Stability and optimal decay estimates for the 3D anisotropic Boussinesq equations

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Abstract

This paper focuses on the three-dimensional(3D) incompressible anisotropic Boussinesq system while the velocity of fluid only involves horizontal dissipation and the temperature has a damping term. By utilizing the structure of the system, the energy methods and the means of bootstrapping argument, we prove the global stability property in the Sobolev space $H^k(\mathbb{R}^3)(k \geq 3)$ of perturbations near the hydrostatic equilibrium. Moreover, we take an effective approach to obtain the optimal decay rates for the global solution itself as well as its derivatives. In this paper, we aim to reveal the mechanism of how the temperature helps stabilize the fluid. Additionally, exploring the stability of perturbations near hydrostatic equilibrium may provide valuable insights into specific severe weather phenomena.

Key Words: Boussinesq equations; stability; optimal decay estimate; anisotropic

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1 Introduction

The Boussinesq equations simulate buoyancy-driven fluids such as atmospheric fronts and oceanic circulation, and have played pivotal roles in the study of Rayleigh–Bénard's convection (see, e.g., [1–5]). This paper focuses on the stability and the optimal decay estimates of

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solutions to the following 3D incompressible anisotropic Boussinesq equations

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu \Delta_h U + \Theta e_3, \ x \in \mathbb{R}^3, t > 0 \\ \partial_t \Theta + U \cdot \nabla \Theta + \eta \Theta = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{1.1}$$

where $U(x,t) = (U_1(x,t), U_2(x,t), U_3(x,t))$ denotes the velocity field, $P = P(x,t)$ the pressure and $\Theta = \Theta(x,t)$ the temperature, $e_3 = (0,0,1)$(the unit vector in the vertical direction), $\nu > 0$ and $\eta > 0$ are the viscosity and damping coefficients, respectively. Here we written $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ stands for the horizontal Laplacian.

Let's emphasize that the anisotropic dissipation assumption is natural in the study of buoyancy-driven fluids. It arises naturally in the modeling of anisotropic fluids such as the rotating fluids in Ekman layers [5]. It appears that, under suitable scaling and in certain physical regimes, some components of the dissipation can become trivial and be ignored, such as the vertical dissipation is negligible as compared to the horizontal dissipation(see, e.g., [13, 20]). More physical backgrounds of anisotropic fluids can be available in [5, 26]. The motivation for studying (1.1) comes from twofold. The first is to reveal the phenomenon that the coupling and interaction of the velocity and the temperature actually stabilizes the fluid. The second is to develop an efficient approach to obtain the optimal decay rates for the anisotropic Boussinesq system concerned here.

For the 3D Boussinesq equations with full dissipation or partial dissipation, the global well-posedness problem has attracted considerable attention from the community of mathematical fluids and significant progress has been made (see, e.g., [6, 10, 12, 17, 18, 21]). When the velocity equation involves full dissipation and the temperature equation is a pure transport equation, Geng and Fan [16] obtained a regularity criterion to get a global (in time) solution. Otherwise, in the particular case of axisymmetric initial data, Abidi, Hmidi and Keraani [33] showed the global well-posedness for the Boussinesq system in $\mathbb{R}^3$. For the velocity fluid and the temperature equation both are full dissipation, Qiu, Du and Yao [15] obtained a blow-up criterion by means of the Littlewood-Paley theory and Bony's paradifferential calculus in Besov spaces. Jiu, Wang and Wu [6] established partial regularity for the appropriate weak solution at dimension $n > 2$ by the De Giorgi iterative approach. Under the assumption that the initial data is axisymmetric without swirl, Miao and Zheng [11] proved the global well-posedness for the 3D Boussinesq equation with horizontal dissipation.

In contrast to the magnitude of research conducted on the well-posedness problem for the 3D Boussinesq equations, the stability and the large-time behavior have been studied relatively little. Dong [25] studied asymptotic stability to the 3D Boussinesq equation in the whole space with a velocity damping term. In addition, the decay rates of the velocity and large-time behavior of the temperature also were given. Wu and Zhang [32] solved the stability and large-time behavior problem with mixed partial dissipation in spatial domain $\Omega = \mathbb{R}^2 \times T$ with $T = [-\frac{1}{2}, \frac{1}{2}]$. Shang and Xu [14] examined the stability and the decay of the corresponding linearized systems of 3D Boussinesq equations with horizontal viscosity and horizontal thermal diffusion. Recently, Ji, Yan and Wu [22] further expanded their results and obtained the optimal decay for the the nonlinear Boussinesq system.

The hydrostatic equilibrium given by

$$U^{(0)} = (0,0,0),\ \Theta^{(0)} = x_3,\ P^{(0)} = \frac{1}{2}x_3^2. \tag{1.2}$$

is a very special steady-state solution of (1.1) with great geophysical and astrophysical importance (see, e.g., [5, 29–31]). To understand the stability and optimal decay rates of perturbations near the hydrostatic equilibrium in (1.2), we consider the equations governing the perturbation (u, θ, p) with $u = U - U^{(0)},\ \theta = \Theta - \Theta^{(0)},\ p = P - P^{(0)}$,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta_h u + \theta e_3,\ x \in \mathbb{R}^3, t > 0 \\ \partial_t \theta + u \cdot \nabla \theta + u_3 + \eta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3}$$

In this paper, we employ the classic energy method and bootstrapping argument (see [19]) to establish the global stability of solution to system (1.3) in $H^k(\mathbb{R}^3) (k \geq 3)$. In the process of the decay estimates, classical tools such as Fourier-splitting method for large-time behavior no longer directly apply to the system concerned here. We develop an effective approach to obtain the optimal decay rates for this partially dissipated system. The specific results as stated in the following theorems.

Theorem 1.1. *Consider the system in (1.3) with $\nu > 0$ and $\eta > 0$. Assume $(u_0, \theta_0) \in H^k(\mathbb{R}^3)$ with $k \geq 3$ satisfies $\nabla \cdot u_0 = 0$, Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^k} + \|\theta_0\|_{H^k} \leq \varepsilon,$$

then (1.3) has a unique global solution $(u, \theta) \in L^\infty(0, \infty; H^k)$ satisfying, for any $t > 0$,

$$\|(u,\theta)(t)\|_{H^k}^2 + \nu \int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 d\tau + \eta \int_0^t \|\theta(\tau)\|_{H^k}^2 d\tau \leq C\varepsilon^2,$$

where $C > 0$ is a positive constant independent of ε and t.

To prove Theorem 1.1, the key is to use the delicate energy estimate to display the following global energy inequality, for any $t > 0$,

$$E(t) \le E(0) + C_0 E^{\frac{3}{2}}(t), \tag{1.4}$$

where C_0 is a positive constant, and

$$E(t) = \sup_{0\le\tau\le t}\{\|u(\tau)\|_{H^k}^2 + \|\theta(\tau)\|_{H^k}^2\} + 2\nu\int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 d\tau + 2\eta\int_0^t \|\theta(\tau)\|_{H^k}^2 d\tau.$$

Once (1.4) is at our disposal, then a direct application of the bootstrapping argument could imply the global stability. In fact, by local well-posedness and our assumptions on the initial data, these estimates are satisfied at least on some (small) time interval (0,T). In our bootstrap approach we assume that the maximal time T with this property is finite. We then show that on that same time interval all estimates hold with improved bounds instead, which implies that the estimates could be extended for a small additional time, contradicting with the maximality of T. More details are given in Section 2.

Next, we explore the optimal decay estimates on the solutions obtained in Theorem 1.1. The exact functional setting for our initial data (u_0, θ_0) is

$$(u_0, \theta_0) \in H^4(\mathbb{R}^3) \cap L^2_{x_3}L^1_{x_1x_2}(\mathbb{R}^3), \quad (\partial_3 u_0, \partial_3\theta_0), (\partial_3^2 u_0, \partial_3^2\theta_0) \in L^2_{x_3}L^1_{x_1x_2}(\mathbb{R}^3).$$

Our main aim is to achieve the optimal decay rates. To gain insight on our problem, we briefly examine the 3D anisotropic heat equation with horizontal dissipation

$$\begin{cases} \partial_t u = \nu\Delta_h u, & x \in \mathbb{R}^3, t > 0 \\ u(x,0) = u_0(x). \end{cases} \tag{1.5}$$

In order to obtain an explicit decay rate of the solution to (1.5), the energy method is no longer sufficient and explicit representation of the solution is necessary, namely

$$u(t) = e^{\nu\Delta_h t}u_0.$$

We can easily check that the solution u and its first-order derivatives obeys the following optimal decay rates, for any $t > 0$,

$$\|u(t)\|_{L^2} = \|\|e^{\nu\Delta_h t}u_0\|_{L^2_{x_1x_2}}\|_{L^2_{x_3}} \le C(\nu t)^{-\frac{1}{2}}\|u_0\|_{L^2_{x_3}L^1_{x_1x_2}}, \tag{1.6}$$

$$\|\nabla_h u(t)\|_{L^2} \le C(\nu t)^{-1}\|u_0\|_{L^2_{x_3}L^1_{x_1x_2}}, \tag{1.7}$$

$$\|\partial_3 u(t)\|_{L^2} \le C(\nu t)^{-\frac{1}{2}}\|\partial_3 u_0\|_{L^2_{x_3}L^1_{x_1x_2}}. \tag{1.8}$$

We are able to show that the solution of the anisotropic Boussinesq equation (1.3) obeys the same decay rates as those for the heat equation (1.5). More precisely, we obtain the following theorem.

Theorem 1.2. *Assume* $(u_0, \theta_0) \in H^4(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *satisfies*

$$(u_0, \theta_0), (\partial_3 u_0, \partial_3 \theta_0), (\partial_3^2 u_0, \partial_3^2 \theta_0) \in L^2_{x_3} L^1_{x_1 x_2}(\mathbb{R}^3).$$

Then there exists a sufficiently small constant $\varepsilon > 0$ *such that, if*

$$\begin{aligned} &\|(u_0, \theta_0)\|_{H^4(\mathbb{R}^3)} + \|(u_0, \theta_0)\|_{L^2_{x_3} L^1_{x_1 x_2}(\mathbb{R}^3)} + \|(\partial_3 u_0, \partial_3 \theta_0)\|_{L^2_{x_3} L^1_{x_1 x_2}(\mathbb{R}^3)} \\ &+ \|(\partial_3^2 u_0, \partial_3^2 \theta_0)\|_{L^2_{x_3} L^1_{x_1 x_2}(\mathbb{R}^3)} \leq \varepsilon. \end{aligned} \tag{1.9}$$

Then the corresponding solution of system (1.3) (u, θ) *obeys the following time decay estimates,*

$$\begin{aligned} &\|(u(t), \theta(t))\|_{H^4} \leq C\varepsilon, && \|(u(t), \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}} \\ &\|(\nabla_h u(t), \nabla_h \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-1}, && \|(\partial_3 u(t), \partial_3 \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}}, \\ &\|(\nabla_h^2 u(t), \nabla_h^2 \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{5}{4}}, && \|(\nabla_h \partial_3 u(t), \nabla_h \partial_3 \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-1}, \\ &\|(\partial_3^2 u(t), \partial_3^2 \theta(t))\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}}. \end{aligned}$$

The decay rates in $\|u\|_{L^2}$, $\|\nabla_h u\|_{L^2}$ and $\|\partial_3 u\|_{L^2}$ are exact the same as those for the heat equation in (1.6),(1.7) and (1.8), and thus are optimal. In addition, we remark that direct energy estimates are not adequate for the proof of Theorem 1.2. Thus we would like to resort the integral representation of (1.3). First, we take the Fourier transform of (1.3), then represent the nonlinear system into an integral form via Duhamel's principle. This form relies on six kernel functions which are degenerate and anisotropic in the frequency space. We perform a detailed spectral analysis in suitably divided subdomains of the frequency space to acquire optimal and precise upper bounds for the kernel functions. Once these bounds are established, we then estimate the optimal decay rates of (u, θ) and its derivatives via the integral form. The detailed estimates are provided in Section 3.

The rest of this paper is divided into two sections. Section 2 applies the energy estimate and bootstrapping argument to prove Theorem 1.1. Our main results about the optimal decay rates are established in Section 3. For more details are displayed, Section 3 is further divided into six subsections. To simply the notation, we shall write $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^3)}$, $\|f\|_{L^p_{x_i}}$ for the L^p-norm in x_i-variable, ∂_i for $\partial_{x_i} (i = 1, 2, 3)$, and $\nabla_h = (\partial_1, \partial_2)$.

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We first introduce several significant tools to be used in the proof. The first lemma provides anisotropic upper bounds for the integral of the triple product. In the aspect of dealing with anisotropic equations, it is a powerful tool. The proof of this lemma can be found in [34].

Lemma 2.1. *Assume that* $f, \partial_1 f, \partial_2 f, \partial_1\partial_2 f, g, \partial_2 g, \partial_3 g, h, \partial_3 h \in L^2$. *Then*

$$\int |fgh|dx \leq C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$
$$\int |fgh|dx \leq C\|f\|_{L^2}^{\frac{1}{4}}\|\partial_1 f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_3 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}.$$

The following Lemma can be shown by making use of the following basic one–dimensional inequality

$$\|g\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|g\|_{L^2}^{\frac{1}{2}}\|g'\|_{L^2}^{\frac{1}{2}}.$$

Lemma 2.2. *The following estimates hold when the right–hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^2}^{\frac{1}{8}}\|\partial_1 f\|_{L^2}^{\frac{1}{8}}\|\partial_2 f\|_{L^2}^{\frac{1}{8}}\|\partial_3 f\|_{L^2}^{\frac{1}{8}}\|\partial_{12} f\|_{L^2}^{\frac{1}{8}}\|\partial_{23} f\|_{L^2}^{\frac{1}{8}}\|\partial_{13} f\|_{L^2}^{\frac{1}{8}}\|\partial_{123} f\|_{L^2}^{\frac{1}{8}}.$$

Consequently,

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{H^1}^{\frac{1}{2}}\|\nabla_h f\|_{H^2}^{\frac{1}{2}}.$$

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. The framework of the proof is the bootstrapping argument. Define the energy functional $E(t)$ by

$$E(t) = \sup_{0\leq\tau\leq t}\{\|u(\tau)\|_{H^k}^2 + \|\theta(\tau)\|_{H^k}^2\} + 2\nu\int_0^t \|\nabla_h u(\tau)\|_{H^k}^2 d\tau + 2\eta\int_0^t \|\theta(\tau)\|_{H^k}^2 d\tau.$$

Our main efforts are devoted to showing that, for a constant $C_0 > 0$ and for $t > 0$,

$$E(t) \leq E(0) + C_0 E^{\frac{3}{2}}(t). \tag{2.1}$$

Once (2.1) is shown, then a direct application of the bootstrapping argument implies that, if

$$E(0) = \|(u_0, \theta_0)\|_{H^k}^2 \leq \frac{1}{16C^2} \quad or \quad \|(u_0, \theta_0)\|_{H^k} \leq \varepsilon := \frac{1}{4C}, \tag{2.2}$$

then,

$$E(t) \leq \frac{1}{8C^2} \quad for\ all\ t > 0. \tag{2.3}$$

In fact, if we make the ansatz that

$$E(t) \leq \frac{1}{4C^2}. \tag{2.4}$$

Inserting (2.4) in (2.1) and invoking (2.2) yields

$$E(t) \leq E(0) + \frac{1}{2}E(t) \quad or \quad E(t) \leq 2E(0) \leq \frac{1}{8C^2},$$

which is only half of the bound in the ansatz in (2.4). Then the bootstrapping argument implies (2.3). Next, we prove the energy inequality (2.1). Due to the equivalence of the norms

$$\|f\|_{H^k}^2 \sim \|f\|_{L^2}^2 + \sum_{i=1}^{3} \|\partial_i^k f\|_{L^2}^2.$$

