

# Asymptotic Behavior of the coupled Klein-Gordon-Schrödinger systems on compact manifolds

César A. Bortot, Thales M. Souza and Janaina P. Zanchetta

**Abstract.** This paper is concerned with a 2-dimensional Klein-Gordon-Schrödinger system subject to two types of locally distributed damping on a compact Riemannian manifold  $\mathcal{M}$  without boundary. Making use of unique continuation property, the observability inequalities, and the smoothing effect due to Aloui, we obtain exponential stability results.

**Mathematics Subject Classification.** Primary 35L70; Secondary 35B40.

**Keywords.** Klein-Gordon-Schrodinger system; Exponential stability; Compact manifolds; Differential equations on manifolds.

## 1. Introduction

The classical Klein-Gordon-Schrödinger system through Yukawa coupling, given by

$$\begin{cases} i\psi_t + \Delta\psi = \phi\psi & \text{in } \Omega \times (0, \infty) \\ \phi_{tt} - \Delta\phi + \mu^2\phi = |\psi|^2 & \text{in } \Omega \times (0, \infty) \\ \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1 & \text{in } \Omega \end{cases} \quad (1.1)$$

where,  $\psi$  is a complex scalar nucleon field while  $\phi$  is a real scalar meson one and the positive constant  $\mu$  represents the mass of a meson. Since it is considered a bounded domain with Dirichlet conditions, the term  $\mu^2\phi$  does not affect the employed arguments in the proof of the asymptotic stability. In our case we will assume fields  $\psi$  and  $\phi$  which has an average value of zero, thus the term  $\mu^2\phi$  does not hinder the multiplier techniques employed in our results of decay rates. So, for simplicity, this term will be omitted. For more details on physical modeling see [47]. The Klein-Gordon-Schrodinger system through Yukawa coupling has been investigated by many authors, recent papers such as Poulou et. al [38, 39], and Cavalcanti et. al [15, 16].

In this paper, we consider the Cauchy problem of the following Klein-Gordon-Schrödinger equations through Yukawa interaction,

$$\begin{cases} i\psi_t + \Delta\psi + i\alpha \mathcal{B}_j(x, \psi) = \phi\psi\chi_\omega & \text{in } \mathcal{M} \times (0, \infty), j = 1, 2 \\ \phi_{tt} - \Delta\phi + a(x)\phi_t = |\psi|^2\chi_\omega & \text{in } \mathcal{M} \times (0, \infty) \\ \psi(0) = \psi_0, \phi(0) = \phi_0, \phi_t(0) = \phi_1 & \text{in } \mathcal{M} \end{cases} \quad (P_j)$$

where  $(\mathcal{M}, g)$  is a bidimensional compact Riemannian manifold without boundary and  $g$  represents your metric,  $\mathcal{B}_j(x, \psi)$  is non-linear locally distributed damping,  $\omega$  is a region on  $\mathcal{M}$  where the dissipative effect is effective, and  $\chi_\omega$  is the characteristic function on  $\omega$ . We study two types of damping, defined by

$$\mathcal{B}_1(x, \psi) = b(x)(1 - \Delta)^{1/2}b(x)\psi \quad \text{and} \quad \mathcal{B}_2(x, \psi) = b(x)(|\psi|^2 + 1)\psi \quad (1.2)$$

We assume that  $a(\cdot), b(\cdot)$  are non-negative functions satisfying

$$\begin{cases} a, b \in W^{1, \infty}(\mathcal{M}) \cap C^\infty(\mathcal{M}) \\ a(x) \geq a_0 > 0 \text{ in } \omega, \quad \text{and} \quad b(x) \geq b_0 > 0 \text{ in } \omega, \end{cases}$$

where  $\omega$  is an open subset of  $\mathcal{M}$  such that  $\text{meas}(\omega) > 0$  satisfying geometric control condition.

We are interested in uniform decay results for Klein-Gordon-Schrödinger equations with the damping effect. The Klein-Gordon-Schrödinger system with the dissipative mechanisms was considered extensively in the literature, for instance, the exponential decay of Klein-Gordon-Schrödinger system with full damping in both equations holds. These results due to Cavalcanti in [15], they have used a perturbed energy method to guarantee exponential decay rates. On the other hand, a uniform decay result holds, considering locally distributed damping into the wave equation and full damping into the Schrödinger equation. This was proven by the authors in [11].

An interesting result of exponential decay considering Klein-Gordon-Schrödinger system with localized damping in both equations are due the authors in [1]. More precisely they consider the following Klein-Gordon-Schrödinger equations

$$\begin{cases} i\psi_t + \Delta\psi + i\alpha b(x)(-\Delta)^{\frac{1}{2}}b(x)\psi = \phi\psi\chi_\omega & \text{in } \Omega \times (0, \infty), (\alpha > 0) \\ \phi_{tt} - \Delta\phi + a(x)\phi_t = |\psi|^2\chi_\omega & \text{in } \Omega \times (0, \infty), \end{cases}$$

where  $\Omega$  is a bounded domain and  $\omega$  is a region of domain with damping effect. Uniform decay rates were have obtained combining multipliers method, integral inequalities of energy, and regularizing effect due to Aloui [3]. Recently in [2], the authors generalize the previous results considering the weaker damped structure  $i\alpha b(x)(|\psi|^2 + 1)\psi$  instead of  $i\alpha b(x)(-\Delta)^{\frac{1}{2}}b(x)\psi$  assumed in [1], making use of the observability inequality in both equations, the linear wave (see [7]) and the Schrödinger equation (see [21, 31]), furthermore, combined with other tools have proven exponential decay as done in [20].

The purpose of the present article is to extend substantially all previous results given by [1] and [2] in the geometric sense and exhibit an important

multiplier function.. Here we study the problem  $(P_j)$ ,  $j = 1, 2$ , on a compact Riemannian manifold with arbitrary metric. In what follows, we would like to explain the relevance of this paper compared to [1, 2]. In fact, these recent articles take advantage of a smoothing effect introduced by Aloui [3] for bounded domains and observability inequalities associated with the linear problems of the wave and Schrödinger equations. The main features of this work are as follows:

(i) We consider the Klein-Gordon-Schrödinger system with localized damping and provide a regular function  $f : \mathcal{M} \rightarrow \mathbb{R}$  that allows us to apply the multiplier method and Unique continuation property. See Theorem 4.1.

(ii) We establish uniform decay rates of both systems. See Theorem 2.4 and 2.5. More specifically, we use multiplier functional combined with the regularizing effect due to Aloui on manifolds. In addition, by using geometric control conditions on  $\omega$ , we obtain observability inequalities in the linear wave and the Schrödinger equation.

It is worth mentioning that the results of unique continuation are closely related to the existence of this multiplier. On the other hand, the effective dissipation region  $\omega$  needs the properties of this mentioned multiplier. The general idea is construct a function  $f : \mathcal{M} \rightarrow \mathbb{R}$  and define an open subset  $V \subset \mathcal{M}$  where  $\text{meas}(V) \geq \text{meas}(\mathcal{M}) - \epsilon$  for every  $\epsilon > 0$ , such that  $f$  satisfies conditions related to it is hessian, gradient, and Laplacian in  $V$ . Thus, we can define  $\omega$  (see Figure 1) as the open subset such that,

$$\omega \supset \supset (\mathcal{M} \setminus V).$$

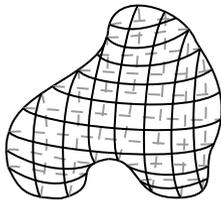


FIGURE 1. Blank, we have an arbitrarily large area, free of dissipative effects while in black, it is in the region  $\omega$  where the dissipative effect is effective, this region can be chosen arbitrarily small, both totally distributed on  $\mathcal{M}$ .

Recently in [14] and [16], the authors exposed a multiplier for the Schrödinger equation and the wave equation, respectively. The main result Theorem 4.1 generalizes the multiplier function, and in this case, we obtain a multiplier that works for both equations simultaneously. Therefore, we can use the relevant results [14, Theorem 4.2] and [16, Theorem 5.1] in the Klein-Gordon-Schrödinger system.

We refer the reader to, for instance, the references about classical Klein-Gordon-Schrödinger system [6, 22, 23, 24]. In general we would like to mention

some nice important papers in connection with Klein-Gordon-Schrödinger equations [5, 8, 10, 25, 26, 27, 35, 36, 37, 42, 43, 46].

The rest of this paper is structured as follows. In section 2 we give the precise assumptions and state our main results, in section 3 we give an idea of the well-posedness, in section 4 we present the construction of multiplier  $f$ , and in section 5 we give the proof of the main theorems.

## 2. Main Result

We shall use standard Sobolev spaces on Riemannian manifolds, for more detail see [32], [44] and [45]. We will give some definitions for sake of completeness. First, we consider the space  $L^2(\mathcal{M})$  of complex-valued function on  $\mathcal{M}$ , with the following real inner product and norm

$$\begin{aligned} L^2(\mathcal{M}) &= \{y : \mathcal{M} \rightarrow \mathbb{C}; \int_{\mathcal{M}} |y|^2 d\mathcal{M} < \infty\} \\ (y, z)_{L^2(\mathcal{M})} &= \operatorname{Re} \int_{\mathcal{M}} y(x) \overline{z(x)} d\mathcal{M}, \\ \|y\|_{L^2(\mathcal{M})}^2 &= (y, y)_{L^2(\mathcal{M})}. \end{aligned}$$

Besides, we consider

$$\begin{aligned} H^1(\mathcal{M}) &= \{y : \mathcal{M} \rightarrow \mathbb{C}; y \in L^2(\mathcal{M}); |\nabla y| \in L^2(\mathcal{M})\}, \\ \mathcal{V} &:= \{y \in H^1(\mathcal{M}); \int_{\mathcal{M}} y(x) d\mathcal{M} = 0\}, \\ H &:= \{y \in L^2(\mathcal{M}); \int_{\mathcal{M}} y(x) d\mathcal{M} = 0\}, \end{aligned}$$

equipped with the norms, respectively

$$\|y\|_{H^1(\mathcal{M})}^2 = \|y\|_{L^2(\mathcal{M})}^2 + \|\nabla y\|_{L^2(\mathcal{M})}^2, \quad \|y\|_{\mathcal{V}}^2 = \|\nabla y\|_{L^2(\mathcal{M})}^2, \quad (2.1)$$

and  $H$  with the standart  $L^2$ -norm.

It is important to note that, from Poincaré's inequality,

$$\|y\|_{L^2(\mathcal{M})}^2 \leq \lambda \|\nabla y\|_{L^2(\mathcal{M})}^2; \quad \forall y \in \mathcal{V}, \quad (2.2)$$

where  $\lambda^{-1}$  is the first eigenvalue of the Laplace-Beltrami operator. Consequently the norms in  $H^1(\mathcal{M})$  and  $\mathcal{V}$  are equivalent.

