Global existence and finite time blow-up for the heat flow of H-system with constant mean curvature

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Abstract: In this paper, we use the modified potential well method to study the long time behaviors of solutions to the heat flow of H-system in a bounded smooth domain of R^2. Global existence and finite time blowup of solutions are proved when the initial energy is in three cases. When the initial energy is low or critical, we not only give a threshold result for the global existence and blowup of solutions, but also obtain the decay rate of the L^2 norm for global solutions. When the initial energy is high, sufficient conditions for the global existence and blowup of solutions are also provided. We extend the recent results which were obtained in [12].
Keywords: H-system, heat flow, potential well method, blow-up

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, and let H be a bounded Lipschitz function on $\mathbb{R}^3$. A map $u \in C^2(\Omega, \mathbb{R}^3)$ is called an H-surface (parametrized over Ω) if u satisfies

$$\Delta u = 2H(u) u_x \wedge u_y, \tag{1.1}$$

where $\wedge$ denotes the wedge product of $\mathbb{R}^3$. System of the general form (1.1) arises from differential geometry and in the calculus of variation. If u is a conformal representation of a surface S in $\mathbb{R}^3$, i.e., $u_x \cdot u_y = 0 = |u_x|^2 - |u_y|^2$, then $H(u)$ is the mean curvature of S at the point u. For $H(u) \equiv const$, the weak solutions of the Dirichlet problem associated to (1.1) correspond to critical points of the energy functional,

$$e(u) = \frac{1}{2}\int_\Omega |\nabla u|^2,$$

under the constraint that the volume functional

$$V_H(u) = \frac{2}{3}\int_\Omega H(u) u \cdot u_x \wedge u_y,$$

is a given constant.

Starting with the pioneering works of Wente [23] in 1969, a very large amount of literature has been devoted to system 1.1. System 1.1 with constant mean curvature H

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has been extensively studied by Wente [23], Hildebrandt [10], Struwe [20], and Brezis and Coron [3, 3]. Hildebrant [10] considered the Plateau problem for surfaces of constant mean curvature, Brezis and Coron [3, 3] obtained the multiple solutions problem of H-surfaces, and Struwe [20] proved the existence of surfaces of constant mean curvature H with free boundaries. For variable H, there are recent works by Caldiroli and Musina [5]and Duzzer and Grotowski [8]. Caldiroli adn Musina in [5] considered system (1.1) with small boundary data, and proved blowup phenomena and nonexistence results. The existence of solutions to the system for non-constant H in higher dimensional compact Riemannian manifolds without boundary was proved by Duzzer-Grotowski [8].

In this paper, we study an initial-boundary value system for the heat flow of the equation of H-surface:

$$\begin{cases} u_t = \Delta u - 2H(u)u_x \wedge u_y, \text{ in } \Omega \times (0, \infty), \\ u|_{t=0} = u_0, \text{ in } \Omega, \\ u|_{\partial\Omega} = \chi, \end{cases} \tag{1.2}$$

where $u_0 \in H^1(\Omega), \chi \in H^{\frac{1}{2}}(\partial\Omega)$, and $u_0|_{\partial\Omega} = \chi$. Throughout this paper, we assume that

$$H(u) \equiv H \equiv const > 0, \text{ and } \chi = 0. \tag{1.3}$$

Struwe [20], by the assumption $|H(u)| \cdot |\chi|_{L^\infty(\partial\Omega)} < 1$, proved that the equations (1.2) with the condition with free boundaries admit a unique solution. By using the theorems and methods in [6, 20], Rey [18] proved that if $u_0(x) \in W^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ and $u_0(x)|_{\partial\Omega} = \chi$, then system (1.2) has a unique global regular solution $u \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega \times (0, +\infty), \mathbb{R}^3)$ under the assumption $|H(u)| \cdot |\chi|_{L^\infty(\partial\Omega)} < 1$. Chen and Levine [7] removed the assumption $|H(u)| \cdot |\chi|_{L^\infty(\partial\Omega)} < 1$, and obtained the existence of regular solution to system (1.2) but added the following assumption

$$\int_\Omega |\nabla u|^2(\cdot, t) \le \int_\Omega |\nabla u|^2(\cdot, s), \quad \text{for } 0 \le s \le t, \tag{1.4}$$

which is the main difference between the heat flow of the equation of H-surface and the heat flow of harmonic maps. The existence of weak solutions and short-time regularity for the H-surface flow were considered by Bögelein, Duzar and Scheven [1, 2]. If $\chi \equiv 0$, Huang, Tan and Wang [12] gave sufficient conditions with low initial energy such that the heat flow develops finite time singularity.

In this article, we consider the heat flow system of H-surface with low initial energy, critical initial energy and high initial energy. The results in our paper will be obtained by the modified potential well method. Potential well method, which was first put forward to consider semi-linear hyperbolic initial boundary value problem by Payne and Sattinger [17, 19] around 1970s, is a powerful tool in studying the long time behaviors of solutions of some evolution equations. The potential well is defined by the level set of energy functional and the derivative functional. It is generally true that solutions starting inside the well are global in time, solutions starting outside the well and at an unstable point blow up in finite time. After the pioneer work of Sattinger and Payne, some authors [13, 14, 15, 16, 21, 24] used the method to study the global existence and nonexistence of solutions for various nonlinear evolution equations with initial boundary value problem. In [15, 16], Liu et al. modified and improved the method by introducing a family of potential wells which include the known potential well as a special case. The modified potential well method has been used to study semilinear pseudo-parabolic equations [24] and fourth-order parabolic equation [9]. In this paper, we use the modified potential well method to obtain global existence and blow up in finite time of solutions when the initial energy is low, critical and high, respectively. When the initial energy is low, similar results are obtained in [12], but our result is more general, moreover, we prove a more precise decay rate of $|u|_2$.

2 Preliminaries

Throughout this paper, we denote the $L^2(\Omega)$ norm, $H_0^1(\Omega)$ norm by $|\cdot|_2$, $\|\cdot\|$, respectively. And $(\cdot,\cdot)$ is used to denote the inner product in $L^2(\Omega)$. In order to state our main results precisely, let us introduce some notations and sets, and then investigate their basic properties.

For $u \in H_0^1(\Omega)$, we set

$$\begin{aligned} E(u) &= e(u) + V_H(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u \cdot u_x \wedge u_y, \\ D(u) &= \int_\Omega |\nabla u|^2 + 2\int_\Omega H(u)uu_x \wedge u_y. \end{aligned} \tag{2.1}$$

The Nehari manifold is defined by

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) : D(u) = 0, u \neq 0\right\}, \tag{2.2}$$

which can be separated into the two unbounded sets

$$\begin{aligned} \mathcal{N}_+ &= \left\{u \in H_0^1(\Omega) : D(u) > 0\right\}, \\ \mathcal{N}_- &= \left\{u \in H_0^1(\Omega) : D(u) < 0\right\}. \end{aligned} \tag{2.3}$$

The potential well and its corresponding set are defined respectively as

$$\begin{aligned} W &= \left\{u \in H_0^1(\Omega) : D(u) > 0, E(u) < d\right\} \cup \{0\}, \\ V &= \left\{u \in H_0^1(\Omega) : D(u) < 0, E(u) < d\right\}, \end{aligned} \tag{2.4}$$

where,

$$d = \min_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{s \geqslant 0} E(su) = \inf_{u \in \mathcal{N}} E(u),$$

is the depth of the potential well W.

