# POINTWISE CONVERGENCE ALONG A TANGENTIAL CURVE FOR THE FRACTIONAL SCHRÖDINGER EQUATION WITH $0 < m < 1$

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Abstract. In this article, we study the pointwise convergence problem about solution to the fractional Schrödinger equation with $0 < m < 1$ along the tangential curve and estimate the capacitary dimension of the divergence set. We extend the results of Cho and Shiraki in [8] for the case $m > 1$ to the case $0 < m < 1$, which is sharp up to the endpoint.

## 1. Introduction

In this paper, we consider the fractional Schrödinger equation on $\mathbb{R}\times\mathbb{R}$ as follows

$$\begin{cases} i\partial_t u + (-\Delta)^{\frac{m}{2}} u = 0, \ (t,x) \in \mathbb{R}\times\mathbb{R} \\ u(x,0) = f(x), \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m > 0$ and $f \in H^s(\mathbb{R})$. The solution to this equation is

$$u(x,t) = e^{it(-\Delta)^{\frac{m}{2}}} f(\xi) = \int \hat{f}(\xi) e^{it|\xi|^m} e^{ix\xi}\, \mathrm{d}\xi.$$

For $m = 2$, Carleson put out a question about exploring the minimal $s$ such that for $\forall f \in H^s$, there holds the pointwise convergence

$$e^{-it\Delta} f(x) \to f(x), \quad \text{for } a.e.\, x \in \mathbb{R}, \tag{1.2}$$

as time $t$ tends to zero.

In dimension one, Carleson [6] proved that the pointwise convergence (1.2) holds if $s \geq \frac{1}{4}$, which is proved to be sharp by Dahlberg and Kenig [10] through constructing the counterexample. In higher dimensions, one can see [3, 4, 5, 9, 11, 12, 13, 22, 25, 27, 28, 29] for details.

Next, we consider the pointwise convergence to the solution of the Schrödinger equation along the tangential curve. We say that the continuous function $\gamma(x,t)$ is Hölder continuous of order $\kappa \in (0,1]$ in $t$ if

$$|\gamma(x,t) - \gamma(x,t')| \leq C_1 |t-t'|^{\kappa}, \quad x \in \mathbb{R}^d, t, t' \in [-1,1], \tag{1.3}$$

and bilipschitz in $x$ if

$$\frac{1}{C_2}|x-x'| \leq |\gamma(x,t) - \gamma(x',t)| \leq C_2 |x-x'|, \quad x, x' \in \mathbb{R}^d, t \in [-1,1]. \tag{1.4}$$

Then we define the set $\Gamma(d,\kappa)$ by

$$\Gamma(d,\kappa) = \{\gamma(x,t) : \gamma(x,0) = x, \text{ and satisfies (1.3)(1.4)}\}. \tag{1.5}$$

Let $\Gamma(\kappa) = \Gamma(1,\kappa)$.

Then, an elementary question is for $\gamma \in \Gamma(\kappa)$, determining the minimal $s$ such that for $\forall f \in H^s(\mathbb{R})$,

$$e^{it(-\Delta)^{\frac{m}{2}}} f(\gamma(x,t)) \to f(x), \quad \text{for } a.e.\, x \in \mathbb{R}, \tag{1.6}$$

as time $t$ tends to zero. Cho-Lee-Varges in [7] deal with the case $\gamma \in \Gamma(\kappa)$ and $m = 2$. Later, Cho and Shiraki considered the case of $m > 1$ in [8].

In the pointwise convergence of the Schrödinger equation with harmonic oscillator potential, Lee and Rogers [14] showed that the pointwise convergence in this case for any $\gamma \in C^1(\mathbb{R}^d \times [-1,1], \mathbb{R}^d)$ with $\gamma(x,0) = x$ is essentially equivalent to that in the vertical case, particularly for $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $\kappa \geq 1$.

We also study the divergence set which consists of all the points for which the pointwise convergence (1.6) fails, and characterize its size in a more precise way than Lebesgue measure.

We say that a measure $\mu$ is $\alpha$-dimensional for $\alpha \in (0, d]$ if

$$\mu(B(x,r)) \leq cr^\alpha, \quad x \in \mathbb{R}^d\, r > 0,$$

where $B(x,r)$ is a ball centered at $x$ with the radius $r$.

Define

$$\mathfrak{D}(\gamma, f) = \{x \in \mathbb{R} : e^{it(-\Delta)^{\frac{m}{2}}} f(\gamma(x,t)) \nrightarrow f(x), \text{ as } t \to 0\}. \tag{1.7}$$

The capacitary dimension of a set $X$ is defined by

$$\dim_c(X) = \{\alpha : \mathcal{M}^\alpha \neq \emptyset\}, \tag{1.8}$$

where

$$\mathcal{M}^\alpha(X) = \{\mu : \mu \text{ is } \alpha - \text{dimensional and } 0 < \mu(X) < \infty\}. \tag{1.9}$$

Sjögren and Sjölin [24] obtained the corresponding results for the estimate of the divergence set for $\gamma(x,t) = x$ and $m > 2$, which was extended to the case $m > 1$ by Barceló, Bennet, Carbery and Rogers in [1]. In higher dimensions, one can refer [2, 11, 12, 15, 16, 17].