Consider the following Hilbert space

$$H^{2m}(\mathcal{M}) = \{y \in L^2(\mathcal{M}); \Delta^m y \in L^2(\mathcal{M})\},$$

equipped with norm

$$\|y\|_{H^{2m}(\mathcal{M})}^2 = \|y\|_{L^2(\mathcal{M})}^2 + \|\Delta^m y\|_{L^2(\mathcal{M})}^2. \quad (2.3)$$

Considering  $f : \mathcal{M} \rightarrow \mathbb{R}$  a sufficiently regular real function and  $H$  a vector field on  $\mathcal{M}$ , we have the following identity (see [16]: p.22)

$$\langle \nabla f, \nabla(H(f)) \rangle_{\mathbf{g}} = \nabla H(\nabla f, \nabla f) + \frac{1}{2} \langle \nabla(|\nabla f|^2), H \rangle_{\mathbf{g}}, \quad (2.4)$$

The following result is also true (see [17], section 4.1):

Let  $\mathcal{M}$  be a compact Riemannian manifold without boundary  $y : \mathcal{M} \rightarrow \mathbb{C} \in H^1(\mathcal{M})$  and  $X$  a vector field of class  $C^1$  on  $\mathcal{M}$ . So, the following statement is valid,

$$\int_{\mathcal{M}} \langle X, \nabla y \rangle d\mathcal{M} = - \int_{\mathcal{M}} (\operatorname{div} X) y d\mathcal{M}. \quad (2.5)$$

Consequently, if  $y \in H^1(\mathcal{M})$  such that  $\Delta y \in L^2(\mathcal{M})$  and  $w \in H^1(\mathcal{M})$  then the following identity is valid:

$$\int_{\mathcal{M}} \langle \nabla y, \nabla w \rangle d\mathcal{M} = - \int_{\mathcal{M}} \Delta y w d\mathcal{M}. \quad (2.6)$$

In the present paper, we consider some crucial assumptions about the dissipative region  $\omega$ , in order to establish the geometric control condition.

*Assumption 1.* We assume that  $a, b \in W^{1,\infty}(\mathcal{M}) \cap C^\infty(\mathcal{M})$  are nonnegative functions such that

$$a(x) \geq a_0 > 0 \text{ and } b(x) \geq b_0 > 0 \text{ in } \omega.$$

In addition,

If  $a(x) \geq a_0 > 0$  in  $\mathcal{M}$ , then we consider  $\chi_\omega \equiv 1$  in  $\mathcal{M}$

If  $b(x) \geq b_0 > 0$  in  $\mathcal{M}$ , then we consider  $\chi_\omega \equiv 1$  in  $\mathcal{M}$ .

**Definition 2.1.** (Geometric Control Condition):  $\omega$  geometrically controls  $\mathcal{M}$ , i.e there exists  $T_0 > 0$ , such that every geodesic of  $\mathcal{M}$  travelling with speed 1 and issued at  $t = 0$ , enters the set  $\omega$  in a time  $t < T_0$ .

*Assumption 2.* We assume that  $\omega$  is an open subset of  $\mathcal{M}$  such that  $\operatorname{meas}(\omega) > 0$  and satisfying the geometric control condition

As a consequence of assumption (2), it follows that there exists a couple  $(\omega, T_0)$ , with  $T_0 > 0$ , such that the following observability inequalities holds:

$$\|\psi_0\|_{L^2(\mathcal{M})}^2 \leq C \int_0^T \int_{\omega} |\psi(x, t)|^2 d\mathcal{M} dt, \quad (2.7)$$

associated with the problem

$$\begin{cases} i\psi_t + \Delta\psi = 0 \text{ in } \mathcal{M} \times (0, T), \\ \psi(0) = \psi_0 \in L^2(\mathcal{M}), \end{cases}$$

and

$$\|\phi_1\|_{L^2(\mathcal{M})}^2 + \|\nabla\phi_0\|_{L^2(\mathcal{M})}^2 \leq C \int_0^T \int_{\omega} |\phi_t(x, t)|^2 d\mathcal{M} dt, \quad (2.8)$$

about problem

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 \text{ in } \mathcal{M} \times (0, T), \\ \phi(0) = \phi_0 \in \mathcal{V}, \\ \phi_t(0) = \phi_1 \in L^2(\mathcal{M}), \end{cases}$$

for some positive constant  $C = C(\omega, T_0)$  and for all  $T > T_0$ .

*Remark 2.2.* Observe that, the couple  $(\omega, T_0)$  satisfies the geometric control condition (GCC, in short) if every geodesic of  $\mathcal{M}$ , traveling with speed 1 and issued at  $t = 0$  enters the open set  $\omega$  before the time  $T_0$ . It is the well-known geometric control condition (GCC) due to Bardos, Lebeau, Rauch [7] and Taylor [40]. Furthermore, observability estimates was extensively studied by many authors. We refer the reader to, for instance, the references [7, 21, 29, 30, 34, 31, 33, 41].

The energy associated to problems  $(P_j)$ ,  $j = 1, 2$ , is defined by

$$E(t) := \frac{1}{2} \int_{\mathcal{M}} (|\psi(x, t)|^2 + |\nabla\phi(x, t)|^2 + |\phi_t(x, t)|^2) d\mathcal{M}. \quad (2.9)$$

Firstly, we observe that a straight forward computation leads to

$$\frac{dE}{dt}(t) + \alpha \int_{\mathcal{M}} \mathcal{B}_i(x, \psi) \bar{\psi} d\mathcal{M} + \int_{\mathcal{M}} a(x) |\phi_t|^2 d\mathcal{M} = \int_{\omega} |\psi|^2 \phi_t d\mathcal{M},$$

multiplying the first equation of  $(P_j)$  by  $\bar{\psi}$ , the second equation by  $\phi_t$ , integrating over  $\mathcal{M}$ , taking a real part where is necessary and making use of Green formula.

Remember that problem  $(P_1)$  concerns regularizing term  $\mathcal{B}_1(x, \psi) = b(x)(1 - \Delta)^{1/2}b(x)\psi$  and problem  $(P_2)$  concerns cubic non-linearity term  $\mathcal{B}_2(x, \psi) = b(x)(|\psi|^2 + 1)\psi$ . Then we have the following energy identities,

$$\begin{aligned} \frac{dE}{dt}(t) + \alpha \int_{\mathcal{M}} b(x) |(1 - \Delta)^{\frac{1}{4}} \psi|^2 d\mathcal{M} \\ + \int_{\mathcal{M}} a(x) |\phi_t|^2 d\mathcal{M} = \int_{\omega} |\psi|^2 \phi_t d\mathcal{M}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \frac{dE}{dt}(t) + \alpha \int_{\mathcal{M}} b(x) (|\psi|^4 + |\psi|^2) d\mathcal{M} \\ + \int_{\mathcal{M}} a(x) |\phi_t|^2 d\mathcal{M} = \int_{\omega} |\psi|^2 \phi_t d\mathcal{M}, \end{aligned} \quad (2.11)$$

regarding  $(P_1)$ , and  $(P_2)$ , respectively. We are now in a position to state the main results.

**Theorem 2.3.** *Suppose Assumption 1 holds. In addition, assume that  $5(2a_0b_0)^{-1} \leq \alpha$  in the problem  $(P_2)$ . Then, given  $(\psi_0, \phi_0, \phi_1) \in \{\mathcal{V} \cap H^2(\mathcal{M})\}^2 \times \mathcal{V}$  problems  $(P_1)$  and  $(P_2)$  has a unique regular solution satisfying*

$$\begin{aligned} \psi &\in L^\infty(0, \infty; \mathcal{V} \cap H^2(\mathcal{M})), \quad \psi' \in L^\infty(0, \infty; L^2(\mathcal{M})), \\ \phi &\in L^\infty(0, \infty; \mathcal{V} \cap H^2(\mathcal{M})), \quad \phi' \in L^\infty(0, \infty; \mathcal{V}), \\ \text{and } \phi'' &\in L^\infty(0, \infty; L^2(\mathcal{M})). \end{aligned} \quad (2.12)$$

Considering the phase space  $\mathcal{H} := \{\mathcal{V} \cap H^2(\mathcal{M})\}^2 \times \mathcal{V}$ , in the next theorem, below, we provide a local uniform decay of the energy. Indeed, we shall consider the initial data taken in bounded sets of  $\mathcal{H}$ , namely,  $\|(\psi_0, \phi_0, \phi_1)\|_{\mathcal{H}} \leq L$ , where  $L$  is a positive constant. This is strongly necessary due to the non linear character of system  $(P_1)$  and since the energy  $E(t)$  is not naturally a non increasing function of the parameter  $t$ . Thus, the constants,  $C$  and  $\gamma$

which appear in (2.13) and (2.14) will depend on  $L > 0$ . We shall denote  $d = d(c, \|b\|_\infty, L)$ , to be fixed in Section 5, where  $c$  comes from the embedding  $D[(1 - \Delta)^{\frac{1}{4}}] \equiv H^{\frac{1}{2}}(\mathcal{M}) \hookrightarrow L^4(\mathcal{M})$ . So, under the above considerations, we can establish the main result concerning uniform stabilization from the problems  $(P_1)$  and  $(P_2)$ .

**Theorem 2.4.** *Suppose that the hypotheses of Theorem 2.3 holds. In addition,  $\alpha > \frac{a_0^{-1}b_0^{-4}d}{2}$  or  $d$  is sufficiently small. Then, there exist  $C, \gamma$  positive constants such that following decay rate holds*

$$E(t) \leq Ce^{-\gamma t}E(0), \quad \text{for all } t \geq 0. \quad (2.13)$$

for every regular solution of problem  $(P_1)$  satisfying (2.12), provided the initial data are taken in bounded sets of  $\mathcal{H}$ .

**Theorem 2.5.** *Suppose that the hypotheses of Theorem 2.3, and Assumption 2 hold. Then, there exist  $C, \gamma$  positive constants such that the following decay rate holds*

$$E(t) \leq Ce^{-\gamma t}E(0), \quad \text{for all } t \geq 0. \quad (2.14)$$

for every regular solution of problem  $(P_2)$  satisfying (2.12), provided the initial data are taken in bounded sets of  $\mathcal{H}$ .

*Remark 2.6.* Observe that geometric control condition is not necessary in Theorem 2.4 regarding  $(P_1)$ . On the other hand, assumption  $5(2a_0b_0)^{-1} \leq \alpha$  is required in the well-posedness of problem  $(P_2)$ . However, this is not a drawback in the method since the constants  $a_0$  and  $b_0$  are used just to localize the dissipative effect according to Assumption 2. On the other hand, to take  $\alpha$  sufficiently large is natural to guarantee the dissipativity of the system. The second one has been considered previously and surely it is much more natural.

