Lyapunov stability of singular planar systems related to dispersion-managed solitons in optical fiber

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Abstract

In this paper, we consider two singular planar differential systems which can describe the evolution of the optical pulse width and chirp for the so-called dispersion-managed solitons. Based on the method of third order approximation in combination with some quantitative information obtained by the upper-lower solutions method and the averaging method, some results on the existence and Lyapunov stability of the periodic solutions are obtained. Moreover, the formula of the first twist coefficient and a stability criterion of a nonlinear differential equation are also established.

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1. Introduction and main results

In data communication systems, due to the widespread use of internet, there is increasing demand to achieve high data transmitting rate in a fiber cable. For this purpose, an approach is to make use of nonlinear light wave communications with suitable periodic to compensate for dispersive and loss effects. The transmission of the optical pulse in a fiber cable is described by the following

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equation (see [16, 28, 29])

$$i\Phi_z - \frac{\eta(z)\Phi_{tt}}{2} + \theta(t)|\Phi|^2\Phi = iW(z)\Phi,$$

where Φ is the complex valued function of the electric field, $\eta(z)$ models the dispersion, $\theta(t)$ models the nonlinear refractive response, $W(z)$ accounts for the effective loss or gain along the fiber, z is the longitudinal coordinate of the fiber line and t is the time. In this paper, it is assumed that the optical fiber has a periodic structure such that the coefficients are periodic. Choose

$$\Phi(z,t) = \omega(z,t)e^{\displaystyle\int_0^z W(s)\mathrm{d}s},$$

which can remove the right hand side term of the above equation to obtain the following Schrödinger equation

$$i\omega_z + l(z)\omega_{tt} + m(z)|\omega|^2\omega = 0, \tag{1.1}$$

where l, m are T-periodic functions. It is universally accepted that the central part of the desired pulse shaped solution to the above equation is described to leading order by

$$\omega(z,t) = \frac{P\big(\frac{t}{u(z)}\big)}{\sqrt{u(z)}}e^{\big(i\frac{v(z)}{u(z)}t^2\big)},$$

where u and v describe the optical pulse width and the chirp of the breathing central part of the optical soliton, P is an input pulse with

$$P(x) = \mu_0 e^{-\dfrac{x^2}{2}}.$$

Then, substituting the above ansatz into the following action functional of the equation (1.1)

$$J(\omega,\tilde{\omega}) = \int \left[\frac{il(z)}{2}(\omega\tilde{\omega}_z - \tilde{\omega}\omega_z)|\omega_t|^2 - \frac{m(z)}{2}|\omega|^4\right]\mathrm{d}t\mathrm{d}z,$$

we can get the following singular planar differential system (see [1] for the detailed derivation)

$$\begin{cases} u' = 4l(z)v, \\ v' = \dfrac{\mu_1 l(z)}{u^3} - \dfrac{\mu_2 m(z)}{u^2}, \end{cases} \tag{1.2}$$

here μ_1 and μ_2 are constants with

$$\mu_1 = \frac{\int |P'(x)|^2 \mathrm{d}x}{\int x^2 |P(x)|^2 \mathrm{d}x}, \quad \mu_2 = \frac{\int |P(x)|^4 \mathrm{d}x}{4 \int x^2 |P(x)|^2 \mathrm{d}x}.$$

In physical applications, one of the main objectives is to study the dynamical behaviors of the system (1.2), including the dynamics properties of periodic solutions. During the past few years, the existence of periodic solutions of the system (1.2) has been studied by a combined variational-topological approach [14], direct calculation and estimates [17], the averaging method and the implicit function theorem [18]. Here, we refer the reader to the classical monograph [33] for a more detailed description of the existence results in [14, 17, 18]. But as far as we know, there are no analytic results available about the Lyapunov stability of periodic solutions of the system (1.2) up to now. Motivated by this, in this paper, we study the Lyapunov stability of periodic solutions of the following more general singular planar differential system

$$\begin{cases} u' = g(t)v, \\ v' = \dfrac{h(t)}{u^l} - \dfrac{p(t)}{u^\alpha} \end{cases} \tag{1.3}$$

and the singular planar differential system with a small parameter

$$\begin{cases} u' = g(t)v, \\ v' = \dfrac{h(t)}{u^l} - \dfrac{\varepsilon p(t)}{u^\alpha}, \end{cases} \tag{1.4}$$

where $l, \alpha > 0$, g, h and p are T-periodic functions, ε is a small parameter. Obviously, the system (1.2) is a particular case of the above two systems. Moreover, the similar systems like (1.3) and (1.4) also have applications in some other physical contexts, such as molecular dynamics [24] and the Bose-Einstein condensates with a periodic control of the scattering length [2, 22].

Since the case $l < \alpha$ may lead to unstable periodic solutions (see [11, 18]). Therefore, throughout this paper, we always assume that $0 < \alpha < l$ and denote κ by

$$\kappa = \frac{1}{l - \alpha}.$$

Moreover, for a given T-periodic function g, we denote

$$g^* = \sup_{t\in[0,T]} g(t), \quad g_* = \inf_{t\in[0,T]} g(t), \quad \bar{g} = \frac{1}{T}\int_0^T g(s)\mathrm{d}s.$$

Now, we state the stability result for the system (1.3).

Theorem 1.1. *Assume that* $g, h, p \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$,

$$g(t) \geq 4g_0, \quad \forall t \in \mathbb{R} \tag{1.5}$$

and

$$\frac{h^* g^*}{h_* g_*} < \left(\frac{5(l+\alpha+1)^2}{3(l^2+l\alpha+\alpha^2+3l+3\alpha+2)}\right)^{\frac{2}{7}}, \tag{1.6}$$

where

$$g_0 = \left(\frac{T\bar{g}}{\pi}\right)^2 \frac{(p^*)^{\kappa(l+1)}}{(h_*)^{\kappa(\alpha+1)}}[l - \alpha\gamma^{\kappa(l+1)}], \quad \gamma = \frac{h_* p_*}{h^* p^*}.$$

Then there exists a constant $\gamma_0 \in (0,1)$ *such that the system* (1.3) *has a stable* T*-periodic solution* (u, v) *if* $\gamma > \gamma_0$.