It suffices to bound $\|(u,\theta)\|_{L^2}$ and $\sum_{i=1}^{3} \|(\partial_i^k u, \partial_i^k \theta)\|_{L^2}$. Firstly, we obtain the global L^2-bound. Dotting the equations in (1.3) by (u,θ) , integrating by parts and using $\nabla \cdot u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(u,\theta)\|_{L^2}^2 + \nu\|\nabla_h u\|_{L^2}^2 + \eta\|\theta\|_{L^2}^2 = 0. \tag{2.5}$$

Then applying the differential operator $\partial_i^k (i = 1, 2, 3)$ to the equations in (1.3), dotting the resulting equations by $(\partial_i^k u, \partial_i^k \theta)$ and integrating by parts, yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\|(\partial_i^k u, \partial_i^k \theta)\|_{L^2}^2 + \nu\sum_{i=1}^{3}\|\partial_i^k \nabla_h u\|_{L^2}^2 + \eta\sum_{i=1}^{3}\|\partial_i^k \theta\|_{L^2}^2 \\ &= I_1 + I_2, \end{aligned} \tag{2.6}$$

where I_1 and I_2 are given by

$$I_1 = -\sum_{i=1}^{3}\int \partial_i^k (u\cdot\nabla u)\cdot\partial_i^k u dx, \qquad I_2 = -\sum_{i=1}^{3}\int \partial_i^k (u\cdot\nabla\theta)\partial_i^k \theta dx.$$

Here we have used the fact that

$$\int \partial_i^k (\theta e_3)\cdot\partial_i^k u dx - \int \partial_i^k u_3 \partial_i^k \theta dx = 0.$$

Collecting (2.5) and (2.6), we have

$$\frac{1}{2}\frac{d}{dt}\|(u,\theta)\|_{H^k}^2 + \nu\|\nabla_h u\|_{H^k}^2 + \eta\|\theta\|_{H^k}^2 = I_1 + I_2. \tag{2.7}$$

To estimate I_1, we decompose it as

$$\begin{aligned} I_1 = &-\sum_{i=1}^{2}\int \partial_i^k (u\cdot\nabla u)\cdot\partial_i^k u dx - \sum_{j=1}^{2}\int \partial_3^k (u_j \partial_j u)\cdot\partial_3^k u dx - \int \partial_3^k (u_3\partial_3 u)\cdot\partial_3^k u dx \\ =&I_{11} + I_{12} + I_{13}. \end{aligned}$$

I_{11} is easy to bound. By the Leibniz Formula and Lemma 2.1,we have

$$\begin{aligned}
I_{11}=&-\sum_{i=1}^{2}\sum_{l=1}^{k}\mathcal{C}_k^l\int\partial_i^l u\cdot\partial_i^{k-l}\nabla u\cdot\partial_i^k u dx\\
\le& C\sum_{i=1}^{2}\sum_{l=1}^{k}\|\partial_i^l u\|_{L^2}^{\frac12}\|\partial_1\partial_i^l u\|_{L^2}^{\frac12}\|\partial_i^{k-l}\nabla u\|_{L^2}^{\frac12}\|\partial_2\partial_i^{k-l}\nabla u\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_3\partial_i^k u\|_{L^2}^{\frac12}\\
\le& C\|u\|_{H^k}\|\nabla_h u\|_{H^k}^2.
\end{aligned}$$

where $\mathcal{C}_k^l$ denotes the combinatorial number,

$$\mathcal{C}_k^l=\frac{k!}{l!(k-l)!}.$$

Using the same decomposition method,

$$\begin{aligned}
I_{12}=&-\sum_{j=1}^{2}\sum_{l=1}^{k}\mathcal{C}_k^l\int\partial_3^l u_j\partial_3^{k-l}\partial_j u\cdot\partial_3^k u dx\\
\le& C\sum_{j=1}^{2}\sum_{l=1}^{k}\|\partial_3^l u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3^l u_j\|_{L^2}^{\frac12}\|\partial_3^{k-l}\partial_j u\|_{L^2}^{\frac12}\|\partial_3^{k-l+1}\partial_j u\|_{L^2}^{\frac12}\|\partial_3^k u\|_{L^2}^{\frac12}\|\partial_2\partial_3^k u\|_{L^2}^{\frac12}\\
\le& C\|u\|_{H^k}\|\nabla_h u\|_{H^k}^2.
\end{aligned}$$

By $\nabla\cdot u=0$ and Lemma 2.1,

$$\begin{aligned}
I_{13}=&-\sum_{l=1}^{k}\mathcal{C}_k^l\int\partial_3^l u_3\partial_3^{k-l}\partial_3 u\cdot\partial_3^k u dx\\
\le& C\sum_{l=1}^{k}\mathcal{C}_k^l\int\partial_3^{l-1}(\nabla_h\cdot u_h)\partial_3^{k-l+1}u\cdot\partial_3^k u dx\\
\le& C\sum_{l=1}^{k}\|\partial_3^{l-1}\nabla_h\cdot u_h\|_{L^2}^{\frac12}\|\partial_3\partial_3^{l-1}\nabla_h\cdot u_h\|_{L^2}^{\frac12}\|\partial_3^{k-l+1}u\|_{L^2}^{\frac12}\|\partial_1\partial_3^{k-l+1}u\|_{L^2}^{\frac12}\|\partial_3^k u\|_{L^2}^{\frac12}\|\partial_2\partial_3^k u\|_{L^2}^{\frac12}\\
\le& C\|u\|_{H^k}\|\nabla_h u\|_{H^k}^2.
\end{aligned}$$

Therefore,

$$I_1\le C\|u\|_{H^k}\|\nabla_h u\|_{H^k}^2. \tag{2.8}$$

Now we turn to estimate I_2. We further decompose it as

$$\begin{aligned}
I_2=&-\sum_{i=1}^{2}\int\partial_i^k(u\cdot\nabla\theta)\partial_i^k\theta dx-\int\partial_3^k(u\cdot\nabla\theta)\partial_3^k\theta dx\\
=&I_{21}+I_{22}.
\end{aligned}$$

To deal with I_{21}, we apply to Young's inequality, Sobolev's inequality, Lemma 2.1 and 2.2,

$$\begin{aligned}
I_{21} =& -\sum_{i=1}^{2}\sum_{l=1}^{k}\mathcal{C}_k^l\int(\partial_i^l u\cdot\partial_i^{k-l}\nabla\theta)\partial_i^k\theta dx\\
=& -\sum_{i=1}^{2}\sum_{l=2}^{k-1}\mathcal{C}_k^l\int(\partial_i^l u\cdot\partial_i^{k-l}\nabla\theta)\partial_i^k\theta dx - k\sum_{i=1}^{2}\int(\partial_i u\cdot\partial_i^{k-1}\nabla\theta)\partial_i^k\theta dx\\
&-\sum_{i=1}^{2}\int(\partial_i^k u\cdot\nabla\theta)\partial_i^k\theta dx\\
\leq& C\sum_{i=1}^{2}\sum_{l=2}^{k-1}\|\partial_i^l u\|_{L^2}^{\frac14}\|\partial_1\partial_i^l u\|_{L^2}^{\frac14}\|\partial_2\partial_i^l u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_i^l u\|_{L^2}^{\frac14}\|\partial_i^{k-l}\nabla\theta\|_{L^2}^{\frac12}\|\partial_3\partial_i^{k-l}\nabla\theta\|_{L^2}^{\frac12}\|\partial_i^k\theta\|_{L^2}\\
&+C\sum_{i=1}^{2}\|\partial_i u\|_{L^\infty}\|\partial_i^{k-1}\nabla\theta\|_{L^2}\|\partial_i^k\theta\|_{L^2}+C\sum_{i=1}^{2}\|\partial_i^k u\|_{L^2}\|\nabla\theta\|_{L^\infty}\|\partial_i^k\theta\|_{L^2}\\
\leq& C(\|u\|_{H^k}+\|\theta\|_{H^k})(\|\nabla_h u\|_{H^k}^2+\|\theta\|_{H^k}^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_{22} =& -\sum_{l=1}^{k}\mathcal{C}_k^l\int(\partial_3^l u\cdot\partial_3^{k-l}\nabla\theta)\partial_3^k\theta dx\\
=& -\sum_{l=2}^{k-1}\mathcal{C}_k^l\int(\partial_3^l u\cdot\partial_3^{k-l}\nabla\theta)\partial_3^k\theta dx - k\int(\partial_3 u\cdot\partial_3^{k-1}\nabla\theta)\partial_3^k\theta dx - \int(\partial_3^k u\cdot\nabla\theta)\partial_3^k\theta dx\\
\leq& C\sum_{l=2}^{k-1}\|\partial_3^l u\|_{L^2}^{\frac14}\|\partial_1\partial_3^l u\|_{L^2}^{\frac14}\|\partial_2\partial_3^l u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3^l u\|_{L^2}^{\frac14}\|\partial_3^{k-l}\nabla\theta\|_{L^2}^{\frac12}\|\partial_3^{k-l+1}\nabla\theta\|_{L^2}^{\frac12}\|\partial_3^k\theta\|_{L^2}\\
&+C\|\partial_3 u\|_{L^\infty}\|\partial_3^{k-1}\nabla\theta\|_{L^2}\|\partial_3^k\theta\|_{L^2}\\
&+C\|\nabla\theta\|_{L^2}^{\frac14}\|\partial_1\nabla\theta\|_{L^2}^{\frac14}\|\partial_3\nabla\theta\|_{L^2}^{\frac14}\|\partial_1\partial_3\nabla\theta\|_{L^2}^{\frac14}\|\partial_3^k u\|_{L^2}^{\frac12}\|\partial_2\partial_3^k u\|_{L^2}^{\frac12}\|\partial_3^k\theta\|_{L^2}\\
\leq& C(\|u\|_{H^k}+\|\theta\|_{H^k})(\|\nabla_h u\|_{H^k}^2+\|\theta\|_{H^k}^2).
\end{aligned}$$

Collecting the bound for I_2, we obtain

$$I_2\leq C(\|u\|_{H^k}+\|\theta\|_{H^k})(\|\nabla_h u\|_{H^k}^2+\|\theta\|_{H^k}^2). \tag{2.9}$$

Inserting (2.8) and (2.9) in (2.7), integrating in time over $[0,t]$, we deduce

$$\begin{aligned}
E(t)&\leq E(0)+C\int_0^t(\|u\|_{H^k}\|\nabla_h u\|_{H^k}^2+(\|u\|_{H^k}+\|\theta\|_{H^k})(\|\nabla_h u\|_{H^k}^2+\|\theta\|_{H^k}^2))d\tau\\
&\leq E(0)+C_0E^{\frac32}(t).
\end{aligned}$$

which is the desired inequality (2.1). This accomplishes the proof of the global stability. It's easy to prove the uniqueness result of Theorem 1.1. Let $(u^{(1)},p^{(1)},\theta^{(1)})$ and $(u^{(2)},p^{(2)},\theta^{(2)})$

be two solutions of equation (1.3) with one of them in the regularity class, say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0,\infty; H^k(\mathbb{R}^3))$ must coincide. In fact, their difference $(\overline{u}, \overline{p}, \overline{\theta})$ with

$$\overline{u} = u^{(2)} - u^{(1)}, \quad \overline{p} = p^{(2)} - p^{(1)}, \quad \overline{\theta} = \theta^{(2)} - \theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t \overline{u} + u^{(2)} \cdot \nabla \overline{u} + \overline{u} \cdot \nabla u^{(1)} = -\nabla \overline{p} + \nu \Delta_h \overline{u} + \overline{\theta} e_3, \\ \partial_t \overline{\theta} + u^{(2)} \cdot \nabla \overline{\theta} + \overline{u} \cdot \nabla \theta^{(1)} + \overline{u}_3 + \eta \overline{\theta} = 0, \\ \nabla \cdot \overline{u} = 0, \\ \overline{u}(x,0) = 0, \overline{\theta}(x,0) = 0. \end{cases} \tag{2.10}$$

Taking the L^2-inner product of (2.10) with $(\overline{u}, \overline{\theta})$, by Lemma 2.1, Young's inequality and the uniformly global bounds for $\|(u^{(1)}, \theta^{(1)})\|_{H^k}$, we deduce

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|(\overline{u}, \overline{\theta})\|_{L^2}^2 + \nu\, \|\nabla_h u\|_{L^2}^2 + \eta \|\theta\|_{L^2}^2 \\ = & -\int (\overline{u} \cdot \nabla u^{(1)}) \cdot \overline{u} dx - \int (\overline{u} \cdot \nabla \theta^{(1)}) \cdot \overline{\theta} dx \\ \leq & C\|\overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \|\overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \\ & + C\|\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \nabla \theta^{(1)}\|_{L^2}^{\frac{1}{4}} \|\partial_3 \nabla \theta^{(1)}\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_3 \nabla \theta^{(1)}\|_{L^2}^{\frac{1}{4}} \|\overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\overline{\theta}\|_{L^2} \\ \leq & C\|\overline{u}\|_{L^2} \|\nabla_h \overline{u}\|_{L^2} + C\|\overline{u}\|_{L^2}^{\frac{1}{2}} \|\nabla_h \overline{u}\|_{L^2}^{\frac{1}{2}} \|\overline{\theta}\|_{L^2} \\ \leq & \frac{\nu}{2}\|\nabla_h \overline{u}\|_{L^2}^2 + C\|(\overline{u}, \overline{\theta})\|_{L^2}^2. \end{aligned}$$

where we have used the fact that

$$\int \overline{\theta} e_3 \cdot \overline{u} dx - \int \overline{u}_3 \overline{\theta} dx = 0.$$

Then we apply the Grönwall's inequality to get the desired global uniqueness,

$$\|\overline{u}\|_{L^2}^2 = \|\overline{\theta}\|_{L^2}^2 = 0.$$

Thus the proof of Theorem 1.1 is completed. □

3 Proof of Theorem 1.2

This section proves Theorem 1.2. We recall several lemmas before proving Theorem 1.2. The first lemma states Minkowski's inequality. It is an elementary tool for exchanging two Lebesgue norms (see, e.g., [23, 35]).

Lemma 3.1. *For a nonnegative measurable function f over $\mathbb{R}^m \times \mathbb{R}^n$. Let $1 \le q \le p \le \infty$. Then*

$$\Big\| \|f\|_{L^q(\mathbb{R}^n)} \Big\|_{L^p(\mathbb{R}^m)} \le \Big\| \|f\|_{L^p(\mathbb{R}^m)} \Big\|_{L^q(\mathbb{R}^n)}.$$

For convenience, we introduce the notation

$$L^q_{x_h}(\mathbb{R}^2) := L^q_{x_1x_2}(\mathbb{R}^2), \quad \|f\|_{L^p_h L^q_{x_3}} = \Big\| \|f\|^q_{L_{x_3}} \Big\|_{L^p_{x_h}},$$

The second Lemma specifies an exact L^p-L^q decay estimate for the general heat operator associated with a fractional Laplacian. Here the fractional Laplacian operator can be defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

The decay rate is stated as following, whose proof can be found in many references (see [24]).

Lemma 3.2. *Let $\alpha > 0$ and $\beta \ge 0$ are real numbers, $1 \le p \le q \le \infty$. Then, for any $t > 0$,*

$$\|\Lambda^\beta e^{-\nu(-\Delta)^\alpha t} f\|_{L^q(\mathbb{R}^d)} \le C t^{-\frac{\beta}{2\alpha} - \frac{d}{2\alpha}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

The following two lemmas offer upper bounds with optimal decay rates for two special integrals (see, e.g., [36, 37]).