Define $S_t f(\gamma(x,t))$ as follows

$$S_t f(\gamma(x,t)) = e^{it(-\Delta)^{\frac{m}{2}}} f(\gamma(x,t)) = \int e^{i(\gamma(x,t)\xi + t|\xi|^m)} \hat{f}(\xi)\, \mathrm{d}\xi. \tag{1.10}$$

We have the following pointwise convergence result for the operator $S_t$ along the tangential curve $(\gamma(x,t), t)$.

**Theorem 1.1.** *Let $0 < m < 1$, $0 < \kappa \leq 1$, $\mu$ be an $\alpha$-dimensional measure, $\gamma \in \Gamma(\kappa)$. If $s > \max\{\frac{1}{2} - \frac{m}{4}, \frac{1-m\alpha\kappa}{2}\}$, then we have*

$$\lim_{t \to 0} S_t f(\gamma(x,t)) = f(x), \quad \mu - a.e.\ x \in \mathbb{R}, \tag{1.11}$$

*for $f \in H^s(\mathbb{R})$.*

Theorem 1.1 follows from the following local maximal estimate for $S_t f(\gamma(x,t))$, which can be seen in Section 3.

**Theorem 1.2.** *Let $0 < m < 1$, $0 < \kappa \leq 1$, and $\mu$ be an $\alpha$-dimensional measure, $\gamma \in \Gamma(\kappa)$. If $s > \max\{\frac{1}{2} - \frac{m}{4}, \frac{1-m\alpha\kappa}{2}\}$, then we have*

$$\left( \int_{-1}^{1} \sup_{t \in [-1,1]} |S_t f(\gamma(x,t))|^2 \, \mathrm{d}\mu x \right)^{\frac{1}{2}} \lesssim \|f\|_{H^s(\mathbb{R})} \tag{1.12}$$

*for $f \in H^s(\mathbb{R})$.*

**Remark 1.3.** (1) *By the Sobolev embedding, we can obtain the local maximal estimate* (1.12) *holds for* $s > \frac{1}{2}$*. Furthermore, if we also consider the effect of the Schrödinger operator* $e^{it(-\Delta)^m}$*, then through the stationary phase we can lower down the regular exponent* $s$ *to* $s > \frac{1}{2} - \frac{m}{4}$*, which is a better result.*

(2) *For* $0 < m < 1$*, the change of the regularity for the exponent* $s$ *for which the pointwise convergence to the initial data along the tangential curve* $\{(\gamma(x,t),t) : t \in [-1,1]\}$ *with respect to* $m$ *is opposite to the vertical case* $\{(x,t) : t \in [-1,1]\}$*. Indeed, as* $m$ *tends to zero, we can see that regularity of* $s$ *for the pointwise convergence* (1.11) *to hold becomes worse, while that of* $s$ *for the pointwise convergence in the vertical case becomes better.*

(3) *In the following section, we can see that the exponent about* $s$ *in Theorem 1.2 is sharp up to the endpoint, that is, there exist* $\gamma \in \Gamma(\kappa)$*,* $\alpha$*-dimensional measure* $\mu$ *and* $f \in H^s(\mathbb{R})$ *with* $s < \max\{\frac{1}{2} - \frac{m}{4}, \frac{1-m\alpha\kappa}{2}\}$ *such that* (1.12) *fails.*

(4) *Cho and Shiraki* [8] *dealt with the case* $m > 1$ *for the corresponding local maximal estimate for the general* $\alpha$*-dimensional measure including the Lebesgue measure, which coincides with Corollary 1.5 as a special case of Theorem 1.2.*

As a direct consequence of Theorem 1.1, we have the following estimate for the divergence set.

**Corollary 1.4.** *Let* $0 < m < 1$*,* $0 < \kappa \leq 1$*,* $\gamma \in \Gamma(\kappa)$*. If* $s > \frac{1}{2} - \frac{m}{4}$*, then we have*

$$\dim_c(\mathfrak{Q}(\gamma, f)) \leq \frac{1-2s}{m\kappa}. \tag{1.13}$$

If we take $\mu = \mathcal{L}$ which is the Lebesgue measure on $\mathbb{R}$. By Theorem 1.2, we have

**Corollary 1.5.** *Let* $0 < m < 1$*,* $0 < \kappa \leq 1$*,* $\gamma \in \Gamma(\kappa)$*. If* $s > \max\{\frac{1}{2} - \frac{m}{4}, \frac{1-m\kappa}{2}\}$*, then we have*

$$\left( \int_{-1}^{1} \sup_{t \in [-1,1]} |S_t f(\gamma(x,t))|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \lesssim \|f\|_{H^s(\mathbb{R})} \tag{1.14}$$

*for* $f \in H^s(\mathbb{R})$*.*

For the operator $e^{it\phi(\sqrt{-\Delta})}$, Y. Niu and Y. Xue[20] studied the pointwise convergence problem along the tangential curve$(\gamma(x,t),t)$ with $t \in [-1,1]$ for the local maximal estimate in Corollary 1.5 for the Lebesgue measure, where $\phi$ satisfies some growth conditions and $|\xi|^m$ with $m \geq 2$ is the special case of $\phi$.