Now let us define the level set

$$E^\alpha = \left\{u \in H_0^1(\Omega) : E(u) < \alpha\right\}. \tag{2.5}$$

Furthermore, by the definition of $E(u), \mathcal{N}, E^\alpha$ and d, we easily know that

$$\mathcal{N}_\alpha = \mathcal{N} \cap E^\alpha \equiv \left\{u \in \mathcal{N} : \|u\| < \sqrt{6\alpha}\right\} \neq \varnothing \quad \text{for all } \alpha > d. \tag{2.6}$$

We now define

$$\lambda_\alpha = \inf\left\{|u|_2 : u \in \mathcal{N}_\alpha\right\}, \quad \Lambda_\alpha = \sup\left\{|u|_2 : u \in \mathcal{N}_\alpha\right\} \quad \text{for all } \alpha > d. \tag{2.7}$$

It is clear that λ_α is nonincreasing and Λ_α is nondecreasing with respect to α . We also introduce the following sets

$$\begin{aligned} \mathcal{B} &= \left\{u_0 \in H_0^1(\Omega) : \text{ the solution } u=u(t) \text{ of (1) blows up in finite time } \right\}, \\ \mathcal{G} &= \left\{u_0 \in H_0^1(\Omega) : \text{ the solution } u=u(t) \text{ of (1) exists for all } t>0\right\}, \\ \mathcal{G}_o &= \left\{u_0 \in G : u(t) \mapsto 0 \text{ in } H_0^1(\Omega) \text{ as } t \to \infty\right\}. \end{aligned} \tag{2.8}$$

For $0<\delta<\frac{3}{2}$, let us define the modified functional and Nehari manifold as follows:

$$\begin{aligned} D_\delta(u) &= \delta\|\nabla u\|_2^2 + 2\int_\Omega H(u)u\cdot u_x\wedge u_y, \\ \mathcal{N}_\delta &= \left\{u\in H_0^1(\Omega) : D_\delta(u)=0, \|u\|\neq 0\right\}, \\ d_\delta &= \inf_{u\in\mathcal{N}_\delta} E(u), \\ r(\delta) &= \frac{2\sqrt{2\pi}\delta}{H}. \end{aligned} \tag{2.9}$$

Then we can define the modified potential wells and their corresponding sets as follows:

$$\begin{aligned} W_\delta &= \left\{u\in H_0^1(\Omega) : D_\delta(u)>0, E(u)<d(\delta)\right\}\cup\{0\}, \\ V_\delta &= \left\{u\in H_0^1(\Omega) : D_\delta(u)<0, E(u)<d(\delta)\right\}, \\ B_\delta &= \left\{u\in H_0^1(\Omega) : \|u\|<r(\delta)\right\}, \\ B_\delta^c &= \left\{u\in H_0^1(\Omega) : \|u\|>r(\delta)\right\}. \end{aligned} \tag{2.10}$$

For future convenience, we give some useful lemmas which will play an important role in the proof of our main results. We first recall the following isoperimetric inequality, whose proof can be found in [4] and [22].

Lemma 2.1 (Isoperimetric inequality)**.** *For any $u\in H_0^1(\Omega;\mathbb{R}^3)$, there holds*

$$\int_\Omega |\nabla u|^2 \geq \sqrt[3]{32\pi}\left|\int_\Omega u\cdot u_x\wedge u_y\right|^{2/3}. \tag{2.11}$$

Lemma 2.2. *Let $u_0\in H_0^1(\Omega)$.*

(1) *If $0<\|u\|<r(\delta)$, then $D_\delta(u)>0$. In particular, if $0<\|u\|_{H_0^1}<r(1)$, then $D(u)>0$;*
(2) *If $D_\delta(u)<0$, then $\|u\|>r(\delta)$. In particular, if $D(u)<0$, then $\|u\|>r(1)$;*
(3) *If $D_\delta(u)=0$, then $\|u\|\geqslant r(\delta)$ or $\|u\|=0$. In particular, if $D(u)=0$, then $\|u\|\geqslant r(1)$ or $\|u\|=0$;*
(4) *If $D_\delta(u)=0$ and $\|u\|\neq 0$, then $E(u)>0$ for $0<\delta<\frac{3}{2}$, $E(u)=0$ for $\delta=\frac{3}{2}$, $E(u)<0$ for $\delta>\frac{3}{2}$.*

Proof. (1) Since $0<\|u\|<r(\delta)$, by the Isoperimetric inequality, we have

$$\left|\int_\Omega u\cdot u_x\wedge u_y\right| \leq \frac{1}{4\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{3}{2}}. \tag{2.12}$$

So from the assumption $0 < \|u\| < r(\delta) = \frac{2\sqrt{2\pi}\delta}{H}$, we obtain

$$\begin{aligned} D_\delta(u) &= \delta\|\nabla u\|_2^2 + 2\int_\Omega H(u)u \cdot u_x \wedge u_y \\ &\geq \delta\|\nabla u\|_2^2 - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{3}{2}} \\ &\geq \|\nabla u\|_2^2\left(\delta - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\right) > 0. \end{aligned} \tag{2.13}$$

(2) By the assumption $D_\delta(u) < 0$ and the Isoperimetric inequality, we have

$$\begin{aligned} 0 > D_\delta(u) &= \delta\|\nabla u\|_2^2 + 2\int_\Omega H(u)u \cdot u_x \wedge u_y \\ &\geq \delta\|\nabla u\|_2^2 - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{3}{2}} \\ &\geq \|\nabla u\|_2^2\left(\delta - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\right). \end{aligned} \tag{2.14}$$

Hence, $\|u\| > r(\delta)$.

(3) By the assumption $D_\delta(u) = 0$ and the Isoperimetric inequality, we have

$$\begin{aligned} 0 = D_\delta(u) &= \delta\|\nabla u\|_2^2 + 2\int_\Omega H(u)u \cdot u_x \wedge u_y \\ &\geq \delta\|\nabla u\|_2^2 - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{3}{2}} \\ &\geq \|\nabla u\|_2^2\left(\delta - \frac{H}{2\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\right). \end{aligned} \tag{2.15}$$

Hence, $\|u\| \geq r(\delta)$ or $u = 0$.

(4) We easily know that

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u \cdot u_x \wedge u_y \\ &= \left(\frac{1}{2} - \frac{\delta}{3}\right)\|u\|^2 + \frac{1}{3}D_\delta(u) \\ &= \left(\frac{1}{2} - \frac{\delta}{3}\right)\|u\|^2. \end{aligned} \tag{2.16}$$

Then using (2.16), we can prove the conclusion. □

Lemma 2.3. *(1) $d(\delta) \geqslant a(\delta)r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{3}, 0 < \delta < \frac{3}{2}$,*

(2) $\lim_{\delta\to 0} d(\delta) = 0, d\left(\frac{3}{2}\right) = 0$ and $d(\delta) < 0$ for $\delta > \frac{3}{2}$,

(3) $d(\delta)$ is increasing on $0 < \delta \leqslant 1$, decreasing on $1 \leqslant \delta \leqslant \frac{3}{2}$ and takes the maximum $d = d(1)$ at $\delta = 1$.

Proof. (1) If $u \in \mathcal{N}_\delta$, by Lemma 2.2 (3), then $\|u\| \geqslant r(\delta)$. Moreover, we can deduce

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u\cdot u_x\wedge u_y \\ &= \left(\frac{1}{2}-\frac{\delta}{3}\right)\|u\|^2 + \frac{1}{3}D_\delta(u) \\ &= \left(\frac{1}{2}-\frac{\delta}{3}\right)\|u\|^2 \geq a(\delta)r^2(\delta). \end{aligned} \tag{2.17}$$

Hence, $d(\delta) \geqslant a(\delta)r^2(\delta)$.

