Identification of an unbounded bi-periodic interface for the inverse fluid-solid interaction problem

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Abstract

This paper is concerned with the inverse scattering of acoustic waves by an unbounded periodic elastic medium in the three-dimensional case. A novel uniqueness theorem is proved for the inverse problem of recovering a bi-periodic interface between acoustic and elastic waves using the near-field data measured only from the acoustic side of the interface, corresponding to a countably infinite number of quasi-periodic incident acoustic waves. The proposed method depends only on a fundamental a priori estimate established for the acoustic and elastic wave fields and a new mixed-reciprocity relation established in this paper for the solutions of the fluid-solid interaction scattering problem.

Keywords: Inverse scattering, bi-periodic interface, uniqueness, fluid-solid interaction.

1 Introduction

This paper is concerned with the scattering of time-harmonic acoustic waves by an unbounded bi-periodic interface in three dimensions. The medium above the interface is assumed to be filled with homogeneous compressible inviscid fluid, and the medium below the interface is occupied by an isotropic linearly elastic solid. Recently, this class of problems have attracted much attention due to practical applications in diverse fields such as modern diffractive optics and nondestructive testing; see e.g., [5, 6] and references therein.

For convenience, we write a point x in $\mathbb{R}^3$ for $(\widetilde{x}, x_3)$ with $\widetilde{x} := (x_1, x_2) \in \mathbb{R}^2$. Let Γ denote a bi-periodic interface described by a $\mathcal{C}^2$-smooth function f. We assume that f is periodic with respect to the variable $\widetilde{x}$, that is, $f(\widetilde{x}, x_3) = f(\widetilde{x} + 2n\pi, x_3)$ for $n := (n_1, n_2) \in \mathbb{Z}^2$. As shown in Figure 1, let Ω^+ denote the unbounded region above Γ, filled with fluid with the real valued

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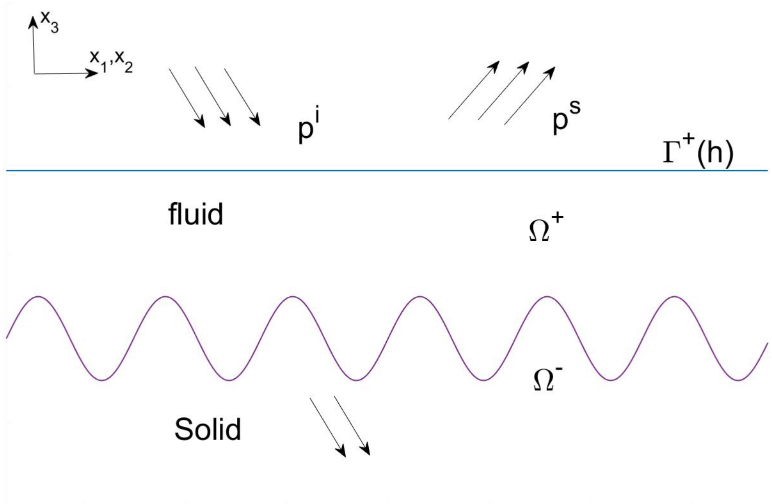

Figure 1: Geometric configuration of the scattering problem

constant mass density $\rho_f > 0$. And let Ω^- denote the unbounded region below Γ, occupied by elastic solid with the real valued constant mass density $\rho_s > 0$ and the lamé constants $\lambda,\ \mu \in \mathbb{R}$ satisfying the condition that $\mu > 0$, $3\lambda + 2\mu > 0$.

Consider the incident wave taking the form

$$p^i(x) = \exp(i\alpha_j \cdot \widetilde{x} - i\eta_j x_3), \quad j \in \mathbb{Z}^2, \tag{1.1}$$

where $\alpha_j = \alpha + j$ with $\alpha = (\alpha_1, \alpha_2) := k(\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2)$ for $\theta_1 \in [0, \pi/2), \theta_2 \in [0, 2\pi)$ and $k \in \mathbb{R}_+$ is the wavenumber, and $\eta_j \in \mathbb{C}$ is given by

$$\eta_j = \sqrt{k^2 - |\alpha_j|^2}, \quad \text{if } |\alpha_j| \le k, \qquad \eta_j = i\sqrt{|\alpha_j|^2 - k^2}, \quad \text{if } |\alpha_j| > k.$$

Under the hypothesis of small amplitude oscillations in both the solid and the fluid, the scattering problem can be formulated as finding the wave field $(p, \mathbf{u})$ such that

$$\begin{cases} \triangle p + k^2 p = 0 & \text{in } \Omega^+, \\ \triangle^* \mathbf{u} + \rho_s \omega^2 \mathbf{u} = 0 & \text{in } \Omega^-, \\ t(\mathbf{u}) = -p\nu & \text{on } \Gamma, \\ \eta \mathbf{u} \cdot \nu = \frac{\partial p}{\partial \nu} & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $\triangle^* := \mu\triangle + (\lambda + \mu)\nabla \mathrm{div}$ denotes the Navier operator, and p is the total field consisting of the incident wave p^i and its scattered wave p^s. In (1.2), the wave field p and $\mathbf{u}$ are coupled on the bi-periodic interface Γ with the transmission coefficient $\eta := \rho_f \omega^2$ and the stress vector

$$t(\mathbf{u}) := 2\mu \frac{\partial \mathbf{u}}{\partial \nu} + \lambda(\mathrm{div}\, \mathbf{u})\nu + \mu\nu \times \mathrm{curl}\, \mathbf{u} \quad \text{on } \Gamma,$$

where ν stands for the normal vector directing into Ω^-.