This paper is organised as follows. In section 2, we give some useful lemmas and elementary tools. In section 3, we show how local maximal estimate in Theorem 1.2 yields the pointwise convergence result in Theorem 1.1. In section 4, we prove a main lemma which is the reduction of the estimates (1.12) in Theorem 1.2. In section 5, we prove the necessary condition for the local maximal estimate (1.12) in Theorem 1.2 to hold.

Finally, we conclude the introduction by giving some notations which will be used throughout this paper. If $A$ and $B$ are two positive quantities, we write $A \lesssim B$ when there exists a constant $C > 0$ such that $A \leq CB$, where the constant will be clear from the context. We use $\mathcal{S}(\mathbb{R})$ to denote the Schwartz class of functions on the Euclidean space $\mathbb{R}$. We denote $I$ to be the interval $[-1,1]$.

## 2. Preliminaries

In this section, we give some useful tools for the later proof in the following sections.

The Frostman lemma in [18, 19] establish the relationship between the capacitary dimension for the Borel set and the corresponding $\alpha$-dimensional measure on it.

**Lemma 2.1** (Frostman)**.** *Let $X \subset \mathbb{R}^d$ be a Borel measure. Then $\dim_c(X) \geq \alpha$ if and only if there exists an $\alpha$-dimensional measure $\mu$ such that $\operatorname{supp}\mu \subset X$ and $0 < \mu(X) < \infty$.*

The following Van der corput lemma plays an important role in the oscillatory integral estimate for (1.12) in Theorem 1.2 in section 4.

**Lemma 2.2** (Van der corput lemma, [26])**.** *Suppose $\phi$ is real-valued and smooth in $(a, b)$, $\psi$ is complex-valued and smooth, and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then*

$$\Big| \int_a^b e^{i\lambda\phi(x)}\psi(x)\, dx \Big| \leq c_k \lambda^{-\frac{1}{k}} \Big[ |\psi(b)| + \int_a^b |\psi'(x)|\, dx \Big] \tag{2.1}$$

*holds when*

(i) $k \geq 2$ *or*
(ii) $k = 1$ *and $\phi'(x)$ is monotonic.*

*The bound $c_k$ is independent of $\phi$ and $\lambda$.*

Cho and Shiraki proved an useful estimate in [8], which we will use it with Lemma 2.2 to prove Theorem 1.2.

**Lemma 2.3** (Cho, Shiraki, [8])**.** *Let $\mu$ be an $\alpha$-dimensional measure for $0 < \alpha \leq 1$. There exists a constant $C$ such that for any interval $[a, b] (\infty < a, b < \infty)$*

$$\begin{aligned} &\Big| \iint \iint g(x,t) h(x',t') \chi_{[a,b]}(x - x')\, \mathrm{d}\mu x \mathrm{d}t \mathrm{d}\mu x' \mathrm{d}t' \Big| \\ &\leq C(b-a)^{\alpha} \|g\|_{L_x^2(\mathrm{d}\mu)L_t^1} \|h\|_{L_{x'}^2(\mathrm{d}\mu)L_{t'}^1}. \end{aligned} \tag{2.2}$$

*Moreover, for $0 < \rho < \alpha$, there exists a constant $C$ such that*

$$\begin{aligned} &\Big| \iint \iint g(x,t) h(x',t') |x - x'|^{-\rho}\, \mathrm{d}\mu x \mathrm{d}t \mathrm{d}\mu x' \mathrm{d}t' \Big| \\ &\leq C \|g\|_{L_x^2(\mathrm{d}\mu)L_t^1} \|h\|_{L_{x'}^2(\mathrm{d}\mu)L_{t'}^1}. \end{aligned} \tag{2.3}$$

*Here, the both integrals are taken over $(x, t), (x', t') \in I \times I$.*

## 3. Reduction argument

3.1. **Proof of Theorem 1.2 $\Rightarrow$ Theorem 1.1.** Since Schwartz functions are dense in $H^s(\mathbb{R})$ with $s \geq 0$, if $f \in H^s(\mathbb{R})$ with $s > \max\{\frac{1}{2} - \frac{m}{4}, \frac{1-m\alpha\kappa}{2}\}$ as in Theorem 1.2, then for $\forall \epsilon > 0$ we can split the function $f$ into two parts as follows $f = g + h$ with $g \in \mathcal{S}(\mathbb{R})$ and $\|h\|_{H^s} < \epsilon$.