(2) We easily know that

$$E(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2 + \frac{2\lambda^3}{3}\int_\Omega H(u)u\cdot u_x\wedge u_y.$$

Hence,

$$\lim_{\lambda\to 0} E(\lambda u) = 0. \tag{2.18}$$

And if we let $\lambda u \in \mathcal{N}_\delta$, then λu satisfies

$$0 = D_\delta(\lambda u) = \delta\lambda^2\|\nabla u\|_2^2 + 2\lambda^3\int_\Omega H(u)u\cdot u_x\wedge u_y.$$

Then, we obtain

$$\lambda = \frac{\delta\|\nabla u\|_2^2}{2\int_\Omega H(u)u\cdot u_x\wedge u_y}, \tag{2.19}$$

which yields

$$\lim_{\delta\to 0}\lambda(\delta) = 0. \tag{2.20}$$

Now (2.18) implies that

$$\lim_{\delta\to 0} E(\lambda u) = \lim_{\lambda\to 0} E(\lambda u) = 0, \tag{2.21}$$

and

$$\lim_{\delta\to 0} d(\delta) = 0. \tag{2.22}$$

It is easy to see that from (2.17)

$$d\left(\frac{3}{2}\right) = 0 \text{ and } d(\delta) < 0 \text{ for } \delta > \frac{3}{2}.$$

The proof is complete.

(3) We need to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \frac{3}{2}$ and for any $u \in \mathcal{N}_{\delta''}$, there is a $v \in \mathcal{N}_{\delta'}$ and a constant $\varepsilon(\delta', \delta'')$ such that $E(v) < E(u) - \varepsilon(\delta', \delta'')$. Indeed, by the definition of (2.19), we easily know that $D_\delta(\lambda(\delta)u) = 0$ and $\lambda(\delta'') = 1$. Let $h(\lambda) = E(\lambda u)$, we have

$$\begin{aligned} \frac{d}{d\lambda}h(\lambda) &= \frac{1}{\lambda}\big((1-\delta)\|\lambda u\|^2 + D_\delta(\lambda u)\big) \\ &= (1-\delta)\lambda\|u\|^2. \end{aligned} \tag{2.23}$$

Take $v = \lambda(\delta')u$, then $v \in \mathcal{N}_{\delta'}$.

For $0<\delta'<\delta''<1$, we obtain

$$
\begin{aligned}
E(u)-E(v) &= h(1)-h\left(\lambda\left(\delta'\right)\right) \\
&> \left(1-\delta''\right) r^{2}\left(\delta''\right) \lambda\left(\delta'\right)\left(1-\lambda\left(\delta'\right)\right) \equiv \varepsilon\left(\delta', \delta''\right).
\end{aligned} \tag{2.24}
$$

For $1<\delta''<\delta'<\frac{3}{2}$, we obtain

$$
\begin{aligned}
E(u)-E(v) &= h(1)-h\left(\lambda\left(\delta'\right)\right) \\
&> \left(\delta''-1\right) r^{2}\left(\delta''\right) \lambda\left(\delta''\right)\left(\lambda\left(\delta'\right)-1\right) \equiv \varepsilon\left(\delta', \delta''\right).
\end{aligned} \tag{2.25}
$$

Hence, the proof is complete.

□

Lemma 2.4. *Let $u_0 \in H_0^1(\Omega)$ and $0<\delta<\frac{3}{2}$. If $E(u) \leqslant d(\delta)$, then we have*

(1) *If $D_\delta(u)>0$, then $\|u\|^2<\frac{d(\delta)}{a(\delta)}$, where $a(\delta)=\frac{1}{2}-\frac{\delta}{3}$. In particular, if $D(u) \leqslant d$ and $D(u)>0$, then*

$$
\|u\|^2<6d. \tag{2.26}
$$

(2) *If $\|u\|^2>\frac{d(\delta)}{a(\delta)}$, then $D_\delta(u)<0$. In particular, if $E(u) \leqslant d$ and*

$$
\|u\|^2>6d, \tag{2.27}
$$

then $D(u)<0$.

(3) *If $D_\delta(u)=0$, then $\|u\|^2 \leqslant \frac{d(\delta)}{a(\delta)}$. In particular, if $E(u) \leqslant d$ and $D(u)=0$, then*

$$
\|u\|^2 \leqslant 6d. \tag{2.28}
$$

Proof. (1)For $0<\delta<\frac{3}{2}$, we see that

$$
\begin{aligned}
E(u) &= \frac{1}{2} \int_{\Omega}|\nabla u|^2+\frac{2}{3} \int_{\Omega} H(u) u \cdot u_x \wedge u_y \\
&= \left(\frac{1}{2}-\frac{\delta}{3}\right)\|u\|^2+\frac{1}{3} D_\delta(u) \\
&= a(\delta)\|u\|^2 \leq d(\delta).
\end{aligned} \tag{2.29}
$$

Therefore,

$$
\|u\|^2<\frac{d(\delta)}{a(\delta)}.
$$

Finally, (2) and (3) follow from (2.29). □

Lemma 2.5. *Let $u \in H_0^1(\Omega)$. We have*

(1) *0 is away from both N and $\mathcal{N}_-$, i.e. dist $(0, \mathcal{N})>0$, dist $(0, N_-)>0$,*
(2) *For any $\alpha>0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega)$.*

Proof. (1) If $u \in \mathcal{N}$, then we have

$$
\begin{aligned}
d \leq E(u) &= \frac{1}{2} \int_{\Omega}|\nabla u|^2+\frac{2}{3} \int_{\Omega} H(u) u \cdot u_x \wedge u_y \\
&= \frac{1}{6} \int_{\Omega}|\nabla u|^2+\frac{1}{3} D(u)=\frac{1}{6} \int_{\Omega}|\nabla u|^2.
\end{aligned} \tag{2.30}
$$

If $u \in \mathcal{N}_-$, then we have

$$\begin{aligned} d \le E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u\cdot u_x \wedge u_y \\ &= \frac{1}{6}\int_\Omega |\nabla u|^2 + \frac{1}{3}D(u) \le \frac{1}{6}\int_\Omega |\nabla u|^2. \end{aligned} \tag{2.31}$$

Hence, 0 is away from both N and $\mathcal{N}_-$, i.e. dist $(0,\mathcal{N}) > 0$, *dist* $(0, N_-) > 0$.

(2) Since $E(u) < \alpha$ and $D(u) > 0$, we obtain

$$\begin{aligned} \alpha > E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u\cdot u_x \wedge u_y \\ &= \frac{1}{6}\int_\Omega |\nabla u|^2 + \frac{1}{3}D(u) > \frac{1}{6}\int_\Omega |\nabla u|^2. \end{aligned} \tag{2.32}$$

Hence, for any $\alpha > 0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega)$. □

3 Low initial energy $E(u_0) < d$.

The goal of this section is to prove Theorem 3.2–3.4. A threshold result for the global solutions and finite time blowup will be given.

Theorem 3.1. *Assume that $u_0 \in H_0^1(\Omega)$, T is the maximal existence time of u, and $0 < e < d, \delta_1 < \delta_2$ are two roots of equation $d(\delta) = e$. We have*

(1) *If $D(u_0) > 0$, all weak solutions u of system (1.2) with $E(u_0) = e$ belong to W_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*
(2) *If $D(u_0) < 0$, all weak solutions u of system (1.2) with $E(u_0) = e$ belong to V_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*

Theorem 3.2. *(Global existence) Assume that $u_0 \in H_0^1(\Omega)$, $E(u_0) < d$, $D(u_0) > 0$. Then system (1.2) has a global solution $u(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ and $u(t) \in W$ for $0 \leqslant t < \infty$.*

Remark 3.1. *Result similar to Theorem 3.2 is obtained in [12]. But our proof is different to [12]. In fact, using the modified potential well method we can obtain the more general conclusion:*

If the assumption $D(u_0) > 0$ is replaced by $D_{\delta_2}(u_0) > 0$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u_0)$, then system (1.2) *admits a global weak solution.*

The following result is obtained in [12]. But our proof is different from the proof in [12]. For the reader's convenience, we will give the detailed proof.