By (1.1), it is noticed that the incident wave $p^i(\cdot)$ satisfies such an α-quasi-periodic condition $p^i(\widetilde{x}+2n\pi, x_3) = e^{i2\alpha\cdot n\pi}p^i(\widetilde{x}, x_3)$ for all $n \in \mathbb{Z}^2$. So, the same condition is also required for the wave field $(p, \mathbf{u})$ in order to obtain the well-posedness of the scattering problem. That is,

$$p(\widetilde{x}+2n\pi, x_3) = e^{i2\alpha\cdot n\pi}p(\widetilde{x}, x_3), \qquad \mathbf{u}(\widetilde{x}+2n\pi, x_3) = e^{i2\alpha\cdot n\pi}\mathbf{u}(\widetilde{x}, x_3). \tag{1.3}$$

Moreover, the upward and downward Rayleigh expansions are imposed on the scattered field p^s and transmitted field $\mathbf{u}$, respectively, in the periodical case:

$$\begin{cases} p^s = \displaystyle\sum_{n\in\mathbb{Z}^2} p_n \exp(i\alpha_n \cdot \widetilde{x} + i\eta_n x_3), \quad x_3 > A_1, \\ \mathbf{u} = \displaystyle\sum_{n\in\mathbb{Z}^2} \left\{ A_{p,n} \begin{pmatrix} \alpha_n^T \\ -\beta_n \end{pmatrix} \exp(i\alpha_n\cdot\widetilde{x} - i\beta_n x_3) + \mathbf{A}_{s,n}\exp(i\alpha_n\cdot\widetilde{x} - i\gamma_n x_3) \right\}, \quad x_3 < A_2, \end{cases} \tag{1.4}$$

where $A_1 := \max(f)$, $A_2 := \min(f)$, the Rayleigh coefficients $p_n \in \mathbb{C}$, $A_{p,n} \in \mathbb{C}$ and $\mathbf{A}_{s,n} \in \mathbb{C}^3$ satisfies the property that $\mathbf{A}_{s,n}\cdot(\alpha_n, -\gamma_n)^T = 0$ for all $n \in \mathbb{Z}^2$. Similar to the definition of η_n, β_n and γ_n are defined by the wavenumber $k_p := \omega\sqrt{\rho_s/(2\mu+\lambda)}$ and $k_s := \omega\sqrt{\rho_s/\mu}$, respectively.

For any fixed $j \in \mathbb{Z}^2$, it has been shown in [9] that the scattering problem (1.1)-(1.4) has a unique solution in the classical H^1-space for all frequencies excluding a discrete set with the only accumulation point at infinity, using the variational method. For convenience, throughout this paper, we always assume that the problem (1.1)-(1.4) is uniquely solvable. The inverse problem we are concerned in this paper with is to determine the shape and location of the bi-periodic interface Γ by the knowledge of the acoustic wave fields p in Ω^+. Lots of uniqueness theorems and numerical methods for periodic interfaces can be found in e.g., [2, 7, 11, 12, 15, 16, 19] and references therein for the full Helmholtz equation or Maxwell's equation case. However, to the best of authors' knowledge, there exists almost no results available on the inverse problem considered in the current paper in the literature. Recently, a factorization method was just studied in [18] for numerically reconstructing the bi-periodic interface, generating the idea of A. Kirsch et al. [10] for the bounded case. Notice that a related uniqueness result [13] was shown in determining a bounded elastic body from the far-field data. The method used in [13] follows from the work [8, 3] with the full Helmholtz equation, based on a technical analysis of both an interior transmission problem in the whole domain and the H^2-regularity of the scattered solution. It seems very hard to extend this idea to the inverse fluid-solid interaction problem in the periodical case, since the interface under consideration is unbounded. It is important for the current topic to mention that a uniqueness result [14] was established for identifying a bounded penetrable solid body with buried objects inside, which depends only on a fundamental a priori estimate of the scattered solution.

In this paper, we shall study the unique determination of a bi-periodic interface by the knowledge of the near-field data only from one side of the interface. A motivation comes from the work of Yang et al. [17] for the full Helmholtz equation in the bounded case. We expect to address a novel version of uniqueness for the inverse problem in the periodical case. To this end, a related scattering problem is first considered with a series of incident point sources in Section 2. Then, a uniform a priori estimate of solutions is proved as the location of the point sources approximate the interface. This, combined with a new mixed-relation established in Section 3, leads to a novel uniqueness result on the inverse problem. That is, the bi-periodic interface can be uniquely recovered by means of acoustic near-field data in the top domain above the interface, generated by a countably infinite number of quasi-periodic incident waves. Particularly, the same result can be also obtained if the elastic near-field data are taken only in the bottom domain below the interface. The method proposed depends only on the H^1-estimate of the elastic wave field and the L^2-estimate of the acoustic wave field, and is thus very simple.

We conclude this section with some notations that will be used in the rest of this paper. For simplicity, we use $\Omega^\pm$ and $\Gamma^\pm$ again to denote the same sets restricted to one period $0 < x_1, x_2 < 2\pi$. We also introduce four subsets of $\Omega^\pm$ for each $h > 0$, denoted by $\Omega^+(h) := \{x \in \Omega^+ : x_3 < A_1 + h\}$, $\Omega^-(h) := \{x \in \Omega^- : x_3 > A_2 - h\}$, $\Gamma^+(h) := \{x \in \Omega^+ : x_3 = A_1 + h\}$ and $\Gamma^-(h) := \{x \in \Omega^- : x_3 = A_2 - h\}$, respectively. Then, let $H^1_\alpha(\Omega^\pm(h))$ and $L^p_\alpha(\Omega^\pm(h))(p \geq 1)$ denote the Sobolev spaces of scalar functions on $\Omega^\pm(h)$ which are assumed to be α-quasi-periodic with respect to the variable $\widetilde{x}$. They are equipped with the norms in the usual Sobolev spaces $H^1(\Omega^\pm(h))$ and $L^p(\Omega^\pm(h))$, respectively, whereas $H^{1/2}_\alpha(\Gamma^\pm(h))$ denote the trace space of $H^1_\alpha(\Omega^\pm(h))$ and $H^{-1/2}_\alpha(\Gamma^\pm(h))$ is the dual space of $H^{1/2}_\alpha(\Gamma^\pm(h))$.