Lemma 3.3. *Assume $0 < s_1 \le s_2$. Then, for some constant $C > 0$,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2} d\tau \le \begin{cases} C(1+t)^{-s_1}, & if s_2 > 1, \\ C(1+t)^{-s_1} ln(1+t), & if s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & if s_2 < 1. \end{cases}$$

Lemma 3.4. *For any $c > 0$ and $s > 0$,*

$$\int_0^t e^{-c(t-\tau)}(1+\tau)^{-s} d\tau \le C(1+t)^{-s}.$$

Now we derive an integral representation of (1.3). First, we would like to eliminate the bad effects of the pressure term and reveal the hidden structure in (1.3). We apply the Helmholtz-Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.3) to obtain

$$\partial_t u = \nu \Delta_h u + \mathbb{P}(\theta e_3) - \mathbb{P}(u \cdot \nabla u). \tag{3.1}$$

By the definition of $\mathbb{P}$,

$$\mathbb{P}(\theta e_3) = \theta e_3 - \nabla\Delta^{-1}\nabla \cdot (\theta e_3) = \begin{bmatrix} -\partial_1\partial_3\Delta^{-1}\theta \\ -\partial_2\partial_3\Delta^{-1}\theta \\ \theta - \partial_3^2\Delta^{-1}\theta \end{bmatrix}. \tag{3.2}$$

Alternatively we can write $\theta-\partial_3^2\Delta^{-1}\theta=\Delta_h\Delta^{-1}\theta$. Inserting (3.2) in (3.1) yields

$$\begin{cases} \partial_t u=\nu\Delta_h u+\begin{bmatrix} -\partial_1\partial_3\Delta^{-1}\theta \\ -\partial_2\partial_3\Delta^{-1}\theta \\ \Delta_h\Delta^{-1}\theta \end{bmatrix}-\mathbb{P}(u\cdot\nabla u), \\ \partial_t\theta=-u_3-\eta\theta-(u\cdot\nabla\theta), \end{cases} \tag{3.3}$$

which separates the linear parts and the nonlinear parts. Then taking the Fourier transform of (3.3), we have

$$\partial_t\begin{bmatrix} \widehat{u} \\ \widehat{\theta} \end{bmatrix}=A\begin{bmatrix} \widehat{u} \\ \widehat{\theta} \end{bmatrix}+\begin{bmatrix} \widehat{M_1} \\ \widehat{M_2} \end{bmatrix}, \tag{3.4}$$

where

$$A=\begin{bmatrix} -\nu|\xi_h|^2 & 0 & 0 & -\frac{\xi_1\xi_3}{|\xi|^2} \\ 0 & -\nu|\xi_h|^2 & 0 & -\frac{\xi_2\xi_3}{|\xi|^2} \\ 0 & 0 & -\nu|\xi_h|^2 & \frac{|\xi_h|^2}{|\xi|^2} \\ 0 & 0 & -1 & -\eta \end{bmatrix},$$

$$u=(u_1,u_2,u_3)^T,\quad M_1=-\mathbb{P}(u\cdot\nabla u)=\begin{bmatrix} -(\mathbb{P}(u\cdot\nabla u))_1 \\ -(\mathbb{P}(u\cdot\nabla u))_2 \\ -(\mathbb{P}(u\cdot\nabla u))_3 \end{bmatrix},\quad M_2=-(u\cdot\nabla\theta).$$

The characteristic polynomial of A is given by

$$(\lambda+\nu|\xi_h|^2)^2(\lambda^2+(\eta+\nu|\xi_h|^2)\lambda+\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2})=0,$$

where $|\xi_h|^2=\xi_1^2+\xi_2^2$, and thus the eigenvalues of A are

$$\lambda_1=\lambda_2=-\nu|\xi_h|^2,\quad \lambda_3=\frac{-(\eta+\nu|\xi_h|^2)-\sqrt{\Gamma}}{2},\quad \lambda_4=\frac{-(\eta+\nu|\xi_h|^2)+\sqrt{\Gamma}}{2},$$

with

$$\Gamma=(\eta+\nu|\xi_h|^2)^2-4(\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}).$$

By Duhamel's principle

$$\begin{bmatrix} \widehat{u}(t) \\ \widehat{\theta}(t) \end{bmatrix}=e^{At}\begin{bmatrix} \widehat{u_0}(t) \\ \widehat{\theta_0}(t) \end{bmatrix}+\int_0^t e^{A(t-\tau)}\begin{bmatrix} \widehat{M_1}(\tau) \\ \widehat{M_2}(\tau) \end{bmatrix}.$$

By computing the corresponding eigenvectors and diagonalizing A, e^{At} can be obtained. Then we have the following integral representation,

$$\begin{aligned} \widehat{u_h}(t)=&e^{\lambda_1 t}\widehat{u_{0h}}+\widehat{K_1}(t)\widehat{u_{03}}+\widehat{K_2}(t)\widehat{\theta_0}-\int_0^t e^{\lambda_1(t-\tau)}(\mathbb{P}(\widehat{u\cdot\nabla u}))_h(\tau)d\tau \\ &-\int_0^t\widehat{K_1}(t-\tau)(\mathbb{P}(\widehat{u\cdot\nabla u}))_3(\tau)d\tau-\int_0^t\widehat{K_2}(t-\tau)(\widehat{u\cdot\nabla\theta})(\tau)d\tau, \end{aligned} \tag{3.5}$$

$$\begin{aligned}\widehat{u_3}(t) =& \widehat{K_3}(t)\widehat{u_{03}} + \widehat{K_4}(t)\widehat{\theta_0}\\ &- \int_0^t \widehat{K_3}(t-\tau)(\mathbb{P}(\widehat{u\cdot\nabla u}))_3(\tau)d\tau - \int_0^t \widehat{K_4}(t-\tau)(\widehat{u\cdot\nabla\theta})(\tau)d\tau,\end{aligned}\tag{3.6}$$

$$\begin{aligned}\widehat{\theta}(t) =& \widehat{K_5}(t)\widehat{u_{03}} + \widehat{K_6}(t)\widehat{\theta_0}\\ &- \int_0^t \widehat{K_5}(t-\tau)(\mathbb{P}(\widehat{u\cdot\nabla u}))_3(\tau)d\tau - \int_0^t \widehat{K_6}(t-\tau)(\widehat{u\cdot\nabla\theta})(\tau)d\tau,\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}&\widehat{K_1}(t) = \frac{\xi_h\xi_3}{|\xi_h|^2}e^{\lambda_1 t} + \frac{\xi_h\xi_3}{|\xi_h|^2}G_2(t) + \nu\xi_h\xi_3 G_1(t), && \widehat{K_2}(t) = -\frac{\xi_h\xi_3}{|\xi|^2}G_1(t),\\ &\widehat{K_3}(t) = -G_2(t) - \nu|\xi_h|^2 G_1(t), && \widehat{K_4}(t) = \frac{|\xi_h|^2}{|\xi|^2}G_1(t),\\ &\widehat{K_5}(t) = -G_1(t), && \widehat{K_6}(t) = G_3(t) + \nu|\xi_h|^2 G_1(t),\end{aligned}$$

with

$$\begin{aligned}&G_1(t) = \frac{e^{\lambda_4 t} - e\lambda_3 t}{\lambda_4-\lambda_3}, \quad G_2(t) = \frac{\lambda_3 e^{\lambda_4 t} - \lambda_4 e\lambda_3 t}{\lambda_4-\lambda_3} = \lambda_3 G_1(t) - e^{\lambda_3 t},\\ &G_3(t) = \frac{\lambda_4 e^{\lambda_4 t} - \lambda_3 e\lambda_3 t}{\lambda_4-\lambda_3} = \lambda_3 G_1(t) + e^{\lambda_4 t}.\end{aligned}$$

We remark that when $\lambda_3 = \lambda_4$, the representation in (3.5), (3.6) and (3.7) remains valid if we replace G_1 by its limiting form

$$G_1(t) = \lim_{\lambda_4\to\lambda_3}\frac{e^{\lambda_4 t} - e^{\lambda_3 t}}{\lambda_4-\lambda_3} = te^{\lambda_3 t}.$$

Next we analyze the behaviors of the kernels $\widehat{K_1}(\xi,t)$ through $\widehat{K_6}(\xi,t)$, which play an important role in the proof of Theorem 1.2. The kernels depend on the Fourier frequencies ξ. We divide the frequency space into subdomains, and the following proposition provides precise and sharp upper bounds in each subdomain.

Proposition 3.1. *We split the domain $\mathbb{R}^3$ into two subdomains, $\mathbb{R}^3 = A_1\bigcup A_2$ with*

$$\begin{aligned}A_1 &:= \left\{\xi\in\mathbb{R}^3:\ \sqrt{\Gamma} \le \frac{\eta+\nu|\xi_h|^2}{2} \ \textit{or}\ \nu\eta|\xi_h|^2 + \frac{|\xi_h|^2}{|\xi|^2} \ge \frac{3}{16}(\eta+\nu|\xi_h|^2)^2\right\},\\ A_2 &:= \left\{\xi\in\mathbb{R}^3:\ \sqrt{\Gamma} > \frac{\eta+\nu|\xi_h|^2}{2} \ \textit{or}\ \nu\eta|\xi_h|^2 + \frac{|\xi_h|^2}{|\xi|^2} < \frac{3}{16}(\eta+\nu|\xi_h|^2)^2\right\}.\end{aligned}$$

Then we have

(I) There exist two constants $C>0$ and $c_0=c_0(\nu,\eta)>0$ such that, for any $\xi\in A_1$,

$$\begin{aligned}
&Re\lambda_3\le -\frac{1}{2}(\eta+\nu|\xi_h|^2),\quad Re\lambda_4\le -\frac{1}{4}(\eta+\nu|\xi_h|^2),\\
&|G_1(t)|\le te^{-\frac{1}{4}(\eta+\nu|\xi_h|^2)},\quad |\widehat{K_1}(t)|\le C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}),\\
&|\widehat{K_i}(t)|\le Ce^{-c_0(1+|\xi_h|^2)t},\quad i=2,3,\dots,6.
\end{aligned}$$

(II)There are two constants $C>0$ and $c_0=c_0(\nu,\eta)>0$ such that, for any $\xi\in A_2$,

$$\begin{aligned}
&\lambda_3\le -\frac{3}{4}(\eta+\nu|\xi_h|^2),\quad \lambda_4\le -\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2},\\
&|G_1(t)|\le 2(\eta+\nu|\xi_h|^2)^{-1}(e^{\lambda_3t}+e^{\lambda_4t}),\\
&|\widehat{K_1}(t)|\le C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),\\
&|\widehat{K_i}(t)|\le C(1+|\xi_h|^2)^{-1}(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),\quad i=2,4,5,\\
&|\widehat{K_j}(t)|\le C(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),\quad j=3,6.
\end{aligned}$$

If we further split A_2 into two subdomains as follows

$$\begin{aligned}
A_{21}&=\{\xi\in A_2,\ \nu|\xi_h|^2\le\eta\},\\
A_{22}&=\{\xi\in A_2,\ \nu|\xi_h|^2>\eta\},
\end{aligned}$$

Then, we have the following more explicit upper bounds

(a)For $\xi\in A_{21}$,

$$\begin{aligned}
&|\widehat{K_1}(t)|\le C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}),\\
&|\widehat{K_i}(t)|\le C(1+|\xi_h|^2)^{-1}(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}),\quad i=2,4,5,\\
&|\widehat{K_j}(t)|\le C(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}),\quad j=3,6.
\end{aligned}$$

(b)For $\xi\in A_{22}$,

$$\begin{aligned}
&|\widehat{K_1}(t)|\le C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2t}+e^{-c_0(1+|\xi_h|^2)t}+e^{-c_0t}),\\
&|\widehat{K_i}(t)|\le C(1+|\xi_h|^2)^{-1}(e^{-c_0(1+|\xi_h|^2)t}+e^{-c_0t}),\quad i=2,4,5,\\
&|\widehat{K_j}(t)|\le C(e^{-c_0(1+|\xi_h|^2)t}+e^{-c_0t}),\quad j=3,6.
\end{aligned}$$

Proof of Proposition 3.1. (I) For $\xi\in A_1$, $\sqrt{\Gamma}\le\frac{\eta+\nu|\xi_h|^2}{2}$. Through the direct estimates, we have

$$Re\lambda_3\le -\frac{1}{2}(\eta+\nu|\xi_h|^2),\quad Re\lambda_4\le -\frac{1}{4}(\eta+\nu|\xi_h|^2).$$

In order to further prove, we divide our consideration into two cases,

(i) For $\Gamma \geq 0$, λ_3 and λ_4 are real numbers. Thus we have

$$|\lambda_3|, |\lambda_4| \leq \frac{3}{4}(\eta + \nu|\xi_h|^2).$$

By the definition of A_1, there exists a constant C such that, for any $\xi \in A_1$, $|\xi_h| \leq C$. Then

$$|\lambda_3|, |\lambda_4| \leq C.$$

In addition, by the mean-value theorem,

$$|G_1(t)| \leq te^{-\frac{1}{4}(\eta+\nu|\xi_h|^2)t}.$$

Then we have

$$|G_2(t)| = |\lambda_3 G_1(t) - e^{\lambda_3}t| \leq Ce^{-c_0(1+|\xi_h|^2)t},$$

and

$$|G_3(t)| = |\lambda_3 G_1(t) + e^{\lambda_4}t| \leq Ce^{-c_0(1+|\xi_h|^2)t}.$$

By the definitions of $\widehat{K_1}(t)$ through $\widehat{K_6}(t)$,

$$|\widehat{K_1}(t)| \leq C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2 t} + e^{-c_0(1+|\xi_h|^2)t}),$$
$$|\widehat{K_i}(t)| \leq Ce^{-c_0(1+|\xi_h|^2)t},\ i = 2, 3, \ldots, 6.$$

where we have used the simple fact that $xe^{-C_1 x} \leq C_2$ for any $x \geq 0$, $C_1 > 0$ and suitable $C_2 > 0$.

(ii) For $\Gamma < 0$, λ_3 and λ_4 are a pair of complex conjugates. More precisely, we have

$$|\lambda_3|, |\lambda_4| = \sqrt{\nu\eta|\xi_h|^2 + \frac{|\xi_h|^2}{|\xi|^2}} \leq \sqrt{\nu\eta|\xi_h|^2 + 1}.$$

As we mentioned before, $|\xi_h| \leq C$ for any $\xi \in A_1$. Therefore

$$|\lambda_3|, |\lambda_4| \leq C.$$

Furthermore, since λ_3 and λ_4 are a pair of complex conjugates,

$$G_1(t) = \frac{e^{\lambda_4 t} - e^{\lambda_3 t}}{\lambda_4 - \lambda_3} = e^{-\frac{1}{2}(\eta+\nu|\xi_h|^2)t}\frac{2\sin(\frac{\sqrt{-\Gamma}}{2}t)}{\sqrt{-\Gamma}}.$$

Using the simple fact that $|\sin\rho| \leq \rho$ for any $\rho \in \mathbb{R}$, we have

$$G_1(t) \leq te^{-\frac{1}{2}(\eta+\nu|\xi_h|^2)t}.$$

The upper bounds for $\widehat{K_1}(t)$ through $\widehat{K_6}(t)$ then follow as before.