Theorem 3.3. *Assume that $u_0 \in H_0^1(\Omega)$, $E(u_0) < d$ and $D(u_0) < 0$. Then the weak solution $u(t)$ of system* (1.2) *blows up in finite time, that is, there exists a $T > 0$ such that*

$$\lim_{t\to T}\int_0^t |u(\tau)|_2 d\tau = +\infty$$

Remark 3.2. *Assume that $u_0 \in H_0^1(\Omega)$,$E(u_0) < d$. When $D(u_0) > 0$, system (1.2) has a global solution. When $D(u_0) < 0$, system* (1.2) *does not admit any global weak solution.*

Theorem 3.4. *Assume that $u_0 \in H_0^1(\Omega)$, $E(u_0) < d$ and $D(u_0) > 0$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u_0)$. Then, for the global weak solution u of system (1.2), it holds*

$$|u|_2^2 \leqslant |u_0|_2^2 \, e^{-2(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{3.1}$$

Remark 3.3. *In comparison with the decay rate of [12], the result of the decay rate of $|u|_2$ in Theorem 3.4 is much more precise.*

In order to prove Theorem 4.1 – 4.3, we need the following lemmas:

Lemma 3.1. *For $0 < T \leq \infty$, assume that $u : \Omega \times [0, T) \to \mathbb{R}^3$ is a weak solution to system (1.2). Then it holds*

$$\int_{t_1}^{t_2} \int_\Omega |u_t|^2 + E(u(t_2)) = E(u(t_1)), \quad \forall t_1, t_2 \in (0, T). \tag{3.2}$$

Proof. Multiplying (1.2) by u_t and integrating over Ω via the integration by parts we get (3.2). $\square$

Lemma 3.2. *If $0 < E(u) < d$ for some $u \in H_0^1(\Omega)$, and $\delta_1 < 1 < \delta_2$ are the two roots of equation $d(\delta) = E(u)$, then the sign of $D_\delta(u)$ doesn't change for $\delta_1 < \delta < \delta_2$.*

Proof. Since $E(u) > 0$, we have $\|u\| \neq 0$. If the sign of $D_\delta(u)$ is changeable for $\delta_1 < \delta < \delta_2$, then we choose $\bar{\delta} \in (\delta_1, \delta_2)$ such that $D_{\bar{\delta}}(u) = 0$. Hence, by the definition of $d(\bar{\delta})$, we can obtain $E(u) \geqslant d(\bar{\delta})$, which contradicts $E(u) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$ (by Lemma 2.3 (3)). $\square$

Definition 3.5. *(Maximal existence time). Assume that $u(t)$ is a weak solution of system (1.2). The maximal existence time T of $u(t)$ is defined as follows:*

1. *If $u(t)$ exists for $0 \leqslant t < \infty$, then $T = +\infty$.*
2. *If there is a $t_0 \in (0, \infty)$ such that $u(t)$ exists for $0 \leqslant t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.*

Proof of theorem 3.1. (1) Let $u(t)$ be any weak solution of system (1.2) with $E(u_0) = e$, $D(u_0) > 0$, and T be the maximal existence time of $u(t)$. Using $E(u_0) = e, D(u_0) > 0$ and Lemma 3.2, we have $D_\delta(u_0) > 0$ and $E(u_0) < d(\delta)$. So $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D(u)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_{\delta_0}$, and $D_{\delta_0}(u(t_0)) = 0$, $\|u(t_0)\| \neq 0$ or $E(u(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |u_\tau|^2 + E(u(t)) = E(u_0) < d(\delta), \; \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.3}$$

we easily know that $E(u(t_0)) \neq d(\delta_0)$. If $D_{\delta_0}(u(t_0)) = 0, \|u(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E(u(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3).

(2) Let $u(t)$ be any weak solution of system (1.2) with $E(u_0) = e$, $D(u_0) < 0$, and T be the maximal existence time of $u(t)$. Using $E(u_0) = e, D(u_0) < 0$ and Lemma 3.2, we have $D_\delta(u_0) < 0$ and $E(u_0) < d(\delta)$. So $u_0 \in V_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $u(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D(u)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such

that $u(t_0) \in \partial V_{\delta_0}$, and $D_{\delta_0}(u(t_0)) = 0$, or $E(u(t_0)) = d(\delta_0)$. From the energy equality (3.3), we easily know that $E(u(t_0)) \neq d(\delta_0)$. If $D_{\delta_0}(u(t_0)) = 0$, and t_0 is the first time such that $D_{\delta_0}(u(t)) = 0$, then $D_{\delta_0}(u(t)) < 0$ for $0 \leqslant t < T$. By Lemma (2.2) (2), we have $\|u(t_0)\| > r(\delta_0)$ for $0 \leqslant t < T$. So, $\|u(t_0)\| > r(\delta_0)$ and $E(u(t_0)) \neq d(\delta_0)$, which contradicts (3.3).

□

Proof of theorem 3.2. From the standard argument in [11], we can prove the local existence result of (1.2) in a more general case of initial value $u_0 \in H_0^1(\Omega)$ and $u \in C^0([0,T_0], H_0^1(\Omega))$.

Using $E(u_0) < d, D(u_0) > 0$ and Lemma 3.2, we have $D_\delta(u_0) > 0$ and $E(u_0) < d(\delta)$. So $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D(u)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0,T)$ such that $u(t_0) \in \partial W_{\delta_0}$, and $D_{\delta_0}(u(t_0)) = 0, \|u(t_0)\| \neq 0$ or $E(u(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |u_\tau|^2 + E(u(t)) = E(u_0) < d(\delta),\ \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.4}$$

we easily know that $E(u(t_0)) \neq d(\delta_0)$. If $D_{\delta_0}(u(t_0)) = 0, \|u(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E(u(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3).

□

Remark 3.4. *If in Theorem 3.2 the condition $D_{\delta_2}(u_0) > 0$ is replaced by $\|u_0\| < r(\delta_2)$, then system (1.2) has a global weak solution $u(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_t(t) \in L^2(0,\infty; H_0^1(\Omega))$ and the following result holds*

$$\|u\| < \frac{d(\delta)}{a(\delta)}, \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty, \tag{3.5}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta), \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty. \tag{3.6}$$

In particular

$$\|u\|^2 < \frac{d(\delta_1)}{a(\delta_1)}, \tag{3.7}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta_1), \quad 0 \leqslant t < \infty. \tag{3.8}$$

Proof of theorem 3.3. We argue by contradiction. Suppose that there would exist a global weak solution $u(t)$. Set

$$f(t) = \int_0^t \int_\Omega |u|^2, t > 0. \tag{3.9}$$

Multiplying (1.2) by u and integrating over $\Omega \times (0,t)$, we get

$$\int_\Omega |u(t)|^2 - \int_\Omega |u_0|^2 = -2 \int_0^t \int_\Omega \left(|\nabla u|^2 + 2H(u) u \cdot u_x \wedge u_y\right). \tag{3.10}$$