2 A priori estimate for the wave fields

In this section, we shall study the scattering problem (1.2)-(1.4) with the incident field p^i induced by a point source

$$G(x,y) = \frac{i}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{1}{\eta_n} \exp(i\alpha_n \cdot (\widetilde{x} - \widetilde{y}) + i\eta_n|x_3 - y_3|), \quad x \neq y. \tag{2.1}$$

located at $y \in \Omega^+$, namely, the α-quasi-periodic Green's function of the Helmholtz equation in the free space satisfying $\triangle G(\cdot,y) + k^2 G(\cdot,y) = -\delta_y(\cdot)$ in $\mathbb{R}^3$. Eliminating the incident field p^i, the scattered field p^s and $\mathbf{u}$ solve the transmission problem

$$\begin{cases} \triangle E + k^2 E = 0 & \text{in } \Omega^+, \\ \triangle^* \mathbf{F} + \rho_s \omega^2 \mathbf{F} = 0 & \text{in } \Omega^-, \\ t(\mathbf{F}) + E\nu = \mathbf{f}_1 & \text{on } \Gamma, \\ \eta \mathbf{F} \cdot \nu - \frac{\partial E}{\partial \nu} = f_2 & \text{on } \Gamma, \end{cases} \tag{2.2}$$

with the α-quasi-periodic condition (1.3) and the Rayleigh expansions (1.4), where $\mathbf{f}_1(\cdot) := -G(\cdot,y)\nu$ and $f_2(\cdot) := \partial_\nu G(\cdot,y)$. The well-posedness of (2.2) can be similarly proved as in [9] by the variational method or the boundary integral equation method. Then, a precise singularity analysis will be explored for the scattered fields p^s and $\mathbf{u}$, as the location of the incident point source approximates the bi-periodic interface.

For a fixed $y_0 \in \Gamma$, define a sequence of points by

$$y_j := y_0 - \frac{\delta}{j}\nu(y_0), \qquad j = 1, 2, 3, \cdots\cdots$$

with a sufficiently small $\delta > 0$ such that $y_j \in \Omega^+$ for all $j \in \mathbb{N}_+$. We will show that $\mathbf{u}$ is uniformly bounded in the H^1-space and p^s has the at most same singularity with the Green's function $G(\cdot, y_j)$ near the point y_0, as $j \to \infty$. To this end, let $E_j(\cdot) := p^s(\cdot; y_j) - G(\cdot, z_j)$ in Ω^+ and $\mathbf{F}_j(\cdot) := \mathbf{u}(\cdot; y_j)$ with z_j defined by $z_j := y_0 + (\delta/j)\nu(y_0)$, $j \in \mathbb{N}_+$. It is easily checked that E_j and $\mathbf{F}_j$ satisfy the transmission problem (2.2), (1.3) and (1.4) with the boundary conditions

$$\mathbf{f}_{1,j}(\cdot) : = -G(\cdot, y_j)\nu - G(\cdot, z_j)\nu, \qquad \text{on } \Gamma \tag{2.3}$$

$$f_{2,j}(\cdot) : = \partial_\nu G(\cdot, y_j) + \partial_\nu G(\cdot, z_j), \qquad \text{on } \Gamma \tag{2.4}$$

for each $j \in \mathbb{N}_+$.

Notice that the difference between G and Φ defines an analytic function in $\mathbb{R}^3$, where $\Phi(x, y) = e^{ik|x-y|}/(4\pi|x-y|)$ denotes the fundamental solution of the Helmholtz equation in the free space. One thus has $\mathbf{f}_{1,j} \in L^{2-\varepsilon_1}_\alpha(\Gamma)^3$ and $f_{2,j} \in L^\infty_\alpha(\Gamma)$ uniformly for all $j \in \mathbb{N}_+$ and any fixed $0 < \varepsilon_1 \le 1$. This allows us to have the following a priori estimates of the solutions.

Theorem 2.1. *Let p_j and $\mathbf{u}_j$ be the solution to the scattering problem (1.2)-(1.4) with the incident point source $p^i(\cdot) = G(\cdot, y_j)$. Then,*

$$\|p_j\|_{L^2_\alpha(\Omega^+(h))} + \|\nabla p_j\|_{L^{3/2-\varepsilon}_\alpha(\Omega^+(h))^3} + \|\mathbf{u}_j\|_{H^1_\alpha(\Omega^-(h))^3} \le C, \tag{2.5}$$

where $\varepsilon > 0$ is chosen with $0 < \varepsilon \le \frac{1}{2}$ and $C > 0$ is independent of $j \in \mathbb{N}_+$.

Proof. Based on the above analysis, it is known that the functions E_j and $\mathbf{F}_j$ satisfy the boundary value problem (2.2) with (1.3), (1.4) and (2.3)-(2.4).

For any fixed $j \in \mathbb{N}_+$, by the Green and Betti's formulas, we deduce that E_j and $\mathbf{F}_j$ solve the variational problem: find $(E_j, \mathbf{F}_j) \in X := H^1_\alpha(\Omega^+(h)) \times H^1_\alpha(\Omega^-(h))^3$ such that

$$A((E_j, \mathbf{F}_j); (\phi, \mathbf{\Psi})) = F((\mathbf{f}_{1,j}, f_{2,j}); (\phi, \mathbf{\Psi})) \quad \text{for all } (\phi, \mathbf{\Psi}) \in X, \tag{2.6}$$