For the divergence set $\mathfrak{D}(\gamma, f)$, we have

$$\mu(\mathfrak{D}(\gamma, f)) \leq \sum_{j\in\mathbb{Z}} \sum_{\lambda=1}^{\infty} \mu(\{x \in I + j : \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \tfrac{1}{\lambda}\}). \tag{3.1}$$

In order to prove the pointwise convergence result, that is $\mu(\mathfrak{D}(\gamma, f)) = 0$, it suffices to prove that

$$\mu(\{x \in I + j : \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \tfrac{1}{\lambda}\}) = 0, \ \text{for}\, \forall\, j \in \mathbb{Z}\, \text{and}\, 1 \leq \lambda < \infty. \tag{3.2}$$

If $j = 0$ and $\lambda \geq 1$, we have

$$\begin{aligned}
&\mu(\{x \in I : \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \tfrac{1}{\lambda}\}) \\
&\lesssim \mu(\{x \in I : \lim_{t\to 0} |S_t f(\gamma(x,t)) - S_t g(\gamma(x,t))| > \tfrac{1}{3\lambda}\}) \\
&\quad + \mu(\{x \in I : \lim_{t\to 0} |S_t g(\gamma(x,t)) - g(x)| > \tfrac{1}{3\lambda}\}) \\
&\quad + \mu(\{x \in I : |g(x) - f(x)| > \tfrac{1}{3\lambda}\}) \\
&\lesssim \mu(\{x \in I : \sup_{t\in I} |S_t f(\gamma(x,t)) - S_t g(\gamma(x,t))| > \tfrac{1}{3\lambda}\}) \\
&\quad + 0 + \mu(\{x \in I : |g(x) - f(x)| > \tfrac{1}{3\lambda}\}) \\
&\lesssim \mu(\{x \in I : \sup_{t\in I} |S_t f(\gamma(x,t)) - S_t g(\gamma(x,t))| > \tfrac{1}{3\lambda}\}),
\end{aligned}$$

where the last inequality is obtained by

$$\begin{aligned}
|g(x) - f(x)| &= |S_0 g(\gamma(x,0)) - S_0 f(\gamma(x,0))| \\
&\leq \sup_{t\in I} |S_t g(\gamma(x,t)) - S_t f(\gamma(x,t))|.
\end{aligned}$$

Using Theorem 1.2 and Chebyshev's inequality, we have

$$\begin{aligned}
&\mu(\{x \in I : \sup_{t\in I} |S_t f(\gamma(x,t)) - S_t g(\gamma(x,t))| > \tfrac{1}{3\lambda}\}) \\
&\lesssim \lambda^2 \|f - g\|_{H^s(\mathbb{R})}^2 = \lambda^2 \|h\|_{H^s(\mathbb{R})}^2 \leq \lambda^2 \epsilon^2,
\end{aligned}$$

which yields that

$$\mu(\{x \in I : \sup_{t\in I} |S_t f(\gamma(x,t)) - S_t g(\gamma(x,t))| > \tfrac{1}{3\lambda}\}) = 0$$

by the arbitrary choice of $\epsilon$.

Then for $\forall j \in \mathbb{Z}$, let $\gamma_j(x,t) = \gamma(x+j,t)$ and $\mu_j(x) = \mu(x+j)$. Since $\gamma_j(x,t)$ also satisfies (1.3) and (1.4), then we can obtain that Theorem 1.2 also holds for $\gamma_j(x,t)$ and $\mu_j(x)$. Hence by the similar argument as above, we can get the pointwise convergence result on the interval $I + j$ with respect to $x$, that is for $\forall j \in \mathbb{Z}$ and $\lambda \geq 1$, there holds

$$\mu(\{x \in I + j : \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \tfrac{1}{\lambda}\}) = 0.$$

3.2. **Proof of Theorem 1.1 $\Rightarrow$ Corollary 1.4.** Let $s > \frac{m}{4}$ and $f \in H^s$, if $\dim_c(\mathfrak{Q}(\gamma, f)) > \frac{1-2s}{m\kappa}$, then by Lemma 2.1, for $\dim_c(\mathfrak{Q}(\gamma, f)) > \alpha > \frac{1-2s}{m\kappa}$, there exists an $\alpha$-dimensional measure $\mu$ such that $0 < \mu(\mathfrak{Q}(\gamma, f)) < \infty$ and $\operatorname{supp}\mu \subset \mathfrak{Q}(\gamma, f)$.

By Theorem 1.1 and $s > \frac{1-m\alpha\kappa}{2}$, we can obtain that $\mu(\mathfrak{Q}(\gamma, f)) = 0$, which contradicts with $0 < \mu(\mathfrak{Q}(\gamma, f)) < \infty$.

Hence, there holds $\dim_c(\mathfrak{Q}(\gamma, f)) \leq \frac{1-2s}{m\kappa}$.