(II) For $\xi \in A_2$, λ_3 and λ_4 are real numbers, and we have

$$\frac{\eta+\nu|\xi_h|^2}{2} < \sqrt{\Gamma} < \eta+\nu|\xi_h|^2.$$

Clearly,

$$\lambda_3 = \frac{-(\eta+\nu|\xi_h|^2)-\sqrt{\Gamma}}{2} < -\frac{3}{4}(\eta+\nu|\xi_h|^2),$$
$$\lambda_4 = \frac{-(\eta+\nu|\xi_h|^2)+\sqrt{\Gamma}}{2} = \frac{2(\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2})}{-(\eta+\nu|\xi_h|^2+\sqrt{\Gamma})} \le -\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}.$$

Then

$$|G_1(t)| = \frac{|e^{\lambda_4 t}-e^{\lambda_3 t}|}{\sqrt{\Gamma}} \le 2(\eta+\nu|\xi_h|^2)^{-1}(e^{\lambda_3 t}+e^{\lambda_4 t})$$
$$\le 2(\eta+\nu|\xi_h|^2)^{-1}(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}).$$

By the bound of Γ, there is

$$|\lambda_3| < \eta+\nu|\xi_h|^2.$$

Consequently, we have

$$|G_2(t)| \le C(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),$$

and

$$|G_3(t)| \le C(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}).$$

Invoking the uniform bound for $|G_1|$, $|G_2|$ and $|G_3|$, we get

$$|\widehat{K_1}(t)| \le \frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2 t}+e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),$$
$$|\widehat{K_i}(t)| \le C(1+|\xi_h|^2)^{-1}(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),\ i=2,4,5.$$
$$|\widehat{K_j}(t)| \le C(e^{-c_0(1+|\xi_h|^2)t}+e^{-\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2}t}),\ j=3,6.$$

Finally, according to the upper bounds for $\widehat{K_1}(t)$ through $\widehat{K_6}(t)$, we can establish more precise upper bounds by further division of A_2 into A_{21} and A_{22}. For $\xi \in A_{21}$, $\nu|\xi_h|^2 \le \eta$, we obtain

$$\frac{\nu\eta|\xi_h|^2+\frac{|\xi_h|^2}{|\xi|^2}}{\eta+\nu|\xi_h|^2} \ge \frac{\nu\eta|\xi_h|^2}{\eta+\nu|\xi_h|^2} \ge \frac{\nu\eta|\xi_h|^2}{2\eta} \ge c_0|\xi_h|^2.$$

For $\xi \in A_{22}$, $\nu|\xi_h|^2 > \eta$, we derive

$$\frac{\nu\eta|\xi_h|^2 + \frac{|\xi_h|^2}{|\xi|^2}}{\eta + \nu|\xi_h|^2} \geq \frac{\nu\eta|\xi_h|^2}{\eta + \nu|\xi_h|^2} \geq \frac{\nu\eta|\xi_h|^2}{2\nu|\xi_h|^2} \geq c_0.$$

This completes the proof of Proposition 3.1. Now we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We prove Theorem 1.2 by the bootstrapping argument. We assume the initial datum (u_0, θ_0) satisfies (1.9) for sufficiently small $\varepsilon > 0$. The bootstrapping argument starts with the ansatz that

$$\begin{aligned}
&\|(u(t),\theta(t))\|_{H^4} \leq C_0\varepsilon, && \|(u(t),\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|(\nabla_h u(t),\nabla_h\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-1}, && \|(\partial_3 u(t),\partial_3\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|(\nabla_h^2 u(t),\nabla_h^2\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-\frac{5}{4}}, && \|(\nabla_h\partial_3 u(t),\nabla_h\partial_3\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-1},\\
&\|(\partial_3^2 u(t),\partial_3^2\theta(t))\|_{L^2} \leq C_0\varepsilon(1+t)^{-\frac{1}{2}},
\end{aligned} \tag{3.8}$$

where C_0 is a constant to be specified in the following proof. We show by using the ansatz and the integral representation of (u, θ) in (3.5), (3.6) and (3.7) that

$$\begin{aligned}
&\|(u(t),\theta(t))\|_{H^4} \leq \frac{C_0}{2}\varepsilon, && \|(u(t),\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|(\nabla_h u(t),\nabla_h\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1}, && \|(\partial_3 u(t),\partial_3\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|(\nabla_h^2 u(t),\nabla_h^2\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{5}{4}}, && \|(\nabla_h\partial_3 u(t),\nabla_h\partial_3\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1},\\
&\|(\partial_3^2 u(t),\partial_3^2\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{3.9}$$

The bootstrapping argument then implies that (3.9) hold for all $t > 0$.

It then suffices to prove (3.9). The first inequality follows directly from Theorem 1.1 with $k = 4$. By Theorem 1.1, we have

$$\|u(t)\|_{H^4} + \|\theta(t)\|_{H^4} \leq C_1\varepsilon.$$

Then $\|(u(t),\theta(t))\|_{H^4} \leq \frac{C_0}{2}\varepsilon$ holds when we take $C_0 \geq 2C_1$. The rest of the implementation relies on the upper bounds in Proposition 3.1. As a special consequence of Proposition 3.1, we have For $\xi \in A_1 \cup A_{21}$,

$$|\widehat{K_1}| \leq C\frac{|\xi_3|}{|\xi_h|}e^{-c_0|\xi_h|^2 t}, \ |\widehat{K_i}| \leq Ce^{-c_0|\xi_h|^2 t}, \ i = 2, 3, \ldots, 6. \tag{3.10}$$

For $\xi \in A_{22}$,

$$|\widehat{K_1}| \leq C\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2 t} + e^{-c_0 t}), \ |\widehat{K_i}| \leq C(e^{-c_0|\xi_h|^2 t} + e^{-c_0 t}), \ i = 2, 3, \ldots, 6. \tag{3.11}$$

Since the proof is long and complicated, for the sake of clarity, the rest of this section is divided into six subsections with each subsection devoted to one of the inequalities in (3.9) except the first which has been proved.

3.1 Estimates of $\|(u(t),\theta(t))\|_{L^2}$

This subsection proves the second inequality in (3.9). To estimate $\|(u(t),\theta(t))\|_{L^2}$, we should deal with it in three subdomains A_1, A_{21} and A_{22} defined in Proposition 3.1. By (3.5), (3.6) and Plancherel's theorem, we have

$$\begin{aligned}
\|u(t)\|_{L^2(\mathbb{R}^3)} =&\|\widehat{u}(t)\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{u_h}(t)\|_{L^2(\mathbb{R}^3)} + \|\widehat{u_3}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq&\|e^{\lambda_1 t}\widehat{u_{0h}}\|_{L^2(\mathbb{R}^3)} + \|\widehat{K_1}(t)\widehat{u_{03}}\|_{L^2(\mathbb{R}^3)} + \|\widehat{K_2}(t)\widehat{\theta_0}\|_{L^2(\mathbb{R}^3)} + \|\widehat{K_3}(t)\widehat{u_{03}}\|_{L^2(\mathbb{R}^3)} \\
&+ \|\widehat{K_4}(t)\widehat{\theta_0}\|_{L^2(\mathbb{R}^3)} + \int_0^t \|e^{\lambda_1(t-\tau)}\mathbb{P}(\widehat{u\cdot\nabla u})_h(\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\mathbb{P}(\widehat{u\cdot\nabla u})_3(\tau)\|_{L^2(\mathbb{R}^3)}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{u\cdot\nabla\theta}(\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
&+ \int_0^t \|\widehat{K_3}(t-\tau)\mathbb{P}(\widehat{u\cdot\nabla u})_3(\tau)\|_{L^2(\mathbb{R}^3)}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\widehat{u\cdot\nabla\theta}(\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
=&J_1 + J_2 + \ldots + J_{10}.
\end{aligned} \tag{3.12}$$

We first bound J_1. By Lemma 3.2, we derive

$$J_1 = \|e^{-c_0|\xi_h|^2 t}\widehat{u_{0h}}\|_{L^2(\mathbb{R}^3)} = \Big\|\|e^{c_0\Delta_h t}u_{0h}\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}} \leq C(1+t)^{-\frac{1}{2}}\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}}.$$

By $\nabla\cdot u_0 = 0$, (3.10), (3.11)and Lemma 3.2, we have

$$\begin{aligned}
J_2 =&\|\widehat{K_1}(t)\widehat{u_{03}}\|_{L^2(A_1\cup A_{21})} + \|\widehat{K_1}(t)\widehat{u_{03}}\|_{L^2(A_{22})} \leq C\|\frac{|\xi_3|}{|\xi_h|}(e^{-c_0|\xi_h|^2 t} + e^{-c_0 t})\widehat{u_{03}}\|_{L^2} \\
\leq&C\|\frac{1}{|\xi_h|}e^{-c_0|\xi_h|^2 t}(-|\xi_h|\cdot\widehat{u_{0h}})\|_{L^2} + Ce^{-c_0 t}\|\frac{1}{|\xi_h|}(-|\xi_h|\cdot\widehat{u_{0h}})\|_{L^2} \\
\leq&C\|e^{c_0\Delta_h t}u_{0h}\|_{L^2} + Ce^{-c_0 t}\|u_{0h}\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}(\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{0h}\|_{L^2}).
\end{aligned}$$

where we have used $e^{-c_0 t} \leq C(1+t)^{-\frac{1}{2}}$ for any $t \geq 0$. Similarly,

$$\begin{aligned}
J_3 =&\|\widehat{K_2}(t)\widehat{\theta_0}\|_{L^2(A_1\cup A_{21})} + \|\widehat{K_2}(t)\widehat{\theta_0}\|_{L^2(A_{22})} \\
\leq&C\|e^{-c_0|\xi_h|^2 t}\widehat{\theta_0}\|_{L^2} + Ce^{-c_0 t}\|\theta_0\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}(\|\theta_0\|_{L^2_{x_3}L^1_{x_h}} + \|\theta_0\|_{L^2}).
\end{aligned}$$

Due to the same bound of $\widehat{K_i}(t),\ i = 2,\ldots,6,\ J_4$ and J_5 can be estimated analogously, we have

$$J_4 \leq C(1+t)^{-\frac{1}{2}}(\|u_{03}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{03}\|_{L^2}),$$

and

$$J_5 \le C(1+t)^{-\frac{1}{2}}(\|\theta_0\|_{L^2_{x_3}L^1_{x_h}} + \|\theta_0\|_{L^2}).$$

By the definition of the projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$, we distinguish the horizontal derivatives from the vertical ones, thus we obtain

$$\begin{aligned}\mathbb{P}(u\cdot\nabla u)_h &= u\cdot\nabla u_h - \nabla_h\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\\ &= u_h\cdot\nabla_h u_h + u_3\partial_3 u_h - \Delta^{-1}\nabla\cdot\nabla\cdot\nabla_h(u\otimes u).\end{aligned} \tag{3.13}$$

Using the boundedness of the Riesz transform on L^2, simultaneously,

$$\|\Delta^{-1}\nabla\cdot\nabla\cdot f\|_{L^2} \le C\|f\|_{L^2}, \tag{3.14}$$

then correspondingly the upper bound of J_6 consists of three parts,

$$\begin{aligned}J_6 \le& \int_0^t \|e^{c_0\Delta_h(t-\tau)}u_h\cdot\nabla_h u_h(\tau)\|_{L^2}d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}u_3\partial_3 u_h(\tau)\|_{L^2}d\tau\\ &+ C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u\otimes u)(\tau)\|_{L^2}d\tau\\ =&J_{61}+J_{62}+J_{63}.\end{aligned}$$

Applying Sobolev's inequality, the ansatz in (3.8), Lemma 3.1, 3.2 and 3.3, we obtain

$$\begin{aligned}J_{61} \le& \int_0^t \Big\|\|e^{c_0\Delta_h(t-\tau)}u_h\cdot\nabla_h u_h(\tau)\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\ \le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\Big\|\|u_h\cdot\nabla_h u_h(\tau)\|_{L^1_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\ \le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\Big\|\|u_h(\tau)\|_{L^2_{x_h}}\|\nabla_h u_h(\tau)\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\ \le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|u_h(\tau)\|_{L^\infty_{x_3}L^2_{x_h}}\|\nabla_h u_h(\tau)\|_{L^2_{x_3}L^2_{x_h}}d\tau\\ \le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|u_h(\tau)\|_{L^2_{x_h}L^\infty_{x_3}}\|\nabla_h u_h(\tau)\|_{L^2}d\tau\\ \le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|u_h(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla_h u_h(\tau)\|_{L^2}d\tau\\ \le& CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\\ \le& CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.\end{aligned}$$

J_{63} contains the good derivative ∇_h and admits the same upper bound as J_{61}. We now turn to J_{62},

$$\begin{aligned}
J_{62} \le& \int_0^t \Big\| \|e^{c_0\Delta_h(t-\tau)} u_3\partial_3 u_h(\tau)\|_{L^2_{x_h}} \Big\|_{L^2_{x_3}} d\tau \\
\le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}} \Big\| \|u_3\partial_3 u_h(\tau)\|_{L^1_{x_h}} \Big\|_{L^2_{x_3}} d\tau \\
\le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}} \|u_3(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_3(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_h(\tau)\|_{L^2} d\tau \\
\le& C\int_0^t (1+t-\tau)^{-\frac{1}{2}} \|u_3(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla_h\cdot u_h(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_h(\tau)\|_{L^2} d\tau \\
\le& CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}} d\tau \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

Combining the upper bounds of J_{61}, J_{62} and J_{63}, we have

$$J_6 \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

We now bound J_7, By observing the constraint form of $\widehat{K_1}(t)$ in (3.10) and (3.11), we need to generate the factor ξ_h from $\mathbb{P}(\widehat{u\cdot\nabla u})_3$. By the definition of $\mathbb{P}$, we have

$$\begin{aligned}
\mathbb{P}(u\cdot\nabla u)_3 =& u\cdot\nabla u_3 - \partial_3\Delta^{-1}\nabla\cdot(u\cdot\nabla u) \\
=& \partial_1(u_1u_3) + \partial_2(u_2u_3) - \partial_3\Delta^{-1}(\partial_1\nabla\cdot(uu_1) + \partial_2\nabla\cdot(uu_2)) \\
& - \partial_3\Delta^{-1}\partial_3\partial_1(u_1u_3) - \partial_3\Delta^{-1}\partial_3\partial_2(u_2u_3) + \Delta^{-1}\Delta_h\partial_3(u_3u_3).
\end{aligned} \tag{3.15}$$

It is clear that each term contains ∂_1 or ∂_2, then Fourier transform gets the desired factor ξ_h. Thus we have

$$\begin{aligned}
J_7 =& \int_0^t \|\widehat{K_1}(t-\tau)\mathbb{P}(\widehat{u\cdot\nabla u})_3\|_{L^2(A_1\cup A_{21})} d\tau + \int_0^t \|\widehat{K_1}(t-\tau)\mathbb{P}(\widehat{u\cdot\nabla u})_3\|_{L^2(A_{22})} d\tau \\
\le& C\int_0^t \| \big(e^{-c_0|\xi_h|^2(t-\tau)} + e^{-c_0(t-\tau)}\big)\big(\xi_3(\widehat{u_1u_3}) + \xi_3(\widehat{u_2u_3}) \\
& + |\xi_3|^2|\xi|^{-2}(\widehat{\nabla\cdot(uu_1)} + \widehat{\nabla\cdot(uu_2)}) + |\xi_3|^2|\xi|^{-2}(\partial_3\widehat{(u_1u_3)} + \partial_3\widehat{(u_2u_3)} + \nabla_h\widehat{(u_3u_3)})\big)\|_{L^2} d\tau \\
\le& C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3(u_hu_3)\|_{L^2} d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\cdot(u_hu_1)\|_{L^2} d\tau \\
& + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\cdot(u_hu_2)\|_{L^2} d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3u_3)\|_{L^2} d\tau \\
& + \int_0^t e^{-c_0(t-\tau)}\|\partial_3(u_hu_3)\|_{L^2} d\tau + \int_0^t e^{-c_0(t-\tau)}\|\nabla_h\cdot(u_hu_1)\|_{L^2} d\tau \\
& + \int_0^t e^{-c_0(t-\tau)}\|\nabla_h\cdot(u_hu_2)\|_{L^2} d\tau + \int_0^t e^{-c_0(t-\tau)}\|\nabla_h(u_3u_3)\|_{L^2} d\tau \\
=& J_{71} + J_{71} + \ldots + J_{78}.
\end{aligned}$$

The progress of constraining J_{71} through J_{74} is much similar to J_6. Thus we obtain

$$J_{71} + J_{72} + J_{73} + J_{74} \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