where the sesquilinear form $A : X \times X \to \mathbb{C}$ is defined as

$$A((E_j, \mathbf{F}_j); (\phi, \mathbf{\Psi})) := \int_{\Omega^+(h)} (\nabla E_j \cdot \nabla\overline{\phi} - k^2 E_j\overline{\phi})dx - \eta\int_\Gamma \mathbf{F}_j \cdot \nu\overline{\phi}ds - \int_{\Gamma^+(h)} T^+ E_j\overline{\phi}ds$$
$$+\eta\left\{\int_{\Omega^-(h)} (\mathcal{E}(\mathbf{F}_j, \overline{\mathbf{\Psi}}) - \rho_s\omega^2\mathbf{F}_j \cdot \overline{\mathbf{\Psi}})dx - \int_\Gamma E_j\nu \cdot \overline{\mathbf{\Psi}}ds - \int_{\Gamma^-(h)} T^-\mathbf{F}_j \cdot \overline{\mathbf{\Psi}}ds\right\}$$

and the right functional $F : X \to \mathbb{C}$ is defined as

$$F((\mathbf{f}_{1,j}, f_{2,j}); (\phi, \mathbf{\Psi})) := -\int_\Gamma f_{2,j}\overline{\phi}ds - \eta\int_\Gamma \mathbf{f}_{1,j} \cdot \overline{\mathbf{\Psi}}ds.$$

In (2.6), $\mathcal{E}(\cdot,\cdot)$ is described by

$$\mathcal{E}(\mathbf{H}_1,\overline{\mathbf{H}_2}) := 2\mu \sum_{i,j=1}^{3} \partial_i H_{1,j}\partial_i \overline{H_{2,j}} + \lambda(\operatorname{div}\mathbf{H}_1)(\operatorname{div}\overline{\mathbf{H}_2}) - \mu \operatorname{curl}\mathbf{H}_1\cdot\operatorname{curl}\overline{\mathbf{H}_2}$$

with $\mathbf{H}_1 := (H_{1,1}, H_{1,2}, H_{1,3})$ and $\mathbf{H}_2 := (H_{2,1}, H_{2,2}, H_{2,3})$, $T^+ : H_\alpha^{\frac{1}{2}}(\Gamma^+(h)) \to H_\alpha^{-\frac{1}{2}}(\Gamma^+(h))$ and $T^- : H_\alpha^{\frac{1}{2}}(\Gamma^+(h))^3 \to H_\alpha^{-\frac{1}{2}}(\Gamma^+(h))^3$ are two Dirichlet-to-Neumann operators defined by $T^+g = \partial_\nu v|_{\Gamma^+(h)}$ and $T^-\mathbf{g} = t(\mathbf{v})|_{\Gamma^-(h)}$, respectively, which are related to the following two Dirichlet problems

$$\begin{cases} \triangle v + k^2 v = 0 & \text{in } x_3 > A_1 + h, \\ v = g & \text{on } \Gamma^+(h), \\ v \text{ satisfies the upward Rayleigh expansion in (1.4)} \end{cases} \tag{2.7}$$

for $g \in H_\alpha^{\frac{1}{2}}(\Gamma^+(h))$ and

$$\begin{cases} \triangle^* \mathbf{v} + \rho_s\omega^2 \mathbf{v} = 0 & \text{in } x_3 < A_2 - h, \\ \mathbf{v} = \mathbf{g} & \text{on } \Gamma^-(h), \\ \mathbf{v} \text{ satisfies the downward Rayleigh expansion in (1.4)} \end{cases} \tag{2.8}$$

for $\mathbf{g} \in H_\alpha^{\frac{1}{2}}(\Gamma^-(h))^3$.

It is noticed from [9, 18] that the operator $-\mathrm{Re}(T^-)$ can be decomposed into two parts: $-\mathrm{Re}(T^-) = T_1 + T_2$ with a positive definite operator T_1 and a finite rank operator T_2. Combining with the positivity of $-\mathrm{Re}(T^+)$ implies that the sesquilinear form A can be divided into $A = A_1 + A_2$ with

$$\begin{aligned} A_1((E_j,\mathbf{F}_j);(\phi,\mathbf{\Psi})) : &= \int_{\Omega^+(h)} (\nabla E_j\cdot\nabla\overline{\phi} + E_j\overline{\phi})dx - \int_{\Gamma^+(h)} T^+E_j\overline{\phi}ds \\ &+\eta\left\{\int_{\Omega^-(h)} (\mathcal{E}(\mathbf{F}_j,\overline{\mathbf{\Psi}}) + \mathbf{F}_j\cdot\overline{\mathbf{\Psi}})dx + \int_{\Gamma^-(h)} T_1\mathbf{F}_j\cdot\overline{\mathbf{\Psi}}ds\right\} \end{aligned}$$

and

$$\begin{aligned} A_2((E_j,\mathbf{F}_j);(\phi,\mathbf{\Psi})) : &= -\int_{\Omega^+(h)} ((1+k^2)E_j\overline{\phi})dx - \eta\int_\Gamma \mathbf{F}_j\cdot\nu\overline{\phi}ds \\ &+\eta\left\{\int_{\Omega^-(h)} (\rho_1\mathbf{F}_j\cdot\overline{\mathbf{\Psi}})dx - \int_\Gamma E_j\nu\cdot\overline{\mathbf{\Psi}}ds + \int_{\Gamma^-(h)} T_2\mathbf{F}_j\cdot\overline{\mathbf{\Psi}}ds\right\}. \end{aligned}$$

Here, $\rho_1 := -(1+\rho_s\omega^2)$. It is easily seen that $A_1(\cdot;\cdot)$ is coercive on $X\times X$, and $A_2(\cdot;\cdot)$ is a compact operator on X due to the compact embedding of H^1 into L^2. Thus, $A(\cdot;\cdot)$ generates a Fredholm operator denoted by $\mathcal{A}$ on X with index 0 by the Riesz representation theorem. Combining the uniqueness of the scattering problem (1.2)-(1.4) yields that $\mathcal{A}$ has a bounded invertible operator $\mathcal{A}^{-1}$ on X.