The proof of Theorem 1.1 will be presented in Section 3 by a stability criterion which will be obtained in Section 2 combined with some quantitative information obtained by the upper-lower solutions method on the reversed order. This is a kind of global result, because the upper-lower solutions method provides us the explicit bounds of the periodic solutions.

We are now ready to state the stability result for the system (1.4).

Theorem 1.2. *Assume that* $g, h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$, $\bar{p} > 0$,

$$\frac{l}{\alpha} \geq \frac{\bar{h}p^*}{\bar{p}h_*} \tag{1.7}$$

and

$$2l^2 + 2\alpha^2 + 7l\alpha + l + \alpha > 1. \tag{1.8}$$

Then the system (1.4) *has a stable* T*-periodic solution* $\big(u, v\big)$ *if* ε *is small enough.*

The proof of Theorem 1.2 will be given in Section 4 by the method of third order approximation in combination with some asymptotic information obtained by the averaging method. This is a kind of perturbative result, because the averaging method provides us the asymptotic information of the periodic solutions.

Theorem 1.1 and 1.2 can directly apply to the system (1.2), as the following corollary shows.

Corollary 1.3. *For the system* (1.2)*, we have the following two results:*

(1) *Assume that* $l, m \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$,

$$\frac{l^*}{l_*} < \Big(\frac{5}{3}\Big)^{\frac{1}{7}}$$

and

$$l(t) \geq 16\left(\frac{T\bar{l}}{\pi}\right)^2 \frac{(\mu_2 m^*)^4}{(\mu_1 l_*)^3}\left[3 - 2\left(\frac{l_* m_*}{l^* m^*}\right)^4\right], \quad \forall t \in \mathbb{R}.$$

Then there exists a constant $\gamma_0 \in (0,1)$ *such that the system* (1.2) *has a stable* T*-periodic solution if*

$$\frac{l_* m_*}{l^* m^*} > \gamma_0.$$

(2) *Assume that* $l \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$, $\bar{m} > 0$*, and*

$$\frac{\bar{l} m^*}{\bar{m} l_*} \leq \frac{3}{2}.$$

Then the system (1.2) *has a stable* T*-periodic solution if* μ_0 *is small enough.*

The method of third order approximation was established by Ortega [23] and Zhang [36] for the second order conservative systems. During the past few years, some progress has been made on this topic, we just refer the reader to [4, 5, 9, 34, 37] for regular differential equations, [7, 8, 19, 20, 31, 32, 35] for singular differential equations and [21] for a piecewise smooth dynamical system. Obviously, the method of third order approximation cannot be directly applied to the system (1.3) and (1.4). To do this, we will establish the formula of the first twist coefficient for their equivalent equations in Section 2. Moreover, a stability criterion is also obtained. It is worth noting that the first twist coefficient of such nonlinear differential equation presented in this paper is the first one available in the literature.

Finally, we note that the study on the existence and dynamical behaviors of periodic solutions of the first order singular planar differential systems is more recent and there are still few works in the literature up to now. See [3, 13, 26] and the references therein. To give more detail, Cheng and Cui [3] used Leray-Schauder alternative principle, Manásevich-Mawhin continuation theorem and fixed point theorem in cones to study the existence of positive periodic solutions of the Basener-Ross system. In [13], a generalized version of the Poincaré-Birkhoff theorem has been applied to study the existence of periodic solutions for a singular Hamiltonian system by Fonda and Sfecci. The existence of periodic solutions of the Steen's planar system has been studied in [26] by the Poincaré-Bohl theorem. To the best of our knowledge, the corresponding analytic results about the Lyapunov stability of periodic solutions of the first order singular planar differential systems has not been established in the literature up to now. Therefore, the results of this paper will fill, at least partially, this gap.

2. The first twist coefficient and a stability criterion

From the first equation of systems (1.3) and (1.4), we have

$$v = \frac{u'}{g(t)}.$$

Then elimination of v in the second equation of systems (1.3) and (1.4) lead to the following two singular differential equations

$$\left(\frac{u'}{g(t)}\right)' - \frac{h(t)}{u^l} + \frac{p(t)}{u^\alpha} = 0 \tag{2.1}$$

and

$$\left(\frac{u'}{g(t)}\right)' - \frac{h(t)}{u^l} + \frac{\varepsilon p(t)}{u^\alpha} = 0. \tag{2.2}$$

It is easy to see that the periodic solutions of equations (2.1) and (2.2) correspond to the periodic solutions of systems (1.3) and (1.4), respectively.

Therefore, we need to study the Lyapunov stability of the above two equations firstly.

In order to apply the method of third order approximation to the equations (2.1) and (2.2), we will establish the formula of the first twist coefficient for the following more general equation

$$\left(\frac{u'}{g(t)}\right)' + f(t,u) = 0, \tag{2.3}$$

where $g \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ and $f \in C^{0,4}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}, \mathbb{R})$. The third order approximation of the above equation is

$$\left(\frac{u'}{g(t)}\right)' + a(t)u + b(t)u^2 + c(t)u^3 + o(u^3) = 0, \tag{2.4}$$

where

$$a(t) = \frac{\partial f}{\partial u}\bigg|_{u=\varphi(t)}, \quad b(t) = \frac{1}{2}\frac{\partial^2 f}{\partial u^2}\bigg|_{u=\varphi(t)}, \quad c(t) = \frac{1}{6}\frac{\partial^3 f}{\partial u^3}\bigg|_{u=\varphi(t)}$$

and φ is a T-periodic solution of (2.3). Obviously, $a, b, c \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$. By [5, (3.10)] or [6, Theorem 2.2], we know that if

$$\int_0^T g(t)\mathrm{d}t \cdot \int_0^T a_+(t)\mathrm{d}t \leq 4, \tag{2.5}$$

then the following linearized equation of (2.4)

$$\left(\frac{u'}{g(t)}\right)' + a(t)u = 0 \tag{2.6}$$

is elliptic, here

$$a_+(t) = \max\{0, a(t)\}.$$

Under the change of time

$$s = \tau(t) = \int_0^t g(s)\mathrm{d}s,$$

equation (2.4) becomes

$$x'' + \tilde{a}(s)x + \tilde{b}(s)x^2 + \tilde{c}(s)x^3 + o(x^3) = 0, \tag{2.7}$$