By $\nabla \cdot u = 0$, Hölder's inequality, Lemma 3.4 and the ansatz (3.8), we have

$$\begin{aligned}
J_{75} \le& C\int_0^t e^{-c_0(t-\tau)}\|u_3\partial_3 u_h\|_{L^2}d\tau + \int_0^t e^{-c_0(t-\tau)}\|u_h(\nabla_h \cdot u_h)\|_{L^2}d\tau\\
\le& C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^4}\|\partial_3 u_h\|_{L^4}d\tau + \int_0^t e^{-c_0(t-\tau)}\|u_h\|_{L^6}\|\nabla_h \cdot u_h\|_{L^3}d\tau\\
\le& C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^2}^{\frac{1}{4}}\|\nabla u_3\|_{L^2}^{\frac{3}{4}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{4}}\|\nabla\partial_3 u_h\|_{L^2}^{\frac{3}{4}}d\tau\\
&+ C\int_0^t e^{-c_0(t-\tau)}\|\nabla u_h\|_{L^2}\|\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}}\|\nabla(\nabla_h \cdot u_h)\|_{L^2}^{\frac{1}{2}}d\tau\\
\le& CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-1}d\tau + CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{3}{2}}d\tau\\
\le& CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

The handling of J_{76}, J_{77} and J_{78} are same as the second term of J_{75}, thus we yield the same constraint result

$$J_{76} + J_{77} + J_{78} \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Combining the upper bound of J_{71} through J_{78}, we derive

$$J_7 \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

By (3.10) and (3.11), J_8 can be separated two parts,

$$\begin{aligned}
J_8 =& \int_0^t \|\widehat{K_2}(t-\tau)\widehat{u\cdot\nabla\theta}\|_{L^2(A_1\cup A_{21})}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{u\cdot\nabla\theta}\|_{L^2(A_{22})}d\tau\\
\le& C\int_0^t \|e^{-c_0|\xi_h|^2(t-\tau)}\widehat{u\cdot\nabla\theta}\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|u\cdot\nabla\theta\|_{L^2}d\tau\\
=& J_{81} + J_{82}.
\end{aligned}$$

Using $u \cdot \nabla\theta = u_h \cdot \nabla_h\theta + u_3\partial_3\theta$, yields

$$J_{81} \le C\int_0^t \|e^{c_0\Delta_h(t-\tau)}u_h \cdot \nabla_h\theta\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}u_3\partial_3\theta\|_{L^2}d\tau.$$

The two terms on the right-hand side can be bounded as J_{61} and J_{62} above. The progress of dealing with J_{82} is similar as the first term of J_{75}. Thus, we obtain

$$J_8 \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

By $\nabla\cdot u=0$ and (3.10), (3.11), we yield

$$\begin{aligned}\mathbb{P}(u\cdot\nabla u)_3 =& u\cdot\nabla u_3-\partial_3\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\\ =&(u_h\cdot\nabla_h)u_3-u_3(\nabla_h\cdot u_h)-\partial_3\Delta^{-1}\nabla\cdot(u_h\cdot\nabla_h u)-\partial_3\Delta^{-1}\nabla\cdot(u_3\partial_3 u). \end{aligned}\tag{3.16}$$

Then

$$\begin{aligned}J_9=&\int_0^t\|\widehat{K_3}(t-\tau)\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2(A_1\cup A_{21})}d\tau+\int_0^t\|\widehat{K_3}(t-\tau)\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2(A_{22})}d\tau\\ \le& C\int_0^t\|e^{-c_0|\xi_h|^2(t-\tau)}\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2}d\tau+\int_0^t\|e^{-c_0(t-\tau)}\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2}d\tau\\ \le& C\int_0^t\|e^{c_0\Delta_h(t-\tau)}(u_h\cdot\nabla_h)u_3\|_{L^2}d\tau+C\int_0^t\|e^{c_0\Delta_h(t-\tau)}u_3(\nabla_h\cdot u_h)\|_{L^2}d\tau\\ &+C\int_0^t\|e^{c_0\Delta_h(t-\tau)}(u_h\cdot\nabla_h)u\|_{L^2}d\tau+C\int_0^t\|e^{c_0\Delta_h(t-\tau)}u_3\partial_3 u\|_{L^2}d\tau\\ &+C\int_0^t e^{-c_0(t-\tau)}\|(u_h\cdot\nabla_h)u_3\|_{L^2}d\tau+C\int_0^t e^{-c_0(t-\tau)}\|u_3(\nabla_h\cdot u_h)\|_{L^2}d\tau\\ &+C\int_0^t e^{-c_0(t-\tau)}\|(u_h\cdot\nabla_h)u\|_{L^2}d\tau+C\int_0^t e^{-c_0(t-\tau)}\|u_3\partial_3 u\|_{L^2}d\tau\\ =&J_{91}+J_{92}+\ldots+J_{98},\end{aligned}$$

where we have used the boundedness of Riesz transform,

$$\|\partial_3\Delta^{-1}\nabla\cdot f\|_{L^2}\le C\|f\|_{L^2}. \tag{3.17}$$

Observing the form of J_{91} through J_{98}, they are easy to be estimated like beforementioned terms, thus we have

$$J_9\le CC_0^2\varepsilon^2(1+t)^{-\frac12}.$$

$\widehat{K_4}$ obeys the same bound as $\widehat{K_2}$, then

$$J_{10}\le CC_0^2\varepsilon^2(1+t)^{-\frac12}.$$

Inserting the uppers bounds of J_1 through J_{10} in (3.12) leads to

$$\|u(t)\|_{L^2}\le C_2(1+t)^{-\frac12}(\|(u_0,\theta_0)\|_{L^2}+\|(u_0,\theta_0)\|_{L^2_{x_3}L^1_{x_h}})+C_3C_0^2\varepsilon^2(1+t)^{-\frac12}. \tag{3.18}$$

Therefore, if we choose C_0 and ε satisfying

$$C_2\le\frac{C_0}{8},\qquad C_3C_0\varepsilon\le\frac18.$$

Using the initial data condition in (1.9), then (3.18) implies

$$\|u(t)\|_{L^2} \le \frac{C_0}{8}\varepsilon(1+t)^{-\frac{1}{2}} + \frac{1}{8}C_0\varepsilon(1+t)^{-\frac{1}{2}} = \frac{C_0}{4}\varepsilon(1+t)^{-\frac{1}{2}}.$$

By the integral representation formula of (3.7), we have

$$\begin{aligned}\|\theta\|_{L^2(\mathbb{R}^3)} \le& \|\widehat{K_5}(t)\widehat{u_{03}}\|_{L^2(\mathbb{R}^3)} + \|\widehat{K_6}(t)\widehat{\theta_0}\|_{L^2(\mathbb{R}^3)} \\ &+ \int_0^t \|\widehat{K_5}(t-\tau)\mathbb{P}\widehat{(u\cdot\nabla u)}_3(\tau)\|_{L^2(\mathbb{R}^3)}d\tau + \int_0^t \|\widehat{K_5}(t-\tau)\widehat{u\cdot\nabla\theta}(\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ =& L_1 + L_2 + L_3 + L_4.\end{aligned}$$

Due to the same bounds of $\widehat{K_i},\ i = 2, 3, \ldots, 6$, L_1 through L_4 have the same upper bound with J_4, J_5, J_9 and J_{10}, respectively. Thus we have

$$\|\theta(t)\|_{L^2} \le \frac{C_0}{4}\varepsilon(1+t)^{-\frac{1}{2}}.$$

Therefore,

$$\|(u(t), \theta(t))\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the second inequality in (3.9).

3.2 Estimates of $\|(\nabla_h u(t), \nabla_h\theta(t))\|_{L^2}$

The goal of this subsection is to prove the third inequality in (3.9). We again make use of the integral representation (3.5), (3.6) and (3.7). Applying ∇_h to (3.5), (3.6) and (3.7), then taking the L^2-norm, we obtain, after using Plancherel's theorem

$$\begin{aligned}\|\nabla_h u(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\nabla_h u}(t)\|_{L^2(\mathbb{R}^3)} \le \|\widehat{\nabla_h u_h}(t)\|_{L^2(\mathbb{R}^3)} + \|\widehat{\nabla_h u_3}(t)\|_{L^2(\mathbb{R}^3)} \\ \le& \|e^{\lambda_1 t}\widehat{\nabla_h u_{0h}}\|_{L^2} + \|\widehat{K_1}(t)\widehat{\nabla_h u_{03}}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\nabla_h\theta_0}\|_{L^2} + \|\widehat{K_3}(t)\widehat{\nabla_h u_{03}}\|_{L^2} \\ &+ \|\widehat{K_4}(t)\widehat{\nabla_h\theta_0}\|_{L^2} + \int_0^t \|e^{\lambda_1(t-\tau)}\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}_h(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|\widehat{K_1}(t-\tau)\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\nabla_h\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|\widehat{K_3}(t-\tau)\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\nabla_h\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\ =& M_1 + M_2 + \ldots + M_{10}.\end{aligned} \tag{3.19}$$

and

$$
\begin{aligned}
\|\nabla_h\theta(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\nabla_h\theta}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq& \|\widehat{K_5}(t)\widehat{\nabla_h u_{03}}\|_{L^2} + \|\widehat{K_6}(t)\widehat{\nabla_h\theta_0}\|_{L^2} + \int_0^t \|\widehat{K_5}(t-\tau)\nabla_h\widehat{\mathbb{P}(u\cdot\nabla u)_3}(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_6}(t-\tau)\nabla_h\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau. \qquad (3.20)
\end{aligned}
$$

By combining the experience gained from J_1 through J_{10} above, we would simplify some of the cumbersome processes in dealing with M_1 through M_{10}. Using Lemma 3.2 and noticing that

$$
M_1 \leq \|e^{-c_0|\xi_h|^2 t}\widehat{\nabla_h u_{0h}}\|_{L^2} = \Big\|\|e^{c_0\Delta_h t}\nabla_h u_{0h}\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}} \leq C(1+t)^{-1}\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}}.
$$

The handling method is the same as in J_2, applying $\nabla\cdot u_0 = 0$, (3.10), (3.11) and Lemma 3.2, we have

$$
\begin{aligned}
M_2 \leq& C\|e^{c_0\Delta_h t}\nabla_h\cdot u_{0h}\|_{L^2} + Ce^{-c_0 t}\|\nabla_h\cdot u_{0h}\|_{L^2} \\
\leq& C(1+t)^{-1}(\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{0h}\|_{H^1}),
\end{aligned}
$$

where we have used the simple fact that $e^{-c_0 t} \leq C(1+t)^{-1}$. Then M_3 through M_5 are obtained easily, we have

$$
M_3 + M_5 \leq C(1+t)^{-1}(\|\theta_0\|_{L^2_{x_3}L^1_{x_h}} + \|\theta_0\|_{H^1}),
$$

and

$$
M_4 \leq C(1+t)^{-1}(\|u_{03}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{03}\|_{H^1}).
$$

As in (3.13), we write

$$
\mathbb{P}(u\cdot\nabla u)_h = u_3\partial_3 u_h + u_h\cdot\nabla_h u_h - \Delta^{-1}\nabla\cdot\nabla\cdot\nabla_h(u\otimes u).
$$

Thus M_6 is further decomposed into three parts,

$$
\begin{aligned}
M_6 \leq & \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\partial_3 u_h)\|_{L^2}d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_h\cdot\nabla_h u_h)\|_{L^2}d\tau \\
& + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\nabla_h(u\otimes u)\|_{L^2}d\tau \\
= & \int_0^t \left\|\|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\partial_3 u_h)\|_{L^2_{x_h}}\right\|_{L^2_{x_3}} d\tau + \int_0^t \left\|\|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_h\cdot\nabla_h u_h)\|_{L^2_{x_h}}\right\|_{L^2_{x_3}} d\tau \\
& + \int_0^t \left\|\|e^{c_0\Delta_h(t-\tau)}\nabla_h\nabla_h(u\otimes u)\|_{L^2_{x_h}}\right\|_{L^2_{x_3}} d\tau \\
\leq & \int_0^t (1+t-\tau)^{-1}\left\|\|u_3\partial_3 u_h\|_{L^1_{x_h}}\right\|_{L^2_{x_3}} d\tau + \int_0^t (1+t-\tau)^{-1}\left\|\|u_h\cdot\nabla_h u_h\|_{L^1_{x_h}}\right\|_{L^2_{x_3}} d\tau \\
& + \int_0^t (1+t-\tau)^{-1}\left\|\|\nabla_h(u\otimes u)\|_{L^1_{x_h}}\right\|_{L^2_{x_3}} d\tau,
\end{aligned}
$$

where we have used the boundedness of Riesz transform in (3.14). The three terms on the right-hand side are parallel to J_6, whereupon we get

$$M_6 \leq CC_0^2\varepsilon^2(1+t)^{-1}.$$

Due to the specificity of $\widehat{K_1}$ in (3.10) and (3.11), we should give M_7 more attentions. During the estimation process $\partial_3\mathbb{P}(u\cdot\nabla u)_3$ is needed to handle. As in (3.15), we can further write

$$
\begin{aligned}
|\partial_3\mathbb{P}(u\cdot\nabla u)_3| \leq & |\partial_1\partial_3(u_1u_3)| + |\partial_2\partial_3(u_2u_3)| + |\partial_3\partial_3\Delta^{-1}(\partial_1\nabla\cdot(uu_1)+\partial_2\nabla\cdot(uu_2))| \\
& + |\partial_3\partial_3\Delta^{-1}(\partial_3\partial_1(u_1u_3)+\partial_3\partial_2(u_2u_3))| + |\partial_3\Delta^{-1}\Delta_h\partial_3(u_3u_3)| \\
\leq & C\big(|\nabla_h\partial_3(u_hu_3)| + |\nabla_h\nabla\cdot(uu_h)| + |\Delta_h(u_3u_3)|\big).
\end{aligned} \tag{3.21}
$$

Thus

$$
\begin{aligned}
M_7 = & \int_0^t \|\widehat{K_1}(t-\tau)\nabla_h\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2(A_1\cup A_{21})}d\tau + \int_0^t \|\widehat{K_1}(t-\tau)\nabla_h\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2(A_{22})}d\tau \\
\leq & C\int_0^t \|e^{-c_0|\xi_h|^2(t-\tau)}\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2}d\tau + \int_0^t e^{-c_0(t-\tau)}\|\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_3\|_{L^2}d\tau \\
\leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\partial_3(u_hu_3)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
& + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\Delta_h(u_3u_3)\|_{L^2}d\tau \\
& + \int_0^t e^{-c_0(t-\tau)}\|\nabla_h\partial_3(u_hu_3)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau + \int_0^t e^{-c_0(t-\tau)}\|\Delta_h(u_3u_3)\|_{L^2}d\tau \\
= & M_{71} + \ldots + M_{78}.
\end{aligned}
$$

M_{71} through M_{73} are similar to those terms in M_6 and admit the same bound. By Hölder's inequality, Sobolev's inequality, Lemma 3.2 and 3.3, the ansatz (3.8), we have

$$\begin{aligned}
M_{74} \leq & C\int_0^t \left\| \|e^{c_0\Delta_h(t-\tau)}\Delta_h(u_3u_3)\|_{L^2_{x_h}} \right\|_{L^2_{x_3}} d\tau \\
\leq & C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \left\| \|u_3u_3\|_{L^1_{x_h}} \right\|_{L^2_{x_3}} d\tau \\
\leq & C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|u_3\|_{L^2}^{\frac{3}{2}} \|\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} d\tau \\
\leq & CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{5}{4}} d\tau \\
\leq & CC_0^2\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