According to the definition of $f(t)$, we have $f'(t)=\int_\Omega |u(t)|^2$ and hence

$$f'(t)=\int_\Omega |u|^2=\int_\Omega |u_0|^2-2\int_0^t\int_\Omega \left(|\nabla u|^2+2H(u)u\cdot u_x\wedge u_y\right), \tag{3.11}$$

and

$$f''(t)=-2\int_\Omega \left(|\nabla u|^2+2H(u)u\cdot u_x\wedge u_y\right)=-2D(u). \tag{3.12}$$

Now using (3.2),(3.12) and

$$\begin{aligned} E(u)&=\frac{1}{2}\int_\Omega |\nabla u|^2+\frac{2}{3}\int_\Omega H(u)u\cdot u_x\wedge u_y\\ &=\frac{1}{6}\|u\|^2+\frac{1}{3}D(u), \end{aligned} \tag{3.13}$$

we can obtain

$$\begin{aligned} f''(t)&=6\int_0^t\int_\Omega u_\tau^2 d\tau+f'(t)-6E\left(u_0\right)\\ &=6\int_0^t\int_\Omega u_\tau^2 d\tau+\int_\Omega |u|^2-6E\left(u_0\right). \end{aligned} \tag{3.14}$$

Note that

$$\begin{aligned} f(t)f''(t)&=f(t)\left[6\int_0^t\int_\Omega u_\tau^2 d\tau+f'(t)-6E\left(u_0\right)\right]\\ &=6\int_0^t\int_\Omega |u|^2\int_0^t\int_\Omega u_\tau^2 d\tau+f(t)f'(t)-6E(u_0)\int_0^t\int_\Omega |u|^2. \end{aligned} \tag{3.15}$$

Hence, we have

$$\begin{aligned} f(t)f''(t)-\frac{3}{2}(f'(t))^2\geq{}& 6\int_0^t\int_\Omega |u|^2\int_0^t\int_\Omega u_\tau^2 d\tau-6\left(\int_0^t\int_\Omega u_\tau\cdot u d\tau\right)^2\\ &+f(t)f'(t)-3f'(t)\int_\Omega u_0^2-6E(u_0)\int_0^t\int_\Omega |u|^2. \end{aligned} \tag{3.16}$$

Making use of the Schwartz inequality, we have

$$f(t)f''(t)-\frac{3}{2}(f'(t))^2\geq f(t)f'(t)-3f'(t)\int_\Omega u_0^2-6E(u_0)\int_0^t\int_\Omega |u|^2. \tag{3.17}$$

Next, we distinguish two case:
(1) If $E\left(u_0\right)\leqslant 0$, then

$$f(t)f''(t)-\frac{3}{2}(f'(t))^2\geq f(t)f'(t)-3f'(t)\int_\Omega u_0^2. \tag{3.18}$$

Now we prove $D(u)<0$ for $t>0$. If not, we must be allowed to choose a $t_0>0$ such that

$D\left(u\left(t_0\right)\right)=0$ and $D(u)<0$ for $0 \leqslant t<t_0$. From Lemma 2.2 (2), we have $\|u\|>r(1)$ for $0 \leqslant t<t_0,\left\|u\left(t_0\right)\right\| \geqslant r(1)$ and $E\left(u\left(t_0\right)\right) \geqslant d$, which contradicts (3.3). From (3.12) we have $f'(t)>0$ for $t \geqslant 0$. From $f'(0)=\int_\Omega\left|u_0\right|^2 \geqslant 0$, we can know that there exists a $t_0 \geqslant 0$ such that $f'\left(t_0\right)>0$. For $t \geqslant t_0$ we have

$$f(t) \geqslant f'\left(t_0\right)\left(t-t_0\right)>f'(0)\left(t-t_0\right). \tag{3.19}$$

Hence, for sufficiently large t , we obtain

$$f(t)>3 \int_\Omega\left|u_0\right|^2, \tag{3.20}$$

then

$$f(t) f''(t)-\frac{3}{2}\left(f'(t)\right)^2>0.$$

(2) If $0<E\left(u_0\right)<d$, then by Theorem 3.1 we have $u(t) \in V_\delta$ for $1<\delta<\delta_2, t \geqslant 0$, and $D_\delta(u)<0$, $\|u\|>r(\delta)$ for $1<\delta<\delta_2, t \geqslant 0$, where δ_2 is the larger root of equation $d(\delta)=E\left(u_0\right).$ Hence, $D_{\delta_2}(u) \leqslant 0$ and $\|u\| \geqslant r\left(\delta_2\right)$ for $t \geqslant 0$. By (3.12), we have

$$\begin{aligned}
f''(t) & =-2 D(u)=2\left(\delta_2-1\right)\|\nabla u\|_2^2-2 D_{\delta_2}(u), \\
& \geqslant 2\left(\delta_2-1\right)|\nabla u|_2=2\left(\delta_2-1\right)\|u\|^2 \geqslant 2\left(\delta_2-1\right) r^2\left(\delta_2\right), \quad t \geqslant 0, \\
f'(t) & \geqslant 2\left(\delta_2-1\right) r^2\left(\delta_2\right) t+f'(0) \geqslant 2\left(\delta_2-1\right) r^2\left(\delta_2\right) t, \quad t \geqslant 0, \\
f(t) & \geqslant\left(\delta_2-1\right) r^2\left(\delta_2\right) t^2, \quad t \geqslant 0 .
\end{aligned} \tag{3.21}$$

Therefore, for sufficiently large t, we infer

$$\frac{1}{2} f(t)>3 \int_\Omega\left|u_0\right|^2, \quad \frac{1}{2} f'(t)>6 E\left(u_0\right). \tag{3.22}$$

Then, (3.17) implies that

$$\begin{aligned}
f(t) f''(t)-\frac{3}{2}\left(f'(t)\right)^2 \geq f(t) f'(t)-3 f'(t) \int_\Omega u_0^2-6 E\left(u_0\right) f(t). \\
=\left(\frac{1}{2} f(t)-3 \int_\Omega\left|u_0\right|^2\right) f'(t) \\
+\left(\frac{1}{2} f'(t)-6 E\left(u_0\right)\right) f(t)>0.
\end{aligned}$$

The remainder of the proof is the same as that in [16]. □

Proof of theorem 3.4. Multiplying (1.2) by v, $v \in L^{\infty}\left(0, \infty ; H_0^1(\Omega)\right)$, we have

$$\left(u_t, v\right)+(\nabla u, \nabla v)=2 H\left(u_x \wedge u_y, v\right). \tag{3.23}$$

Letting $v=u$, (3.23) implies that

$$\frac{1}{2} \frac{d}{d t}|u|_2^2+D(u)=0, \quad 0 \leqslant t<\infty. \tag{3.24}$$

From $0<E\left(u_0\right)<d, D\left(u_0\right)>0$ and Lemma 3.1, we have $u(t) \in W_\delta$ for $\delta_1<\delta<\delta_2$ and $0 \leqslant t<\infty$, where $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=E\left(u_0\right).$ Hence, we obtain $D_\delta(u) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{\delta_1}(u) \geqslant 0$ for $0 \leqslant t < \infty$. So, (3.24) gives

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + (1-\delta_1)|u|_2^2 + D_\delta(u) = 0, \quad 0 \leqslant t < \infty. \tag{3.25}$$

Now (3.24) implies that

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + (1-\delta_1)|u|_2^2 \leq 0, \quad 0 \leqslant t < \infty. \tag{3.26}$$

and

$$|u|_2^2 \leqslant |u_0|_2^2 - 2(1-\delta_1)\int_0^t |u(\tau)|^2 d\tau, \quad 0 \leqslant t < \infty. \tag{3.27}$$

By Gronwall's inequality, we have

$$|u|_2^2 \leqslant |u_0|_2^2\, e^{-2(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{3.28}$$

□

4 Critical initial energy $E(u_0) = d$.

The goal of this section is to prove Theorem 4.1–4.3.