where $x(s) = u(\tau^{-1}(s))$ and

$$\tilde{a}(s) = \frac{a(\tau^{-1}(s))}{g(\tau^{-1}(s))}, \quad \tilde{b}(s) = \frac{b(\tau^{-1}(s))}{g(\tau^{-1}(s))}, \quad \tilde{c}(s) = \frac{c(\tau^{-1}(s))}{g(\tau^{-1}(s))},$$

which are $\tilde{T}$-periodic with

$$\tilde{T} = \tau(T) = \int_0^T g(t)\mathrm{d}t = T\bar{g}.$$

Then, by [23, 36], the first twist coefficient of the equation (2.7) is

$$\begin{aligned}\beta = &\iint_{[0,\tilde{T}]^2} \tilde{b}(s)\tilde{b}(\nu)r^3(s)r^3(\nu)\chi_\theta(|\tilde{\phi}(s) - \tilde{\phi}(\nu)|)\mathrm{d}s\mathrm{d}\nu \\ &- \frac{3}{8}\int_0^{\tilde{T}} \tilde{c}(s)r^4(s)\mathrm{d}s,\end{aligned} \tag{2.8}$$

where r is a $\tilde{T}$-periodic solution of the following equation

$$r'' + \tilde{a}(s)r = \frac{1}{r^3},$$

$\tilde{\phi}$ and χ_θ are defined by

$$\tilde{\phi}(t) = \int_0^t \frac{1}{r^2(s)}\mathrm{d}s$$

and

$$\chi_\theta(\xi) = \frac{3\cos(\xi - \frac{\theta}{2})}{16\sin(\frac{\theta}{2})} + \frac{\cos 3(\xi - \frac{\theta}{2})}{16\sin(\frac{3\theta}{2})}, \quad \xi \in [0, \theta],$$

with $\theta = \tilde{T}\varrho$ and ϱ is the rotation number of the following equation

$$x'' + \tilde{a}(s)x = 0.$$

Obviously, the time rescaling does not change the sign of the first twist coefficient of the equation (2.4). Substituting $s = \tau(t)$ and $\nu = \tau(s)$ into (2.8), then the first twist coefficient of the equation (2.4) can be written as, up to a positive factor,

$$\begin{aligned}\beta = &\iint_{[0,T]^2} b(t)b(s)R^3(t)R^3(s)\chi_\theta(|\phi(t) - \phi(s)|)\mathrm{d}t\mathrm{d}s \\ &- \frac{3}{8}\int_0^T c(t)R^4(t)\mathrm{d}t,\end{aligned}$$

where $R(t) = r(\tau(t))$ and $\phi(t) = \tilde{\phi}(\tau(t))$.

If the linear equation (2.6) is elliptic and $\beta \neq 0$, then the T-periodic solution φ of (2.3) is of twist type. According to Moser's invariant curve theorem [27], the twist periodic solution is Lyapunov stable.

Moreover, we obtain the following stability criterion.

Theorem 2.1. *Suppose that* φ *is a* T*-periodic solution of* (2.3) *such that*

(A$_1$) $0 < a_{_*} \leq a^* \leq g_* \left(\dfrac{\pi}{2T\bar{g}}\right)^2$;

(A$_2$) $c_{_*} > 0$;

(A$_3$) $10a_{_*}^{\frac{3}{2}}|b|_{_*}^2 g_*^{\frac{7}{2}} > 9(a^*)^{\frac{5}{2}}c^*(g^*)^{\frac{7}{2}}$,

Then φ *is of twist type.*

Proof. By the fact that $\tau^{-1}(s)$ is increasing in s and $\tilde{T} = \tau(T) = 2T\bar{g}$, we get

$$\tilde{a}^* = \sup_{s\in[0,\tilde{T}]} \frac{a(\tau^{-1}(s))}{g(\tau^{-1}(s))} = \sup_{t\in[0,T]} \frac{a(t)}{g(t)} \leq \frac{a^*}{g_*},$$

$$\tilde{a}_* = \inf_{s\in[0,\tilde{T}]} \frac{a(\tau^{-1}(s))}{g(\tau^{-1}(s))} = \inf_{t\in[0,T]} \frac{a(t)}{g(t)} \geq \frac{a_*}{g^*}.$$

Analogously, we have

$$|\tilde{b}|_* \geq \frac{|b|_*}{g^*}, \quad \tilde{c}_* \geq \frac{c_*}{g^*}, \quad \tilde{c}^* \leq \frac{c^*}{g_*}.$$

Using the above inequalities, we notice that if (A$_1$)-(A$_3$) hold, then we have

(i) $0 < \tilde{a}_* \leq \tilde{a}^* < (\frac{\pi}{2\tilde{T}})^2$;

(ii) $\tilde{c}_* > 0$;

(iii) $10(\tilde{a}_*)^{\frac{3}{2}}|\tilde{b}|_*^2 > 9(\tilde{a}^*)^{\frac{5}{2}}\tilde{c}^*$.

Then, by [31, Theorem 3.1], we know that the trivial solution $x = 0$ of (2.7) is of twist type. By the fact that the time rescaling does not affect the stability of equation (2.4), we conclude that the trivial solution $u = 0$ of (2.4) is also of twist type, that is, φ is of twist type. □

Here, we should point out that the Theorem 3.1 in [31] deals with the 2π-periodic equations, but, we find that the assumptions (i)-(iii) of Theorem 3.1 in [31] remain unchanged for an arbitrary period T when we check the details of its proof. Therefore, we can use it directly in the above proof.

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1 by the stability criterion obtained in Section 2 and the upper-lower solutions method on the reversed order [12, 30]. The proof falls naturally into three steps.

Step 1: Existence and the explicit bounds.

Based on the upper-lower solutions method on the reversed order, we obtain the following results about the existence and explicit bounds of periodic solutions of the system (1.3).