The process of dealing with M_{75} is somewhat cumbersome, and we need divide it into four parts. Using $\nabla \cdot u = 0$, we have

$$\begin{aligned}
M_{75} \leq & C\int_0^t e^{-c_0(t-\tau)}\|u_3\nabla_h\partial_3 u_h\|_{L^2} d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h u_h \nabla_h \cdot u_h\|_{L^2} d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3 u_h \nabla_h u_3\|_{L^2} d\tau + C\int_0^t e^{-c_0(t-\tau)}\|u_h \nabla_h \nabla_h \cdot u_h\|_{L^2} d\tau \\
& = M_{751} + M_{752} + M_{753} + M_{754}.
\end{aligned}$$

Applying Hölder's inequality, Sobolev's inequality and Lemma 3.4, we derive

$$\begin{aligned}
M_{751} + M_{754} \leq & C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^6}\|\nabla_h\partial_3 u_h\|_{L^3} d\tau + C\int_0^t e^{-c_0(t-\tau)}\|u_h\|_{L^6}\|\nabla_h\nabla_h \cdot u_h\|_{L^3} d\tau \\
\leq & C\int_0^t e^{-c_0(t-\tau)}\|\nabla u_3\|_{L^2}\|\nabla_h\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\partial_3 u_h\|_{L^2}^{\frac{1}{2}} d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\nabla u_h\|_{L^2}\|\nabla_h\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\nabla_h \cdot u_h\|_{L^2}^{\frac{1}{2}} d\tau \\
\leq & CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-1} d\tau + CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-\frac{9}{8}} d\tau \\
\leq & CC_0^2\varepsilon^2(1+t)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
M_{752} + M_{753} \leq & C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h u_h\|_{L^4}^2 d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3 u_h\|_{L^4}\|\nabla_h u_3\|_{L^4} d\tau \\
\leq & C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u_h\|_{L^2}^{\frac{3}{2}} d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3 u_h\|_{L^2}^{\frac{1}{4}}\|\nabla\partial_3 u_h\|_{L^2}^{\frac{3}{4}}\|\nabla_h u_3\|_{L^2}^{\frac{1}{4}}\|\nabla\nabla_h u_3\|_{L^2}^{\frac{3}{4}} d\tau \\
\leq & CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-2} d\tau + CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-\frac{3}{2}} d\tau \\
\leq & CC_0^2\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

Combining the bound of M_{751} through M_{754}, we have

$$M_{75} \leq CC_0^2\varepsilon^2(1+t)^{-1}.$$

The terms in M_{76} through M_{78} can also be bounded like as those in M_{75}, the details are omitted. As a consequence, we have

$$M_7 \leq CC_0^2\varepsilon^2(1+t)^{-1}.$$

By (3.10), (3.11) and $u \cdot \nabla\theta = u_h \cdot \nabla_h\theta + u_3\partial_3\theta$, we obtain

$$\begin{aligned} M_8 \leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_h \cdot \nabla_h\theta)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\partial_3\theta)\|_{L^2}d\tau \\ & + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h u \cdot \nabla\theta\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|u \cdot \nabla_h\nabla\theta\|_{L^2}d\tau \end{aligned}$$

The four terms on the right-hand side are similar to some terms in M_6 and M_7 and admit the same bound. By the definition of $\mathbb{P}$ and $\nabla \cdot u = 0$, we have

$$\begin{aligned} \nabla_h\mathbb{P}(u \cdot \nabla u)_3 = & \nabla_h(u \cdot \nabla u_3) - \nabla_h\partial_3\Delta^{-1}\nabla \cdot (u \cdot \nabla u) \\ = & \nabla_h(u_h \cdot \nabla_h u_3) - \nabla_h(u_3\nabla_h \cdot u_h) - \partial_3\Delta^{-1}\nabla \cdot \nabla_h(u_h \cdot \nabla_h u) \\ & - \partial_3\Delta^{-1}\nabla \cdot \nabla_h(u_3\partial_3 u). \end{aligned} \tag{3.22}$$

By (3.10), (3.11) and the boundedness of the Riesz transform in (3.17), we have

$$\begin{aligned} M_9 \leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\nabla_h \cdot u_h)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_h \cdot \nabla_h u)\|_{L^2}d\tau \\ & + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\partial_3 u)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h(u_3\nabla_h \cdot u_h)\|_{L^2}d\tau \\ & + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h(u_h \cdot \nabla_h u)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h(u_3\partial_3 u)\|_{L^2}d\tau \end{aligned}$$

The estimation of these items using the same method as before are relatively easy. We can obtain the same constraint results, namely

$$M_9 \leq CC_0^2\varepsilon^2(1+t)^{-1}.$$

By (3.10) and (3.11), we can see M_{10} is same to M_8. Collecting the bounds from M_1 to M_{10} and inserting them in (3.19), we obtain, after using the initial data condition in (1.9),

$$\|\nabla_h u\|_{L^2} \leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2(1+t)^{-1}.$$

The estimate for $\|\nabla_h\theta\|_{L^2}$ using (3.20) is very similar and we omit the details. Therefore,

$$\|(\nabla_h u(t), \nabla_h\theta(t))\|_{L^2} \leq C_4\varepsilon(1+t)^{-1} + C_5C_0^2\varepsilon^2(1+t)^{-1}.$$

If we choose C_0 and ε satisfying

$$C_4 \leq \frac{C_0}{4}, \quad C_5C_0\varepsilon \leq \frac{1}{4},$$

then

$$\|(\nabla_h u(t), \nabla_h\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1}.$$

This completes the proof of the third inequality in (3.9).

3.3 Estimates of $\|(\partial_3 u(t), \partial_3\theta(t))\|_{L^2}$

We now verify the upper bound for $\|(\partial_3 u, \partial_3\theta)\|_{L^2}$ in (3.9). Applying ∂_3 to (3.5), (3.6) and (3.7), then taking L^2-norm, after using Plancherel's theorem. We have

$$\begin{aligned}
\|\partial_3 u(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\partial_3 u}(t)\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{\partial_3 u_h}(t)\|_{L^2(\mathbb{R}^3)} + \|\widehat{\partial_3 u_3}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq& \|e^{\lambda_1 t}\widehat{\partial_3 u_{0h}}\|_{L^2} + \|\widehat{K_1}(t)\widehat{\partial_3 u_{03}}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\partial_3\theta_0}\|_{L^2} + \|\widehat{K_3}(t)\widehat{\partial_3 u_{03}}\|_{L^2} \\
&+ \|\widehat{K_4}(t)\widehat{\partial_3\theta_0}\|_{L^2} + \int_0^t \|e^{\lambda_1(t-\tau)}\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_h(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\partial_3\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_3}(t-\tau)\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\partial_3\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
=& N_1 + N_2 + \ldots + N_{10}.
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\|\partial_3\theta(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\partial_3\theta}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq& \|\widehat{K_5}(t)\widehat{\partial_3 u_{03}}\|_{L^2} + \|\widehat{K_6}(t)\widehat{\partial_3\theta_0}\|_{L^2} + \int_0^t \|\widehat{K_5}(t-\tau)\partial_3\mathbb{P}\widehat{(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_6}(t-\tau)\partial_3\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau.
\end{aligned} \tag{3.24}$$

In fact, N_1 to N_5 can be shown by repeating the process for J_1 to J_5 with $\partial_3 u$ and $\partial_3\theta$ replacing u and θ, respectively. Thus, we have

$$N_1 + \ldots + N_5 \leq C(1+t)^{-\frac{1}{2}}\big(\|(\partial_3 u_0, \partial_3\theta_0)\|_{L^2_{x_3}L^1_{x_h}} + \|(u_0,\theta_0)\|_{H^1}\big).$$

As in (3.13), we can write

$$\begin{aligned}\partial_3\mathbb{P}(u\cdot\nabla u)_h =&\partial_3 u\cdot\nabla u_h+u\cdot\partial_3\nabla u_h-\partial_3\nabla_h\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\\ =&u_3\partial_{33}u_h+(\partial_3 u_h\cdot\nabla_h u_h-\nabla_h\cdot u_h\partial_3 u_h)+u_h\cdot\nabla_h\partial_3 u_h-\partial_3\Delta^{-1}\nabla\cdot\nabla_h(u\cdot\nabla u).\end{aligned}\tag{3.25}$$

Combining the boundedness of Riesz transform in (3.17), corresponding N_6 is then divided into four terms,

$$\begin{aligned}N_6\le&\int_0^t\|e^{c_0\Delta_h(t-\tau)}u_3\partial_{33}u_h\|_{L^2}d\tau+\int_0^t\|e^{c_0\Delta_h(t-\tau)}(\partial_3 u_h\cdot\nabla_h u_h-\nabla_h\cdot u_h\partial_3 u_h)\|_{L^2}d\tau\\ &+\int_0^t\|e^{c_0\Delta_h(t-\tau)}u_h\cdot\nabla_h\partial_3 u_h\|_{L^2}d\tau+\int_0^t\|e^{c_0\Delta_h(t-\tau)}\nabla_h(u\cdot\nabla u)\|_{L^2}d\tau\\ =&N_{61}+N_{62}+N_{63}+N_{64}.\end{aligned}$$

By Sobolev's inequality, Lemma 3.1, 3.2, 3.3, and the ansatz (3.8), we have

$$\begin{aligned}N_{61}=&\int_0^t\left\|\|e^{c_0\Delta_h(t-\tau)}u_3\partial_{33}u_h\|_{L^2_{x_h}}\right\|_{L^2_{x_3}}d\tau\\ \le&C\int_0^t(1+t-\tau)^{-\frac12}\left\|\|u_3\partial_{33}u_h\|_{L^1_{x_h}}\right\|_{L^2_{x_3}}d\tau\\ \le&C\int_0^t(1+t-\tau)^{-\frac12}\|u_3\|_{L^2}^{\frac12}\|\nabla_h\cdot u_h\|_{L^2}^{\frac12}\|\partial_{33}u_h\|_{L^2}d\tau\\ \le&CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-\frac12}(1+\tau)^{-\frac54}d\tau\\ \le&CC_0^2\varepsilon^2(1+t)^{-\frac12}.\end{aligned}$$

N_{62} can be dealt with similarly,

$$\begin{aligned}N_{62}\le&C\int_0^t\left\|\|e^{c_0\Delta_h(t-\tau)}\partial_3 u_h\cdot\nabla_h u_h\|_{L^2_{x_h}}\right\|_{L^2_{x_3}}d\tau\\ \le&C\int_0^t(1+t-\tau)^{-\frac12}\left\|\|\partial_3 u_h\cdot\nabla_h u_h\|_{L^1_{x_h}}\right\|_{L^2_{x_3}}d\tau\\ \le&C\int_0^t(1+t-\tau)^{-\frac12}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\partial_{33}u_h\|_{L^2}^{\frac12}\|\nabla_h u_h\|_{L^2}d\tau\\ \le&CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-\frac12}(1+\tau)^{-\frac32}d\tau\\ \le&CC_0^2\varepsilon^2(1+t)^{-\frac12}.\end{aligned}$$

We apply the same processing method and yield the uniform upper bound about N_{63} and N_{64}. By estimating J_7, we can obtain

$$\begin{aligned}|\widehat{K_1}(t)\mathbb{P}\widehat{(u\cdot\nabla u)}_3|\le&C(e^{-c_0|\xi_h|^2t}+e^{-c_0t})\big(|\partial_3\widehat{(u_hu_3)}|+|\nabla_h\cdot\widehat{(u_hu_1)}|\\ &+|\nabla_h\cdot\widehat{(u_hu_2)}|+|\nabla_h\widehat{(u_3u_3)}|\big),\end{aligned}\tag{3.26}$$

then

$$\begin{aligned}
N_7 \leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_{33}(u_hu_3)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
&+ C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3\nabla_h(u_3u_3)\|_{L^2}d\tau \\
&+ C\int_0^t e^{-c_0(t-\tau)}\|\partial_{33}(u_hu_3)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
&+ C\int_0^t e^{-c_0(t-\tau)}\|\partial_3\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + \int_0^t e^{-c_0(t-\tau)}\|\partial_3\nabla_h(u_3u_3)\|_{L^2}d\tau \\
=& N_{71} + \ldots + N_{78}.
\end{aligned}$$

N_{71} through N_{74} can be estimated easily. In order to estimate N_{75} more accurately, after using $\nabla\cdot u = 0$, we divide it into three parts for better clarity,

$$\begin{aligned}
N_{75} \leq & C\int_0^t e^{-c_0(t-\tau)}\|u_3\partial_{33}u_h\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3u_h(\nabla_h\cdot u_h)\|_{L^2}d\tau \\
&+ C\int_0^t e^{-c_0(t-\tau)}\|u_h\partial_3(\nabla_h\cdot u_h)\|_{L^2}d\tau \\
=& N_{751} + N_{752} + N_{753}.
\end{aligned}$$

By Hölder's inequality, Sobolev's inequality, Lemma 3.4 and the ansatz (3.8), we have

$$\begin{aligned}
N_{751} + N_{753} \leq & C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^6}\|\partial_{33}u_h\|_{L^3}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|u_h\|_{L^6}\|\partial_3\nabla_h\cdot u_h\|_{L^3}d\tau \\
\leq & C\int_0^t e^{-c_0(t-\tau)}\|\nabla u_3\|_{L^2}\|\partial_{33}u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_{33}u_h\|_{L^2}^{\frac{1}{2}}d\tau \\
&+ C\int_0^t e^{-c_0(t-\tau)}\|\nabla u_h\|_{L^2}\|\partial_3\nabla_h\cdot u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\nabla_h\cdot u_h\|_{L^2}^{\frac{1}{2}}d\tau \\
\leq & CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-\frac{3}{4}}d\tau + CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+t)^{-1}d\tau \\
\leq & CC_0^2\varepsilon^2((1+t)^{-\frac{1}{2}}.
\end{aligned}$$

N_{752} is same as M_{743}. N_{76} to N_{78} can be estimated by similar progress. In general, we have

$$N_7 \leq CC_0^2\varepsilon^2((1+t)^{-\frac{1}{2}}.$$

The terms in N_8 thorough N_{10} can also be bounded similarly and the details are omitted. Collecting the bounds from N_1 to N_{10} and inserting them in (3.23), we obtain, after using the initial data condition in (1.9),

$$\|\partial_3u(t)\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{1}{2}} + CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

The estimate for $\|\partial_3\theta(t)\|_{L^2}$ using (3.24) is very similar and we omit the details. Therefore

$$\|(\partial_3 u(t), \partial_3 u(t))\|_{L^2} \le C_6\varepsilon(1+t)^{-\frac{1}{2}} + C_7C_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{3.27}$$

If we choose C_0 and ε stisfying

$$C_6 \le \frac{C_0}{4}, \quad C_7C_0\varepsilon \le \frac{1}{4},$$

then (3.27) implies

$$\|(\partial_3 u(t), \partial_3 u(t))\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the fourth inequality in (3.9).

