ESTIMATES OF MEAN-TYPE FRACTIONAL INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS

MUHAMMAD SAMRAIZ, ZAHIDA PERVEEN, SAJID IQBAL, SAIMA NAHEED, AND THABET ABDELJAWAD

Abstract. In this article, we established a wide range of fractional mean-type integral inequalities for notable Hilfer fractional derivative using twice differentiable convex and s-convex functions for $s \in (0,1]$ with related identities. Also the results for Caputo fractional derivatives are derived as a special case of our general results.

1. Introduction

The subject of fractional calculus has achieved a significant prominence during most recent couple of years due to its demonstrated applications in the field of science and engineering. This offers useful strategies to solve differential and integral equations see the books [1,2]. Fractional calculus has been applied in different areas of science, engineering, financial mathematics, applied sciences, bio engineering etc.

Mathematical inequalities significantly important in the study of mathematics and related fields. Now a days, fractional integral inequalities are fruitful in generating the uniqueness of solutions for fractional partial differential equations. They also provide boudedness of the solutions of fractional boundary value problems. These recommendations have inspired various researchers in the field of integral inequalities to inquire the extensions by involving fractional calculus operators.

The convex functions are utilized to create numerous inequalities in literature [3–8]. Hermite-Hadamard's inequality [9] is one of the most important classical inequalities as it has a rich geometrical meaning and applications [10–13]. Hermite-Hadamard's double inequality is one of the most widely studied concerning convex functions. The inequality is defined as follows:

Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $\theta, \zeta \in I$ with $\theta < \zeta$. Then

$$\psi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{1}{\zeta-\theta}\int_{\theta}^{\zeta}\psi(\nu)d\nu \leq \frac{\psi(\theta)+\psi(\zeta)}{2}. \tag{1.1}$$

If ψ is concave then the inequalities (1.1) hold in reverse direction. For particular choices of function ψ, some classical inequalities for means can be derived from (1.1), (see [14]). The principle point of this paper is to infer Hermite Hadamard-type integral inequalities for Hilfer fractional derivative.

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2. Preliminaries

This section includes some preliminary facts.

Definition 2.1. *([15])Let $\psi : [\theta, \zeta] \to \mathbb{R}$ is said to be convex if the inequality*

$$\psi(\nu\gamma + (1-\nu)\beta) \leq \nu\psi(\gamma) + (1-\nu)\psi(\beta),$$

holds for $\gamma, \beta \in [\theta, \zeta]$ and $\nu \in [0, 1]$.

The definition of classical Riemann-Liouville fractional derivative (see [16, Chapter 4]) is given as:

Definition 2.2. *Let $\Phi \in L^1[\theta, \zeta]$, then the right-sided and left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$ are defined by*

$$D^{\gamma}_{\theta+}\psi(\nu) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{d\nu}\right)^n \int_{\theta}^{\nu} (\nu - \tau)^{n-\gamma-1}\psi(\tau)d\tau,$$

and

$$D^{\gamma}_{\zeta^-}\psi(\nu) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{d\nu}\right)^n \int_{\nu}^{\zeta} (\tau - \nu)^{n-\gamma-1}\psi(\tau)d\tau,$$

where

$$n = [\gamma] + 1, \;\; \nu \in [\theta, \zeta].$$

Let $x > \theta > 0$. By $L^1(\theta, x)$, we denote the space of all Lebesgue integrable functions on the interval (θ, x). For any $\psi \in L^1(\theta, x)$ the Riemann-Liouville fractional integral of ψ of order γ is defined by

$$(I^{\gamma}_{\theta_+}\psi)(\nu) = \frac{1}{\Gamma(\nu)}\int_{\theta}^{\nu}(x-\tau)^{\gamma-1}\psi(\tau)d\tau = (\psi * K_{\gamma})(\nu), \quad \nu \in [\theta, x], \quad (\gamma > 0), \tag{2.1}$$