### 3. Existence and uniqueness

The well-posedness of the problems  $(P_1)$  and  $(P_2)$  was studied in [1, 2], through Faedo-Galerkin's method, in the bounded domains case. The same results on a compact Riemannian manifold without boundary will follow with the same arguments. For the sake of completeness, we comment some steps of the proof. In what follows, for simplicity, we will denote  $u_t = u'$ . Let us represent by  $\{\omega_\nu\}$  a basis in  $\mathcal{V} \cap H^2(\mathcal{M})$  formed by the eigenfunctions of  $-\Delta$ , by  $V_m$  the subspace of  $\mathcal{V} \cap H^2(\mathcal{M})$  generated by the first  $m$  vectors and by

$$\psi_m(t) = \sum_{i=1}^m g_{im}(t)\omega_i, \quad \phi_m(t) = \sum_{i=1}^m h_{im}(t)\omega_i,$$

where  $\{\psi_m(t), \phi_m(t)\}$  is the solution to the following approximate problem

$$\left\{ \begin{array}{l} (\psi'_m(t), u) + i(\nabla\psi_m(t), \nabla u) + \alpha(b(x)(1 - \Delta)^{\frac{1}{2}}b(x)\psi_m(t), u) \\ \qquad \qquad \qquad = -i(\phi_m(t)\psi_m(t)\chi_\omega, u), \quad \forall u \in V_m, \\ (\phi''_m(t), v) + (\nabla\phi_m(t), \nabla v) + (a(x)\phi'_m(t), v) \\ \qquad \qquad \qquad = (|\psi_m(t)|^2\chi_\omega, v), \quad \forall v \in V_m, \\ \psi_m(0) = \psi_{0m} \rightarrow \psi_0, \quad \phi_m(0) = \phi_{0m} \rightarrow \phi_0 \quad \text{in } \mathcal{V} \cap H^2(\mathcal{M}), \\ \phi'_m(0) = \phi_{1m} \rightarrow \phi_1 \quad \text{in } \mathcal{V}. \end{array} \right. \quad (3.1)$$

regarding problem  $(P_1)$ . On the other point view, we consider the solution to the following approximate problem

$$\left\{ \begin{array}{l} (\psi'_m(t), u) + i(\nabla\psi_m(t), \nabla u) + \alpha(b(x)|\psi_m(t)|^2\psi_m(t), u) \\ \qquad \qquad \qquad + \alpha(b(x)\psi_m(t), u) = -i(\phi_m(t)\psi_m(t)\chi_\omega, u), \quad \forall u \in V_m, \\ (\phi''_m(t), v) + (\nabla\phi_m(t), \nabla v) + (a(x)\phi'_m(t), v) \\ \qquad \qquad \qquad = (|\psi_m(t)|^2\chi_\omega, v), \quad \forall v \in V_m, \\ \psi_m(0) = \psi_{0m} \rightarrow \psi_0, \quad \phi_m(0) = \phi_{0m} \rightarrow \phi_0 \quad \text{in } \mathcal{V} \cap H^2(\mathcal{M}), \\ \phi'_m(0) = \phi_{1m} \rightarrow \phi_1 \quad \text{in } \mathcal{V}. \end{array} \right. \quad (3.2)$$

regarding problem  $(P_2)$

The approximate systems (3.1) and (3.2) are finite systems of ordinary differential equations which has a solution in  $[0, t_m[$ . The extension of the solution to the whole interval  $[0, T]$ , for all  $T > 0$ , is a consequence of the first a priori estimate. The proof of Theorem 2.3 is divided into three steps. In Step 1 solution of the approximate problem. In Step 2 a priori estimates for  $\{\psi_m(t), \phi_m(t)\}$ . In Step 3 passage fo limits.

The proof follows the same basic steps as the one of [1] (see Theorem 2.1) and [2] (see Theorem 2.2) since Sobolev's immersions for compact Riemannian manifolds hold (see [9]). Therefore, we will omit the details of the proof.

*Uniqueness:* Fix  $j = 1$  or  $2$ . Let  $\{\psi_1, \phi_1\}$  and  $\{\psi_2, \phi_2\}$  solutions do problem  $(P_j)$ . Then, the uniqueness follows defining  $z = \psi_1 - \psi_2$  and  $w = \phi_1 - \phi_2$  and repeating verbatim the same arguments already used in the first estimates.

#### 4. Definition of effective dissipation region $\omega$ and construction of multiplier $f$

In the course of this work, we will strongly use a regular function  $f : \mathcal{M} \rightarrow \mathbb{R}$  that satisfies certain properties in a subset  $V$  with regular boundary such that  $meas(V) \geq meas(\mathcal{M}) - \epsilon$ . The function existence will define the multiplier, which is also crucial for Unique Continuation Principle applications of the Schrödinger and Wave equations. The subset  $\omega \subset \mathcal{M}$  where dissipation is effective satisfies

$$\omega \supset \supset (\mathcal{M} \setminus V).$$

In this section, we construct the multiplier  $f : \mathcal{M} \rightarrow \mathbb{R}$  and we find precisely region  $V$ . This multiplier will satisfy specific conditions, in view of

problems  $(P_1)$  and  $(P_2)$ . We based the construction following the ideas in [14] and [16].

We omit the metric  $g$  when there is no possibility of misunderstandings. Now, we are in conditions to state and prove the main result of this section. In what follows, the objects on  $\mathcal{M}$  will be denoted by usual symbols and the objects on the tangent space will be denoted by calligraphic symbols.

**Theorem 4.1.** *Let  $(\mathcal{M}^n, g)$  be a compact Riemannian manifold without boundary of dimension  $n$  and Riemannian metric  $g$  and fix  $\epsilon > 0$ . Therefore, there are  $C_0, C_1$  positive constants and a regular function  $f : \mathcal{M} \rightarrow \mathbb{R}$  and a open subset  $V \subset \mathcal{M}$ , with regular boundary  $\partial V$ , satisfying:*

- (i)  $\text{meas}(V) \geq \text{meas}(\mathcal{M}) - \epsilon$ .
- (ii)  $\Delta f = C_0 > 0$  on  $V$ .
- (iii)  $\nabla^2 f(v, v) \geq c|v|^2$ , for every vector  $v$  on a tangent space of  $V$ .
- (iv)  $|\nabla f| \geq C_1 > 0$  on  $V$ .
- (v)  $|\nabla f|$  is bounded on  $\mathcal{M}$ .
- (vi) There is a positive function  $\xi : \mathcal{M} \rightarrow \mathbb{R}$ ,  $\nabla \xi(x) = 0, \forall x \in V$ , such that

$$C_2 \int_0^T \int_V \varphi_t^2 + |\nabla \varphi|^2 d\mathcal{M}dt \leq \int_0^T \int_V \left( \frac{\Delta f}{2} - \xi \right) \varphi_t^2 d\mathcal{M}dt + \int_0^T \int_V \nabla^2 f(\nabla \varphi, \nabla \varphi) + \left( \xi - \frac{\Delta f}{2} \right) |\nabla \varphi|^2 d\mathcal{M}dt, \quad (4.1)$$

is satisfied for some positive constant  $C_2 > 0$  and all admissible function  $\varphi$ .

We divided the Theorem 4.1 proof into several steps. First, we show some preliminary constructions.

**4.0.1. Some formulas in a coordinate system.** Let  $\mathcal{M}$  be a Riemannian manifold,  $p \in \mathcal{M}$  and let  $U \subset \mathcal{M}$  a neighborhood of  $p$ . Let  $(x_1, \dots, x_n)$  be a coordinate system on  $U$ . Denote the components of the Riemannian metric concerning this coordinate system by  $g_{ij}$ . Suppose that  $g_{ij}(p) = \delta_{ij}$ . We denote the components of the inverse matrix of  $(g_{ij})$  by  $g^{ij}$ . Thus, if  $f \in C^1(U)$ , then the gradient of  $f$  is given by

$$(\nabla f)_i = \sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x_j}. \quad (4.2)$$

The Hessian of  $f \in C^2(U)$  is given by

$$(\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^n \frac{\partial f}{\partial x_k} \Gamma_{ij}^k \quad (4.3)$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols. We denote the Hessian of  $f$  by  $\nabla^2 f$ . Finally the Laplacian of  $f$  is the trace of the Hessian with respect to the metric  $g$  and it is given by

$$\Delta f = \sum_{i,j=1}^n \left( \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^n \frac{\partial f}{\partial x_k} \Gamma_{ij}^k \right) g^{ij}. \quad (4.4)$$

Observe that  $\Gamma_{ij}^k(p) = 0$  because we are dealing with normal coordinates.

#### 4.1. Construction of the local multiplier

**Lemma 4.2.** *Let  $\mathcal{M}$  be a Riemannian manifold and consider  $p \in \mathcal{M}$ . There exist a neighborhood  $\mathbb{V}_p$  of  $p$  and a smooth function  $f : \mathbb{V}_p \rightarrow \mathbb{R}$  such that  $|\nabla f| \geq C > 0$ ,  $\nabla^2 f$  is positive definite,  $\Delta f$  is a (positive) constant and, the inequality (4.1) on  $\mathbb{V}_p$  holds.*

*Proof:* The main idea proof is combine Lemma 5.1 in [14] and Lemma 6.1 in [16]. Indeed, from Lemma 5.1 in [14], for any  $p \in \mathcal{M}$  there is a neighborhood  $V_p$  of  $p$ , and smooth function  $f : V_p \rightarrow \mathbb{R}$  such that  $|\nabla f| \geq C_1 > 0$ ,  $\nabla^2 f$  is positive-definite and  $\Delta f$  is a positive constant. That is, there are  $\gamma_1, \gamma_2$  positive constants such that

$$\begin{aligned} \Delta f(q) &= \gamma_1 \\ \nabla^2 f(q)(v, v) &\geq \gamma_2 |v|^2, \quad \forall v \in T_q \mathcal{M} \end{aligned}$$

for all  $q \in V_p$ .

Note that  $\gamma_2 \geq 1$ . In fact, we have  $f \approx f_0$  in the  $C^2$ -norm sense ([14], see Subsection 5.2), where

$$f_0(x) = \sum_{i=1}^n x_i^2 + 3x_1 - 1$$

Thus,

$$\nabla^2 f_0(p)(v, v) = 2g_E(p)(v, v) \geq \gamma_2 g_E(p)(v, v) \quad \forall v \in T_p \mathcal{M}$$

where  $1 \leq \gamma_2 < 2$ , and  $g_E(\cdot)$  Euclidian metric. Therefore, by continuity, the choice of  $V_p$  is such that

$$\nabla^2 f(q)(v, v) \geq \gamma_2 g(q)(v, v), \quad \forall v \in T_q \mathcal{M}$$

for all  $q \in V_p$ .

It remains to verify (4.1) on a neighborhood of  $p$ . That is, to find a neighborhood  $\mathbb{V}_p$  of  $p$ , eventually  $\mathbb{V}_p \subset V_p$ , such that

$$\begin{aligned} C_2 \int_0^T \int_{\mathbb{V}_p} \varphi_t^2 + |\nabla \varphi|^2 d\mathcal{M} dt &\leq \int_0^T \int_{\mathbb{V}_p} \left( \frac{\Delta f}{2} - \xi \right) \varphi_t^2 d\mathcal{M} dt \\ &+ \int_0^T \int_{\mathbb{V}_p} \nabla^2 f(\nabla \varphi, \nabla \varphi) + \left( \xi - \frac{\Delta f}{2} \right) |\nabla \varphi|^2 d\mathcal{M} dt, \end{aligned} \quad (4.5)$$

with  $\xi$  a positive function and  $C_2 > 0$ .