## 4. Proof of Theorem 1.2

Take a function $\psi(\xi) \in C_c^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp} \psi(\xi) \subset \{\xi \in \mathbb{R} : \tfrac{1}{2} \leq |\xi| \leq 2\}, \ \psi(\mathbb{R}) \subset [0,1].$$

Let $\psi_k(\xi) = \psi(\frac{\xi}{2^{k-1}})$, and use $\psi(\xi)$ to obtain the Littlewood-Paley decomposition, that is

$$\varphi_0(\xi) + \sum_{k\geq 1} \psi_k(\xi) = 1,$$

where $\varphi_0(\xi) \in C_c^\infty(\mathbb{R})$ satisfies that

$$\text{supp } \varphi_0(\xi) \subset [-1,1],\ \varphi_0(\mathbb{R}) \subset [0,1],\ \varphi_0(\xi) = 1 \text{ on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

We define the operator $\Delta_k$ by

$$\begin{aligned}
\widehat{\Delta_0 f}(\xi) &= \varphi_0(\xi)\widehat{f}(\xi),\\
\widehat{\Delta_k f}(\xi) &= \psi_k(\xi)\widehat{f}(\xi),\ \text{for } k \geq 1.
\end{aligned}$$

With this decomposition and the triangle inequality, we can get

$$\begin{aligned}
&\|S_t f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)} L_t^\infty(I)\\
&= \|\sum_{k\geq 0} S_t \Delta_k f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)} L_t^\infty(I)\\
&\lesssim \|S_t \Delta_0 f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)} L_t^\infty(I)\\
&\quad + \sum_{k\geq 1} \|S_t \Delta_k f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)} L_t^\infty(I).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\|S_t \Delta_0 f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)} L_t^\infty(I)\\
&\lesssim \int \varphi_0(\xi)|\hat{f}(\xi)|\,\mathrm{d}\xi\\
&\lesssim (\int |\hat{f}(\xi)|^2\,\mathrm{d}\xi)^{\frac{1}{2}}\\
&= \|f\|_{L^2},
\end{aligned} \tag{4.1}$$

then it is remained to estimate the local maximal estimate for $S_t\Delta_k f(\gamma(x,t))$, $k \geq 1$.

Let $s_* = \min\{\frac{m}{4}, \frac{m\alpha\kappa}{2}\}$.

**Lemma 4.1.** *Let $\mu$ be an $\alpha$-dimensional measure, $\gamma(x,t) \in \Gamma(\kappa)$. Then we have*

$$\|S_t f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)L_t^\infty(I)} \lesssim \lambda^{\frac{1}{2}-s_*+\epsilon}\|f\|_{L^2(\mathbb{R})}, \tag{4.2}$$

*where $f \in L^2$ and is supported in $[-2\lambda, -\frac{\lambda}{2}] \cup [\frac{\lambda}{2}, 2\lambda]$ for $\lambda > 1$.*

We postpone the proof of this lemma, and first look at how we get the results in Theorem 1.2 by Lemma 4.1. By Lemma 4.1, we have

$$\begin{aligned}
&\sum_{k\geq 1} \|S_t \Delta_k f(\gamma(x,t))\|_{L^2(I,\mathrm{d}\mu)L_t^\infty(I)}\\
&\lesssim \sum_{k\geq 1} 2^{k(\frac{1}{2}-s_*+\epsilon)}\|\Delta_k f\|_{L^2}\\
&\lesssim \sum_{k\geq 1} 2^{-k\epsilon}\|f\|_{H^{\frac{1}{2}-s_*+2\epsilon}}\\
&\lesssim \|f\|_{H^{\frac{1}{2}-s_*+2\epsilon}}.
\end{aligned} \tag{4.3}$$

Combining (4.1) and (4.3), we can obtain the proof of Theorem 1.2.

Now we turn to the proof of Lemma 4.1.

**The proof of Lemma 4.1.** Let

$$Tf = \chi(x,t)\int e^{i\gamma(x,t)\xi}e^{it|\xi|^m}f(\xi)\psi(\tfrac{\xi}{\lambda})\,\mathrm{d}\xi, \tag{4.4}$$

where $\chi(x,t)$ is the characteristic function on $I\times I$.

In order to prove Lemma (4.1), we just need to prove that

$$\|Tf\|_{L^2_x(\mathrm{d}\mu)L^\infty_t} \lesssim \lambda^{\frac{1}{2}-s_*+\epsilon}\|f\|_{L^2}. \tag{4.5}$$

Indeed, by Plancherel' theorem, we obtain

$$\|S_tf(\gamma(x,t))\|_{L^2_x(I,\mathrm{d}\mu)L^\infty_t(I)} = \|T\hat{f}\|_{L^2_x(\mathrm{d}\mu)L^\infty_t} \lesssim \lambda^{\frac{1}{2}-s_*+\epsilon}\|\hat{f}\|_{L^2} = \lambda^{\frac{1}{2}-s_*+\epsilon}\|f\|_{L^2}. \tag{4.6}$$

By the $TT^*$ method, it suffices to prove

$$\|T^*g\|_{L^2} \lesssim \lambda^{\frac{1}{2}-s_*+\epsilon}\|g\|_{L^2_x(d\mu)L^1_t}, \tag{4.7}$$

where

$$T^*g = \psi(\tfrac{\xi}{\lambda})\int e^{-i\gamma(x,t)\xi}e^{-it|\xi|^m}g(x,t)\chi(x,t)\,\mathrm{d}\mu\mathrm{d}t. \tag{4.8}$$

Next we prove (4.7).

