) that

$$\int_0^t -\langle u, v'\rangle + B(u, v, t) = (u(0), v(0)), \tag{30}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (27) we obtain

$$\int_0^t -\langle u_m, v'\rangle + B(u_m, v, t)dt = (u_0, v(0)), \tag{31}$$

we use again (30), we obtain

$$\int_0^t -\langle u, v'\rangle + B(u, v, t)dt = (u_0, v(0)), \tag{32}$$

since $u_m(0) \to u_0$ in $L^2(\Omega)$. Comparing (30) and (32), we conclude $u(0) = u_0$. □

Theorem 3. (Uniqueness of weak solutions.) A weak solution of problem $(P1)$ is unique.

Proof. We suppose there exists two weak solution u_1 and $u_2$and we put $U = u_2 - u_1$ then U is also a solution of problem $(P1)$ with $U_0 = (u_2 - u_1)(0) \equiv 0$. Setting $v = U$ in identity (18) we are

$$\frac{d}{dt}\left(\frac{1}{2} \parallel U \parallel_{L^2(\Omega)}^2\right) + B(U, U, t) = 0,$$

from Lemma (1), we have $B(U,U,t) \geq \beta \parallel U \parallel^2_{H^1_0(\Omega)} \geq 0$, so $\frac{d}{dt}(\frac{1}{2} \parallel U \parallel^2_{L^2(\Omega)}) \leq 0$, then integrate with respect to t to find

$$\parallel U \parallel^2_{L^2(\Omega)} \leq \parallel U_0 \parallel^2_{L^2(\Omega)} = 0,$$

Thus $U \equiv 0$. □

2.2 | Existence and uniqueness of weak solution of problem (P2)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem $(P2)$

Definition 2. We say $c \in L^2(0,T; H^1_0(\Omega))$ with $c_t \in L^2(0,T; H^{-1}(\Omega))$ is a weak solution of the problem $(P2)$ if and only if

$$\langle c_t, q\rangle + L(c,q,t) = 0, \tag{33}$$

where

$$L(c,q,t) = \int_\Omega [(\nabla c \nabla q) + \tau c q + \rho u q + K v c q] dx, \tag{34}$$

for all $q \in H^1_0(\Omega)$,$0 \leq t \leq T$,and

$$c(0,x) = c_0 \in L^2(\Omega). \tag{35}$$

Remark 2. Note that $c \in C([0,T]; L^2(\Omega))$ as $c \in L^2(0,T; H^1_0(\Omega))$ and $c_t \in L^2(0,T; H^{-1}(\Omega))$ Then equality (35) makes sense.

To demonstrate existence of weak solution of problem (P1) we use the Galerkin method, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H^1_0(\Omega), \\ \forall m; w_t ....w_m \text{ its linearly independent }, \\ \text{ the finite linear combination of } w_i \text{ are dense in } H^1_0(\Omega). \end{cases} \tag{36}$$

We are looking for $c_m = c_m(t)$ solution <<approached>> of the problem in the form

$$c_m(t) = \sum_{i=1}^{m} d_{im}(t) w_i, \tag{37}$$

the d_{im} to be determined by the conditions:

$$\begin{cases} \langle c'_m, w_j\rangle + L(c_m, w_j, t) = 0, \\ 1 \leq j \leq m. \end{cases} \tag{38}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$c_m(0) = c_{0m},\ c_{0m} = \sum_{i=1}^{m} \beta_{im} w_j \to c_0 \text{ in } H^1_0(\Omega), \text{ when } m \to \infty. \tag{39}$$

We now propose to send m to infinity and to show a subsequence of our solutions c_m approximation problems (38) and (39) converges towards a weak solution of the problem $(P2)$. For this we need uniform estimates.

2.2.1 | Energy estimates

We propose now to send m to infinity and show a subsequence of our solutions c_m of the approximation problems (38) and (39) converges to a weak solution of problem $(P2)$. For this we will need some uniform estimates.