Lemma 3.1. *Suppose that* $g,\ h,\ p \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$ *and*

$$g(t) \geq g_0, \quad \forall t \in \mathbb{R}. \tag{3.1}$$

Then the system (1.3) *has a* T*-periodic solution* (u, v) *such that*

$$\Big(\frac{h_*}{p^*}\Big)^{\kappa} < u(t) < \Big(\frac{h^*}{p_*}\Big)^{\kappa}, \quad \forall t \in \mathbb{R}. \tag{3.2}$$

Proof. By the same change of time $s = \tau(t)$ which was defined in Section 2, we can rewrite the equation (2.1) as

$$y''(s) + \frac{1}{\tilde{g}(s)}\Big(-\frac{\tilde{h}(s)}{y^l(s)} + \frac{\tilde{p}(s)}{y^{\alpha}(s)}\Big) = 0, \tag{3.3}$$

where

$$y(s) = u(\tau^{-1}(s)), \quad \tilde{h}(s) = h(\tau^{-1}(s)),$$

$$\tilde{p}(s) = p(\tau^{-1}(s)), \quad \tilde{g}(s) = g(\tau^{-1}(s))$$

are $\tilde{T}$-periodic with $\tilde{T} = T\bar{g}$. It is noticeable that

$$\psi_1(s) = \Big(\frac{\tilde{h}_*}{\tilde{p}^*}\Big)^{\kappa}$$

and

$$\psi_2(s) = \Big(\frac{\tilde{h}^*}{\tilde{p}_*}\Big)^{\kappa}$$

are constant strict upper and lower solution of the equation (3.3) with the reserved order $\psi_2 > \psi_1$, respectively. Set

$$F(s, y) = \frac{1}{\tilde{g}(s)}\Big(-\frac{\tilde{h}(s)}{y^l(s)} + \frac{\tilde{p}(s)}{y^{\alpha}(s)}\Big).$$

By the upper-lower solutions method on the reversed order, we get that the equation (3.3) has a $\tilde{T}$-periodic solution if

$$\begin{aligned} F_y(s,y) &= \frac{1}{\tilde{g}(s)}\left(\frac{l\tilde{h}(s)}{y^{l+1}(s)} - \frac{\alpha\tilde{p}(s)}{y^{\alpha+1}(s)}\right) \\ &\le \frac{1}{\tilde{g}(s)}\left(\frac{l\tilde{h}(s)}{\psi_1^{l+1}(s)} - \frac{\alpha\tilde{p}(s)}{\psi_2^{\alpha+1}(s)}\right) \\ &\le \frac{1}{\tilde{g}_*}\left(\frac{l\tilde{h}^*}{(\frac{\tilde{h}_*}{\tilde{p}^*})^{\kappa(l+1)}} - \frac{\alpha\tilde{p}_*}{(\frac{\tilde{h}^*}{\tilde{p}_*})^{\kappa(\alpha+1)}}\right) \\ &\le (\frac{\pi}{\tilde{T}})^2, \quad \text{for} \quad \text{every} \quad y \in [\psi_1, \psi_2]. \end{aligned} \tag{3.4}$$

By the facts that $\tau^{-1}(s)$ is increasing in s, $\tau^{-1}(0) = 0$ and $\tau^{-1}(\tilde{T}) = T$, we have

$$\tilde{h}^* = \sup_{s\in[0,\tilde{T}]} h(\tau^{-1}(s)) = \sup_{s\in[0,T]} h(t) = h^*$$

and

$$\tilde{h}_* = \inf_{s\in[0,\tilde{T}]} h(\tau^{-1}(s)) = \inf_{s\in[0,T]} h(t) = h_*.$$

Similarly, we have $\tilde{p}^* = p^*$, $\tilde{p}_* = p_*$ and $\tilde{g}_* = g_*$. Then, (3.4) becomes

$$\frac{lh^*(p^*)^{\kappa(l+1)}}{(h_*)^{\kappa(l+1)}} - \frac{\alpha(p_*)^{\kappa(l+1)}}{(h^*)^{\kappa(\alpha+1)}} \le g_*\left(\frac{\pi}{T\bar{g}}\right)^2. \tag{3.5}$$

A simple calculation shows that (3.5) is equivalent to (3.1). Thus, if (3.1) holds, then (3.3) has a $\tilde{T}$-periodic solution y such that

$$(\frac{\tilde{h}_*}{\tilde{p}^*})^\kappa = \psi_1(s) < y(s) < \psi_2(s) = (\frac{\tilde{h}^*}{\tilde{p}_*})^\kappa.$$

Accordingly, we show that the equation (2.1) has a T-periodic solution $u(t) = y(\tau(t))$ such that (3.2) holds, that is, the system (1.3) has a T-periodic solution (u, v). □

Step 2: We show that u is Lyapunov stable.

We will apply Theorem 2.1 to prove that the T-periodic solution u of (2.1) obtained in Lemma 3.1 is of twist type and therefore is Lyapunov stable.

Lemma 3.2. *Suppose that* (1.5) *and* (1.6) *hold. Then there exists a constant* $\gamma_0 \in (0,1)$ *such that for every* $\gamma > \gamma_0$*, the* T*-periodic solution* u *of the equation* (2.1) *obtained in Lemma 3.1 is of twist type.*

Proof. We first compute the coefficients in (2.4) for the equation (2.1)

$$a(t) = \frac{lh(t)}{u^{l+1}} - \frac{\alpha p(t)}{u^{\alpha+1}},$$

$$b(t) = \frac{-l(l+1)h(t)}{2u^{l+2}} + \frac{\alpha(\alpha+1)p(t)}{2u^{\alpha+2}}, \tag{3.6}$$

$$c(t) = \frac{l(l+1)(l+2)h(t)}{6u^{l+3}} - \frac{\alpha(\alpha+1)(\alpha+2)p(t)}{6u^{\alpha+3}}. \tag{3.7}$$

Taking into account (3.2), we have

$$a(t) > \frac{lh(t)}{(\frac{h^*}{p_*})^{\kappa(l+1)}} - \frac{\alpha p(t)}{(\frac{h_*}{p^*})^{\kappa(\alpha+1)}}$$

and

$$a(t) < \frac{lh(t)}{(\frac{h_*}{p^*})^{\kappa(l+1)}} - \frac{\alpha p(t)}{(\frac{h^*}{p_*})^{\kappa(\alpha+1)}}. \tag{3.8}$$