3.4 Estimates of $\|(\nabla_h^2 u(t), \nabla_h^2\theta(t))\|_{L^2}$

This subsection establishes the fifth inequality in (3.9). Applying ∇_h^2 to (3.5), (3.6) and (3.7), then taking L^2-norm, after using Plancherel's theorem. We have

$$\begin{aligned}
\|\nabla_h^2 u(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\nabla_h^2 u}(t)\|_{L^2(\mathbb{R}^3)} \le \|\widehat{\nabla_h^2 u_h}(t)\|_{L^2(\mathbb{R}^3)} + \|\widehat{\nabla_h^2 u_3}(t)\|_{L^2(\mathbb{R}^3)} \\
\le& \|e^{\lambda_1 t}\widehat{\nabla_h^2 u_{0h}}\|_{L^2} + \|\widehat{K_1}(t)\widehat{\nabla_h^2 u_{03}}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\nabla_h^2\theta_0}\|_{L^2} + \|\widehat{K_3}(t)\widehat{\nabla_h^2 u_{03}}\|_{L^2} \\
&+ \|\widehat{K_4}(t)\widehat{\nabla_h^2\theta_0}\|_{L^2} + \int_0^t \|e^{\lambda_1(t-\tau)}\nabla_h^2\widehat{\mathbb{P}(u\cdot\nabla u)}_h(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\nabla_h^2\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\nabla_h^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_3}(t-\tau)\nabla_h^2\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\nabla_h^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
=& O_1 + O_2 + \ldots + O_{10}.
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|\nabla_h^2\theta(t)\|_{L^2(\mathbb{R}^3)} =& \|\widehat{\nabla_h^2\theta}(t)\|_{L^2(\mathbb{R}^3)} \\
\le& \|\widehat{K_5}(t)\widehat{\nabla_h^2 u_{03}}\|_{L^2} + \|\widehat{K_6}(t)\widehat{\nabla_h^2\theta_0}\|_{L^2} + \int_0^t \|\widehat{K_5}(t-\tau)\nabla_h^2\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_6}(t-\tau)\nabla_h^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau.
\end{aligned} \tag{3.29}$$

We start with $\|\nabla_h^2 u(t)\|_{L^2}$, by Lemma 3.2, we have

$$O_1 \le \Big\| \|e^{c_0\Delta_h t}\nabla_h^2 u_{0h}\|_{L^2_{x_h}} \Big\|_{L^2_{x_3}} \le C(1+t)^{-\frac{3}{2}}\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}}.$$

By the upper bounds of $\widehat{K_i},\ i=1,\dots,6$ in (3.10), (3.11) and Lemma 3.2, we obtain

$$
\begin{aligned}
O_2 \leq & C\|e^{-c_0|\xi_h|^2t}|\xi_3|\widehat{\nabla_h u_{03}}\|_{L^2} + Ce^{-c_0t}\||\xi_3|\widehat{\nabla_h u_{03}}\|L^2\\
\leq & C\|e^{c_0\Delta_h t}\nabla_h\nabla_h\cdot u_{0h}\|_{L^2} + Ce^{-c_0t}\|\nabla_h\nabla_h\cdot u_{0h}\|_{L^2}\\
\leq & C(1+t)^{-\frac{3}{2}}\big(\|u_{0h}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{0h}\|_{H^2}\big),
\end{aligned}
$$

where we have used $e^{-c_0t}\leq C(1+t)^{-\frac{3}{2}}$. Analogously, we yield

$$
O_3+O_5 \leq C(1+t)^{-\frac{3}{2}}\big(\|\theta_0\|_{L^2_{x_3}L^1_{x_h}} + \|\theta_0\|_{H^2}\big),
$$

and

$$
O_4 \leq C(1+t)^{-\frac{3}{2}}\big(\|u_{03}\|_{L^2_{x_3}L^1_{x_h}} + \|u_{03}\|_{H^2}\big).
$$

Next, we estimate the nonlinear terms. By the expression of $\mathbb{P}(u\cdot\nabla u)_h$ in (3.13) and using the fact that the boundedness of Riesz transform in (3.14), we have

$$
\begin{aligned}
O_6 \leq & \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2(u_h\cdot\nabla_h u_h)\|_{L^2}d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2(u_3\partial_3 u_h)\|_{L^2}d\tau\\
& + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2\nabla_h(u\otimes u)\|_{L^2}d\tau\\
= & O_{61}+O_{62}+O_{63}.
\end{aligned}
$$

By Hölder's inequality, Sobolev's inequality, Lemma 3.1, 3.2 and 3.3, we have

$$
\begin{aligned}
O_{61} = & \int_0^t \Big\|\|e^{c_0\Delta_h(t-\tau)}\nabla_h^2(u_h\cdot\nabla_h u_h)\|_{L^2_h}\Big\|_{L^2_{x_3}} d\tau\\
\leq & C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\Big\|\|u_h\cdot\nabla_h u_h\|_{L^1_h}\Big\|_{L^2_{x_3}} d\tau\\
\leq & C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h u_h\|_{L^2}d\tau\\
\leq & CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{2}}d\tau\\
\leq & CC_0^2\varepsilon^2(1+t)^{-\frac{3}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
O_{62} \leq & C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|u_3\|_{L^2}^{\frac{1}{2}}\|\nabla_h\cdot u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}d\tau\\
\leq & CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{5}{4}}d\tau\\
\leq & CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
$$

O_{63} is same as O_{61}, combining O_{61} to O_{63}, we have

$$
O_6 \leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.
$$

By the upper bound of $\widehat{K_1}$ in (3.10) and (3.11), there is

$$O_7 \leq \int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\partial_3\mathbb{P}(u\cdot\nabla u)_3\|_{L^2}d\tau + \int_0^t e^{-c_0(t-\tau)}\|\nabla_h\partial_3\mathbb{P}(u\cdot\nabla u)_3\|_{L^2}d\tau.$$

Using the beforementioned estimate with $|\partial_3\mathbb{P}(u\cdot\nabla u)_3|$ in (3.21), we have

$$\begin{aligned}
O_7 \leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2\partial_3(u_hu_3)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
& + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\Delta_h\nabla_h(u_3u_3)\|_{L^2}d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h^2\partial_3(u_hu_3)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h^2\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h^2\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\Delta_h\nabla_h(u_3u_3)\|_{L^2}d\tau \\
= & O_{71} + \ldots + O_{78}.
\end{aligned}$$

The estimates for O_{71} through O_{74} can be obtained similarly. We now deal with O_{75} to O_{78}. By the ansatz in (3.8) and Lemma 2.2,

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2}^{\frac{1}{8}}\|\nabla_hu\|_{L^2}^{\frac{1}{4}}\|\partial_3u\|_{L^2}^{\frac{1}{8}}\|\nabla_h^2u\|_{L^2}^{\frac{1}{8}}\|\nabla_h\partial_3u\|_{L^2}^{\frac{1}{4}}\|\partial_{123}u\|_{L^2}^{\frac{1}{8}} \leq CC_0\varepsilon(1+t)^{-\frac{25}{32}}. \quad (3.30)$$

Using $\nabla\cdot u = 0$, Hölder's inequality, Sobolev inequality, we consider the norm

$$\begin{aligned}
\|\nabla_h^2\partial_3(u_hu_3)\|_{L^2} \leq & \|u_3\nabla_h^2\partial_3u_h\|_{L^2} + \|\nabla_h^2u_h\partial_3u_3\|_{L^2} + \|\nabla_h\partial_3u_h\nabla_hu_3\|_{L^2} \\
& + \|\nabla_hu_h\nabla_h\partial_3u_3\|_{L^2} + \|\partial_3u_h\nabla_h^2u_3\|_{L^2} + \|u_h\nabla_h^2\partial_3u_3\|_{L^2} \\
\leq & \|u_3\|_{L^\infty}\|\nabla_h^2\partial_3u_h\|_{L^2} + C\|\nabla_h^2u_h\|_{L^3}\|\nabla_h\cdot u_h\|_{L^6} \\
& + \|\nabla_h\partial_3u_h\|_{L^3}\|\nabla_hu_3\|_{L^6} + \|\nabla_h^2u_3\|_{L^2}\|\partial_3u_h\|_{L^\infty} + \|u_h\|_{L^\infty}\|\nabla_h^3\cdot u_h\|_{L^2} \\
\leq & C\|u_3\|_{L^\infty}\|\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla^2\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}} + C\|\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\cdot u_h\|_{L^2} \\
& + C\|\nabla_h\partial_3u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h\partial_3u_h\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_hu_3\|_{L^2} \\
& + C\|\nabla_h^2u_3\|_{L^2}\|\partial_3u_h\|_{L^2}^{\frac{1}{4}}\|\nabla^2\partial_3u_h\|_{L^2}^{\frac{3}{4}} + C\|u_h\|_{L^\infty}\|\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla^2\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, O_{75} can be divided into five parts. By the ansatz in (3.8), upper (3.30) and Lemma 3.4, we have

$$\begin{aligned}
O_{751} + O_{755} = & C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^\infty}\|\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla^2\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}d\tau \\
& + C\int_0^t e^{-c_0(t-\tau)}\|u_h\|_{L^\infty}\|\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}}\|\nabla^2\nabla_h^2u_h\|_{L^2}^{\frac{1}{2}} \\
\leq & CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{45}{32}}d\tau \\
\leq & CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.
\end{aligned}$$

Similarly,

$$O_{752} + O_{753} + O_{754} \leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.$$

Consequently,

$$O_{75} \leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.$$

The terms in O_{76} throuogh O_{78} can also be bounded similarly and the details are omitted. Whether $\xi \in A_1 \cup A_{21}$ or $\xi \in A_{22}$, O_8 through O_{10} are easily estimated like O_7. Namely

$$O_8 + O_9 + O_{10} \leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.$$

In general, collecting the bounds from O_1 to O_{10} and inserting them in (3.28), after using the initial data condition in (1.9), we obtain

$$\|\nabla_h^2 u\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{5}{4}} + CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.$$

The estimate for $\|\nabla_h^2\theta(t)\|_{L^2}$ using (3.29) is very similar and we omit the details. Therefore

$$\|(\nabla_h^2 u(t), \nabla_h^2\theta(t))\|_{L^2} \leq C_8\varepsilon(1+t)^{-\frac{5}{4}} + C_9C_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}.$$

If we choose C_0 and ε satisfying

$$C_8 \leq \frac{C_0}{4}, \quad C_9C_0\varepsilon \leq \frac{1}{4},$$

then

$$\|(\nabla_h^2 u(t), \nabla_h^2)\theta(t)\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{5}{4}}.$$

This completes the proof of the fifth inequality in (3.9).

3.5 Estimates of $\|(\nabla_h\partial_3 u(t), \nabla_h\partial_3\theta(t))\|_{L^2}$

This subsection establishes the sixth inequality in (3.9). Applying $\nabla_h\partial_3$ to (3.5), (3.6) and (3.7), then taking L^2-norm, after using Plancherel's theorem. We have

$$\begin{aligned}
\|\nabla_h\partial_3 u(t)\|_{L^2} =& \|\widehat{\nabla_h\partial_3 u}(t)\|_{L^2} \leq \|\widehat{\nabla_h\partial_3 u_h}(t)\|_{L^2} + \|\widehat{\nabla_h\partial_3 u_3}(t)\|_{L^2} \\
\leq& \|e^{\lambda_1 t}\widehat{\nabla_h\partial_3 u_{0h}}\|_{L^2} + \|\widehat{K_1}(t)\widehat{\nabla_h\partial_3 u_{03}}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\nabla_h\partial_3\theta_0}\|_{L^2} \\
&+ \|\widehat{K_3}(t)\widehat{\nabla_h\partial_3 u_{03}}\|_{L^2} + \|\widehat{K_4}(t)\widehat{\nabla_h\partial_3\theta_0}\|_{L^2} + \int_0^t \|e^{\lambda_1(t-\tau)}\nabla_h\partial_3\widehat{\mathbb{P}(u\cdot\nabla u)}_h\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\nabla_h\partial_3\widehat{\mathbb{P}(u\cdot\nabla u)}_3\|_{L^2}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\nabla_h\widehat{\partial_3(u\cdot\nabla\theta)}\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_3}(t-\tau)\nabla_h\partial_3\widehat{\mathbb{P}(u\cdot\nabla u)}_3\|_{L^2}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\nabla_h\widehat{\partial_3(u\cdot\nabla\theta)}\|_{L^2}d\tau \\
=& P_1 + P_2 + \ldots + P_{10}.
\end{aligned} \tag{3.31}$$

and

$$
\begin{aligned}
\|\nabla_h\partial_3\theta(t)\|_{L^2} =&\|\widehat{\nabla_h\partial_3}\theta(t)\|_{L^2}\\
&\le\|\widehat{K_5}(t)\widehat{\nabla_h\partial_3 u_{03}}\|_{L^2}+\|\widehat{K_6}(t)\widehat{\nabla_h\partial_3\theta_0}\|_{L^2}+\int_0^t\|\widehat{K_5}(t-\tau)\nabla_h\partial_3\widehat{\mathbb{P}(u\cdot\nabla u)_3}\|_{L^2}d\tau\\
&\quad+\int_0^t\|\widehat{K_6}(t-\tau)\nabla_h\widehat{\partial_3(u\cdot\nabla\theta)}\|_{L^2}d\tau. \qquad (3.32)
\end{aligned}
$$

This terms contain the good derivative ∇_h, it can be used to improve the decay rates. But we can't apply the bad derivative ∂_3. Next, let's look at the specific handling process. Using Lemma 3.2, we have

$$
P_1=\Big\|\|e^{c_0\Delta_h t}\nabla_h\partial_3 u_{0h}\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}\le C(1+t)^{-1}\|\partial_3 u_{0h}\|_{L^2_{x_3}L^1_{x_h}}.
$$

By the upper bounds of $\widehat{K_1}$ in (3.10), (3.11) and $\nabla\cdot u_0$=0, we have

$$
\begin{aligned}
P_2\le&C\Big\|\|e^{c_0\Delta_h t}\partial_3\nabla_h\cdot u_{0h}\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}+Ce^{-c_0t}\|\partial_3\nabla_h\cdot u_{0h}\|_{L^2}\\
\le&C(1+t)^{-1}\big(\|\partial_3 u_{0h}\|_{L^2_{x_3}L^1_{x_h}}+\|u_{0h}\|_{H^2}\big),
\end{aligned}
$$

where we have used $(1+t)e^{-c_0t}\le C$. By the same technique, we obtain

$$
\begin{aligned}
P_3+P_5\le C(1+t)^{-1}\big(\|\partial_3\theta_0\|_{L^2_{x_3}L^1_{x_h}}+\|\theta_0\|_{H^2}\big),\\
P_4\le C(1+t)^{-1}\big(\|\partial_3 u_{03}\|_{L^2_{x_3}L^1_{x_h}}+\|u_{03}\|_{H^2}\big).
\end{aligned}
$$

Applying the estimate of $\partial_3\mathbb{P}(u\cdot\nabla u)_h$ in (3.25) and the boundedness of the Riesz transform in (3.14), thus

$$
\begin{aligned}
P_6\le&\int_0^t\|e^{c_0\Delta_h t}\nabla_h(u_3\partial_{33}u_h)\|_{L^2}d\tau+\int_0^t\|e^{c_0\Delta_h t}\nabla_h(\partial_3 u_h\cdot\nabla_h u_h-\nabla_h\cdot u_h\partial_3 u_h)\|_{L^2}d\tau\\
&+\int_0^t\|e^{c_0\Delta_h t}\nabla_h(u_h\cdot\nabla_h\partial_3 u_h)\|_{L^2}d\tau+\int_0^t\|e^{c_0\Delta_h t}\nabla_h^2(u\cdot\nabla u)\|_{L^2}d\tau\\
=&P_{61}+P_{62}+P_{63}+P_{64}.
\end{aligned}
$$

By Hölder's inequality, Lemma 3.1, 3.2, 3.3 and the ansatz in (3.8), we derive

$$
\begin{aligned}
P_{61}=&\int_0^t\Big\|\|e^{c_0\Delta_h(t-\tau)}\nabla_h(u_3\partial_{33}u_h)\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\
\le&C\int_0^t(1+t-\tau)^{-1}\Big\|\|u_3\partial_{33}u_h\|_{L^1_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\
\le&C\int_0^t(1+t-\tau)^{-1}\|u_3\|_{L^2}^{\frac12}\|\nabla_h\cdot u_h\|_{L^2}^{\frac12}\|\partial_{33}u_h\|_{L^2}d\tau\\
\le&CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-1}(1+\tau)^{-\frac54}d\tau\\
\le&CC_0^2\varepsilon^2(1+t)^{-1}.
\end{aligned}
$$