Theorem 4. (Energy estimates.) They exists a constant C depending only on Ω, T such that

$$\begin{aligned} \max\nolimits_{0\leq t\leq T} \parallel c_m \parallel_{L^2(\Omega)} + \parallel c_m \parallel_{L^2(0,T;H^1_0(\Omega))} \\ + \parallel c'_m \parallel_{L^2(0,T;H^{-1}(\Omega))} \leq C \parallel c_0 \parallel_{L^2(\Omega)} . \end{aligned} \tag{40}$$

Proof. In order to prove the estimation (40) we will estime each terms in the left side of (38) one by one as follows:

1. Multiplying equation (38) by $d_{jm}(t)$ and summing for j we find

$$\langle c'_m, w_j\rangle + B(c_m, c_m, t) = 0, \tag{41}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\| c_m \|^2_{L^2(\Omega)}] + L(c_m(t), c_m(t)) = 0, \tag{42}$$

and we put $\| v \| = \sqrt{L(v,v)}$ (= is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{d}{dt}(\| c_m \|^2_{L^2(\Omega)}) + \| c_m \|^2_{H_0^1(\Omega)} = 0, \tag{43}$$

we have

$$\frac{d}{dt}(\| c_m \|^2_{L^2(\Omega)}) \leq 0,$$

and we have

$$\| c_m \|^2_{L^2(\Omega)} \leq \| c_m(0) \|^2_{L^2(\Omega)} \leq \| c_0 \|^2_{L^2(\Omega)}, \tag{44}$$

so we are

$$\max_{0\leq t\leq T} \| c_m \|_{L^2(\Omega)} \leq \| c_0 \|_{L^2(\Omega)} . \tag{45}$$

2. Integrate inequality (43) from 0 to T and we use (45) to find

$$\| c_m \|^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \| c_m \|^2_{H_0^1(\Omega)} \, dt. \tag{46}$$

3. Fix any $v \in H_0^1(\Omega)$, with $\| v \|^2_{H_0^1(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, ..., m)$. we use (38) from all $0 \leq t \leq T$ that

$$(c_m^{'}, v^1) + L(c_m, v^1, t) = 0.$$

Then (37) implies

$$(c_m^{'}, v) = \langle c_m^{'}, v\rangle = \langle c_m^{'}, v^1\rangle = -L(c_m, v^1, t),$$

consequently

$$| \langle c_m^{'}, v\rangle | \leq C \| c_m \|^2_{H_0^1(\Omega)},$$

and as

$$\| v^1 \|^2_{H_0^1(\Omega)} \leq \| v \|^2_{H_0^1(\Omega)} \leq 1,$$

thus

$$\| c_m^{'} \|_{H^{-1}(\Omega)} \leq C \| c_m \|_{H_0^1(\Omega)},$$

and therefore

$$\| c_m^{'} \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| c_m^{'} \|^2_{H^{-1}(\Omega)} \, dt \leq C \int_0^T \| c_m \|^2_{H_0^1(\Omega)} \, dt \leq C \| c_0 \|^2_{L^2(\Omega)} .$$

□

2.2.2 | Existence and uniqueness of weak solution

Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P2)$.

Theorem 5. (Existence of weak solution.) Under hypothesis (6), there exists a weak solution of $(P2)$.

Proof. According to the energy estimates (40), we see that the sequence $\{c_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{c_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ Consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $c \in L^2(0,T;H_0^1(\Omega))$ with $c^{'} \in L^2(0,T;H^{-1}(\Omega))$, such that

$$\begin{aligned} c_m &\to c \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ c_m^{'} &\to c^{'} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{47}$$

2. Next fix an integer N and choose a function $v \in C^1(0,T;H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{48}$$

where $\left\{d^{(k)}\right\}_{k=1}^{N}$ are given smooth functions. We choose $m \geq N$, multiply equation (38) by $d^{(k)}(t)\forall K = 1...N$, and then integrate with respect to t to find

$$\int_0^t \langle c_m^{'}, v\rangle + L(c_m, v, t)dt = 0, \tag{49}$$

we recall (47) to find upon passing to weak limits that

$$\int_0^t \langle c^{'}, v\rangle + L(c, v, t)dt = 0, \ \forall v \in L^2(0, T; H_0^1(\Omega)). \tag{50}$$