where $K_{\gamma}(\nu) = \frac{\nu^{\gamma-1}}{\Gamma(\gamma)}$. The integral on the right side of (2.1) exists for almost $\nu \in [\theta, x]$ and $I^{\gamma}_{\theta_+}\psi \in L^1(\theta, x)$. The space of all continuous differentiable functions up to order m, on $[\theta, x]$ is presented by $C^m[\theta, x]$. By $AC[\theta, x]$, we mean the space of all absolutely continuous functions on $[\theta, x]$. The space $AC^m[\theta, x]$, denote the all those functions $\psi \in C^m[\theta, x]$ with $\psi^{(m-1)} \in AC[\theta, x]$. By $L_{\infty}(\theta, x)$, we denote the space of all measurable functions essentially bounded on $[\theta, x]$. Let $\mu > 0, m = [\mu]+1$ and $f \in AC^m[a, b]$. The Caputo derivative of order $\gamma > 0$ is defined as

$$({}^C D^{\gamma}_{\theta_+}\psi)(\nu) = \left(I^{m-\gamma}_{\theta_+}\frac{d^m}{d\nu^m}\psi\right)(\nu) = \frac{1}{\Gamma(m-\gamma)}\int_{\theta}^{\nu}(\nu-\tau)^{m-\gamma-1}\frac{d^m}{d\nu^m}\psi(\tau)d\tau.$$

Definition 2.3. *[17] Let $\psi \in L^1[\theta, \zeta], \psi * K_{(1-\beta)(1-\gamma)} \in AC^1[\theta, \zeta]$. The fractional derivative operator $D^{\gamma,\beta}_{\theta+}$ of order $0 < \gamma < 1$ and type $0 < \beta \leq 1$ with respect to $\nu \in [\theta, \zeta]$ is defined by*

$$\left(D^{\gamma,\beta}_{\theta+}\psi\right)(\nu) := I^{\beta(1-\gamma)}_{\theta+}\frac{d}{d\nu}\left(I^{(1-\beta)(1-\gamma)}_{\theta+}\psi(\nu)\right), \tag{2.2}$$

whenever the right hand side exists. The derivative (2.2) is usually called Hilfer fractional derivative.

The more general integral representation of equation (2.2) given in [17] is defined by:

Let $\psi \in L^1[\theta,\zeta], \psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta,\zeta], n-1<\gamma<n, 0<\beta\le 1, n\in\mathbb{N}$. Then

$$\left(D_{\theta+}^{\gamma,\beta}\psi\right)(\nu) = \left(I_{\theta+}^{\beta(n-\gamma)}\frac{d^n}{d\nu^n}\left(I_{\theta+}^{(1-\beta)(n-\gamma)}\psi(\nu)\right)\right), \tag{2.3}$$

which coincide with (2.2) for $n=1$.
Specially for $\beta=0$, $D_{\theta+}^{\gamma,0}\psi = D_{\theta+}^{\gamma}\psi$ is a Riemann-Liouville fractional derivative of order γ, and for $\beta=1$ it is a Caputo fractional derivative $D_{\theta+}^{\gamma,1}\psi = {}^{C}D_{\theta+}^{\gamma}\psi$ of order γ. Applying the properties of Riemann-Liouville integral the relation (2.3) can be rewritten in the form:

$$\begin{aligned}\left(D_{\theta+}^{\gamma,\beta}\psi\right)(\nu) &= \left(I_{\theta+}^{\beta(n-\gamma)}\left(\left(D_{\theta+}^{n-(1-\beta)(n-\gamma)}\psi\right)(\nu)\right)\right)\\ &= \frac{1}{\Gamma(\beta(n-\gamma))}\int_{\theta}^{\nu}(\nu-\tau)^{\beta(n-\gamma)-1}\left(\left(D_{\theta+}^{\gamma+\beta(n-\gamma)}\psi\right)(\tau)\right)d\tau. \end{aligned} \tag{2.4}$$

The geometric arithmetically s-convex function given in [18] stated as follows:

Definition 2.4. *Let $\psi : I\subset\mathbb{R}^+\to\mathbb{R}^+$ and $s\in(0,1]$. A function ψ is geometric-arithmetically s-convex function on I if for every $\gamma,\beta\in I$ and $\nu\in[0,1]$, we have*

$$\psi(\gamma^{\nu}\beta^{1-\nu}) \le \nu^s(\psi(\gamma)) + (1-\nu)^s\psi(\beta).$$

The following lemma was given by Liao et al. [18].

Lemma 2.5. *For $\theta\in[0,1]$, γ, $\beta>0$, we have*

$$\theta\gamma + (1-\theta)\beta \ge \beta^{1-\theta}\gamma^{\theta}.$$

Deng *et al.* [19] prove the following lemma.