$$\begin{aligned}
\|T^*g\|^2_{L^2} &= \int \psi(\tfrac{\xi}{\lambda})^2 \iint\iint e^{-i[(\gamma(x,t)-\gamma(x',t'))\xi+(t-t')|\xi|^m]} \\
&\qquad \chi(x,t)g(x,t)\chi(x',t')\overline{g}(x',t')\,\mathrm{d}\mu x\mathrm{d}t\mathrm{d}\mu x'\mathrm{d}t'\mathrm{d}\xi \\
&= \lambda\int \psi(\xi)^2 \iint\iint e^{-i[(\gamma(x,t)-\gamma(x',t'))\lambda\xi+\lambda^m(t-t')|\xi|^m]} \\
&\qquad \chi(x,t)g(x,t)\chi(x',t')\overline{g}(x',t')\,\mathrm{d}\mu x\mathrm{d}t\mathrm{d}\mu x'\mathrm{d}t'\mathrm{d}\xi \\
&= \int_W\int_{W'} \chi(w)g(w)\chi(w')\overline{g}(w')K_\lambda(w,w')\,\mathrm{d}w\mathrm{d}w' \\
&= \sum_{i=1}^{3}\iint_{V_i} \chi(w)g(w)\chi(w')\overline{g}(w')K_\lambda(w,w')\,\mathrm{d}w\mathrm{d}w' \\
&= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,
\end{aligned}$$

where $w=(x,t)$, $w'=(x',t')$, $W=I\times I$, $W'=I\times I$, and

$$K_\lambda(w,w') = \lambda\int e^{-i\phi(\lambda\xi)}\psi(\xi)^2\,\mathrm{d}\xi \tag{4.9}$$

with

$$\phi(\xi) = (\gamma(x,t)-\gamma(x',t'))\xi+(t-t')|\xi|^m. \tag{4.10}$$

The set $V_i, i=1,2,3$ are defined by

$$\begin{aligned}
V_1 &= \{(w,w')\in W\times W' : |x-x'|\le \lambda^{-\frac{2s_*}{\alpha}}\}, \\
V_2 &= \{(w,w')\in W\times W' : |x-x'| > \lambda^{-\frac{2s_*}{\alpha}}, \tfrac{1}{C_2}|x-x'|\ge 2C_1|t-t'|^\kappa\}, \\
V_3 &= \{(w,w')\in W\times W' : |x-x'| > \lambda^{-\frac{2s_*}{\alpha}}, \tfrac{1}{C_2}|x-x'| < 2C_1|t-t'|^\kappa\}.
\end{aligned}$$

The estimate (4.7) is reduced to prove

$$|\mathcal{I}_i| \lesssim \lambda^{1-2s_*+\epsilon}\|g\|_{L^2_x(d\mu)L^2_t}, \quad \text{for } i=1,2,3. \tag{4.11}$$

**Case 1, Estimate for $\mathcal{I}_1$.** In this case, it is easy to see that

$$|K_\lambda(w,w')| \lesssim \lambda, \tag{4.12}$$

which yields that

$$\begin{aligned}|\mathcal{I}_1| &\lesssim \lambda \int \chi(w)|g(w)|\chi_{[-\lambda^{-\frac{2s_*}{\alpha}},\lambda^{-\frac{2s_*}{\alpha}}]}(x-x')\chi(w')|g(w')|\,\mathrm{d}\mu x\mathrm{d}t\mathrm{d}\mu x'\mathrm{d}t' \\ &\lesssim \lambda^{1-2s_*}\|g\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.\end{aligned} \tag{4.13}$$

**Case 2, Estimate for $\mathcal{I}_2$.** By the definition of the set $V_2$, we have

$$\begin{aligned}|\gamma(x,t)-\gamma(x',t')| &\geq |\gamma(x,t)-\gamma(x',t)| - |\gamma(x',t)-\gamma(x',t')| \\ &\geq \frac{1}{C_2}|x-x'| - C_1|t-t'|^\kappa \\ &\gtrsim |x-x'|.\end{aligned}$$

Split the integral $K_\lambda(w,w')$ into two parts as follows

$$\begin{aligned}K_\lambda(w,w') &= \lambda\int_{U_1} e^{-i\phi(\lambda)}\psi(\xi)^2\,\mathrm{d}\xi + \lambda\int_{U_2} e^{-i\phi(\lambda)}\psi(\xi)^2\,\mathrm{d}\xi \\ &= E_1 + E_2,\end{aligned}$$

where

$$\begin{aligned}U_1 &= \{\xi \in \operatorname{supp}\psi : |x-x'| \geq 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}, \\ U_2 &= \{\xi \in \operatorname{supp}\psi : |x-x'| < 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}.\end{aligned}$$