As functions of the form (48) are dense in $L^2(0, T; H_0^1(\Omega))$. Hence in particular

$$\langle c', v\rangle + L(c, v, t) = 0, \ \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0, T] \tag{51}$$

and as (35) and from Remark (2) we have $c \in C(0, T; L^2(\Omega))$.
3. In order to prove for prouver $c(0) = c_0$, we first note from (35) that

$$\int_0^t -\langle c, v'\rangle + L(c, v, t)dt = (c(0), v(0)), \tag{52}$$

for each $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (49) we obtain

$$\int_0^t -\langle c_m, v'\rangle + B(c_m, v, t)dt = (c_0, v(0)), \tag{53}$$

we use again (52), we obtain

$$\int_0^t -\langle c, v'\rangle + B(c, v, t)dt = (c_0, v(0)), \tag{54}$$

since $c_m(0) \to c_0$ in $L^2(\Omega)$. Comparing (52) and (54), we conclude $c(0) = c_0$. □

Theorem 6. (Uniqueness of weak solutions.) A weak solution of problem $(P2)$ is unique.

Proof. We suppose there exists two weak solution c_1 et c_2 and we put that
$C = c_2 - c_1$ then C is also a solution of $(P2)$ with $C_0 = (c_2 - c_1)(0) \equiv 0$. Setting $v = C$ in identity (51) we have

$$\frac{d}{dt}(\frac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) + L(C, C, t) = 0,$$

and as $\parallel C \parallel = \sqrt{L(C, C)}$ (= norm in $H_0^1(\Omega)$), there $L(C, C, t) = \parallel C \parallel^2_{H_0^1(\Omega)} \geq 0$, then we have

$$\frac{d}{dt}(\frac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) \leq 0,$$

then integrate with respect to t to find

$$\parallel C \parallel^2_{L^2(\Omega)} \leq \parallel C_0 \parallel^2_{L^2(\Omega)} = 0,$$

then $C \equiv 0$. □

2.3 | Existence and uniqueness of weak solution of the problem (P3)

In subsection, we state and prove existence and uniqueness of weak solution result of the problem $(P3)$

Definition 3. We say $v \in L^2(0, T; H_0^1(\Omega))$ with $v_t \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution to the problem $(P3)$ if and only if

$$\langle v_t, \Psi\rangle + A(v, \Psi, t) = 0, \tag{55}$$

when

$$A(v, \Psi, t) = \int_\Omega \nabla v \nabla \Psi dx, \tag{56}$$

for all $\Psi \in H_0^1(\Omega)$,$0 \leq t \leq T$,and

$$v(0,x) = v_0 \in L^2(\Omega). \tag{57}$$

Remark 3. Note that $v \in C([0,T]; L^2(\Omega))$ as $v \in L^2(0,T; H_0^1(\Omega))$ and $v_t \in L^2(0,T; H^{-1}(\Omega))$ Then equality (57) makes sense.

Before proving the existence and uniqueness of weak solution of problem $(P3)$, we need the following lemma:

Lemma 2. i)For all $\Psi \in H_0^1(\Omega)$ and $A(v,\Psi,t)$is continuous in $H_0^1(\Omega) \times H_0^1(\Omega)$, there exists a constant positive Csuch that

$$| A(v,\Psi,t) | \leq C \parallel v \parallel_{H_0^1(\Omega)} \parallel \Psi \parallel_{H_0^1(\Omega)} . \tag{58}$$

ii)For any $v \in H_0^1(\Omega)$ Then there exists a constant positive β such that

$$\beta \parallel v \parallel_{H_0^1(\Omega)} \leq A(v,v,t), \ \forall v \in H_0^1(\Omega). \tag{59}$$

Proof. i) We use the Cauchy-Schwarz inequality on (56) we obtain

$$| A(u,v,t) | \leq \parallel \nabla v \parallel_{L^2(\Omega)} \parallel \nabla \Psi \parallel_{L^2(\Omega)},$$

and

$$| A(u,v,t) | \leq C \parallel v \parallel_{H^1(\Omega)} \parallel \Psi \parallel_{H^1(\Omega)} .$$

ii) The expression of $A(v,v,t)$ becomes

$$A(v,v,t) = \int_\Omega (\nabla v)^2 dx = \parallel \nabla v \parallel^2_{L^2(\Omega)},$$

finally, Poincares inequality, gives $A(v,v,t) \geq \beta \parallel v \parallel^2_{H_0^1(\Omega)}$. □