Then, we can verify that if

$$\gamma > (\frac{\alpha}{l})^{\frac{1}{\kappa(l+1)}} := \gamma_1,$$

then

$$\begin{aligned} a_* &> \frac{lh_*}{(\frac{h^*}{p_*})^{\kappa(l+1)}} - \frac{\alpha p^*}{(\frac{h_*}{p^*})^{\kappa(\alpha+1)}} \\ &= h_*\Big[l(\frac{p_*}{h^*})^{\kappa(l+1)} - \alpha(\frac{p^*}{h_*})^{\kappa(l+1)}\Big] \\ &= h_*(\frac{p^*}{h_*})^{\kappa(l+1)}\Big[l\gamma^{\kappa(l+1)} - \alpha\Big] \\ &> 0. \end{aligned} \tag{3.9}$$

Combining (1.5) with (3.8), we have

$$\begin{aligned}
a^* &< \frac{lh^*}{(\frac{h_*}{p^*})^{\kappa(l+1)}} - \frac{\alpha p_*}{(\frac{h^*}{p_*})^{\kappa(\alpha+1)}} \\
&= \frac{lh^*(p^*)^{\kappa(l+1)}}{(h_*)^{\kappa(l+1)}} - \frac{\alpha (p_*)^{\kappa(l+1)}}{(h^*)^{\kappa(\alpha+1)}} \\
&= h^*\left(\frac{p^*}{h_*}\right)^{\kappa(l+1)}\left(l-\alpha\gamma^{\kappa(l+1)}\right) \qquad (3.10)\\
&\le g_*(\frac{\pi}{2T\bar{g}})^2.
\end{aligned}$$

Then the above two inequalities show that (A_1) of Theorem 2.1 is satisfied. By the estimates (3.2) and (3.7), one can find that if

$$\gamma > \left(\frac{\alpha(\alpha+1)(\alpha+2)}{l(l+1)(l+2)}\right)^{\frac{1}{\kappa(l+3)}} := \gamma_2,$$

then

$$\begin{aligned}
c_* &> \frac{l(l+1)(l+2)h_*}{6(\dfrac{h^*}{p_*})^{\kappa(l+3)}} - \frac{\alpha(\alpha+1)(\alpha+2)p^*}{6(\dfrac{h_*}{p^*})^{\kappa(\alpha+3)}} \\
&= h_*\left(\frac{p^*}{h_*}\right)^{\kappa(l+3)}\Big[\frac{l(l+1)(l+2)}{6}\gamma^{\kappa(l+3)} - \frac{\alpha(\alpha+1)(\alpha+2)}{6}\Big] \qquad (3.11)\\
&> 0.
\end{aligned}$$

We thus get that (A_2) of Theorem 2.1 holds. Analogously, we have

$$\begin{aligned}
c^* &< \frac{l(l+1)(l+2)h^*}{6(\dfrac{h_*}{p^*})^{k(l+3)}} - \frac{\alpha(\alpha+1)(\alpha+2)p_*}{6(\dfrac{h^*}{p_*})^{k(\alpha+3)}} \\
&= h^*\left[\frac{l(l+1)(l+2)}{6}\left(\frac{p^*}{h_*}\right)^{\kappa(l+3)} - \frac{\alpha(\alpha+1)(\alpha+2)}{6}\left(\frac{p_*}{h^*}\right)^{k(l+3)}\right] \\
&= h^*\left(\frac{p^*}{h_*}\right)^{k(l+3)}\Big[\frac{l(l+1)(l+2)}{6} - \frac{\alpha(\alpha+1)(\alpha+2)}{6}\gamma^{\kappa(l+3)}\Big]. \qquad (3.12)
\end{aligned}$$

Substituting the estimates (3.2) into (3.6), we are able to obtain that if

$$\gamma > \left(\frac{\alpha(\alpha+1)}{l(l+1)}\right)^{\frac{1}{\kappa(l+2)}} := \gamma_3,$$

then

$$
\begin{aligned}
b(t) &< -\frac{l(l+1)h_*}{2(\frac{h^*}{p_*})^{\kappa(l+2)}} + \frac{\alpha(\alpha+1)p^*}{2(\frac{h_*}{p^*})^{\kappa(\alpha+2)}} \\
&= h_*\Big(- \frac{l(l+1)}{2}(\frac{p_*}{h^*})^{\kappa(l+2)} + \frac{\alpha(\alpha+1)}{2}(\frac{p^*}{h_*})^{\kappa(l+2)}\Big) \\
&= h_*(\frac{p^*}{h_*})^{\kappa(l+2)}\Big(- \frac{l(l+1)}{2}\gamma^{\kappa(l+2)} + \frac{\alpha(\alpha+1)}{2}\Big) \\
&< 0,
\end{aligned}
$$

which yields

$$
|b|_* > h_*(\frac{p^*}{h_*})^{\kappa(l+2)}\Big(\frac{l(l+1)}{2}\gamma^{\kappa(l+2)} - \frac{\alpha(\alpha+1)}{2}\Big). \tag{3.13}
$$

By virtue of (3.9), (3.10), (3.12) and (3.13), a simple computation shows that the condition (A_3) of Theorem 2.1 holds if

$$
G_1(\gamma) > G_2(\gamma), \tag{3.14}
$$

where

$$
G_1(\gamma) = 5(h_*g_*)^{\frac{7}{2}}(l\gamma^{\kappa(l+1)} - \alpha)^{\frac{3}{2}}\big[l(l+1)\gamma^{\kappa(l+2)} - \alpha(\alpha+1)\big]^2,
$$

$$
G_2(\gamma) = 3(h^*g^*)^{\frac{7}{2}}(l - \alpha\gamma^{\kappa(l+1)})^{\frac{5}{2}}[l(l+1)(l+2) - \alpha(\alpha+1)(\alpha+2)\gamma^{\kappa(l+3)}].
$$