Using the same beforementioned conditions and technique, P_{62} to P_{64} can be obtained the same bound, the details are omitted. By the bound of $|\widehat{K}_1\mathbb{P}(u\cdot\nabla u)_3|$ in (3.26), P_7 can be divided into eight parts,

$$\begin{aligned}
P_7 \le& C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\partial_{33}(u_hu_3)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\partial_3\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau\\
&+C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h\partial_3\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\nabla_h^2\partial_3(u_3u_3)\|_{L^2}d\tau\\
&+C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h\partial_{33}(u_hu_3)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h\partial_3\nabla_h\cdot(u_hu_1)\|_{L^2}d\tau\\
&+C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h\partial_3\nabla_h\cdot(u_hu_2)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\nabla_h^2\partial_3(u_3u_3)\|_{L^2}d\tau\\
=&P_{71}+\ldots+P_{78}.
\end{aligned}$$

P_{71} through P_{74} are easily by applying the same method with P_{61}, and we will not repeat these cumbersome details. By Hölder's inequality, we consider the norm

$$\begin{aligned}
\|\nabla_h\partial_{33}(u_hu_3)\|_{L^2}\le&\|u_3\nabla_h\partial_{33}u_h\|_{L^2}+\|\nabla_h\partial_3u_h\nabla_h\cdot u_h\|_{L^2}+\|\partial_{33}u_h\nabla_hu_3\|_{L^2}\\
&+\|\nabla_hu_h\partial_3\nabla_h\cdot u_h\|_{L^2}+\|\partial_3u_h\nabla_h^2\cdot u_h\|_{L^2}+\|u_h\partial_3\nabla_h^2\cdot u_h\|_{L^2}\\
\le&\|u_3\|_{L^\infty}\|\nabla_h\partial_{33}u_h\|_{L^2}+C\|\nabla_h\partial_3u_h\|_{L^2}\|\nabla_hu_h\|_{L^\infty}+\|\partial_{33}u_h\|_{L^\infty}\|\nabla_hu_3\|_{L^2}\\
&+\|\partial_3u_h\|_{L^\infty}\|\nabla_h^2\cdot u_h\|_{L^2}+\|u_h\|_{L^\infty}\|\partial_3\nabla_h^2\cdot u_h\|_{L^2}\\
\le&\|u_3\|_{L^\infty}\|\nabla_hu_h\|_{L^2}^{\frac13}\|\nabla^3\nabla_hu_h\|_{L^2}^{\frac23}+C\|\nabla_h\partial_3u_h\|_{L^2}\|\nabla_hu_h\|_{L^2}^{\frac14}\|\nabla^2\nabla_hu_h\|_{L^2}^{\frac34}\\
&+\|\partial_{33}u_h\|_{L^2}^{\frac14}\|\nabla^2\partial_{33}u_h\|_{L^2}^{\frac34}\|\nabla_hu_3\|_{L^2}+\|\partial_3u_h\|_{L^2}^{\frac14}\|\nabla^2\partial_3u_h\|_{L^2}^{\frac34}\|\nabla_h^2\cdot u_h\|_{L^2}\\
&+\|u_h\|_{L^\infty}\|\nabla_h^2\cdot u_h\|_{L^2}^{\frac12}\|\nabla^2\nabla_h^2\cdot u_h\|_{L^2}^{\frac12}.
\end{aligned}$$

Then P_{75} is divided into six parts. We need to deal them respectively. By the estimate of $\|u\|_{L^\infty}$ in (3.30), the ansatz in (3.8) and Lemma 3.4, we have

$$\begin{aligned}
P_{751}\le&C\int_0^t e^{-c_0(t-\tau)}\|u_3\|_{L^\infty}\|\nabla_hu_h\|_{L^2}^{\frac13}\|\nabla^3\nabla_hu_h\|_{L^2}^{\frac23}d\tau\\
\le&CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}\big((1+\tau)^{-\frac{107}{96}}\big)d\tau\\
\le&CC_0^2\varepsilon^2(1+t)^{-1}.
\end{aligned}$$

Apparently, the rest of P_{752} to P_{756} can obtain the upper bound more easily. The terms in P_{76} through P_{78} can also be bounded similarly and the details are omitted. Therefore, we have

$$P_7\le CC_0^2\varepsilon^2(1+t)^{-1}.$$

Applying the upper bounds for $\widehat{K_2}$, $\widehat{K_3}$ and $\widehat{K_4}$ in (3.10), (3.11), and the estimate in (3.21). After separating each term, every part in P_8 through P_{10} can be dealt using the same technique like P_7. Here we only write the result,

$$P_8 + P_9 + P_{10} \leq CC_0^2\varepsilon^2(1+t)^{-1}.$$

Inserting the bounds from P_1 through P_{10} in (3.31), and using the initial data in (1.9), then we have

$$\|\nabla_h\partial_3 u\|_{L^2} \leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2(1+t)^{-1}.$$

The estimate for $\|\nabla_h\partial_3\theta\|_{L^2}$ in (3.32) is similar and we omit the details. Therefore,

$$\|(\nabla_h\partial_3 u(t), \nabla_h\partial_3\theta(t))\|_{L^2} \leq C_{10}\varepsilon(1+t)^{-1} + C_{11}C_0^2\varepsilon^2(1+t)^{-1}. \tag{3.33}$$

If we choose C_0 and ε satisfying

$$C_{10} \leq \frac{C_0}{4}, \quad C_{11}C_0\varepsilon \leq \frac{1}{4},$$

then (3.33) implies

$$\|(\nabla_h\partial_3 u(t), \nabla_h\partial_3\theta(t))\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1}.$$

This completes the proof of the sixth inequality in (3.9).

3.6 Estimates of $\|(\partial_3^2 u(t), \partial_3^2\theta(t))\|_{L^2}$

This subsection establishes the seventh inequality in (3.9). Applying ∂_3^2 to (3.5), (3.6) and (3.7), then taking L^2-norm, after using Plancherel's theorem. We have

$$\begin{aligned}
\|\partial_3^2 u(t)\|_{L^2(\mathbb{R}^3)} =&\|\widehat{\partial_3^2 u}(t)\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{\partial_3^2 u_h}(t)\|_{L^2(\mathbb{R}^3)} + \|\widehat{\partial_3^2 u_3}(t)\|_{L^2(\mathbb{R}^3)} \\
\leq&\|e^{\lambda_1 t}\widehat{\partial_3^2 u_{0h}}\|_{L^2} + \|\widehat{K_1}(t)\widehat{\partial_3^2 u_{03}}\|_{L^2} + \|\widehat{K_2}(t)\widehat{\partial_3^2\theta_0}\|_{L^2} \\
&+ \|\widehat{K_3}(t)\widehat{\partial_3^2 u_{03}}\|_{L^2} + \|\widehat{K_4}(t)\widehat{\partial_3^2\theta_0}\|_{L^2} + \int_0^t \|e^{\lambda_1(t-\tau)}\partial_3^2\widehat{\mathbb{P}(u\cdot\nabla u)_h}(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\partial_3^2\widehat{\mathbb{P}(u\cdot\nabla u)_3}(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_2}(t-\tau)\partial_3^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
&+ \int_0^t \|\widehat{K_3}(t-\tau)\partial_3^2\widehat{\mathbb{P}(u\cdot\nabla u)_3}(\tau)\|_{L^2}d\tau + \int_0^t \|\widehat{K_4}(t-\tau)\partial_3^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau \\
=&Q_1 + Q_2 + \ldots + Q_{10}.
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\|\partial_3^2\theta(t)\|_{L^2(\mathbb{R}^3)} =&\|\widehat{\partial_3^2\theta}(t)\|_{L^2(\mathbb{R}^3)}\\ \leq&\|\widehat{K_5}(t)\widehat{\partial_3^2 u_{03}}\|_{L^2} + \|\widehat{K_6}(t)\widehat{\partial_3^2\theta_0}\|_{L^2} + \int_0^t \|\widehat{K_5}(t-\tau)\partial_3^2\widehat{\mathbb{P}(u\cdot\nabla u)}_3(\tau)\|_{L^2}d\tau\\ &+\int_0^t \|\widehat{K_6}(t-\tau)\partial_3^2\widehat{(u\cdot\nabla\theta)}(\tau)\|_{L^2}d\tau. \end{aligned}\tag{3.35}$$

This terms contain the bad derivative ∂_3, and we can't deal with it. Let's look at the specific handling process in $\|\partial_3^2 u\|_{L^2}$. In fact, Q_1 through Q_5 can be shown by repeating the process of J_1 through J_5 with $\partial_{33}u$ and $\partial_{33}\theta$ replacing u and θ, respectively, namely

$$Q_1+\ldots+Q_5 \leq C(1+t)^{-\frac{1}{2}}\big(\|(\partial_3^2 u_0,\partial_3^2\theta_0)\|_{L^2_{x_3}L^1_{x_h}} + \|(u_0,\theta_0)\|_{H^2}\big).$$

By the estimate of $\mathbb{P}(u\cdot\nabla u)_h$ in (3.13), and $\nabla\cdot u=0$, we have

$$\partial_3^2\mathbb{P}(u\cdot\nabla u)_h = \partial_3^2(u_h\cdot\nabla_h u_h) + \partial_3^2(u_3\partial_3 u_h) - \partial_3^2\Delta^{-1}\nabla\cdot\nabla\cdot\nabla_h(u\otimes u).$$

Combining the boundness of Riesz transform in (3.14), Q_6 can be divided into three parts,

$$\begin{aligned}Q_6 \leq&\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2(u_h\cdot\nabla_h u_h)\|_{L^2}d\tau + \int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2(u_3\partial_3 u_h)\|_{L^2}d\tau\\ &+\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2\nabla_h(u\otimes u)\|_{L^2}d\tau\\ =&Q_{61}+Q_{62}+Q_{63}.\end{aligned}$$

We further divide Q_{61} into three parts,

$$\begin{aligned}Q_{61} \leq& C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2 u_h\cdot\nabla_h u_h\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3 u_h\cdot\partial_3\nabla_h u_h\|_{L^2}d\tau\\ &+C\int_0^t \|e^{c_0\Delta_h(t-\tau)}u_h\cdot\partial_3^2\nabla_h u_h\|_{L^2}d\tau\\ =&Q_{611}+Q_{612}+Q_{613}.\end{aligned}$$

By Hölder's inequality, Lemma 3.1, 3.2, 3.3 and the ansztz (3.8), we have

$$\begin{aligned}Q_{611} =& C\int_0^t \Big\|\|e^{c_0\Delta_h(t-\tau)}\partial_3^2 u_h\cdot\nabla_h u_h\|_{L^2_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\ \leq& C\int_0^t 1+t-\tau)^{-\frac{1}{2}}\Big\|\|\partial_3^2 u_h\cdot\nabla_h u_h\|_{L^1_{x_h}}\Big\|_{L^2_{x_3}}d\tau\\ \leq& C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|\partial_3^2 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3^3 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h u_h\|_{L^2}d\tau\\ \leq& CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}d\tau\\ \leq& CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.\end{aligned}$$

By the same technique, Q_{612}, Q_{613} and each term in Q_{62} and Q_{63} can be estimated, and obtained same decay rate. Therefore,

$$Q_6 \leq CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Using the beforementioned bound in (3.26), we have

$$\begin{aligned} |\widehat{K_1}(t)\partial_3^2\mathbb{P}\widehat{(u\cdot\nabla u)}_3| \leq C(e^{-c_0|\xi_h|^2 t}+e^{-c_0 t})\big(|\partial_3^3\widehat{(u_h u_3)}|+|\partial_3^2\nabla_h\cdot\widehat{(u_h u_1)}| \\ +|\partial_3^2\nabla_h\cdot\widehat{(u_h u_2)}|+|\partial_3^2\nabla_h\widehat{(u_3 u_3)}|\big). \end{aligned} \tag{3.36}$$

Then Q_7 can be divided into eight parts,

$$\begin{aligned} Q_7 \leq & C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^3(u_h u_3)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2\nabla_h\cdot(u_h u_1)\|_{L^2}d\tau \\ & + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2\nabla_h\cdot(u_h u_2)\|_{L^2}d\tau + C\int_0^t \|e^{c_0\Delta_h(t-\tau)}\partial_3^2\nabla_h(u_3 u_3)\|_{L^2}d\tau \\ & + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3^3(u_h u_3)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3^2\nabla_h\cdot(u_h u_1)\|_{L^2}d\tau \\ & + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3^2\nabla_h\cdot(u_h u_2)\|_{L^2}d\tau + C\int_0^t e^{-c_0(t-\tau)}\|\partial_3^2\nabla_h(u_3 u_3)\|_{L^2}d\tau \\ = & Q_{71}+\ldots+Q_{78}. \end{aligned}$$

Q_{71} through Q_{74} can be dealt by the same method with Q_6, and we have

$$Q_{71}+\ldots+Q_{74} \leq CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Q_{75} through Q_{78} need more attentions. By Hölder's inequality and Sobolev's inequality, the norm

$$\begin{aligned} \|\partial_3^3(u_h u_3)\|_{L^2} \leq & \|u_3\partial_3^3 u_h\|_{L^2} + \|\partial_3^2 u_h\nabla_h\cdot u_h\|_{L^2} + \|\partial_3 u_h\partial_3\nabla_h\cdot u_h\|_{L^2} \\ \leq & C\|u_3\|_{L^3}\|\partial_3^3 u_h\|_{L^6} + \|\partial_3^2 u_h\|_{L^4}\|\nabla_h\cdot u_h\|_{L^4} + \|\partial_3 u_h\|_{L^\infty}\|\partial_3\nabla_h\cdot u_h\|_{L^2} \\ \leq & C\|u_3\|_{L^2}^{\frac{1}{2}}\|\nabla u_3\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3^3 u_h\|_{L^2} + \|\partial_3^2 u_h\|_{L^2}^{\frac{1}{4}}\|\nabla\partial_3^2 u_h\|_{L^2}^{\frac{3}{4}}\|\nabla_h\cdot u_h\|_{L^2}^{\frac{1}{4}}\|\nabla\nabla_h\cdot u_h\|_{L^2}^{\frac{3}{4}} \\ & + \|\partial_3 u_h\|_{L^2}^{\frac{1}{4}}\|\nabla^2\partial_3 u_h\|_{L^2}^{\frac{3}{4}}\|\partial_3\nabla_h\cdot u_h\|_{L^2}. \end{aligned}$$

Using Lemma 3.4 and the ansatz in (3.8), we have

$$\begin{aligned} Q_{75} \leq & CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{1}{2}}d\tau + CC_0^2\varepsilon^2\int_0^t e^{-c_0(t-\tau)}(1+\tau)^{-\frac{9}{8}}d\tau \\ \leq & CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

Analogously, Q_{76} through Q_{78} are easily to obtain the upper bounds which are same as Q_{75}. Thus, we have

$$Q_7 \leq CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Due to the upper bounds of $\widehat{K_2}$, $\widehat{K_3}$ and $\widehat{K_4}$ in (3.10), (3.11), and the estimate of $|\partial_3\mathbb{P}(u\cdot\nabla u)_3|$ in (3.21), we can get the same bound by using the similar technique like Q_7, namely

$$Q_8 + Q_9 + Q_{10} \le CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

Collecting the bounds of Q_1 through Q_{10} and inserting them in (3.34), after using the initial data in (2.2), we have

$$\|\partial_3^2 u\|_{L^2} \le C\varepsilon(1+t)^{-\frac{1}{2}} + CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}.$$

The estimate for $\|\partial_3^2\theta\|_{L^2}$ in (3.35) is similar and we omit the details. Therefore,

$$\|(\partial_3^2 u(t), \partial_3^2\theta(t))\|_{L^2} \le C_{12}\varepsilon(1+t)^{-\frac{1}{2}} + C_{13}C_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{3.37}$$

If we choose C_0 and ε satisfying

$$C_{12} \le \frac{C_0}{4}, \quad C_{13}C_0\varepsilon \le \frac{1}{4},$$

then (3.37) implies

$$\|(\partial_3^2 u(t), \partial_3^2\theta(t))\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the seventh inequality in (3.9) and thus, the proof of Theorem 1.2.

□

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