Theorem 4.1. *(Global existence) Assume that $u_0 \in H_0^1(\Omega)$,$E(u_0) = d$ and $D(u_0) \geqslant 0$. Then system (1.2) has a global weak solution $u(t) \in L^\infty(0,\infty; H_0^1(\Omega))$ and $u(t) \in \overline{W} = W \cup \partial W$ for $0 \leqslant t < \infty$*

Lemma 4.1. *Assume that $u \in H_0^1(\Omega)$, $\|\nabla u\|_2^2 \neq 0$, and $D(u) \geq 0$. Then:*

(1) $\lim_{\lambda\to 0} E(\lambda u) = 0, \lim_{\lambda\to+\infty} E(\lambda u) = -\infty$,

(2) *On the interval $0 < \lambda < \infty$, there exists a unique $\lambda^* = \lambda^*(u)$, such that*

$$\frac{d}{d\lambda}E(\lambda u)|_{\lambda=\lambda^*} = 0, \tag{4.1}$$

(3) *$E(\lambda u)$ is increasing on $0 \leqslant \lambda \leqslant \lambda^*$, decreasing on $\lambda^* \leqslant \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$,*

(4) *$D(\lambda u) > 0$ for $0 < \lambda < \lambda^*, D(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$, and $D(\lambda^* u) = 0$.*

Proof. (1) Firstly, the assumption $D(u) \geq 0$ implies that

$$\int_\Omega H(u)u \cdot u_x \wedge u_y < 0.$$

From the definition of $E(u)$, i.e.

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u \cdot u_x \wedge u_y,$$

and we see that

$$E(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2 + \frac{2\lambda^3}{3}\int_\Omega H(u)u \cdot u_x \wedge u_y.$$

Hence, we have

$$\lim_{\lambda\to 0} E(\lambda u)=0 \quad \text{and} \quad \lim_{\lambda\to+\infty} E(\lambda u)=-\infty. \tag{4.2}$$

(2) It is easy to show that

$$\frac{d}{d\lambda}E(\lambda u)=\lambda\int_\Omega |\nabla u|^2+2\lambda^2\int_\Omega H(u)u\cdot u_x\wedge u_y,$$

which leads to the conclusion.

(3) By Lemma 4.1 (2), one has

$$\begin{aligned}&\frac{d}{d\lambda}E(\lambda u)>0 \text{ for } 0<\lambda<\lambda^*,\\ &\frac{d}{d\lambda}E(\lambda u)<0 \text{ for } \lambda^*<\lambda<\infty,\end{aligned} \tag{4.3}$$

which leads to the conclusion.

(4) The conclusion follows from

$$D(\lambda)=\frac{d}{d\lambda}E(\lambda u)=\lambda\int_\Omega |\nabla u|^2+2\lambda^2\int_\Omega H(u)u\cdot u_x\wedge u_y.$$

□

Proof of theorem 4.1. Firstly, $E(u_0)=d$ implies that $\|u_0\|_{H_0^1}\neq 0$. Choose a sequence $\{\lambda_m\}$ such that $0<\lambda_m<1$, $m=1,2,\ldots$ and $\lambda_m\to 1$ as $m\to\infty$. Let $u_{0m}=\lambda_m u_0$. We consider the following initial problem

$$\begin{cases} u_t=\Delta u-2H(u)u_x\wedge u_y, \text{ in } \Omega\times(0,\infty),\\ u|_{t=0}=u_{0m}, \text{ in } \Omega,\\ u|_{\partial\Omega}=\chi, t>0,\end{cases} \tag{4.4}$$

From $D(u_0)\geqslant 0$ and Lemma 4.1, we have $\lambda^*=\lambda^*(u_0)\geqslant 1$. Thus, we get $D(u_{0m})=D(\lambda_m u_0)>0$ and $E(u_{0m})=E(\lambda_m u_0)<E(u_0)=d$. From Theorem 3.2, it follows that for each m problem (4.4) admits a global weak solution $u_m(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_{mt}(t)\in L^2(0,\infty;H_0^1(\Omega))$ and $u_m(t)\in W$ for $0\leqslant t<\infty$ satisfying

$$(u_{m,t},v)+(\nabla u_{m,t},\nabla v)=2(H(u)u_{m,x}\wedge u_{m,y},v), \quad \text{for all } v\in H_0^1(\Omega), t>0. \tag{4.5}$$

$$\int_0^t\int_\Omega |u_{m,\tau}|^2+E(u_m(t))=E(u_{0m})<d, \quad 0\leqslant t<\infty, \tag{4.6}$$

which implies that

$$E(u_m)=\frac{1}{6}\|u_m\|^2+\frac{1}{3}D(u_m). \tag{4.7}$$

So, one has

$$\int_0^T |u_{m\tau}|_2^2\,d\tau+\frac{1}{6}\|u_m\|_{H_0^1}^2<d, \quad 0\leqslant t<\infty. \tag{4.8}$$

The remainder of the proof is similar to the proof of Theorem 3.2. □

Theorem 4.2. *(Blow-up) Assume that $u_0\in H_0^1(\Omega)$, $E(u_0)=d$ and $I(u_0)>0$, Then the existence time of weak solution for system (1.2) is finite.*

Proof of theorem 4.2. Let $u(t)$ be any weak solution of system (1.2) with $E(u_0) = d$ and $DI(u_0) < 0$, T be the existence time of $u(t)$. We next prove $T < \infty$. We argue by contradiction. Suppose that there would exist a global weak solution $u(t)$. Set

$$f(t) = \int_0^t \int_\Omega |u|^2, t > 0. \tag{4.9}$$

Multiplying (1.2) by u and integrating over $\Omega \times (0,t)$, we get

$$\int_\Omega |u(t)|^2 - \int_\Omega |u_0|^2 = -2\int_0^t \int_\Omega \left(|\nabla u|^2 + 2H(u)u \cdot u_x \wedge u_y\right). \tag{4.10}$$

According to the definition of $f(t)$, we have $f'(t) = \int_\Omega |u(t)|^2$ and hence

$$f'(t) = \int_\Omega |u|^2 = \int_\Omega |u_0|^2 - 2\int_0^t \int_\Omega \left(|\nabla u|^2 + 2H(u)u \cdot u_x \wedge u_y\right), \tag{4.11}$$

and

$$f''(t) = -2\int_\Omega \left(|\nabla u|^2 + 2H(u)u \cdot u_x \wedge u_y\right) = -2D(u). \tag{4.12}$$

Now using (3.2),(4.12) and

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2}{3}\int_\Omega H(u)u \cdot u_x \wedge u_y \\ &= \frac{1}{6}\|u\|^2 + \frac{1}{3}D(u), \end{aligned} \tag{4.13}$$

we can obtain

$$\begin{aligned} f''(t) &= 6\int_0^t \int_\Omega u_\tau^2 d\tau + f'(t) - 6d \\ &= 6\int_0^t \int_\Omega u_\tau^2 d\tau + \int_\Omega |u|^2 - 6d. \end{aligned} \tag{4.14}$$

Note that

$$\begin{aligned} f(t)f''(t) &= f(t)\left[6\int_0^t \int_\Omega u_\tau^2 d\tau + f'(t) - 6E(u_0)\right] \\ &= 6\int_0^t \int_\Omega |u|^2 \int_0^t \int_\Omega u_\tau^2 d\tau + f(t)f'(t) - 6d\int_0^t \int_\Omega |u|^2. \end{aligned} \tag{4.15}$$