Moreover, by (1.6), we get

$$
\begin{aligned}
G_1(1) - G_2(1) =& 5(h_*g_*)^{\frac{7}{2}}(l-\alpha)^{\frac{3}{2}}(l(l+1) - \alpha(\alpha+1))^2 \\
& - 3(h^*g^*)^{\frac{7}{2}}(l-\alpha)^{\frac{5}{2}}[l(l+1)(l+2) - \alpha(\alpha+1)(\alpha+2)] \\
=& 5(h_*g_*)^{\frac{7}{2}}(l-\alpha)^{\frac{7}{2}} \cdot \\
& \Big[(l+\alpha+1)^2 - \frac{3}{5}(\frac{h^*g^*}{h_*g_*})^{\frac{7}{2}}(l^2 + l\alpha + \alpha^2 + 3l + 3\alpha + 2)\Big] \\
>& 0.
\end{aligned}
$$

Then, by continuity, the above inequality shows that there exists a constant $\gamma_4 \in (0,1)$ such that (3.14) holds if

$$
\gamma > \gamma_4.
$$

Fix

$$\gamma_0 = \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\},$$

then the relations (3.9), (3.10), (3.11) and (3.14) imply that all conditions of Theorem 2.1 are satisfied if

$$\gamma > \gamma_0.$$

Applying Theorem 2.1, we get that the T-periodic solution u of the equation (2.1) obtained in Lemma 3.1 is of twist type if $\gamma > \gamma_0$. $\square$

Step 3: We verify that v is Lyapunov stable.

Suppose that $u(t) = u(t, u(0), u'(0))$ is a stable T-periodic solution of the equation (2.1) which was obtained in Lemma 3.2. Let $u_1(t) = u_1(t, u_1(0), u_1'(0))$ be a new solution of the equation (2.1). According to the definition of Lyapunov stability, we know that for any given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|u(0) - u_1(0)| + |u'(0) - u_1'(0)| < \delta_1, \tag{3.15}$$

then

$$|u(t) - u_1(t)| + |u'(t) - u_1'(t)| < g_* \varepsilon. \tag{3.16}$$

By Lemma 3.1, we get that the system (1.3) has a T-periodic solution

$$(u(t), v(t)) = \big(u(t, u(0), v(0)), v(t, u(0), v(0))\big)$$

with

$$v(t) = \frac{u'(t)}{g(t)}.$$

Let

$$(u_1(t), v_1(t)) = \big(u_1(t, u_1(0), v_1(0)), v_1(t, u_1(0), v_1(0))\big)$$

be a new solution of the system (1.3). By the relation (3.15) and the following facts

$$v(0) = \frac{u'(0)}{g(0)} \quad \text{and} \quad v_1(0) = \frac{u_1'(0)}{g(0)},$$

we know that there exists a $\delta_2 > 0$ such that

$$|u(0) - u_1(0)| + |v(0) - v_1(0)| < \delta_2,$$

then by (3.16), we have

$$\begin{aligned}|v(t)-v_1(t)| =&\frac{1}{g(t)}|u'(t)-u_1'(t)|\\ \leq&\frac{1}{g_*}|u'(t)-u_1'(t)|\\ <&\varepsilon,\end{aligned}$$

which means that the T-periodic solution v is Lyapunov stable.

Up to now, the proof of Theorem 1.1 is finished.

4. Proof of Theorem 1.2

In this section, by the third order approximation and the averaging method [15, 18, 25], we will verify the correctness of the Theorem 1.2. The proof will also be divided into three steps.

Step 1: Existence and some location information.

By the averaging method, we obtain the following result on the existence and the asymptotic behavior of periodic solutions of the system (1.4).

Lemma 4.1. *Suppose that $\bar{h}\bar{p} > 0$. Then there exists a $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, the system (1.3) has a T-periodic solution $\big(u(t,\varepsilon), v(t,\varepsilon)\big)$ which satisfies*

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{\kappa}u(t,\varepsilon), \varepsilon^{\frac{\kappa(1-l)}{2}}v(t,\varepsilon)\right) = (\mu^{\kappa}, 0), \quad \textit{uniformly}\ \ \textit{in}\ t, \tag{4.1}$$

where

$$\mu = \frac{\bar{h}}{\bar{p}}.$$

Proof. To aplly the averaging theory, we rewrite the system (1.4) into a perturbative system. Doing the rescaling of variables

$$\begin{cases} u = \varepsilon^{-\kappa}x, \\ v = \varepsilon^{\frac{\kappa(l-1)}{2}}y, \end{cases}$$

we can reform the system (1.3) as follows

$$\begin{cases} x' = \epsilon g(t) y, \\ y' = \epsilon \left(\dfrac{h(t)}{x^l} - \dfrac{p(t)}{x^\alpha} \right). \end{cases} \tag{4.2}$$

Obviously, the averaged system of the above system is

$$\begin{cases} \xi' = \epsilon \bar{g} \eta, \\ \eta' = \epsilon \left(\dfrac{\bar{h}}{\xi^l} - \dfrac{\bar{p}}{\xi^\alpha} \right), \end{cases}$$

where $\epsilon = \varepsilon^{\frac{\kappa(l+1)}{2}}$. After some calculations, we notice that the above averaged system has a unique constant solution

$$(\xi_0, \eta_0) = (\mu^\kappa, 0),$$

which is non-degenerate. Then the averaging method (see [15, Section V.3] or [18, Section 2]) shows that there exists $\epsilon^* > 0$ such that the system (4.2) has a T-periodic solution $\big(x(t,\epsilon), y(t,\epsilon)\big)$ for $0 < \epsilon < \epsilon^*$ and

$$\big(x(t,\epsilon), y(t,\epsilon)\big) \to (\mu^\kappa, 0), \quad \text{uniformly} \quad \text{as} \quad \epsilon \to 0^+.$$

Going back to the original variables, we prove that there exists a $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, the system (1.3) has a T-periodic solution $\big(u(t,\varepsilon), v(t,\varepsilon)\big)$ which satisfies (4.1). □

Step 2: We show that u is Lyapunov stable.

We will prove that the T-periodic solution u obtained in Lemma 4.1 is Lyapunov stable.