On the other hand, recalling that $\mathbf{f}_{1,j} \in L^{2-\varepsilon_1}_\alpha(\Gamma)^3$ and $f_{2,j} \in L^\infty_\alpha(\Gamma)$ uniformly for all $j \in \mathbb{N}_+$ and any fixed $0 < \varepsilon_1 \leq 1$, then choosing $\varepsilon_1 = 1/2$, we immediately deduce that $\mathbf{f}_{1,j} \in L^{3/2}_\alpha(\Gamma)^3$ and $f_{2,j} \in L^\infty_\alpha(\Gamma)$ uniformly for all $j \in \mathbb{N}_+$. Furthermore, it follows from [1, Theorem 5.36] that, for any $\psi \in H^1_\alpha(\Omega^\pm(h))$,

$$\|\psi\|_{L^4_\alpha(\Gamma)} \leq C\|\psi\|_{H^1_\alpha(\Omega^\pm(h))},$$

where $C > 0$ is independent of $\psi \in H^1_\alpha(\Omega^\pm(h))$. So we observe from the definition of the function F that it defines a family of bounded linear functionals $\mathcal{F}_j$ on X, that is, there exists some $C > 0$, independent of $j \in \mathbb{N}_+$, such that

$$|\mathcal{F}_j(\phi, \mathbf{\Psi})| = |F((\mathbf{f}_{1,j}, f_{2,j}); (\phi, \mathbf{\Psi}))| \leq C(\|\phi\|_{H^1_\alpha(\Omega^+(h))} + \|\mathbf{\Psi}\|_{H^1_\alpha(\Omega^-(h))^3}).$$

Now, we arrive from this equality and the boundedness of $\mathcal{A}^{-1}$ at

$$\|E_j\|_{H^1_\alpha(\Omega^+(h))} + \|\mathbf{F}_j\|_{H^1_\alpha(\Omega^-(h))^3} \leq C \tag{2.9}$$

uniformly for all $j \in \mathbb{N}_+$. Recalling that $E_j(\cdot) = p(\cdot; y_j) - G(\cdot; y_j) - G(\cdot, z_j)$ in Ω^+ and $\mathbf{F}_j = \mathbf{u}_j$ in Ω^- yields the required estimate (2.5), due to the weak singularity of the Green's function $G(\cdot, \cdot)$. This completes the proof. □

3 Uniqueness of the inverse problem

In this section, we shall study the inverse fluid-solid interaction problem in the bi-periodical case. The uniqueness result is proved for recovering the bi-periodic interface by making use of the acoustic near-field data. The proposed method depends on the H^1-estimate and a new mixed-reciprocity relation of solutions of the scattering problem.

The next lemma is related to the mixed-reciprocity relation between the incident plane wave in (1.1) and the incident point source in (2.1). To see this, we define $\hat{\alpha} := -\alpha$ and consider the $\hat{\alpha}$-quasi-periodic scattering problem

$$\begin{cases} \triangle \hat{p} + k^2 \hat{p} = 0 & \text{in } \Omega^+, \\ \triangle^* \hat{\mathbf{u}} + \rho_s \omega^2 \hat{\mathbf{u}} = 0 & \text{in } \Omega^-, \\ t(\hat{\mathbf{u}}) = -\hat{p}\nu & \text{on } \Gamma, \\ \eta \hat{\mathbf{u}} \cdot \nu = \frac{\partial \hat{p}}{\partial \nu} & \text{on } \Gamma, \end{cases} \tag{3.1}$$

with $\hat{p} := \hat{p}^i + \hat{p}^s$ in Ω^+ and $\hat{\mathbf{u}}$ in Ω^- satisfying the $\hat{\alpha}$-quasi-periodic condition

$$\hat{p}(\widetilde{x} + 2n\pi, x_3) = e^{i2\hat{\alpha}\cdot n\pi}\hat{p}(\widetilde{x}, x_3), \qquad \hat{\mathbf{u}}(\widetilde{x} + 2n\pi, x_3) = e^{i2\hat{\alpha}\cdot n\pi}\hat{\mathbf{u}}(\widetilde{x}, x_3) \tag{3.2}$$

and the upward and downward Rayleigh expansions

$$\begin{cases} \hat{p}^s = \displaystyle\sum_{n\in\mathbb{Z}^2} \hat{p}_n \exp(i\hat{\alpha}_n \cdot \widetilde{x} + i\hat{\eta}_n x_3), \quad x_3 > A_1, \\ \hat{\mathbf{u}} = \displaystyle\sum_{n\in\mathbb{Z}^2} \left\{ \hat{A}_{p,n} \begin{pmatrix} \hat{\alpha}_n^T \\ -\hat{\beta}_n \end{pmatrix} \exp(i\hat{\alpha}_n \cdot \widetilde{x} - i\hat{\beta}_n x_3) + \hat{\mathbf{A}}_{s,n} \exp(i\hat{\alpha}_n \cdot \widetilde{x} - i\hat{\gamma}_n x_3) \right\}, \quad x_3 < A_2. \end{cases} \tag{3.3}$$

Here, $\hat{\alpha}_n, \hat{\beta}_n, \hat{\eta}_n, \hat{\gamma}_n$ are defined by $\alpha_n, \beta_n, \eta_n, \gamma_n$ with α replaced by $\hat{\alpha}$, respectively.

If $\hat{p}^i(\cdot) = \hat{G}(\cdot, z_0)$ with $z_0 \in \Omega^+$, we indicate the dependence of the wave fields on the location of the point source by writing $\hat{p}^s(\cdot; z_0)$ and $\hat{\mathbf{u}}(\cdot; z_0)$. Moreover, let $p(\cdot; m)$ and $\mathbf{u}(\cdot; m)$ be the solution of (1.2)-(1.4) corresponding to the incident wave $p^i(x; m) = e^{i\alpha_m\cdot\widetilde{x} - i\eta_m x_3}$ for $m \in \mathbb{Z}^2$. Then, one can conclude the following result.