To demonstrate existence of a weak solution of $(P3)$ we use the Galerkin method we suppose that $w_k = w_k(x)$ are smooth functions checking:

$$\begin{cases} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent,} \\ \text{the finite linear combination of } w_i \text{ are dense in } H_0^1(\Omega). \end{cases} \tag{60}$$

we are looking for $v_m = v_m(t)$ solution <<approached>> of the problem in the form

$$v_m(t) = \sum_{i=1}^{m} l_{im}(t) w_i, \tag{61}$$

the l_{im} to be determined by the conditions:

$$\begin{cases} \langle v'_m, w_j \rangle + A(v_m, w_j, t) = 0, \\ \qquad 1 \leq j \leq m. \end{cases} \tag{62}$$

The system of nonlinear differential equations is to be completed by the initial condition:

$$v_m(0) = v_{0m}, \ v_{0m} = \sum_{i=1}^{m} \alpha_{im} w_j \to v_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty. \tag{63}$$

We now propose to send m to infinity and to show a subsequence of our solutions v_m of approximation problems (62) and (63) converges to a weak solution of problem $(P3)$. For this we will need uniform estimates.

2.3.1 | Energy estimates

Theorem 7. (Energy estimates.) There is a constant C depending only on Ω, T such that

$$\max_{0 \leq t \leq T} \parallel v_m \parallel_{L^2(\Omega)} + \parallel v_m \parallel_{L^2(0,T;H_0^1(\Omega))} + \tag{64}$$

$$\parallel v'_m \parallel_{L^2(0,T;H^{-1}(\Omega))} \leq C \parallel v_0 \parallel_{L^2(\Omega)} . \tag{65}$$

Proof. Multiplying equation (62) index j by $l_{jm}(t)$ and we are in j he comes :

$$\langle v'_m, w_j \rangle + A(v_m, v_m, t) = 0, \tag{66}$$

where

$$\frac{1}{2}\frac{d}{dt}[\| v_m \|^2_{L^2(\Omega)}] + A(v_m, v_m, t) = 0, \tag{67}$$

we have according to the lemma (2) there is a constant $\beta > 0$ such That

$$\beta \| v_m \|^2_{H^1_0(\Omega)} \leq B(_m, v_m, t), \forall 0 \leq t \leq T, \tag{68}$$

so

$$\frac{d}{dt}(\| v_m \|^2_{L^2(\Omega)}) + \beta \| v_m \|^2_{H^1_0(\Omega)} \leq 0, \tag{69}$$

therefore

$$\| v_m \|^2_{L^2(\Omega)} \leq \| v_m(0) \|^2_{L^2(\Omega)} \leq \| v_0 \|^2_{L^2(\Omega)}, \tag{70}$$

then we have

$$\max_{0\leq t\leq T} \| v_m \|_{L^2(\Omega)} \leq \| v_0 \|_{L^2(\Omega)} . \tag{71}$$

Integrated inequality (69) from 0 to T and we use (71) find

$$\| v_m \|^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \| v_m \|^2_{H^1_0(\Omega)} dt, \tag{72}$$

fixed everything $u \in H^1_0(\Omega)$, with $\| u \|^2_{H^1_0(\Omega)} \leq 1$, and write $u = u^1 + u^2$
where $u^1 \in (w_k)_{k=1}^{k=m}$ and $(u^2, w_k) = 0,\ (k = 1, ..., m)$. we use (62) from all $0 \leq t \leq T$ that

$$(v'_m, u^1) + A(v_m, u^1, t) = 0,$$

when (61) we find

$$(v'_m, u) = (v'_m, u^1) = -A(v_m, u^1, t),$$

therefore

$$| (v'_m, u) | \leq C \| v_m \|^2_{H^1_0(\Omega)}$$

and as

$$\| u^1 \|^2_{H^1_0(\Omega)} \leq \| u \|^2_{H^1_0(\Omega)} \leq 1,$$

therefore

$$\| v'_m \|_{H^{-1}(\Omega)} \leq C \| v_m \|_{H^1_0(\Omega)},$$

and when

$$\| v'_m \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| v'_m \|^2_{H^{-1}(\Omega)} dt \leq C \int_0^T \| v_m \|^2_{H^1_0(\Omega)} dt \leq C \| v_0 \|^2_{L^2(\Omega)} .$$

□

2.3.2 | Existence and uniqueness of weak solution

Then we pass the limit as $m \to \infty$, to build a weak solution for the initial problem condition $(P3)$.