For $E_1$, we have

$$\begin{aligned}|\partial_\xi \phi(\lambda\xi)| &\geq \lambda|(\gamma(x,t)-\gamma(x',t')| - m\lambda^m|t-t'||\xi|^{m-1} \\ &\gtrsim \lambda(|x-x'| - \tfrac{1}{4}|x-x'|) \\ &\gtrsim \lambda|x-x'| \\ &\gtrsim \lambda\lambda^{-\frac{2s_*}{\alpha}} \\ &\geq 1,\end{aligned}$$

since $\frac{2s_*}{\alpha} = \min\{\frac{m}{2\alpha}, m\kappa\} \leq 1$. Thus by Lemma 2.2, we have

$$|E_1| \lesssim \lambda(\lambda|x-x'|)^{-1}. \tag{4.14}$$

For $E_2$, since

$$\begin{aligned}|\partial_\xi^2 \phi(\lambda\xi)| &= m(1-m)\lambda^m|t-t'||\xi|^{m-2} \\ &\gtrsim \lambda|x-x'| \\ &\gtrsim \lambda\lambda^{-\frac{2s_*}{\alpha}} \\ &\geq 1,\end{aligned}$$

where the last inequality is also obtained by $\frac{2s_*}{\alpha} \leq 1$, then we have by Lemma 2.2

$$|E_2| \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}. \tag{4.15}$$

Combining the estimates (4.14) and (4.15) implies that

$$\begin{aligned}|K_\lambda| &\lesssim \lambda(\lambda|x-x'|)^{-1} + \lambda(\lambda|x-x'|)^{-\frac{1}{2}} \\ &\lesssim \lambda(\lambda|x-x'|)^{-\frac{m}{2}} \\ &\lesssim \lambda^{1-2s_*+\epsilon}|x-x'|^{-\min\{\frac{m}{2},m\alpha\kappa\}+\epsilon}.\end{aligned}$$

Since $0<\min\{\frac{m}{2},m\alpha\kappa\}-\epsilon<\alpha$, then we have

$$|I_2| \lesssim \lambda^{1-2s_*+\epsilon}\|g\|^2_{L^2_t(\mathrm{d}\mu)L^1_t}. \tag{4.16}$$

**Case 3, Estimate for** $\mathcal{I}_3$. By the definition of the set $V_3$, we have

$$\begin{aligned}|\partial^2_\xi\phi(\lambda\xi)| &= m(1-m)\lambda^m|t-t'||\xi|^{m-2} \\ &\gtrsim \lambda^m|x-x'|^{\frac{1}{\kappa}} \\ &\gtrsim \lambda^m\lambda^{-\frac{2s_*}{\alpha\kappa}} \\ &\geq 1,\end{aligned}$$

since $\frac{2s_*}{\alpha\kappa}=\min\{\frac{m}{2\alpha\kappa},m\}\leq m$.

By $\frac{2s_*}{m}=\min\{\frac{1}{2},\alpha\kappa\}\leq\frac{1}{2}$ and Lemma 2.2, we have

$$\begin{aligned}|K_\lambda(w,w')| &\lesssim \lambda(\lambda^m|x-x'|^{\frac{1}{\kappa}})^{-\frac{1}{2}} \\ &\lesssim \lambda(\lambda^m|x-x'|^{\frac{1}{\kappa}})^{-\frac{2s_*}{m}} \\ &= \lambda^{1-2s_*}|x-x'|^{-\frac{2s_*}{m\kappa}} \\ &\lesssim \lambda^{1-2s_*+\epsilon}|x-x'|^{-\frac{2s_*}{m\kappa}+\epsilon}.\end{aligned}$$

Since $0<\frac{2s_*}{m\kappa}-\epsilon=\min\{\frac{1}{2\kappa},\alpha\}-\epsilon<\alpha$, then we have

$$|\mathcal{I}_3| \lesssim \lambda^{1-2s_*+\epsilon}\|g\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}. \tag{4.17}$$

The proof is completed. □

## 5. Sharpness of the Sobolev index $s$ in Theorem 1.2

In this section, we study the necessary condition for the Sobolev index $s$ for the local maximal estimate in the Theorem 1.2 to hold.

Take $\gamma(x,t)=x-t^\kappa$, $\mathrm{d}\mu x=|x|^{-1+\alpha}\mathrm{d}x$, and $\phi\in\mathcal{S}(\mathbb{R})$ be supported in a small neighborhood of the origin.

**Case 1,** $s\geq\frac{1-m\alpha\kappa}{2}$**.** Take

$$\hat{f}_1(\xi)=\phi(\lambda^{-\frac{1}{m}}\xi),$$

then

$$\begin{aligned}|S_tf_1(\gamma(x,t))| &= \left|\int e^{i[(x-t^\kappa)\xi+t|\xi|^m]}\hat{f}_1(\xi)\,\mathrm{d}\xi\right| \\ &= \lambda^{\frac{1}{m}}\left|\int e^{ih_1(\xi)}\phi(\xi)\,\mathrm{d}\xi\right|,\end{aligned}$$

where

$$h_1(\xi)=(x-t^\kappa)\lambda^{\frac{1}{m}}\xi+\lambda t|\xi|^m$$

If $x\in(0,\frac{1}{100}\lambda^{-\kappa})$ and $t=t(x)=x^{\frac{1}{\kappa}}$, then

$$0<t\lesssim\lambda^{-1}$$

and

$$|h(\xi)| \lesssim 0 + \lambda\lambda^{-1} \le \tfrac{1}{3} \tag{5.1}$$

which yields that

$$|S_t f_1(\gamma(x, t(x)))| \gtrsim \chi_{(0,\frac{1}{100}\lambda^{-\kappa})}(x). \tag{5.2}$$