Lemma 3.1. *For $z_0 \in \Omega^+$, let $\hat{p}_n(z_0)$ be the Rayleigh coefficients of $\hat{p}^s(\cdot; z_0)$. Then*

$$p^s(z_0; m) = -8\pi^2 i\hat{\eta}_{-m}\hat{p}_{-m}(z_0) \quad \text{for all } m \in \mathbb{Z}^2. \tag{3.4}$$

Proof. Due to $z_0 \in \Omega^+$, we can first define a small ball denoted by $B_\delta(z_0)$, which is centered at z_0 with radius $\delta \in \mathbb{R}_+$ such that $B_\delta(z_0) \subseteq \Omega^+$ for sufficiently small $\delta > 0$. It then follows from the definition of the domain $\Omega^+(h)$ that we can choose some $h_0 > 0$ such that $B_\delta(z_0) \subseteq \Omega^+(h_0)$. Now, an application of the Green's theorem yields that

$$\begin{aligned} 0 &= \int_{\Omega^+(h_0)\backslash\overline{B_\delta(z_0)}} \left[\triangle p(x; m)\hat{p}(x, z_0) - p(x; m)\triangle\hat{p}(x, z_0)\right] dx \\ &= \left\{\int_\Gamma - \int_{\Gamma^+(h_0)} - \int_{\partial B_\delta(z_0)}\right\} \left[\frac{\partial p(x; m)}{\partial\nu(x)}\hat{p}(x, z_0) - p(x; m)\frac{\partial\hat{p}(x, z_0)}{\partial\nu(x)}\right] ds(x) \\ &= : I - II - III \end{aligned} \tag{3.5}$$

In the above equality, the integrals vanish on two vertical lines due to the quasi-periodicity of the functions $p(x; m)$ and $\hat{p}(x, z_0)$.

Using the transmission conditions on Γ and applying the Betti's formula for the elastic fields $\mathbf{u}(x; m)$ and $\hat{\mathbf{u}}(x; z_0)$ in $\Omega^-(h_0)$, we have

$$\begin{aligned} I &= \eta\int_\Gamma \left[t(\mathbf{u}(x; m))\hat{\mathbf{u}}(x; z_0) - \mathbf{u}(x; m)t(\hat{\mathbf{u}}(x; z_0))\right] ds(x) \\ &= \eta\int_{\Gamma^-(h_0)} \left[t(\mathbf{u}(x; m))\hat{\mathbf{u}}(x; z_0) - \mathbf{u}(x; m)t(\hat{\mathbf{u}}(x; z_0))\right] ds(x) \\ &= 0. \end{aligned} \tag{3.6}$$

Here, the downward Rayleigh expansions for $\mathbf{u}(x; m)$ and $\hat{\mathbf{u}}(x; z_0)$ as well as the fact that $\beta_n(\alpha) = \hat{\beta}_{-n}(\hat{\alpha}), \gamma_n(\alpha) = \hat{\gamma}_{-n}(\hat{\alpha})$ for all $n \in \mathbb{Z}^2$ have been used in deriving the last equality of (3.6).

To estimate the terms II and III, it is found by the upward Rayleigh expansions for both $p^s(x; m)$ and $\hat{p}(x; z_0)$, and also $\eta_n(\alpha) = \hat{\eta}_{-n}(\hat{\alpha})$ that

$$\int_{\Gamma^+(h_0)} \left[\frac{\partial p^s(x; m)}{\partial\nu(x)}\hat{p}(x, z_0) - p^s(x; m)\frac{\partial\hat{p}(x, z_0)}{\partial\nu(x)}\right] ds(x) = 0. \tag{3.7}$$

On the other hand, since both $p(x; m)$ and $\hat{p}^s(x; z_0)$ are smooth and satisfy the Helmholtz equations in $B_\delta(z_0)$, it is deduced from the Green's theorem that

$$\int_{\partial B_\delta(z_0)} \left[\frac{\partial p(x; m)}{\partial\nu(x)}\hat{p}^s(x, z_0) - p(x; m)\frac{\partial\hat{p}^s(x, z_0)}{\partial\nu(x)}\right] ds(x) = 0. \tag{3.8}$$

Combining (3.7)-(3.8) and (3.5), we then have

$$\begin{aligned}
&\int_{\Gamma^+(h_0)}\left[\frac{\partial p^i(x;m)}{\partial x_3}\hat{p}(x,z_0)-p^i(x;m)\frac{\partial \hat{p}(x,z_0)}{\partial x_3}\right]ds(x)\\
&\qquad\qquad = \int_{\partial B_\delta(z_0)}\left[p(x;m)\frac{\partial \hat{G}(x,z_0)}{\partial \nu(x)}-\hat{G}(x,z_0)\frac{\partial p(x;m)}{\partial \nu(x)}\right]ds(x)\\
&\qquad\qquad = p(z_0;m), \qquad \text{as } \delta\to 0.
\end{aligned} \tag{3.9}$$

Recalling that $\hat{p}(x,z_0)=\hat{G}(x,z_0)+\hat{p}^s(x;z_0)$ leads from the representation formula of p^i and the Rayleigh expansion of $\hat{p}^s$ to that

$$p^i(z_0;m)=\int_{\Gamma^+(h_0)}\left[\frac{\partial p^i(x;m)}{\partial x_3}\hat{G}(x,z_0)-p^i(x;m)\frac{\partial \hat{G}(x,z_0)}{\partial x_3}\right]ds(x),$$

and

$$\begin{aligned}
&\int_{\Gamma^+(h_0)}\left[\frac{\partial p^i(x;m)}{\partial x_3}\hat{p}^s(x,z_0)-p^i(x;m)\frac{\partial \hat{p}^s(x,z_0)}{\partial x_3}\right]ds(x)\\
=\;& -i\sum_{n\in\mathbb{Z}^2}(\hat{\eta}_n+\eta_m)\hat{p}_n(z_0)\int_0^{2\pi}\int_0^{2\pi}e^{i(\hat{\alpha}_n+\alpha_m)\cdot\widetilde{x}}dx_1dx_2\cdot\exp(i(\hat{\eta}_n-\eta_m)h_0)\\
=\;& -8\pi^2 i\hat{\eta}_{-m}\hat{p}_{-m}(z_0),
\end{aligned}$$

where the fact that $\hat{\alpha}_n+\alpha_m=n+m$ and $\eta_\ell(\alpha)=\hat{\eta}_{-\ell}(\hat{\alpha})$ has been made use of in order to derive the above equalities. Together with these two equalities, we finally arrive by (3.9) at the required equality (3.4). The proof is thus complete. $\square$