Theorem 8. Under hypothesis (7), There is a weak solution of problem $(P3)$.

Proof. According to energy estimates (65), we see that the sequence $\{v_m\}_{m=1}^{\infty}$is bounded in $L^2(0, T; H^1_0(\Omega))$ and $\{v'_m\}_{m=1}^{\infty}$ is bounded in $L^2(0; T; H^{-1}(\Omega))$ Therefore, there is a subsequence which is also noted by$\{v_m\}_{m=1}^{\infty}$ and a function $v \in L^2(0, T; H^1_0(\Omega))$, with $v' \in L^2(0; T; H^{-1}(\Omega))$, such that

$$v_m \to v \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \tag{73}$$

$$v'_m \to v' \text{ weakly in } L^2(0; T; H^{-1}(\Omega)). \tag{74}$$

Next fix an integer N and takes a function $u \in C^1(0, T; H^1_0(\Omega))$ under the form

$$u(t) = \sum_{k=1}^{N} g^{(k)}(t) w_k, \tag{75}$$

where $\left\{g^{(k)}\right\}_{k=1}^{N}$ are given smooth functions and we choose $m \geq N$, multipling the equation (62) by $g^{(k)}(t)$ and $\forall K = 1...N$, then integrate with respect to t to find

$$\int_0^t \langle v_m^{'}, u\rangle + A(v_m, u, t)dt = 0, \tag{76}$$

we recall and (74) to find passing low limits that

$$\int_0^t \langle v^{'}, u\rangle + A(v, u, t)dt = 0, \ \forall u \in L^2(0, T; H_0^1(\Omega)), \tag{77}$$

as functions of the form (75) are dense in $L^2(0, T; H_0^1(\Omega))$. In particular

$$\langle v', u\rangle + A(v, u, t) = 0, \ \forall u \in H_0^1(\Omega) \text{ and } \forall t \in [0, T] \tag{78}$$

and as (57) and from Remark(3) we have $v \in C(0, T; L^2(\Omega))$.
To prove $v(0) = v_0$, we first note of (10) that

$$\int_0^t -\langle v, u'\rangle + A(v, u, t) = (v(0), u(0)), \tag{79}$$

for each $u \in C^1(0, T; H_0^1(\Omega))$ with $u(T) = 0$. Similary, from (76) we obtian

$$\int_0^t -\langle v_m, u'\rangle + A(v_m, u, t)dt = (v_0, u(0)), \tag{80}$$

we use again (73), we obtian

$$\int_0^t -\langle v, u'\rangle + A(v, u, t)dt = (v_0, u(0)), \tag{81}$$

since $v_m(0) \to v_0$ in $L^2(\Omega)$. Comparing (80) and (81), we conclude $v(0) = v_0$. □

Theorem 9. (Uniqueness of weak solutions.) A weak solution of problem $(P3)$ is unique.

Proof. We suppose there exists two weak solution v_1 and v_2 and we put that $V = v_2 - v_1$ then V is also a solution of $(P3)$ with $V_0 = (v_2 - v_1)(0) \equiv 0$. Setting $u = V$ in identity (78), we have

$$\frac{d}{dt}\left(\frac{1}{2} \parallel V \parallel^2_{L^2(\Omega)}\right) + A(V, V, t) = 0,$$

for lemma (2), we have $A(V, V, t) \geq \beta \parallel V \parallel^2_{H_0^1(\Omega)} \geq 0$, so $\frac{d}{dt}(\frac{1}{2} \parallel V \parallel^2_{L^2(\Omega)}) \leq 0$,
then integrate with respect to t to find

$$\parallel V \parallel^2_{L^2(\Omega)} \leq \parallel V_0 \parallel^2_{L^2(\Omega)} = 0,$$

thus $V \equiv 0$.
Then we have the existence and uniqueness of weak solution to chemetaxis model coupled with heat equation. □

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References

1. M. D. Betterton and M. P. Brenner, Collapsing bacterial cylinders, Phys. Rev. E, 6.

2. S. Boulaaras, R. Guefaifia, and N. Mezouar, Global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms, App. Anal. (2020), in press. https://doi.org/10.1080/00036811.2020.1760250.