Following Lemma 6.1 in [16] consider

$$\xi = \frac{\gamma_1}{2} - \frac{1}{2} \quad \text{and} \quad C_2 = \frac{1}{4}. \quad (4.6)$$

Thus, let  $\kappa$  be the smooth field of symmetric bilinear form on  $V_p$  given by

$$\kappa(p)(X, Y) = \nabla^2 f(p)(X, Y) + \left( \frac{\gamma_1}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) g(p)(X, Y)$$

where  $X$  and  $Y$  are vector fields on  $V_p$ . Since  $\nabla^2 f(p)(X, X) \geq \gamma_2 g(p)(X, X)$  and  $\gamma_2 \geq 1$ , we obtain

$$\kappa(p)(X, X) \geq \left( \gamma_2 - \frac{3}{4} \right) g(p)(X, X) \geq \frac{1}{4} g(p)(X, X).$$

Therefore  $\kappa$  is positive definite bilinear form on  $p$ . Then, there exist a neighborhood  $\hat{U}_p \subset V_p$  of  $p$  such that  $\kappa$  is positive definite. Consequently

$$\int_0^T \int_{\hat{U}_p} \nabla^2 f(\nabla\varphi, \nabla\varphi) + \left( \frac{\gamma_1}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) |\nabla\varphi|^2 d\mathcal{M}dt \geq 0. \quad (4.7)$$

On the other hand, observe that

$$\left( \frac{\Delta f(p)}{2} - \frac{\gamma_1}{2} + \frac{1}{4} \right) = \frac{1}{4} > 0.$$

So, there is a neighborhood  $\tilde{U}_p \subset V_p$  of  $p$  such that

$$\int_0^T \int_{\tilde{U}_p} \left( \frac{\Delta f}{2} - \frac{\gamma_1}{2} + \frac{1}{4} \right) \varphi_t^2 d\mathcal{M}dt \geq 0. \quad (4.8)$$

To the end, define  $\mathbb{V}_p = \hat{U}_p \cap \tilde{U}_p$ . So, from (4.6), (4.7) and (4.8),

$$\begin{aligned} \int_0^T \int_{\mathbb{V}_p} \nabla^2 f(\nabla\varphi, \nabla\varphi) + \left( \xi - \frac{\Delta f}{2} - C_2 \right) |\nabla\varphi|^2 d\mathcal{M}dt \\ + \int_0^T \int_{\mathbb{V}_p} \left( \frac{\Delta f}{2} - \xi - C_2 \right) \varphi_t^2 d\mathcal{M}dt \geq 0 \end{aligned}$$

Therefore  $f|_{\mathbb{V}_p} : \mathbb{V}_p \rightarrow \mathbb{R}$  satisfies all conditions of Lemma 4.2.

#### 4.2. Construction of the multiplier in a wide domain

The extension of the local construction to a wide domain follows using arguments on the compactness of  $\mathcal{M}$ , in local construction of multiplier, and the existence of mollifier smoothing. These steps can be found in [14] (see Theorem 5.5) and [16] (see Theorem 6.6). Therefore, fix  $\varepsilon > 0$  there exist a smooth functions  $\xi, f : \mathcal{M} \rightarrow \mathbb{R}$  and an open subset  $V \subset \mathcal{M}$  with smooth boundary satisfying (i)-(vi). Following steps, we get  $|\nabla f|$  is continuous on compact  $\mathcal{M}$  then, (v) holds.

It is worth mentioning other papers in connection with the extension of the local construction to a wide domain, [12, 13, 17, 18].

*Remark 4.3.* It is important to note if  $\mathcal{M}$  has regions with negative sectional curvature and satisfies  $k_1 \leq \sec_g \leq k_2 < 0$ . Then, it is possible to find open subsets, precisely, within those regions such that can still be free of dissipative effect, further reducing the  $\omega$  region where the dissipative effect is effective (see Figure 2).

This fact because in these regions it is possible to obtain limitations on the Hessian of the multiplier  $f$  to get the inequality 4.1. Furthermore, the class of manifolds called Warped Products some examples of this property. For

more details of the proof and examples see [13] (Theorem 5.1 and subsection 5.2.2).

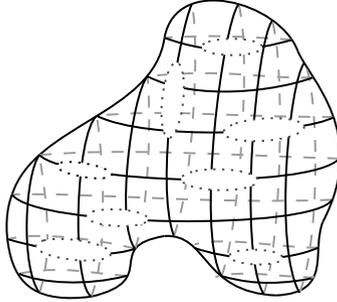


FIGURE 2. Riemannian manifold with open subsets satisfying curvature conditions, which allow to reduce the effective damping region  $\omega$  (in black).

## 5. Uniform Decay Rates

In this section we prove the results stated in Section 2

### 5.1. Problem $(P_1)$

From Theorem 2.3, we consider regular solutions  $(\psi(t), \phi(t), \phi_t(t)) \in \mathcal{H}$  of the problem  $(P_1)$ . Denote  $\psi_t = \psi'$ , and  $\phi_t = \phi'$  to simplify notation. Moreover, we consider bounded initial data, that is,  $\|(\psi_0, \phi_0, \phi_1)\|_{\mathcal{H}} \leq L$ , where  $L > 0$ . Thus, the energy identity (2.10) holds,

$$\begin{aligned} E'(t) + \alpha \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} \\ + \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} = \int_{\omega} |\psi|^2 \phi' d\mathcal{M}. \end{aligned} \quad (5.1)$$

Next, we will analyze the last term on the RHS of (5.1). From Assumption 1, inequality of Gagliardo-Nirenberg and making use of the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left| \int_{\omega} |\psi|^2 \phi' d\mathcal{M} \right| &\leq \frac{a_0^{-1} b_0^{-4}}{2} \int_{\mathcal{M}} |b(x) \psi|^4 d\mathcal{M} + \frac{1}{2} \int_{\Omega} a(x) |\phi'|^2 d\mathcal{M} \\ &\leq \frac{a_0^{-1} b_0^{-4}}{2} \|b(\cdot) \psi\|_{L^4(\mathcal{M})}^2 \|b(\cdot) \psi\|_{L^4(\mathcal{M})}^2 + \frac{1}{2} \int_{\mathcal{M}} a(x) |\phi'|^2 dx \\ &\leq \frac{a_0^{-1} b_0^{-4} c \|b\|_{\infty} \|\psi\|_2 \|\nabla \psi\|_2}{2} \|(1 - \Delta)^{\frac{1}{4}} b(\cdot) \psi\|_2^2 + \frac{1}{2} \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \\ &\leq \frac{a_0^{-1} b_0^{-4} d}{2} \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} + \frac{1}{2} \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \end{aligned} \quad (5.2)$$

where  $d := c\|b\|_\infty L$

Combining (5.1) and (5.2) and considering  $\alpha$  large enough, more specifically,  $\beta := \alpha - \frac{\alpha^{-1}b_0^{-4}d}{2} > 0$ , we obtain

$$E'(t) \leq -k \left[ \int_{\mathcal{M}} a(x)|\phi'|^2 d\mathcal{M} + \int_{\mathcal{M}} |(1-\Delta)^{\frac{1}{4}}b(x)\psi|^2 d\mathcal{M} \right] \quad (5.3)$$

where  $k = \min\{\frac{1}{2}, \beta\}$ .

*Remark 5.1.* From (5.3) we deduce two facts: (I) the map  $t \in (0, \infty) \mapsto E(t)$  is non increasing, and, in addition, (II) we have the following inequality of the energy

$$\begin{aligned} E(t_2) - E(t_1) \\ \leq -k \int_{t_1}^{t_2} \left[ \int_{\mathcal{M}} a(x)|\phi'|^2 d\mathcal{M} + \int_{\mathcal{M}} |(1-\Delta)^{\frac{1}{4}}b(x)\psi|^2 d\mathcal{M} \right] dt, \end{aligned} \quad (5.4)$$

for  $0 \leq t_1 \leq t_2 < +\infty$ , which is crucial in the proof. We observe that in order to transform the energy  $E(t)$  in a non increasing function we could have considered  $d$  small enough instead of taking  $\alpha$  sufficiently large, which would imply to take the initial data sufficiently small. Well, in any case, some kind of tribute must be paid in order to obtain uniform decay rates of the energy to the present system.

In order to prove Theorem 2.4 we proceed in several steps.

**Step 1.** Considering the multiplier  $f : \mathcal{M} \rightarrow \mathbb{R}$ , and function  $\xi \in W^{1,\infty}(\mathcal{M})$  as stated in the previous section, we get

$$\begin{aligned} C_2 \int_0^T E(t) dt &\leq \frac{|\chi|}{2} + \frac{C_5}{2} [E(0) - E(T)] + 4\varepsilon \int_0^T E(t) dt \\ &+ \frac{C_2}{2} \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt + \frac{\|\xi\|_\infty}{8\varepsilon} \int_0^T \int_{\omega} |\phi|^2 d\mathcal{M} dt \\ &+ \frac{C^*}{2} \int_0^T \int_{\mathcal{M} \setminus V} |\nabla \phi|^2 d\mathcal{M} dt, \end{aligned} \quad (5.5)$$

where  $C_2, C_5, C^*$  and  $\varepsilon$  are positive constants and

$$\chi = \left[ \int_{\mathcal{M}} \left( \frac{|\psi|^2}{2} + \phi' \langle \nabla f, \nabla \phi \rangle + \xi \phi \left( \phi' + \frac{\phi a}{2} \right) \right) d\mathcal{M} \right]_0^T$$

In fact, multiplying the first equation of problem  $(P_1)$  by  $\bar{\psi}$  and the second equation by  $\langle q, \nabla \phi \rangle_{\mathbf{g}}$ , where  $q \in (W^{1,\infty}(\mathcal{M}))^n$  (for simplicity, we have omitted the notation of the metric  $\mathbf{g}$ ), using the identity (2.4) and following

Proposition 4.1 in [16] we deduce the following identity:

$$\begin{aligned}
& \left[ \int_{\mathcal{M}} \left( \frac{|\psi|^2}{2} + \phi' \langle q, \nabla \phi \rangle \right) d\mathcal{M} \right]_0^T + \frac{1}{2} \int_0^T \int_{\mathcal{M}} (\operatorname{div} q) [|\phi'|^2 - |\nabla \phi|^2] d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \nabla q \langle \nabla \phi, \nabla \phi \rangle d\mathcal{M} dt - \int_0^T \int_{\omega} |\psi|^2 \langle q, \nabla \phi \rangle d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} a(x) \phi' \langle q, \nabla \phi \rangle d\mathcal{M} dt \\
& + \alpha \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} dt = 0. \tag{5.6}
\end{aligned}$$

Employing (5.6) with  $f : \mathcal{M} \rightarrow \mathbb{R}$  of class  $C^\infty$  as in Theorem 4.1 and replacing  $q = \nabla f$  in (5.6), follows

$$\begin{aligned}
& \left[ \int_{\mathcal{M}} \left( \frac{|\psi|^2}{2} + \phi' \langle \nabla f, \nabla \phi \rangle \right) d\mathcal{M} \right]_0^T + \frac{1}{2} \int_0^T \int_{\mathcal{M}} \Delta f [|\phi'|^2 - |\nabla \phi|^2] d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \nabla^2 f \langle \nabla \phi, \nabla \phi \rangle d\mathcal{M} dt - \int_0^T \int_{\omega} |\psi|^2 \langle \nabla f, \nabla \phi \rangle d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} a(x) \phi' \langle \nabla f, \nabla \phi \rangle d\mathcal{M} dt \\
& + \alpha \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} dt = 0. \tag{5.7}
\end{aligned}$$

Fix  $\xi \in W^{1,\infty}(\mathcal{M})$  and multiply the second equation of problem  $(P_1)$  by  $\xi \phi$ , then we have the following identity:

$$\begin{aligned}
& \left[ \int_{\mathcal{M}} \phi \xi \left( \phi' + \frac{\phi a}{2} \right) d\mathcal{M} \right]_0^T = \int_0^T \int_{\omega} |\psi|^2 \xi \phi d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \xi [|\phi'|^2 - |\nabla \phi|^2] d\mathcal{M} dt - \int_0^T \int_{\mathcal{M}} \phi \langle \nabla \phi \cdot \nabla \xi \rangle d\mathcal{M} dt. \tag{5.8}
\end{aligned}$$