Lemma 4.2. *Suppose that $g,\ h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^+)$, $\bar{p} > 0$, (1.7) and (1.8) hold. Then the T-periodic solution u obtained in Lemma 4.1 is of twist type if ε is small enough. Therefore, u is Lyapunov stable.*

Proof. It is readily seen that the periodic solutions of the equation (2.2) corresponds to the periodic solutions of the system (1.4). We will apply the method of third order approximation to the equation (2.2). We first compute the coefficients in (2.4) for the equation (2.2)

$$a(t) = \frac{lh(t)}{u^{l+1}} - \frac{\varepsilon\alpha p(t)}{u^{\alpha+1}},$$

$$b(t) = \frac{\varepsilon\alpha(\alpha+1)p(t)}{2u^{\alpha+2}} - \frac{l(l+1)h(t)}{2u^{l+2}}$$

and

$$c(t) = \frac{l(l+1)(l+2)h(t)}{6u^{l+3}} - \frac{\varepsilon\alpha(\alpha+1)(\alpha+2)p(t)}{6u^{\alpha+3}},$$

here $u(t) = u(t, \varepsilon)$. By the limit (4.1), the calculations show that

$$\lim_{\varepsilon\to 0} a(t)\varepsilon^{-\kappa(l+1)} = \frac{lh(t)}{\mu^{\kappa(l+1)}} - \frac{\alpha p(t)}{\mu^{\kappa(\alpha+1)}}, \tag{4.3}$$

$$\lim_{\varepsilon\to 0} b(t)\varepsilon^{-\kappa(l+2)} = \frac{\alpha(\alpha+1)p(t)}{2\mu^{\kappa(\alpha+2)}} - \frac{l(l+1)h(t)}{2\mu^{\kappa(l+2)}} \tag{4.4}$$

and

$$\lim_{\varepsilon\to 0} c(t)\varepsilon^{-\kappa(l+3)} = \frac{l(l+1)(l+2)h(t)}{6\mu^{\kappa(l+3)}} - \frac{\alpha(\alpha+1)(\alpha+2)p(t)}{6\mu^{\kappa(\alpha+3)}}. \tag{4.5}$$

Then by (1.7) and (4.3), we get

$$\lim_{\varepsilon\to 0} a(t)\varepsilon^{-\kappa(l+1)} \geq \frac{lh_*}{\mu^{\kappa(l+1)}} - \frac{\alpha p^*}{\mu^{\kappa(\alpha+1)}} \geq 0,$$

that is, $a(t) \geq 0$. Then, by a simple computation, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^{-\kappa(l+1)} \int_0^T g(t)\mathrm{d}t \cdot \int_0^T a_+(t)\mathrm{d}t &= T\bar{g}\int_0^T \left(\frac{lh(t)}{\mu^{\kappa(l+1)}} - \frac{\alpha p(t)}{\mu^{\kappa(\alpha+1)}}\right)\mathrm{d}t \\ &= T^2\bar{g}\left(\frac{l\bar{h}}{\mu^{\kappa(l+1)}} - \frac{\alpha\bar{p}}{\mu^{\kappa(\alpha+1)}}\right) \\ &= T^2\bar{g}(l-\alpha)\frac{\bar{p}^{\kappa(l+1)}}{\bar{h}^{\kappa(\alpha+1)}},\end{aligned}$$

which leads to that (2.5) holds if ε is small enough. From this, we deduce that the linearized equation of (2.2) is elliptic if ε is small enough.

By the limit (4.3) and $\tilde{T} = T\bar{g}$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\bar{\tilde{a}}}{\varepsilon^{\kappa(l+1)}} &= \lim_{\varepsilon\to 0}\frac{1}{\tilde{T}\varepsilon^{\kappa(l+1)}}\int_0^{\tilde{T}}\frac{a(\tau^{-1}(s))}{g(\tau^{-1}(s))}\mathrm{d}s \\
&= \lim_{\varepsilon\to 0}\frac{1}{\tilde{T}\varepsilon^{\kappa(l+1)}}\int_0^{T}a(t)\mathrm{d}t \\
&= \frac{1}{\tilde{T}}\int_0^T\left(\frac{lh(t)}{\mu^{\kappa(l+1)}}-\frac{\alpha p(t)}{\mu^{\kappa(\alpha+1)}}\right)\mathrm{d}t \\
&= \frac{T}{\tilde{T}}\left(\frac{l\bar{h}}{\mu^{\kappa(l+1)}}-\frac{\alpha\bar{p}}{\mu^{\kappa(\alpha+1)}}\right) \\
&= \frac{(l-\alpha)}{\bar{g}}\frac{\bar{p}^{\kappa(l+1)}}{\bar{h}^{\kappa(\alpha+1)}}.
\end{aligned} \tag{4.6}$$

Applying [10, Corollary 4.1], as $\|\tilde{a}\| \to 0^+$, we get

$$\theta = \tilde{T}\rho(\tilde{a}) = \tilde{T}(\bar{\tilde{a}})^{\frac{1}{2}}(1 + O(\|\tilde{a}\|)),$$

$$R(t) = r(\tau(t)) = (\bar{\tilde{a}})^{-\frac{1}{4}}(1 + O(\|\tilde{a}\|)), \quad \text{uniformly in } t.$$

Combining the above two equalities with (4.6), we obtain

$$\lim_{\varepsilon\to 0}\frac{\theta}{\sqrt{\varepsilon^{\kappa(l+1)}}} = T\sqrt{\bar{g}(l-\alpha)}\sqrt{\frac{\bar{p}^{\kappa(l+1)}}{\bar{h}^{\kappa(\alpha+1)}}} \tag{4.7}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\kappa(l+1)}{4}}R(t) = \sqrt[4]{\frac{\bar{g}}{l-\alpha}}\sqrt[4]{\frac{\bar{h}^{\kappa(\alpha+1)}}{\bar{p}^{\kappa(l+1)}}}. \tag{4.8}$$

Analysis similar to that in [8, Section 3] shows that

$$\chi_\theta(\iota) = \frac{5}{12\theta}\left(1 + O(\theta^2)\right) = \frac{5}{12}\frac{\bar{\tilde{a}}^{-\frac{1}{2}}}{\tilde{T}} + O(\bar{\tilde{a}}), \quad \text{as} \quad \bar{\tilde{a}} \to 0^+. \tag{4.9}$$