Hence, we have

$$\begin{aligned} f(t)f''(t) - \frac{3}{2}(f'(t))^2 \geq{}& 6\int_0^t \int_\Omega |u|^2 \int_0^t \int_\Omega u_\tau^2 d\tau - 6\left(\int_0^t \int_\Omega u_\tau \cdot u d\tau\right)^2 \\ &+ f(t)f'(t) - 3f'(t)\int_\Omega u_0^2 - 6d\int_0^t \int_\Omega |u|^2. \end{aligned} \tag{4.16}$$

Hence, according to (4.16) and the Schwartz inequality, we obtain

$$
\begin{aligned}
f(t)f''(t) - \frac{3}{2}(f'(t))^2 \geq f(t)f'(t) - 3f'(t)\int_\Omega u_0^2 - 6df(t). \\
= \left(\frac{1}{2}f(t) - 3\int_\Omega |u_0|^2\right) f'(t) \\
+ \left(\frac{1}{2}f'(t) - 2(p+1)d\right) f(t) > 0.
\end{aligned} \tag{4.17}
$$

On the other hand, from $E(u_0) = d > 0, D(u_0) < 0$ and the continuity of $E(u)$ and $D(u)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $E(u(t_1)) > 0$ and $D(u) < 0$ for $0 \leqslant t \leqslant t_1$. Hence $(u_t, u) = -D(u) > 0, |u_t|_2 > 0$ for $0 \leqslant t \leqslant t_1$. So, using the continuity of $\int_0^t |u_\tau|_2^2 \, d\tau$, we can choose a t_1 such that

$$
0 < d_1 = d - \int_0^{t_1} |u_\tau|_2^2 \, d\tau < d. \tag{4.18}
$$

And by (3.4), we get

$$
0 < E(u(t_1)) = d - \int_0^{t_1} |u_\tau|_2^2 \, d\tau = d_1 < d. \tag{4.19}
$$

So we can choose $t = t_1$ as the initial time, then we obtain $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2), t_1 \leqslant t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Thus we get $D_\delta(u) < 0$ and $\|u\| > r(\delta)$ for $\delta \in (1, \delta_2), t_1 \leqslant t < \infty$, and $D_{\delta_2}(u) \leqslant 0, \|u\| \geqslant r(\delta_2)$ for $t_1 \leqslant t < \infty$. Thus (4.12) implies that

$$
\begin{aligned}
f''(t) &= -2D(u) = 2(\delta_2 - 1)\|\nabla u\|_2^2 - 2D_{\delta_2}(u), \\
&\geqslant 2(\delta_2 - 1)|\nabla u|_2 = 2(\delta_2 - 1)\|u\|^2 \geqslant 2(\delta_2 - 1) r^2(\delta_2), \quad t \geqslant t_1, \\
f'(t) &\geqslant 2(\delta_2 - 1) r^2(\delta_2)(t - t_1) + f'(t_1) \geqslant 2(\delta_2 - 1) r^2(\delta_2)(t - t_1), \quad t \geqslant 0, \\
f(t) &\geqslant (\delta_2 - 1) r^2(\delta_2)(t - t_1)^2 + M(t_1) > (\delta_2 - 1) r^2(\delta_2)(t - t_1)^2, \quad t \geqslant t_1.
\end{aligned} \tag{4.20}
$$

Therefore, for sufficiently large t, we infer

$$
\frac{1}{2}f(t) > 3\int_\Omega |u_0|^2, \quad \frac{1}{2}f'(t) > 6d. \tag{4.21}
$$

Then, (4.17) implies that

$$
\begin{aligned}
f(t)f''(t) - \frac{3}{2}(f'(t))^2 \geq f(t)f'(t) - 3f'(t)\int_\Omega u_0^2 - 6E(u_0)f(t). \\
= \left(\frac{1}{2}f(t) - 3\int_\Omega |u_0|^2\right) f'(t) \\
+ \left(\frac{1}{2}f'(t) - 2(p+1)E(u_0)\right) f(t) > 0
\end{aligned}
$$

The remainder of the proof is the same as that in [16].

□

Theorem 4.3. *Assume that $u_0 \in H_0^1(\Omega)$, $E(u_0) = d$ and $D(u_0) > 0$, $\delta_1 < \delta_2$ are the two* *roots of equation $d(\delta) = E(u_0)$. Then, for the global weak solution u of system (1.2), it holds*

$$|u|_2^2 \leqslant |u_0|_2^2 \, e^{-2(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{4.22}$$

Proof of theorem 4.3. We first know that system (1.2) has a global weak solution from Theorem 4.2. Futhermore, Using Theorem 3.3, Theorem 4.2 and (3.3), if $u(t)$ is a global weak solution of system (1.2) with $E(u_0) = d$, $D(u_0) > 0$, then must have $D(u) \geq 0$ for $0 \leq t < +\infty$. Next, we distinguish two case:

(1) Suppose that $D(u) > 0$ for $0 \leqslant t < \infty$. Multiplying (1.2) by v, $v \in L^\infty(0, \infty; H_0^1(\Omega))$, we have

$$(u_t, v) + (\nabla u, \nabla v) = 2H(u_x \wedge u_y, v). \tag{4.23}$$

Letting $v = u$, (4.23) implies that

$$\frac{1}{2}\frac{d}{dt}\,|u|_2^2 = -D(u) < 0, \quad 0 \leqslant t < \infty. \tag{4.24}$$

Since $|u_t|_2 > 0$, we have that $\int_0^t |u_\tau|^2 \, d\tau$ is increasing for $0 \leqslant t < \infty$. By choosing any $t_1 > 0$ and letting

$$d_1 = d - \int_0^{t_1} |u_\tau|_2^2 d\tau \tag{4.25}$$

From (3.3), if follows that $0 < E(u) \leq d_1 < d$, and $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E(u_0)$. Hence, we obtain $D_{\delta_1}(u) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{\delta_1}(u) \geqslant 0$ for $t_1 \leqslant t < \infty$. So, (4.24) gives

$$\frac{1}{2}\frac{d}{dt}\,|u|_2^2 + (1-\delta_1)\,|u|_2^2 + D_\delta(u) = 0, \quad t_1 \leqslant t < \infty. \tag{4.26}$$

Now (4.24) implies that

$$\frac{1}{2}\frac{d}{dt}\,|u|_2^2 + (1-\delta_1)\,|u|_2^2 \leq 0, \quad t_1 \leqslant t < \infty. \tag{4.27}$$

and

$$|u|_2^2 \leqslant |u_0|_2^2 - 2\,(1-\delta_1)\int_0^t |u(\tau)|^2 d\tau, \quad t_1 \leqslant t < \infty. \tag{4.28}$$

By Gronwalls inequality, we have

$$|u|_2^2 \leqslant |u_0|_2^2 \, e^{-2(1-\delta_1)t}, \quad t_1 \leqslant t < \infty. \tag{4.29}$$

□

(2) Suppose that there exists a $t_1 > 0$ such that $D(u(t_1)) = 0$ and $D(u) > 0$ for $0 \leqslant t < t_1$. Then, $|u_t|_2 > 0$ and $\int_0^t |u_\tau|_2^2 \, d\tau$ is increasing for $0 \leqslant t < t_1$. By (4.25) we have

$$E(u(t_1)) = d - \int_0^{t_1} |u_\tau|_2^2 d\tau < d, \tag{4.30}$$

and $\|u(t_1)\| = 0$. Then, we have that $u(t) \equiv 0$ for $t_1 \leqslant t < \infty$.

Hence, the proof is complete.

5 High initial energy $E(u_0) > d$.

In this section, we investigate the conditions to ensure the existence of global solutions or blow-up solutions to system (1.2) with $E(u_0) > d$.