3. S. Boulaaras, R. Guefaifia and B. Cherif, T. Radwan Existence result for a Kirchhoff elliptic system involving p-Laplacian operator with variable parameters and additive right hand side via sub and super solution methods, December 2020AIMS Mathematics 6(3):2315-2329, DOI: 10.3934/math.2021140.

4. E. O. Budrene and H. C. Berg, Complex patterns formed by motile cells of Escherichia coli, Nature (London), 349 (1991), pp. 630–633.

5. E. O. Budrene and H. C. Berg, Dynamics of formation of symmetrical patterns by chemotactic bacteria, Nature (London), 376 (1995), pp. 49–53.A.

6. M. P. Brenner, L. Levitov, and E. O. Budrene, Physical mechanisms for chemotactic pattern formation by bacteria, Biophys. J., 74 (1995), pp. 1677–1693.

7. S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., 56 (1981), pp.217–237.

8. A.Guesmia and N. Daili, About the existence and uniqueness of solution to fractional Burgers equation, Acta Universitatis Apulensis 21(2010), 161 – 170.

9. A. Guesmia and N. Daili, Existence and uniqueness of an entropy solution for Burgers equations, Applied Mathematical Sciences 2(33) (2008),1635 – 1664.

10. M. A. Herrero and J. J. L. Velazquez, A blow-up mechanism for a chemotaxis model, Ann. Scoula Norm. Pisa IV, 35 (1997) pp. 633–683.

11. M. A. Herrero, E. Medina, and J. J. L. Velazquez, Finite-time aggregation into a single point in a reaction-diffusion system, Nonlinearity, 10 (1997) pp. 1739–1754.

12. M. A. Herrero, E. Medina, and J. J. L. Velazquez, Self-similar blow-up for a reactiondiffusion system, Journal of Computational and Applied Mathematics, 97 (1998) pp. 99–119.

13. T. Hillen and A. Potapov, The one-dimensional chemotaxis model: global existence and asymptotic profile, Math. Methods. Appl. Sci., 27 (2004), pp. 1783–1801.4 (2001),pp. 061904.

14. D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), pp. 52–107.

15. W. Jager and S. Luckhaus , On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., 329 (1992), pp. 819–824.

16. E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J.Theor. Biol., 26 (1970), pp. 399âĂŞ415.

17. C. Messikh, A. Guesmia & S. Saadi Global Existence and Uniqueness of the Weak Solution in Keller Segel Model 2249-4626 & Print ISSN: 0975-5896.

18. H. Mohamed, S. Boulaaras and Y. Bouizem, R. Guefaifia, Existence result for a Kirchhoff elliptic system with variable parameters and additive right-hand side via sub-and supersolution method, Springer, August 2020, 10.1186/s13661-020-01435-z.

19. J. D. Murray, Mathematical Biology, 3rd. ed. (Vol. I and II) Springer, New York, NY, 2002.

20. T. Nagai Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl.,5 (1995), pp. 581–601.

21. K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations,Funkcial. Ekvac., 44 (2001), pp.441–469.

22. K. Zennir, Results on existence for generalized nD Navier-Stokes equations, Open Mathematics, org/10.1515/math-2019-0137.

23. H. Dridi, K. Zennir and S. Alodhaibi, S. Alkhalaf, Nonexistence of Global Solutions for Coupled System of Pseudoparabolic Equations with Variable Exponents and Weak Memories, in Journal of Function Spaces , February 2021, 10.1155/2021/5573959.

24. M. Biomy, K, Zennir and A. Himadan, Local and Global Existence of Solution for Love Type, in journal mathematics November 2020. Waves with Past History

25. K. Zennir, A. Boukarou and R. Nasser Alkhudayr, Global well-posedness for coupled system of MKDV equations in analytic spaces, March 2021, Journal of Function Spaces.

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