Combining (5.8) with (5.7) we have

$$\begin{aligned}
& \chi + \int_0^T \int_{\mathcal{M}} \left( \frac{\Delta f}{2} - \xi \right) |\phi'|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} \left( \xi - \frac{\Delta f}{2} \right) |\nabla \phi|^2 d\mathcal{M} dt \\
& - \int_0^T \int_{\omega} |\psi|^2 \langle \nabla f, \nabla \phi \rangle d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} a(x) \phi' \langle \nabla f, \nabla \phi \rangle d\mathcal{M} dt \\
& + \alpha \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 dx dt - \int_0^T \int_{\omega} \xi |\psi|^2 \phi d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \nabla^2 f \langle \nabla \phi, \nabla \phi \rangle d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} \phi \langle \nabla \phi, \nabla \xi \rangle d\mathcal{M} dt = 0. \tag{5.9}
\end{aligned}$$

This is the precise moment when the inequality *(vi)* of Theorem 4.1 is essential. That is, making use of such property in (5.9) and adding the term

$C_2 \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt$  on both sides of the inequality, we obtain

$$\begin{aligned}
2C_2 \int_0^T E(t) dt &\leq C^* \int_0^T \int_{\mathcal{M} \setminus V} [|\phi'|^2 + |\nabla\phi|^2] d\mathcal{M} dt \\
&+ C_2 \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt + |\chi| + \int_0^T \int_{\omega} |\psi|^2 |\nabla f| |\nabla\phi| d\mathcal{M} dt \\
&+ \int_0^T \int_{\mathcal{M}} a(x) |\phi'| |\nabla f| |\nabla\phi| d\mathcal{M} dt + \alpha \int_0^T \int_{\mathcal{M}} |(1-\Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} dt \\
&+ \int_0^T \int_{\omega} \xi |\psi|^2 |\phi| d\mathcal{M} dt + \int_0^T \int_{\mathcal{M} \setminus V} |\phi| |\nabla\phi| |\nabla\xi| d\mathcal{M} dt, \tag{5.10}
\end{aligned}$$

where  $C^* > 0$  is a constant that depends on  $C_2$ ,  $\xi$  and  $f$ .

Now, we are going to estimate some terms in (5.10). In the estimate for  $I_1$ ,  $I_2$ ,  $I_3$ , and  $I_4$ , we make use integral Cauchy-Schwarz inequality, the numerical Hölder inequality, Poincaré inequality, (5.4), and considering the inequality  $ab \leq \frac{1}{4\varepsilon} a^2 + \varepsilon b^2$ . Moreover, from Theorem 4.1 item (v), denote

$$C_3 := \max_{x \in \mathcal{M}} |\nabla f(x)|.$$

*Estimate for  $I_1 := \int_0^T \int_{\omega} |\psi|^2 |\nabla f| |\nabla\phi| d\mathcal{M} dt$ .*

$$\begin{aligned}
|I_1| &\leq \frac{C_3^2}{4\varepsilon} \int_0^T \int_{\mathcal{M}} |\psi|^4 d\mathcal{M} dt + 2\varepsilon \int_0^T E(t) dt \\
&\leq \frac{C_3^2 d}{4\varepsilon b_0^4 k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) dt. \tag{5.11}
\end{aligned}$$

*Estimate for  $I_2 := \int_0^T \int_{\omega} |\psi|^2 \phi d\mathcal{M} dt$ .*

$$|I_2| \leq \frac{\lambda d}{4\varepsilon b_0^4 k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) dt. \tag{5.12}$$

*Estimate for  $I_3 := -\alpha \int_0^T \int_{\mathcal{M}} |(1-\Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} dt$ .*

$$|I_3| \leq \frac{\alpha}{k} [E(0) - E(T)]. \tag{5.13}$$

*Estimate for  $I_4 := -\int_0^T \int_{\mathcal{M}} a(x) |\phi'| |\nabla f| |\nabla\phi| d\mathcal{M} dt$ .*

$$|I_4| \leq \frac{\|a\|_{\infty} C_3^2}{4\varepsilon k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) dt. \tag{5.14}$$

Combining (5.10)-(5.14) we get

$$\begin{aligned}
2C_2 \int_0^T E(t) dt &\leq C^* \int_0^T \int_{\mathcal{M} \setminus V} |\phi'|^2 + |\nabla \phi|^2 d\mathcal{M} dt \\
&+ C_2 \int_0^T \int_{\mathcal{M}} |\psi|^2 dx dt + |\chi| + C_4 [E(0) - E(T)] \quad (5.15) \\
&+ \int_0^T \int_{\mathcal{M} \setminus V} |\phi| |\nabla \phi| |\nabla \xi| d\mathcal{M} dt + 6\varepsilon \int_0^T E(t) dt,
\end{aligned}$$

where  $C_4 = \left[ \frac{C_3^2 d}{4\varepsilon b_0^4 k} + \frac{\|a\|_\infty C_3^2}{4\varepsilon k} + \frac{\alpha}{k} + \frac{\lambda d}{4\varepsilon b_0^4 k} \right]$ . Note that

$$\begin{aligned}
\int_0^T \int_{\mathcal{M} \setminus V} |\phi| |\nabla \phi| |\nabla \xi| d\mathcal{M} dt &\leq \|\xi\|_\infty \int_0^T \int_{\mathcal{M} \setminus V} \frac{1}{4\varepsilon} |\phi|^2 + \varepsilon |\nabla \phi|^2 d\mathcal{M} dt \\
&\leq \frac{\|\xi\|_\infty}{4\varepsilon} \int_0^T \int_{\omega} |\phi|^2 d\mathcal{M} dt + 2\varepsilon \int_0^T E(t) dt. \quad (5.16)
\end{aligned}$$

From (5.15) and (5.16) we conclude

$$\begin{aligned}
2C_2 \int_0^T E(t) dt &\leq C^* \int_0^T \int_{\mathcal{M} \setminus V} |\phi'|^2 + |\nabla \phi|^2 d\mathcal{M} dt \\
&+ |\chi| + C_2 \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt \\
&+ C_4 [E(0) - E(T)] + 8\varepsilon \int_0^T \int_{\mathcal{M}} E(t) dt \\
&+ \frac{\|\xi\|_\infty}{4\varepsilon} \int_0^T \int_{\omega} |\phi|^2 d\mathcal{M} dt. \quad (5.17)
\end{aligned}$$

*Estimate for  $I_5 := C^* \int_0^T \int_{\mathcal{M} \setminus V} |\phi'|^2 d\mathcal{M} dt$ .* Considering (5.4) we obtain

$$|I_5| \leq \frac{C^*}{a_0 k} [E(0) - E(T)]. \quad (5.18)$$

Finally, from (5.17) and (5.18) we conclude (5.5) where  $C_5 := \left\{ C_4 + \frac{C^*}{a_0 k} \right\}$ .

The key idea in next step is the construction of a ‘‘cut-off’’ function on a neighborhood of  $\mathcal{M} \setminus V$  to estimate  $\int_0^T \int_{\mathcal{M} \setminus V} |\nabla \phi|^2 d\mathcal{M} dt$ . Following [16] (see p.955) we obtain a function  $\eta \in W^{1,\infty}(\mathcal{M})$  satisfying

$$\begin{cases} \eta = 1 \text{ a.e. in } \hat{\omega}; \\ \eta = 0 \text{ a. e. in } \mathcal{M} \setminus \omega; \\ 0 \leq \eta \leq 1 \text{ a.e. in } \mathcal{M} \text{ and } \frac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega). \end{cases} \quad (5.19)$$

where  $\hat{\omega}$  is a region on  $\mathcal{M}$  such that  $\mathcal{M} \setminus V \subset \subset \hat{\omega} \subset \subset \omega$ .

**Step 2.** Under above considerations, we estimate

$$\begin{aligned} \frac{C^*}{2} \int_0^T \int_{\mathcal{M} \setminus V} |\nabla \phi|^2 d\mathcal{M} dt &\leq C^* |Z| + C_6 [E(0) - E(T)] \\ + 2\varepsilon \int_0^T E(t) dt + \frac{C^*}{2} \left\| \frac{|\nabla \eta|^2}{\eta} \right\|_{L^\infty(\omega)} \int_0^T \int_\omega |\phi|^2 d\mathcal{M} dt. \end{aligned} \quad (5.20)$$

where  $\varepsilon, C_6 > 0$  and

$$Z := \left[ \int_\omega \phi \eta \left( \phi' + \frac{\phi a}{2} \right) d\mathcal{M} \right]_0^T \quad (5.21)$$

In fact, taking  $\xi = \eta$  in identity (5.8) and multiplying by  $C^*$ , it results in

$$\begin{aligned} C^* Z &= C^* \int_0^T \int_\omega |\psi|^2 \eta \phi d\mathcal{M} dt + C^* \int_0^T \int_\omega \eta [|\phi'|^2 - |\nabla \phi|^2] d\mathcal{M} dt \\ &\quad - C^* \int_0^T \int_\omega \phi (\nabla \phi \cdot \nabla \eta) d\mathcal{M} dt \end{aligned} \quad (5.22)$$

Next, let us analyze the terms on the RHS of (5.22).

*Estimate for  $L_1 := C^* \int_0^T \int_\omega |\psi|^2 \eta \phi d\mathcal{M} dt$ .* Analogously to the above estimates, it follows that

$$|L_1| \leq \frac{\lambda d (C^*)^2}{4\varepsilon b_0^4 k} [E(0) - E(T)] + 2\varepsilon \int_0^T E(t) dt. \quad (5.23)$$

*Estimate for  $L_2 := C^* \int_0^T \int_{\mathcal{M}} \eta |\phi'|^2 d\mathcal{M} dt$ .*

$$|L_2| \leq \frac{C^*}{a_0 k} [E(0) - E(T)]. \quad (5.24)$$

*Estimate for  $L_3 := -C^* \int_0^T \int_\omega \phi (\nabla \phi \cdot \nabla \eta) d\mathcal{M} dt$ .* From (5.19), we can write

$$\begin{aligned} |L_3| &\leq \frac{C^*}{2} \int_0^T \int_\omega \eta |\nabla \phi|^2 d\mathcal{M} \\ &\quad + \frac{C^*}{2} \left\| \frac{|\nabla \eta|^2}{\eta} \right\|_{L^\infty(\omega)} \int_0^T \int_\omega |\phi|^2 d\mathcal{M} dt. \end{aligned} \quad (5.25)$$

From (5.22)-(5.25) we have

$$\begin{aligned} C^* \int_0^T \int_\omega \eta |\nabla \phi|^2 d\mathcal{M} dt &\leq C^* |Z| + C_6 [E(0) - E(T)] \\ &\quad + 2\varepsilon \int_0^T E(t) dt + \frac{C^*}{2} \int_0^T \int_\omega \eta |\nabla \phi|^2 d\mathcal{M} \\ &\quad + \frac{C^*}{2} \left\| \frac{|\nabla \eta|^2}{\eta} \right\|_{L^\infty(\omega)} \int_0^T \int_\omega |\phi|^2 d\mathcal{M} dt, \end{aligned} \quad (5.26)$$

where  $C_6 := \left[ \frac{\lambda d(C^*)^2}{4\varepsilon b_0^4} + \frac{C^*}{a_0 k} \right]$ . Therefore, from (5.26) and having in mind that  $\eta \equiv 1$  on  $\hat{\omega} \supset \supset M \setminus V$  we conclude Step 2, that is, (5.20) holds.