Lemma 5.1. *For any $\alpha > d, \lambda_\alpha$ and Λ_α defined in (2.7) satisfy*

$$0 < \lambda_\alpha \le \Lambda_\alpha < +\infty. \tag{5.1}$$

Proof. (1) By Hölder inequality, fundamental inequality and $u \in \mathcal{N}$, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u|^2 &= \left|\int_\Omega H(u)u \cdot u_x \wedge u_y\right| \\ &\le H\left(\int_\Omega u^2\right)^{\frac{1}{2}}\left(\int_\Omega |u_x \wedge u_y|^2\right)^{\frac{1}{2}} \\ &\le CH\left(\int_\Omega u^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{5.2}$$

Then from Lemma 2.5 (1), we have $\lambda_\alpha > 0$.

(2) Using the isoperimetric inequality and $u \in \mathcal{N}$, we have

$$\frac{1}{2|H|}\int_\Omega |\nabla u|^2 = \left|\int_\Omega u \cdot u_x \wedge u_y\right| \le \frac{1}{4\sqrt{2\pi}}\left(\int_\Omega |\nabla u|^2\right)^{\frac{3}{2}}. \tag{5.3}$$

So we have $\|u\| \le \frac{|H|}{2\sqrt{2\pi}}$, which leads to the conclusion. □

Theorem 5.1. *Suppose that $E(u_0) > d$, then we have*

(1) *If $u_0 \in \mathcal{N}_+$ and $|u_0|_2 \le \lambda_{E(u_0)}$, then $u_0 \in \mathcal{G}_0$,*
(2) *If $u_0 \in \mathcal{N}_-$ and $|u_0|_2 \ge \Lambda_{E(u_0)}$, then $u_0 \in \mathcal{B}$.*

Proof. The maximal existence time of the solutions to system (1.2) with initial value u_0 is denoted by T_0. If the solution is global, i.e. $T(u_0) = +\infty$, the limit set of u_0 is denoted by ω_0.

(1) Suppose that $u_0 \in \mathcal{N}_+$ with $|u_0|_2 \le \lambda_{E(u_0)}$. We firstly prove that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T(u_0))$. Assume, on the contrary, that there exists a $t_0 \in (0, T(u_0))$ such that $u(t) \in \mathcal{N}_+$ for $0 \le t < t_0$ and $u(t_0) \in \mathcal{N}$. It follows from $D(u(t)) = -\int_\Omega u_t(x,t)u(x,t)\mathrm{d}x$ that $u_t(x,t) \ne 0$ for $(x,t) \in \Omega \times (0, t_0)$. Recording to(3.2) we then have $E(u(t_0)) < E(u_0)$, which implies that $u(t_0) \in E^{E(u_0)}$. Therefore, $u(t_0) \in \mathcal{N}^{E(u_0)}$. Recalling the definition of $\lambda_{E(u_0)}$, we get

$$|u(t_0)|_2 \ge \lambda_E(u_0). \tag{5.4}$$

Since $D(u(t)) > 0$ for $t \in [0, t_0)$, we obtain from (3.24) that

$$|u(t_0)|_2 < |u_0|_2 \le \lambda_{E(u_0)}. \tag{5.5}$$

which contradicts (5.4). Hence, $u(t) \in \mathcal{N}_+$ which shows that $u(t) \in E^{E(u_0)}$ for all $t \in [0, T(u_0))$. Now Lemma 3.2 (2) implies that the orbit $\{u(t)\}$ remains bounded in $H_0^1(\Omega)$ for $t \in [0, T(u_0))$ so that $T(u_0) = \infty$. Assume that ω is an arbitrary element in $\omega(u_0)$. Then by (3.2) and (3.24) we obtain

$$|\omega|_2 > \Lambda_{E(u_0)}, \quad E(\omega) < E(u_0), \tag{5.6}$$

which, according to the definition of $\lambda_{E(u_0)}$ again, implies that $\omega\left(u_0\right) \cap N=\emptyset$. So, $\omega\left(u_0\right)=\{0\}$, i.e. $u_0 \in \mathcal{G}_0$

(2) Suppose that $u_0 \in \mathcal{N}_{-}$ with $\left|u_0\right|_2 \geq \Lambda_{E(u_0)}$. We now prove that $u(t) \in \mathcal{N}_{-}$ for all $t \in\left[0, T\left(u_0\right)\right)$. Assume, on the contrary, that there exists a $t^0 \in\left(0, T\left(u_0\right)\right)$ such that $u(t) \in \mathcal{N}_{-}$ for $0 \leq t<t^0$ and $u(t^0) \in \mathcal{N}$. Similarly to case (1), one has $E(u(t^0))<E\left(u_0\right)$, which implies that $u(t^0) \in E^{E(u_0)}$. Therefore, $u(t^0) \in \mathcal{N}^{E(u_0)}$. Recalling the definition of $\Lambda_{E(u_0)}$, we infer

$$|u(t^0)|_2 \leq \Lambda_{E(u_0)} \tag{5.7}$$

On the other hand, from (3.24) and the fact that $D(u(t))<0$ for $t \in[0, t^0)$, we obtain

$$|u(t^0)|_2>|u_0|_2 \geq \Lambda_{E(u_0)}, \tag{5.8}$$

which contradicts (5.7).

Assume that $T\left(u_0\right)=\infty$. Then for each $\omega \in \omega\left(u_0\right)$, it follows from by (3.2) and (3.24) that

$$\|\omega\|_2>\Lambda_{E(u_0)}, \quad E(\omega)<E\left(u_0\right). \tag{5.9}$$

Noting the definition of $\Lambda_{E(u_0)}$ again, we have $\omega\left(u_0\right) \cap N=\emptyset$. Hence, it is holded that $\omega\left(u_0\right)=\{0\}$, which contradicts Lemma 3.2 (1). Therefore, $T\left(u_0\right)<\infty$ and we can complete this proof.

□

Theorem 5.2. *Assume that $u_0 \in H_0^1(\Omega)$ satisfies*

$$E(u_0) \leq |u_0|_2<-\frac{1}{3} \int_{\Omega} H(u_0) u_0 \cdot u_{0, x} \wedge u_{0, y}, \tag{5.10}$$

Then, $u_0 \in \mathcal{N}_{-} \cap \mathcal{B}$.

Proof. Firstly, we observe

$$\begin{aligned}
E(u_0) & =\frac{1}{2} \int_{\Omega}|\nabla u_0|^2+\frac{2}{3} \int_{\Omega} H(u_0) u_0 \cdot u_{0, x} \wedge u_{0, y} \\
& =\frac{1}{2} D(u_0)-\frac{1}{3} \int_{\Omega} H(u_0) u_0 \cdot u_{0, x} \wedge u_{0, y}
\end{aligned} \tag{5.11}$$

Thus, we have

$$E(u_0)+\frac{1}{3} \int_{\Omega} H(u_0) u_0 \cdot u_{0, x} \wedge u_{0, y}=\frac{1}{2} D(u_0)<0, \tag{5.12}$$

which shows that $u_0 \in \mathcal{N}_{-}$. Then for any $u \in \mathcal{N}_{E(u_0)}$, one has

$$-2 \int_{\Omega} H(u) u \cdot u_x \wedge u_y=\|u\|^2 \leq 6 E(u_0).$$

Taking supremum over $\mathcal{N}_{E(u_0)}$ and (5.10), by Theorem 5.1 we can deduce

$$|u_0|_2 \geq \Lambda_{E(u_0)}.$$

Thus, $u_0 \in \mathcal{N}_{-} \cap \mathcal{B}$. □

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