Lemma 3.1 provides a connection of the solutions of the scattering problems with different incident wave fields. More precisely, the α-quasi-periodic scattered solution with the incident wave (1.1) can be transferred into the Rayleigh coefficient of the -α-quasi-periodic solution with the incident point source. This corresponds to the mixed-reciprocity relation in the bounded and periodical cases (cf. [4, 15]), and can thus simplify the proof of uniqueness on the inverse fluid-solid interaction problem for the bi-periodic structure.

Let Γ and $\widetilde{\Gamma}$ denote the two different bi-periodic interfaces described by the functions f and $\widetilde{f}$, respectively. We choose $h>0$ such that $h>\max\{A_1,\widetilde{A_1}\}$ and define the measurement sets

$$\mathcal{C}_\Gamma(h):=\{p^s(x;m)|_{\Gamma(h)}:\ m\in\mathbb{Z}^2\},\qquad \mathcal{C}_{\widetilde{\Gamma}}(h):=\{\widetilde{p}^s(x;m)|_{\Gamma(h)}:\ m\in\mathbb{Z}^2\} \tag{3.10}$$

with $\Gamma(h):=\{x\in\mathbb{R}^3:x_3=h\}$, where $p^s(\cdot;m)$ and $\widetilde{p}^s(\cdot;m)$ are the scattered solutions to (1.2)-(1.4) with respect to Γ and $\widetilde{\Gamma}$ for the same incident wave $p^i(x;m)=e^{i\alpha_m\cdot\widetilde{x}-i\eta_m x_3}$, $m\in\mathbb{Z}^2$. We are now in a position to state the main uniqueness result for determining the shape and location of the bi-periodic interface in this paper.

Theorem 3.2. *Assume that $\mathcal{C}_\Gamma(h)=\mathcal{C}_{\widetilde{\Gamma}}(h)$. Then $\Gamma=\widetilde{\Gamma}$.*

Proof. We will prove the assertion by contradiction. Suppose that $\Gamma \neq \widetilde{\Gamma}$. Without loss of generality, we can choose some point $z^* \in \Gamma \setminus \widetilde{\Gamma}$ such that $f(z^*) > \widetilde{f}(z^*)$. Define the sequence

$$z_j := z^* - \frac{\varepsilon_0}{j}\nu(z^*) \quad \text{for } j = 1, 2, \cdots\cdots \tag{3.11}$$

with sufficiently small $\varepsilon_0 > 0$ such that $z_j \in B_{\varepsilon_1}(z^*) \subseteq \widetilde{\Omega}^+$ for all $j \in \mathbb{N}_+$ and some $\varepsilon_1 > 0$.

Consider the $\hat{\alpha}$-quasi-periodic scattering problem (3.1)-(3.3) with the two different bi-periodic interfaces Γ and $\widetilde{\Gamma}$, induced by the same incident point source $\hat{p}^i = \hat{G}(\cdot, z_j)$. Let $(\hat{p}(\cdot; z_j), \hat{\mathbf{u}}(\cdot; z_j))$ and $(\hat{\widetilde{p}}(\cdot; z_j), \hat{\widetilde{\mathbf{u}}}(\cdot; z_j))$ denote the corresponding solutions to (3.1)-(3.3). One has from Lemma 3.1 that

$$p^s(z_j; m) = -8\pi^2 i\hat{\eta}_{-m}\hat{p}_{-m}(z_j) \;\text{ and }\; \widetilde{p}^s(z_j; m) = -8\pi^2 i\hat{\eta}_{-m}\hat{\widetilde{p}}_{-m}(z_j) \tag{3.12}$$

for all $m \in \mathbb{Z}^2$, where $\hat{p}_{-m}(z_j)$ and $\hat{\widetilde{p}}_{-m}(z_j)$ are the Rayleigh coefficients of $\hat{p}^s(\cdot; z_j)$ and $\hat{\widetilde{p}}^s(\cdot; z_j)$, respectively. Due to the assumption that $\mathcal{C}_\Gamma(h) = \mathcal{C}_{\widetilde{\Gamma}}(h)$, it is concluded that $\hat{p}_{-m}(z_j) = \hat{\widetilde{p}}_{-m}(z_j)$, $m \in \mathbb{Z}^2$, which means by the Rayleigh expansions and the unique continuation principle that

$$\hat{p}(\cdot; z_j) = \hat{\widetilde{p}}(\cdot; z_j) \quad \text{for each } j \in \mathbb{N}_+ \tag{3.13}$$

in the common domain $\Omega^+ \cap \widetilde{\Omega}^+$.