Then we have

$$\begin{aligned}
&\|\sup_{t\in I} |S_t f_1(\gamma(x, t(x)))|\|_{L^2_x(I,\mathrm{d}\mu)}\\
\gtrsim\ &\lambda^{\frac{1}{m}} \|\chi_{(0,\frac{1}{100}\lambda^{-\kappa})}\|_{L^2_x(I,\mathrm{d}\mu)}\\
\gtrsim\ &\lambda^{\frac{1}{m}} \lambda^{-\frac{\alpha\kappa}{2}}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\|f_1\|_{H^s(\mathbb{R})} &\sim \left( \int (1 + |\xi|^2)^s |\phi(\lambda^{-\frac{1}{m}}\xi)|^2 \,\mathrm{d}\xi \right)^{\frac{1}{2}}\\
&\lesssim \lambda^{\frac{1}{2m}} \lambda^{\frac{s}{m}}.
\end{aligned}$$

Combining the calculations above and Theorem (1.2), we have

$$\lambda^{\frac{1}{m}} \lambda^{-\frac{\alpha\kappa}{2}} \lesssim \lambda^{\frac{1}{2m}} \lambda^{\frac{s}{m}}. \tag{5.3}$$

Let $\lambda \to \infty$, and we can see that it is necessary that

$$\tfrac{1}{m} - \tfrac{\alpha\kappa}{2} \le \tfrac{1}{2m} + \tfrac{s}{m},$$

that is

$$s \ge \tfrac{1-m\alpha\kappa}{2}. \tag{5.4}$$

**Case 2,** $s \ge \frac{1}{2} - \frac{m}{4}$**.** Take

$$\hat{f}_2(\xi) = \lambda^{-(2-m)} \phi(\lambda^{-(2-m)}\xi + \lambda^m), \tag{5.5}$$

then

$$\begin{aligned}
|S_t f_2(\gamma(x, t))| &= \left| \int e^{i[(x-t^\kappa)\xi + t|\xi|^m]} \hat{f}_2(\xi) \,\mathrm{d}\xi \right|\\
&= \left| \int e^{ih_2(\xi)} \phi(\xi) \,\mathrm{d}\xi \right|
\end{aligned}$$

where

$$\begin{aligned}
h_2(\xi) &= (x - t^\kappa)\lambda^{2-m}\xi + \lambda^{(2-m)m} t(\lambda^m - \xi)^m\\
&= (x - t^\kappa)\lambda^{2-m}\xi + \lambda^{2m} t(1 - \tfrac{\xi}{\lambda^m})^m.
\end{aligned}$$

By Taylor's formula, we have

$$(1 - \tfrac{\xi}{\lambda^m})^m = 1 - \tfrac{m\xi}{\lambda^m} + \tfrac{m(m-1)}{2}\tfrac{\xi^2}{\lambda^{2m}} + O(\lambda^{-3m}|\xi|^3),$$

then

$$h_2(\xi) = \lambda^{2m} t + \lambda^m((x - t^\kappa)\lambda^{2-2m} - mt)\xi + \tfrac{m(m-1)}{2} t\xi^2 + O(\lambda^{-m} t|\xi|^3).$$

For $x \in (0, \frac{1}{100})$, take $t(x)$ such that $x = t(x)^\kappa + m\lambda^{2m-2}t(x)$. Let $\tau(t) = t^\kappa + m\lambda^{2m-2}t$, then $\tau^{-1}(x) = t(x)$ which is bijection and increasing with respect to $x$. Thus we have

$$0 = \tau^{-1}(0) < \tau^{-1}(x) = t(x) < \tau^{-1}(\tfrac{1}{100}) \lesssim \tfrac{1}{100^\kappa}, \tag{5.6}$$

then

$$|h_2(\xi) - \lambda^{2m} t| \lesssim 0 + \tfrac{1}{100^\kappa} + O(\lambda^{-m}) \le \tfrac{1}{3}. \tag{5.7}$$

Hence

$$|S_t f_2(\gamma(x, t(x)))| \gtrsim \chi_{(0, \frac{1}{100})}(x),$$

which yields that

$$\| \sup_{t \in I} |S_t f_2(\gamma(x,t))| \|_{L^2_x(I, \mathrm{d}\mu)} \gtrsim 1.$$

It is easy to see that

$$\begin{aligned} \|f_2\|_{H^s(\mathbb{R})} &\sim \left( \int (1+|\xi|^2)^s |\lambda^{-(2-m)} \phi(\lambda^{-(2-m)}\xi + \lambda^m)|^2 \right)^{\frac{1}{2}} \\ &\lesssim \lambda^{2s} \lambda^{-\frac{1}{2}(2-m)}. \end{aligned}$$

Combining the calculations above and Theorem (1.2), we have

$$1 \lesssim \lambda^{2s} \lambda^{-\frac{1}{2}(2-m)}.$$

Let $\lambda \to \infty$, and we can see that it is necessary that

$$2s - \tfrac{1}{2}(2-m) \ge 0,$$

which is

$$s \ge \tfrac{1}{2} - \tfrac{m}{4}.$$

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