Fix

$$\beta_1 = \iint_{[0,T]^2} b(t)b(s)R^3(t)R^3(s)\chi_\theta(|\varphi(t)-\varphi(s)|)\mathrm{d}t\mathrm{d}s$$

and

$$\beta_2 = \frac{3}{8}\int_0^T c(t)R^4(t)\mathrm{d}t.$$

Consider the following limit

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\beta_1}{\varepsilon^{2\kappa}} = \quad & \lim_{\varepsilon\to 0}\iint_{[0,T]^2}\left[\frac{b(t)}{\varepsilon^{\kappa(l+2)}}\right]\left[\frac{b(s)}{\varepsilon^{\kappa(l+2)}}\right]\left[\varepsilon^{\frac{3}{4}\kappa(l+1)}R^3(t)\right] \\
\times \quad & \left[\varepsilon^{\frac{3}{4}\kappa(l+1)}R^3(s)\right]\left[\varepsilon^{\frac{1}{2}\kappa(l+1)}\chi_\theta(|\varphi(t)-\varphi(s)|)\right]dtds.
\end{aligned}$$

By inserting the limits (4.3)-(4.5), (4.7) and (4.8) into the above limit, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\beta_1}{\varepsilon^{2\kappa}} &= \iint_{[0,T]^2}\frac{1}{4}\left[\frac{\alpha(\alpha+1)p(t)}{\mu^{\kappa(\alpha+2)}}-\frac{l(l+1)h(t)}{\mu^{\kappa(l+2)}}\right]\left[\frac{\alpha(\alpha+1)p(s)}{\mu^{\kappa(\alpha+2)}}-\frac{l(l+1)h(s)}{\mu^{\kappa(l+2)}}\right]\\
&\quad\times\left(\frac{\bar{g}}{(l-\alpha)}\right)^{3/2}\left[\frac{\bar{h}^{\kappa(\alpha+1)}}{\bar{p}^{\kappa(l+1)}}\right]^{3/2}\frac{5}{12\tilde{T}}\sqrt{\frac{\bar{g}}{(l-\alpha)}}\sqrt{\frac{\bar{h}^{\kappa(\alpha+1)}}{\bar{p}^{\kappa(l+1)}}}dtds\\
&= \frac{5}{48T}\frac{\bar{g}}{(l-\alpha)^2}\frac{\bar{h}^{2\kappa(\alpha+1)}}{\bar{p}^{2\kappa(l+1)}}\iint_{[0,T]^2}\left[\frac{\alpha(\alpha+1)p(t)}{\mu^{\kappa(\alpha+2)}}-\frac{l(l+1)h(t)}{\mu^{\kappa(l+2)}}\right]\\
&\quad\times\left[\frac{\alpha(\alpha+1)p(s)}{\mu^{\kappa(\alpha+2)}}-\frac{l(l+1)h(s)}{\mu^{\kappa(l+2)}}\right]dtds\\
&= \frac{5}{48T}\frac{\bar{g}}{(l-\alpha)^2}\frac{\bar{h}^{2\kappa(\alpha+1)}}{\bar{p}^{2\kappa(l+1)}}\left[\frac{\alpha(\alpha+1)T\bar{p}}{\mu^{\kappa(\alpha+2)}}-\frac{l(l+1)T\bar{h}}{\mu^{\kappa(l+2)}}\right]^2\\
&= \frac{5T\bar{g}}{48\mu^{2\kappa}}(l+\alpha+1)^2.
\end{aligned}$$

Similar arguments apply to β_2 gives

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\beta_2}{\varepsilon^{2\kappa}} &= \lim_{\varepsilon\to 0}\int_0^T\frac{c(t)}{\varepsilon^{\kappa(l+3)}}\cdot\varepsilon^{\kappa(l+1)}R^4(t)dt\\
&= \int_0^T\frac{1}{6}\left[\frac{l(l+1)(l+2)h(t)}{\mu^{\kappa(l+3)}}-\frac{\alpha(\alpha+1)(\alpha+2)p(t)}{\mu^{\kappa(\alpha+3)}}\right]\cdot\frac{\bar{g}}{l-\alpha}\frac{\bar{h}^{\kappa(\alpha+1)}}{\bar{p}^{\kappa(l+1)}}dt\\
&= \frac{T\bar{g}}{6(l-\alpha)}\frac{\bar{h}^{\kappa(\alpha+1)}}{\bar{p}^{\kappa(l+1)}}\left[\frac{l(l+1)(l+2)\bar{h}}{\mu^{\kappa(l+3)}}-\frac{\alpha(\alpha+1)(\alpha+2)\bar{p}}{\mu^{\kappa(\alpha+3)}}\right]\\
&= \frac{T}{6(l-\alpha)}\left[\frac{l(l+1)(l+2)}{\mu^{2\kappa}}-\frac{\alpha(\alpha+1)(\alpha+2)}{\sigma^{2\kappa}}\right]\\
&= \frac{T\bar{g}}{\mu^{2\kappa}}\frac{l^2+l\alpha+\alpha^2+3l+3\alpha+2}{6}.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\beta}{\varepsilon^{2\kappa}} &= \lim_{\varepsilon\to 0}\frac{\beta_1-\frac{3}{8}\beta_2}{\varepsilon^{2\kappa}}\\
&\geq \frac{T\bar{g}}{\mu^{2\kappa}}\left[\frac{5(l+\alpha+1)^2}{48}-\frac{l^2+l\alpha+\alpha^2+3l+3\alpha+2}{16}\right]\\
&= \frac{T\bar{g}}{48\mu^{2\kappa}}(2l^2+2\alpha^2+7l\alpha+l+\alpha-1).
\end{aligned}$$

Therefore, if (1.8) holds, we have

$$\lim_{\varepsilon\to 0}\frac{\beta}{\varepsilon^{2\kappa}}>0,$$

that is, $\beta>0$ when ε is small enough. □

Step 3: We verify that v is Lyapunov stable.

Repeating the step 3 of the proof of Theorem 1.1, we can verify that v is Lyapunov stable.

Up to now, the proof of Theorem 1.2 is completed.

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