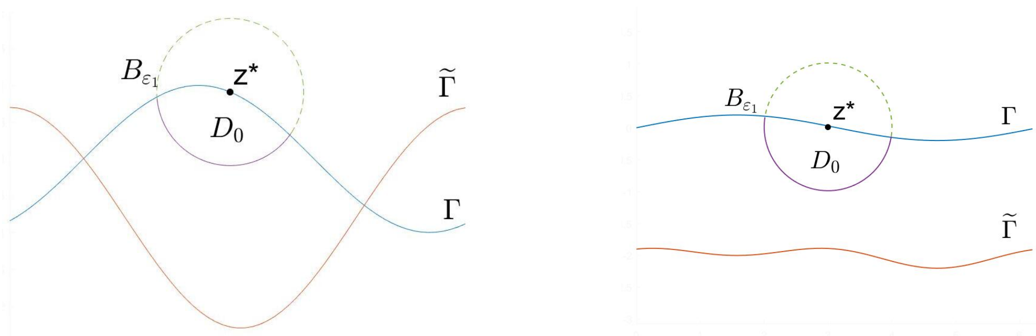

Figure 2: Geometric configuration for the choice of the domain D_0

Let $D_0 := B_{\varepsilon_1}(z^*) \cap \Omega^-$ (See Figure 2) with sufficiently small $\varepsilon_1 > 0$ so that D_0 is of Lipschitz class due to the $\mathcal{C}^2$ regularity of Γ. Define $v_j := \hat{\widetilde{p}}(\cdot; z_j)$ and $\mathbf{w}_j := \hat{\mathbf{u}}(\cdot; z_j)$ in D_0. It is found that v_j and $\mathbf{w}_j$ satisfy the following boundary value problem

$$\begin{cases} \triangle v_j - a_1 v_j = g_{1,j} & \text{in } D_0, \\ \triangle^* \mathbf{w}_j - a_2 \mathbf{w}_j = \mathbf{g}_{2,j} & \text{in } D_0, \\ t(\mathbf{w}_j) + v_j \nu = \mathbf{h}_{1,j} & \text{on } \partial D_0, \\ \eta \mathbf{w}_j \cdot \nu - \frac{\partial v_j}{\partial \nu} = h_{2,j} & \text{on } \partial D_0 \end{cases} \tag{3.14}$$

with the right terms and boundary data:

$$\begin{aligned} &g_{1,j} := -(a_1 + k^2)\hat{\widetilde{p}}(\cdot; z_j), \quad \mathbf{g}_{2,j} := -(a_2 + \rho_s\omega^2)\hat{\mathbf{u}}(\cdot; z_j), \\ &\mathbf{h}_{1,j} := t(\hat{\mathbf{u}}(\cdot; z_j)) + \hat{\widetilde{p}}(\cdot; z_j)\nu, \quad h_{2,j} := \eta\hat{\mathbf{u}}(\cdot; z_j) \cdot \nu - \partial\hat{\widetilde{p}}(\cdot; z_j)/\partial\nu. \end{aligned}$$

Next, we shall show that $g_{1,j}$, $\mathbf{g}_{2,j}$, $\mathbf{h}_{1,j}$ and $h_{2,j}$ are bounded uniformly for all $j \in \mathbb{N}_+$ in the corresponding function spaces. It first follows from the equality (3.13) and the transmission conditions on Γ that

$$\mathbf{h}_{1,j} = 0, \quad h_{2,j} = 0 \qquad \text{on } \Gamma \cap B_{\varepsilon_1}(z^*) \tag{3.15}$$

for all $j \in \mathbb{N}_+$. Using Theorem 2.1, it is then known that $\hat{\mathbf{u}}(\cdot; z_j)$ in D_0 is bounded in the sense of H^1-norm uniformly for all $j \in \mathbb{N}_+$. That is, there exists $C_1 > 0$, independent of $j \in \mathbb{N}_+$, such that $\|\hat{\mathbf{u}}(\cdot; z_j)\|_{H^1(D_0)^3} \le C_1$. Furthermore, by Theorem 2.1, it is also known that $\hat{\widetilde{p}}(\cdot; z_j)$ in D_0 is bounded in the sense of L^2-norm uniformly for all $j \in \mathbb{N}_+$, e.g, $\|\hat{\widetilde{p}}(\cdot; z_j)\|_{L^2(D_0)} \le C_2$ for some fixed $C_2 > 0$. Moreover, the positive distance between z^* and $\widetilde{\Gamma}$ leads to that $\|\hat{\widetilde{p}}^s(\cdot; z_j)\|_{H^s(D_0)} \le C_3$ with one fixed $C_3 > 0$ and $s \ge 1$ uniformly for $j \in \mathbb{N}_+$, due to the well-posedness of (3.1)-(3.3) associated with the bi-periodic interface $\widetilde{\Gamma}$. Then, $\|\hat{\widetilde{p}}(\cdot; z_j)\|_{H^s(D_0 \backslash B_{\varepsilon_2}(z^*))} \le C_4$ ($s \ge 1$) for another constant $C_4 > 0$, independent of $j \in \mathbb{N}_+$. Therefore, we have from the trace theorem and (3.15) that

$$\|g_{1,j}\|_{L^2(D_0)} + \|\mathbf{g}_{2,j}\|_{L^2(D_0)^3} + \|\mathbf{h}_{1,j}\|_{H^{-1/2}(\partial D_0)^3} + \|h_{2,j}\|_{H^{-1/2}(\partial D_0)} \le C_5 \tag{3.16}$$

uniformly for all $j \in \mathbb{N}_+$.

Recalling $v_j = \hat{\widetilde{p}}(\cdot; z_j)$ and using the well-posedness of (3.14) in the variational sense (cf. [13]), we thus arrive at

$$C \ge \|v_j\|_{H^1(D_0)} = \|\hat{\widetilde{p}}(\cdot; z_j)\|_{H^1(D_0)} \ge \|\hat{G}(\cdot, z_j)\|_{H^1(D_0)} - \|\hat{\widetilde{p}}^s(\cdot, z_j)\|_{H^1(D_0)}$$

uniformly for a fixed constant $C > 0$. Obviously, this is a contradiction due to the fact that $\|\hat{\widetilde{p}}^s(\cdot; z_j)\|_{H^1(D_0)}$ is uniformly bounded for all $j \in \mathbb{N}$ and $\|\hat{G}(\cdot, z_j)\|_{H^1(D_0)} \to \infty$ as $j \to \infty$. Therefore, $\Gamma = \widetilde{\Gamma}$, which completes the proof of the theorem. $\square$

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