Before Step 3, combine (5.20) from Step 1 and (5.5) from Step 2, then

$$C_2 \int_0^T E(t) dt \leq \frac{|\chi|}{2} + C_7[E(0) - E(T)] + 6\varepsilon \int_0^T E(t) dt \\ + C^*|Z| + C_8 \int_0^T \int_{\omega} |\phi|^2 d\mathcal{M} dt + \frac{C_2}{2} \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt, \quad (5.27)$$

where  $C_7 := \left[ \frac{C_5}{2} + C_6 \right]$  and  $C_8 := \left[ \frac{C^*}{2} \left\| \frac{|\nabla \eta|^2}{\eta} \right\|_{L^\infty(\omega)} + \frac{\|\xi\|_\infty}{8\varepsilon} \right]$ .

Also note that,

$$\frac{|\chi|}{2} + C^*|Z| \leq C_9[E(0) + E(T)]$$

where  $C_9$  is a positive constant. Taking  $\varepsilon$  small enough such that  $C_{10} := C_2 - 6\varepsilon > 0$  we obtain

$$C_{10} \int_0^T E(t) dt \leq 2C_{11} E(0) + C_{12} \left( \int_0^T \int_{\mathcal{M}} |\phi|^2 + |\psi|^2 d\mathcal{M} dt \right), \quad (5.28)$$

where  $C_{11} := \max\{C_7, C_9\}$  and  $C_{12} := \max\{C_8, C_2/2\}$ . Therefore,

$$\int_0^T E(t) dt \leq C E(0) + C \left[ \int_0^T \int_{\mathcal{M}} |\phi|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt \right] \quad (5.29)$$

where  $C$  is a positive constant which depends on

$$\max_{x \in \mathcal{M}} |\xi(x)|, \max_{x \in \mathcal{M}} |\nabla f(x)|, \|a\|_\infty, \|b\|_\infty, \lambda, k, a_0, b_0, d.$$

*Step 3.* Let  $T_0 > 0$  considered sufficiently large for our purpose. We will prove the following lemma:

**Lemma 5.2.** *For all  $T > T_0$  there exists a positive constant  $C = C(T)$  such that if  $\{\psi, \phi\}$  is the regular solution of  $(P_1)$  with initial data  $(\psi_0, \phi_0, \phi_1) \in \mathcal{H}$  we have*

$$\int_0^T \int_{\mathcal{M}} |\phi|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi|^2 d\mathcal{M} dt \quad (5.30) \\ \leq C(T) \left[ \int_0^T \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi|^2 d\mathcal{M} dt \right].$$

*Proof.* We argue by contradiction. Suppose that (5.30) is not verified and let  $(\psi_k(0), \phi_k(0), \phi'_k(0)) \in \mathcal{H}$  be a sequence of initial data where the corresponding solutions  $\{\psi_k, \phi_k\}$  with  $E_k(0)$  uniformly bounded in  $k$ , verifies

$$\lim_{k \rightarrow +\infty} \frac{\int_0^T \int_{\mathcal{M}} |\phi_k|^2 dx \mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt}{\int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt} \\ = +\infty. \quad (5.31)$$

Since  $E_k(t)$  is non-increasing and  $E_k(0)$  remains bounded then, we obtain a subsequence, still denoted by  $\{\psi_k, \phi_k\}$  which verifies

$$\psi_k \rightharpoonup \psi \text{ weak star in } L^\infty(0, T; L^2(\mathcal{M})), \quad (5.32)$$

$$\phi_k \rightharpoonup \phi \text{ weak star in } L^\infty(0, T; \mathcal{V}), \quad (5.33)$$

$$\phi'_k \rightharpoonup \phi' \text{ weak star in } L^\infty(0, T; L^2(\mathcal{M})). \quad (5.34)$$

We also have, employing compactness results (see Theorem 5.1 in Lions [32]) that

$$\phi_k \rightarrow \phi \text{ strongly in } L^2(0, T; L^2(\mathcal{M})). \quad (5.35)$$

Now, from (5.31), (5.32) and (5.33) we deduce that

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt = 0, \quad (5.36)$$

$$\lim_{k \rightarrow +\infty} \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt = 0, \quad (5.37)$$

On the other hand, from Assumption 1, namely,  $b(x) \geq b_0 > 0$  in  $\omega$ , taking (5.37) into account and considering  $D[(-\Delta)^{\frac{1}{4}}] \equiv H^{\frac{1}{2}}(\mathcal{M}) \hookrightarrow L^4(\mathcal{M})$  for  $n = 2$ , we deduce

$$\lim_{k \rightarrow \infty} \int_0^T \int_{\omega} |\psi_k|^4 d\mathcal{M} dt = 0. \quad (5.38)$$

and

$$\lim_{k \rightarrow \infty} \int_0^T \int_{\omega} |\psi_k|^2 d\mathcal{M} dt = 0. \quad (5.39)$$

From now on let us focus our attention on the coupled wave equation

$$\phi''_k - \Delta \phi_k + a(x) \phi'_k = |\psi_k|^2 \chi_\omega \text{ in } \mathcal{M} \times (0, T) \quad (5.40)$$

Let us divide our proof in two cases (in what concerns the limit  $\phi$  above):

(a)  $\phi \neq 0$ .

Passing to the limit when  $k \rightarrow +\infty$  in (5.40) taking into account the above convergence, we deduce that

$$\begin{cases} \phi'' - \Delta \phi = 0 \text{ in } \mathcal{M} \times (0, T) \\ \phi' = 0 \text{ a. e. in } \mathcal{M} \times (0, T), \end{cases} \quad (5.41)$$

and for  $\phi' = v$ , we obtain, in the distributional sense that

$$\begin{cases} v'' - \Delta v = 0 \text{ in } \mathcal{M} \times (0, T) \\ v = 0 \text{ a. e. in } \omega \times (0, T). \end{cases} \quad (5.42)$$

From Theorem 5.1 in [16] we conclude that  $v \equiv 0$ , that is,  $\phi' \equiv 0$ . Returning to (5.41) we obtain the following elliptic equation for almost everywhere  $t \in (0, T)$ :

$$\begin{cases} -\Delta \phi = 0 \text{ in } \mathcal{M} \\ \phi' = 0 \text{ in } \omega, \end{cases} \quad (5.43)$$

Multiplying (5.43) by  $\phi$  we deduce that  $\int_{\mathcal{M}} |\nabla \phi|^2 d\mathcal{M} = 0$ , which implies that  $\phi \equiv 0$  in  $\mathcal{V} \hookrightarrow L^2(\mathcal{M})$  a.e.  $t \in (0, T)$ , which is a contradiction.

Now, we consider the other case when

(b)  $\phi \equiv 0$ .

Defining

$$c_k := \left[ \int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt \right]^{1/2} \quad (5.44)$$

$$\hat{\phi}_k = \frac{1}{c_k} \phi_k, \quad \hat{\psi}_k = \frac{1}{c_k} \psi_k, \quad (5.45)$$

we obtain

$$\int_0^T \int_{\mathcal{M}} |\hat{\phi}_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\hat{\psi}_k|^2 d\mathcal{M} dt = 1. \quad (5.46)$$

Besides,

$$\begin{aligned} \hat{E}_k(t) &= \frac{1}{2} \left[ \int_{\mathcal{M}} |\hat{\psi}_k|^2 d\mathcal{M} + \int_{\mathcal{M}} |\hat{\phi}'_k|^2 d\mathcal{M} + \int_{\mathcal{M}} |\nabla \hat{\phi}_k|^2 d\mathcal{M} \right] \\ &= \frac{1}{2c_k^2} \left[ \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} + \int_{\mathcal{M}} |\phi'_k|^2 d\mathcal{M} + \int_{\mathcal{M}} |\nabla \phi_k|^2 d\mathcal{M} \right], \end{aligned}$$

that is,

$$\hat{E}_k(t) = \frac{E_k(t)}{c_k^2}. \quad (5.47)$$

On the other hand, integrating (5.1) over  $(0, T)$ , we deduce

$$\begin{aligned} E_k(T) &= E_k(0) - \alpha \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt \\ &\quad - \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 \phi'_k d\mathcal{M} dt. \end{aligned} \quad (5.48)$$

From the fact that  $E_k(t) \geq E_k(T)$  for all  $t \in [0, T]$  and taking (5.48) into account, we obtain

$$\begin{aligned} \int_0^T E_k(t) dt &\geq T E_k(T) = T E_k(0) - \alpha T \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt \\ &\quad - T \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt + T \int_0^T \int_{\omega} |\psi_k|^2 \phi'_k d\mathcal{M} dt. \end{aligned} \quad (5.49)$$

Combining (5.29), (5.49) and making use of Cauchy-Schwarz inequality and considering the Assumption 1, we infer

$$\begin{aligned} T E_k(0) &\leq [2(\alpha + a_0^{-1} b_0^{-4} d + 3)] T \left\{ \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt \right. \\ &\quad \left. + \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt \right\} \\ &\quad + C E_k(0) + C \int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt + C \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt. \end{aligned}$$

The last inequality yields for a large  $T$ ,

$$\begin{aligned}
 E_k(0) \leq C(T, a_0, b_0, \alpha, d) & \left\{ \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt \right. \\
 & + \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt \\
 & \left. + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt \right\} \quad (5.50)
 \end{aligned}$$

Having in mind that  $E_k(t) \leq E_k(0)$  for all  $t \in [0, T]$ , applying inequality (5.50) and dividing both sides by  $c_k^2$  it holds that

$$\begin{aligned}
 \frac{E_k(t)}{c_k^2} \leq C & \left\{ \frac{\int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt}{c_k^2} \right. \\
 & \left. + \frac{\int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt}{c_k^2} + 1 \right\} \quad (5.51)
 \end{aligned}$$

where  $C = C(T, a_0, b_0, \alpha, d)$ . Since in view of (5.31) we have

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \frac{\int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt}{\int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt} \\
 = 0, \quad (5.52)
 \end{aligned}$$

then, from (5.51) there exists  $M > 0$  such that

$$\frac{E_k(t)}{c_k^2} \leq C(T, a_0, b_0, \alpha, d)(M + 1), \text{ for all } t \in [0, T] \text{ and for all } k \in \mathbf{N}.$$

Consequently, from (5.47) it results that

$$\hat{E}_k(t) \leq C(T, a_0, b_0, \alpha, d)(M + 1), \text{ for all } t \in [0, T] \text{ and for all } k \in \mathbf{N} \quad (5.53)$$

Then, in particular, from (5.52) we deduce

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \int_0^T \int_{\mathcal{M}} a(x) |\hat{\phi}'_k|^2 d\mathcal{M} dt \\
 = \lim_{k \rightarrow +\infty} \frac{\int_0^T \int_{\mathcal{M}} a(x) |\phi'_k|^2 d\mathcal{M} dt}{\int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt} = 0, \quad (5.54)
 \end{aligned}$$

and

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \hat{\psi}_k|^2 d\mathcal{M} dt \\
 = \lim_{k \rightarrow +\infty} \frac{\int_0^T \int_{\mathcal{M}} |(1 - \Delta)^{\frac{1}{4}} b(x) \psi_k|^2 d\mathcal{M} dt}{\int_0^T \int_{\mathcal{M}} |\phi_k|^2 d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} |\psi_k|^2 d\mathcal{M} dt} = 0, \quad (5.55)
 \end{aligned}$$

and from (5.53), for a subsequence  $\{\hat{\psi}_k, \hat{\phi}_k\}$ , we obtain

$$\hat{\psi}_k \rightharpoonup \hat{\psi} \text{ weak star in } L^\infty(0, T; L^2(\mathcal{M})), \quad (5.56)$$

$$\hat{\phi}_k \rightharpoonup \hat{\phi} \text{ weak star in } L^\infty(0, T; \mathcal{V}), \quad (5.57)$$

$$\hat{\phi}'_k \rightharpoonup \hat{\phi}' \text{ weak star in } L^\infty(0, T; L^2(\mathcal{M})), \quad (5.58)$$

$$\hat{\phi}_k \rightarrow \hat{\phi} \text{ strongly in } L^2(0, T; L^2(\mathcal{M})). \quad (5.59)$$

In addition,  $\hat{\phi}_k$  satisfies the equation

$$\begin{cases} \hat{\phi}''_k - \Delta \hat{\phi}_k + a(x)\hat{\phi}'_k = \frac{|\psi_k|^2}{c_k} \text{ in } \mathcal{M} \times (0, T) \\ \hat{\phi}'_k \rightarrow 0 \text{ a. e. in } L^2(0, T; L^2(\omega)). \end{cases} \quad (5.60)$$

Passing to the limit when  $k \rightarrow +\infty$  taking the above convergences and (5.55) into account, we get

$$\begin{cases} \hat{\phi}'' - \Delta \hat{\phi} = 0 \text{ in } \mathcal{M} \times (0, T) \\ \hat{\phi}' = 0 \text{ a. e. in } \omega \times (0, T). \end{cases} \quad (5.61)$$

Then,  $v = \hat{\phi}'$  verifies, in the distributional sense

$$\begin{cases} v'' - \Delta v = 0 \text{ in } \mathcal{M} \times (0, T) \\ v = 0 \text{ a. e. in } \omega \times (0, T). \end{cases} \quad (5.62)$$

From Theorem 5.1 in [16] follow that  $v = \hat{\phi}' = 0$ . Returning to (5.61) follow that  $\hat{\phi} = 0$ .

Moreover,  $\hat{\psi}_k$  satisfies the equation

$$\begin{cases} i\hat{\psi}'_k + \Delta \hat{\psi}_k + ib(x)(1 - \Delta)^{\frac{1}{2}}b(x)\hat{\psi}_k = \hat{\phi}_k \psi_k \chi_\omega \text{ in } \mathcal{M} \times (0, T) \\ \hat{\psi}_k(0) = \hat{\psi}_k^0 \text{ in } \mathcal{M}. \end{cases} \quad (5.63)$$

Now, we will use the effect smoothing effect due to Aloui given in [4], theorem 1. Indeed, since  $\hat{\psi}_k$  satisfies (5.63), we have that  $(\hat{\psi}_k)$  satisfies the integral equation

$$\hat{\psi}_k = S(t)\hat{\psi}_k(0) + \int_0^T S(T - \tau)F(\psi_k)(\tau) d\tau, \quad (5.64)$$

where  $S(t)$  is the semigroup generated by

$$\begin{aligned} A : D(A) = \mathcal{V} \cap H^2(\mathcal{M}) &\rightarrow L^2(\mathcal{M}) \\ y &\mapsto Ay := i\Delta y - b(x)(1 - \Delta)^{\frac{1}{2}}b(x)y, \end{aligned}$$

and  $F(\psi_k) := \hat{\phi}_k \psi_k \chi_\omega$ . We observe that

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} |F(\psi_k)|^2 d\mathcal{M} dt &\leq \int_0^T \left( \int_{\omega} |\hat{\phi}_k|^4 d\mathcal{M} \right)^{1/2} \left( \int_{\omega} |\psi_k|^4 d\mathcal{M} \right)^{1/2} dt \\ &= \int_0^T \|\hat{\phi}_k(t)\|_{L^4(\omega)}^2 \|\psi_k(t)\|_{L^4(\omega)}^2 dt \\ &\leq C \|\hat{\phi}_k\|_{L^\infty(0,T;\mathcal{V})}^2 \int_0^T \|\psi_k(t)\|_{L^4(\omega)}^2 dt. \end{aligned} \quad (5.65)$$

Then from (5.38), (5.57) and (5.65), we obtain

$$F(\psi_k) \longrightarrow 0 \text{ in } L^2(0, T; L^2(\mathcal{M})). \quad (5.66)$$

Applying smoothing effect due to Aloui given in [4], Theorem 1, we have, for any  $\theta \in C_0^\infty(0, T)$ ,

$$\|\theta \hat{\psi}_k\|_{L^2(0,T;H^{\frac{1}{2}}(\mathcal{M}))} \leq C \left( \|\hat{\psi}_k^0\|_{L^2(\mathcal{M})} + \|F(\psi_k)\|_{L^2(0,T;L^2(\mathcal{M}))} \right). \quad (5.67)$$

Let  $\varepsilon > 0$  and  $\theta \in C^\infty(0, T)$ ;  $0 \leq \theta \leq 1$  such that

$$\theta(t) = 1 \text{ in } [\varepsilon, T - \varepsilon] \text{ and } \theta(t) = 0 \text{ in } \left[0, \frac{\varepsilon}{2}\right] \cup \left[T - \frac{\varepsilon}{2}, T\right]. \quad (5.68)$$

Therefore, from (5.66), we obtain the following estimative:

$$\begin{aligned} \|\hat{\psi}_k\|_{L^2(0,T;H^{\frac{1}{2}}(\mathcal{M}))} &\leq \|\hat{\psi}_k\|_{L(0,\varepsilon,H^{\frac{1}{2}}(\mathcal{M}))} + \|\theta \hat{\psi}_k\|_{L^2(\varepsilon,T-\varepsilon,H^{\frac{1}{2}}(\mathcal{M}))} \\ &+ \|\hat{\psi}_k\|_{L^2(T-\varepsilon,T,H^{\frac{1}{2}}(\mathcal{M}))} \leq 2\varepsilon \max_{t \in [0,T]} \|\hat{\psi}_k\|_{H^{\frac{1}{2}}(\mathcal{M})} + C \|\hat{\psi}_k^0\|_{L^2(\mathcal{M})} + C, \quad \forall \varepsilon > 0. \end{aligned}$$

This allows us to conclude

$$\{\hat{\psi}_k\} \text{ is bounded in } L^2(0, T; H^{\frac{1}{2}}(\mathcal{M})). \quad (5.69)$$

Recalling (5.63), we note that for a.e  $t \in (0, T)$

$$\hat{\psi}'_k = i\Delta \hat{\psi}_k + ib(x)(-\Delta)^{\frac{1}{2}} b(x) \hat{\psi}_k - i\hat{\phi}_k \psi_k \chi_\omega \in [\mathcal{V} \cap H^2(\mathcal{M})]'. \quad (5.70)$$

Note that  $\Delta : L^2(\mathcal{M}) \rightarrow [\mathcal{V} \cap H^2(\mathcal{M})]'$  is linear and continuous, so

$$\|\Delta \hat{\psi}_k(t)\|_{[\mathcal{V} \cap H^2(\mathcal{M})]'} \leq C \|\hat{\psi}'_k\|_{L^2(\mathcal{M})}. \quad (5.71)$$

So from (5.56) we have that  $i\Delta \hat{\psi}_k$  is limited in  $L^\infty(0, T; [\mathcal{V} \cap H^2(\mathcal{M})]')$  and consequently from (5.70), (5.55) and (5.66) we conclude

Therefore,

$$\{\hat{\psi}'_k\} \text{ is bounded in } L^2(0, T; [\mathcal{V} \cap H^2(\mathcal{M})]'). \quad (5.72)$$

Combining (5.69), (5.72) and the embedding chain  $H^{\frac{1}{2}}(\mathcal{M}) \xrightarrow{c} L^2(\mathcal{M}) \hookrightarrow [\mathcal{V} \cap H^2(\mathcal{M})]'$ , due to Aubin-Lions Theorem (see Theorem 5.1 in Lions [32]), we have,

$$\hat{\psi}_k \rightarrow \hat{\psi} \text{ strongly in } L^2(0, T; L^2(\mathcal{M})). \quad (5.73)$$

Moreover, from (5.55) and (5.73) we get,

$$\hat{\psi}_k \rightarrow \tilde{\psi} \text{ strongly in } L^2(0, T; L^2(\mathcal{M})), \quad (5.74)$$

where

$$\tilde{\psi} = \begin{cases} \hat{\psi}, & \text{in } \mathcal{M} \setminus \omega \\ 0, & \text{in } \omega. \end{cases} \quad (5.75)$$

Note that if  $\hat{\psi} = 0$ , in  $\mathcal{M}$  so from (5.59), (5.74), (5.46) and observing that  $\hat{\phi} = 0$  we have a contradiction.

On the other hand, if  $\hat{\psi} \neq 0$ , by passing limit on (5.63), we have, thanks to (5.65) and (5.55), that

$$\begin{cases} i\hat{\psi}' + \Delta\hat{\psi} = 0 & \text{in } \mathcal{M} \times (0, T) \\ \hat{\psi} = 0 & \text{a. e. in } \omega \times (0, T). \end{cases} \quad (5.76)$$

from Theorem 4.2 in [14] we conclude that  $\hat{\psi} = 0$  a.e. in  $\mathcal{M}$ . Therefore we have  $\hat{\psi} = 0$  and  $\hat{\phi} = 0$ , which is a contradiction by (5.59), (5.74), and (5.46). Thus, Lemma 5.3 is proved.  $\square$

Therefore, from (5.29) and (5.30), making use of the standard arguments we conclude exponential stability. For more details see [19, 20, 28].

## 5.2. Problem ( $P_2$ )

From Theorem 2.3 we consider regular solutions  $(\psi(t), \phi(t), \phi_t(t)) \in \mathcal{H}$  to problem ( $P_2$ ), such that  $\|\{\psi_0, \phi_0, \phi_1\}\|_{\mathcal{H}} \leq L$ , where  $L > 0$ . Note that

$$\left| \int_{\omega} |\psi|^2 \phi' dx \right| \leq \frac{1}{2a_0b_0} \int_{\mathcal{M}} b(x) |\psi|^4 d\mathcal{M} + \frac{1}{2} \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M}$$

Thus, from (2.11),

$$\begin{aligned} E'(t) &\leq - \left( \alpha - \frac{1}{2a_0b_0} \right) \int_{\mathcal{M}} b(x) |\psi|^4 d\mathcal{M} - \alpha \int_{\mathcal{M}} b(x) |\psi|^2 d\mathcal{M} \\ &\quad - \frac{1}{2} \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \\ &\leq -k \left[ \int_{\mathcal{M}} b(x) [|\psi|^4 + |\psi|^2] d\mathcal{M} + \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \right] \end{aligned} \quad (5.77)$$

where  $k = \min \left\{ \alpha - \frac{1}{2a_0b_0}, \frac{1}{2} \right\}$ . Remember that  $\alpha \geq \frac{5}{2a_0b_0}$ , that is,  $k > 0$ .

Therefore

- (I) The map  $t \in (0, \infty) \mapsto E(t)$  is non increasing,
- (II) We have the following inequality of the energy

$$\begin{aligned} E(t_2) - E(t_1) &\leq -k \int_{t_1}^{t_2} \left[ \int_{\mathcal{M}} b(x) [|\psi|^4 + |\psi|^2] d\mathcal{M} + \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \right] dt, \end{aligned} \quad (5.78)$$

In order to prove the exponential decay we will consider the following lemma

**Lemma 5.3.** *For all  $T > T_0$  there exists a positive constant  $C = C(T)$  such that if  $\{\psi, \phi\}$  is the regular solution of  $(P_2)$  with initial data  $(\psi_0, \phi_0, \phi_1)$  taken in limited of  $\mathcal{H}$ , we have*

$$E(0) \leq C \int_0^T \left[ \int_{\mathcal{M}} b(x) [|\psi|^4 + |\psi|^2] d\mathcal{M} + \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \right] dt. \quad (5.79)$$

The proof of the lemma follows the same steps as in [2] (see Lemma 4.1), that is, using the observability inequalities of the wave equation and of the Schrödinger equation, however, here, using the unique continuation principle given in [16] and [14] instead of the Holmgren theorem. Indeed, to use these principles, the existence of the multiplier  $f$  satisfying the conditions of Theorem 4.1 is fundamental.

To end this section. Fix  $T_0$  large enough. From (5.78)

$$E(T_0) - E(0) \leq -k \int_0^{T_0} \underbrace{\left[ \int_{\mathcal{M}} b(x) [|\psi|^4 + |\psi|^2] d\mathcal{M} + \int_{\mathcal{M}} a(x) |\phi'|^2 d\mathcal{M} \right]}_{D(t)} dt,$$

Then, from Lemma 5.3,

$$E(T_0) \leq E(0) \leq C \int_0^{T_0} D(t) dt \leq -\frac{C}{k} E(T_0) + \frac{C}{k} E(0).$$

So,

$$E(T_0) \leq \sigma E(0)$$

where  $\sigma := \frac{C}{k+C} \in (0, 1)$ . Finally, following standard arguments, we conclude exponential stability.

### Acknowledgment

The authors would like to thank Marcelo Moreira Cavalcanti and Ryuichi Fukuoka for fruitful discussions.

### References

- [1] A. F. Almeida, M. M. Cavalcanti and J. P. Zanchetta, *Exponential decay for the coupled Klein-Gordon-Schrödinger equations with locally distributed damping*, Communications on Pure and Applied Analysis., **17**, (2018), p. 2039-2061.
- [2] A. F. Almeida, M. M. Cavalcanti, J. P. Zanchetta . *Exponential stability for the coupled Klein-Gordon-Schrödinger equations with locally distributed damping*, Evolution Equations and Control Theory., **8**, (2019), p. 847-865.
- [3] L. Aloui, *Smoothing effect for regularized Schrödinger equation on bounded domains*, Asymptot. Anal., **59**, (2008), 179-193.
- [4] L. Aloui, *Smoothing effect for regularized Schrödinger equation on compact manifolds*, Collect. Math., **59**, (2008), no. 1, 53-62.
- [5] A. Bachelot, *Problème de Cauchy pour des systèmes hyperboliques semi-linéaires*, Ann. Inst. H. Poincaré Anal. non Linéaire, **1**,(1984), 453-478.

- [6] J. B. Baillon and J. M. Chadam, The Cauchy problem for the coupled Klein-Gordon-Schrödinger equations, in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations*, North Holland, Amsterdam, (1978), 37-44.
- [7] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim, **30**, (1992), 1024-1065.
- [8] C. Banquet, L. C. F. Ferreira and E. J. Villamizar-Roa, *On existence and scattering theory for the Klein-Gordon-Schrödinger system in an infinite  $L^2$ -norm setting*, Ann. Mat. Pura Appl., **194**, (2015), 781-804.
- [9] J. Bertrand and K. Sandeep, Adams inequality on pinched Hadamard manifolds, 2020. <https://arxiv.org/abs/1809.00879v3>
- [10] P. Biler, *Attractors for the system of Schrödinger and Klein-Gordon equations with Yukawa coupling*, SIAM J. Math. Anal., **21**, (1990), 1190-1212.
- [11] V. Bisognin, M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Soriano; *Uniform decay for the Klein-Gordon-Schrödinger equations with locally distributed damping*, NoDEA, Nonlinear differ. equ. appl., **15**, (2008), 91-113.
- [12] C. A. Bortot and M. M. Cavalcanti, *Asymptotic stability for the damped Schrödinger equation on noncompact Riemannian manifolds and exterior domains*, Comm. Partial Differential Equations, **39**, (2014), 1791-1820.
- [13] C. A. Bortot, M. M. Cavalcanti, V. N. Domingos Cavalcanti and P. Piccione. *Exponential Asymptotic Stability for the Klein Gordon Equation on Non-compact Riemannian Manifolds*, Applied Mathematics and Optimization, **78**, (2018), 219-265.
- [14] C. A. Bortot, W. J. Corrêa, R. Fukuoka and T. M. Souza. *Exponential stability for the locally damped defocusing Schrödinger equation on compact manifold*, Communications on Pure and Applied Analysis, **19**, (2020), 1367-1386.
- [15] M. Cavalcanti and V. Domingos Cavalcanti, *Global existence and uniform decay for the coupled Klein-Gordon-Schrödinger equations*, NoDEA, Nonlinear differ. equ. appl., **7**, (2000), 285-307 .
- [16] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka, J. A. Soriano, *Asymptotic stability of the wave equation on compact manifolds and locally distributed damping: a sharp result*, Arch. Rational Mech. Anal., **197**, (2010), 925-964.
- [17] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka and J. A. Soriano, *Uniform Stabilization of the wave equation on compact surfaces and locally distributed damping*, Transactions of AMS, **361**, (2009), 4561-4580.
- [18] M. M. Cavalcanti, V. N. Domingos Cavalcanti, R. Fukuoka, J. A. Soriano, *Uniform stabilization of the wave equation on compact manifolds and locally distributed damping - a sharp result*, Journal of Mathematical Analysis and Applications, **351**, (2009), 661-674.
- [19] M. M. Cavalcanti, V. N. Domingos Cavalcanti and I. Lasiecka, *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction*, Journal of Differential Equations, **236**, (2007), 407-459.
- [20] M. Daoulatli, I. Lasiecka, D. Toundykov, *Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without*

- growth restrictions*, Discrete Continuous Dynamical Systems - S, **2**, (2009), 67-94.
- [21] B. Dehman, P. Gérard, and G. Lebeau, *Stabilization and control for the non-linear Schrödinger equation on a compact surface*, Math Z., **254**, (2006), 729-749.
- [22] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations I*, Bull. Sci. Engrg. Res. Lab. Waseda Univ., **69**, (1975), 51-62.
- [23] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations II*, J. Math. Analysis Applic., **66**, (1978), 358-378.
- [24] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon Schrödinger equations III - Higher order interaction, decay and blow-up*, Math. Japonica, **24**, (1979), 307-321.
- [25] Y. Han, *On the Cauchy problem for the coupled Klein-Gordon-Schrödinger system with rough data*, Discret. Contin. Dyn. Syst., **12**, (2005), 233-242.
- [26] N. Hayashi, *Global strong solutions of coupled Klein-Gordon-Schrödinger equations*, Funkcialaj Ekvacioj, **29**, 299-307, (1986).
- [27] H. Lange and B. Wang, *Attractors for the Klein-Gordon-Schrödinger equation*, J. Math. Phys., **40**, (1999), 2445-2457.
- [28] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential and Integral Equations, **6**, (1993), 507-533.
- [29] C. Laurent, *Internal control of the Schrödinger equation*, Mathematical Control & Related Fields, **4**, (2014), 161-186.
- [30] C. Laurent and M. Leautaud, *Uniform observability estimates for linear waves*, ESAIM: COCV, **22**, (2016), 1097-1136.
- [31] G. Lebeau, *Contrôle de l'équation de Schrödinger. (french) [control of the Schrödinger equation]*, J. Math. Pures Appl., **71**, (1992), 267-291.
- [32] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires*, Dunod, Paris, (1969).
- [33] J. L. Lions, *Contrôlabilité exacte, perturbations et Stabilisation de systèmes distribués*, Tome 1, Masson, Paris, (1988).
- [34] E. Machtyngier, *Exact controllability for the Schrödinger equation*, SIAM J. Control Optim., **32**, no1, (1994), 24-34.
- [35] A. Mostafa, O. Goubet and A. Hakim, *Regularity of the attractor for a coupled Klein-Gordon-Schrödinger system*, Diff. Integral Equ., **16**, (2003), 573-581.
- [36] M. Ohta, *Stability of Stationary States for the Coupled Klein-Gordon-Schrödinger Equations*, Nonlinear Analysis, Theory , Methods and Appl., **27**, (1996), 455-461.
- [37] H. Pecher, *Some new well-posedness results for the Klein-Gordon-Schrödinger system*, Differential Integral Equations, **25**, (2012), 117-142.
- [38] M. N. Poulou and N. M. Stavrakakis, *Global attractor for a system of Klein-Gordon-Schrödinger type in all  $R$* , Nonlinear Anal., **74**, (2011), 2548-2562.
- [39] M. N. Poulou and N. M. Stavrakakis, *Uniform decay for a local dissipative Klein-Gordon-Schrödinger type system*, Electron. J. Differential Equations, **179**, (2012), 1-16.

- [40] J. Rauch and M. Taylor, *Decay of solutions to nondissipative hyperbolic systems on compact manifolds*, Comm. Pure Appl. Math., **28**, (1975), 501-523.
- [41] J. Le Rousseau, G. Lebeau, P. Terpolilli, E. Trélat, *Geometric control condition for the wave equation with a time-dependent observation domain*, Anal. PDE, **10**, (2017), 983-1015.
- [42] A. Shimomura, *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions*, J. Math. Sci. Univ. Tokyo, **10**, (2003), 661-685.
- [43] A. Shimomura, *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions II*, Hokkaido Math. J., **34**, (2005), 405-433.
- [44] M. Spivak, "A Comprehensive Introduction to Differential Geometry," Publish or Perish, INC., Houston, 1999.
- [45] M. Taylor, "Partial Differential Equations," Springer, Berlin, 1991.
- [46] B. Wang, *Classical global solutions for non-linear Klein-Gordon-Schrödinger equations*, Math. Methods Appl. Sci., **20**, (1997), 599-616.
- [47] H. Yukawa, *On the interaction of elementary particles. I*, Progress of Theoretical Physics Supplement, **1**, (1955), 1-10.

César A. Bortot  
Department of Mobility Engineering  
Federal University of Santa Catarina  
89219-600  
Joinville, SC, Brazil.  
e-mail: [c.bortot@ufsc.br](mailto:c.bortot@ufsc.br)

Thales M. Souza  
Department of Mobility Engineering  
Federal University of Santa Catarina  
89219-600  
Joinville, SC, Brazil.  
e-mail: [thales.maier@ufsc.br](mailto:thales.maier@ufsc.br)

Janaina P. Zanchetta  
Department of Mathematics  
State University of Maringá  
87020-900  
Maringá, PR, Brazil.  
e-mail: [janainazanchetta@yahoo.com.br](mailto:janainazanchetta@yahoo.com.br)