Lemma 2.6. *For $\theta\in[0,1]$, we have*

$$\begin{aligned}(1-\theta)^{\gamma} &\le 2^{1-\gamma}-\theta^{\gamma},\ \gamma\in[0,1],\\ (1-\theta)^{\gamma} &\ge 2^{1-\gamma}-\theta^{\gamma},\ \gamma\in[1,\infty).\end{aligned}$$

3. Main Results

This section includes several mid-point type fractional integral inequalities involving Hilfer fractional derivative. The first main result for the fractional derivative is presented in the following theorem.

Theorem 3.1. *Let $\psi\in L^1[\theta,\zeta]$, $\psi * K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta,\zeta]\to\mathbb{R}$ be a positive function with $0\le\theta<\zeta$, $n-1<\gamma<n$, $0<\beta\le 1$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\in L^1[\theta,\zeta]$. If $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi$ is convex function on $[\theta,\zeta]$, then following inequality for fractional derivative holds:*

$$\begin{aligned}D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) &\le \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]\\ &\le D_{\theta+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \end{aligned}\tag{3.1}$$

Proof. We define functions $\tilde{\psi}(\nu) = \psi(\theta+\zeta-\nu)$, $\nu \in [\theta, \zeta]$ and $\Phi(\nu) = \psi(\nu)+\tilde{\psi}(\nu)$, $\nu \in [\theta, \zeta]$. Since $D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi$ is convex on $[\theta, \zeta]$, therefore with $\mu = \frac{1}{2}$, we have

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{x+y}{2}\right) \leq \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(x) + D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(y)}{2}.$$

Choosing $x = \nu\theta + (1-\nu)\zeta$ and $y = (1-\nu)\theta + \nu\zeta$, we get

$$2D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right) \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta) + D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi((1-\nu)\theta+\nu\zeta)$$
$$= D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta+(1-\nu)\zeta).$$

Now, we multiply both sides of above inequality by $\nu^{\beta(n-\gamma)-1}$ and then integrating the resulting inequality with respect to ν over $[0, 1]$, we have

$$\frac{1}{\beta(n-\gamma)}D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \int_0^1 \nu^{\beta(n-\gamma)-1}D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta+(1-\nu)\zeta)d\nu. \tag{3.2}$$

Substituting $u = \nu\theta + (1-\nu)\zeta$, (3.2) becomes

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}}D_{\theta^+}^{\gamma,\beta}\Phi(\zeta). \tag{3.3}$$

Similarly for the choice

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{x+y}{2}\right) \leq \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(x) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(y)}{2},$$

we get

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}}D_{\zeta^-}^{\gamma,\beta}\Phi(\theta). \tag{3.4}$$

By adding (3.3) and (3.4), we obtain

$$\begin{aligned} &D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi\left(\frac{\theta+\zeta}{2}\right) \\ &\leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}}\Big[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\Big], \end{aligned} \tag{3.5}$$

which proves the left half part of inequality (3.1).

For the proof of the second half, we first note that if $D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi$ is convex, then for $\nu \in [0, 1]$, yields

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta) \leq \nu D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta) + (1-\nu)D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta)$$
$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi((1-\nu)\theta+\nu\zeta) \leq (1-\nu)D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta) + \nu D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta).$$

By adding above two inequalities, we have

$$D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta+(1-\nu)\zeta) \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta). \tag{3.6}$$

Similarly

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta+\nu\zeta) \leq D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned}D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta+(1-\nu)\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta+\nu\zeta)\\ \leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta).\end{aligned} \tag{3.8}$$

Now, first, we multiply both sides of (3.8) by $\nu^{\beta(n-\gamma)-1}$ and then we integrate the resulting inequality with respect to ν over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 \nu^{\beta(n-\gamma)-1}D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\nu\theta+(1-\nu)\zeta)d\nu\\ &+\int_0^1 \nu^{\beta(n-\gamma)-1}D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi((1-\nu)\theta+\nu\zeta)d\nu\\ &\leq [D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)]\int_0^1 \nu^{\beta(n-\gamma)-1}.\end{aligned}$$

Substituting $u=\nu\theta+(1-\nu)\zeta$ and $v=(1-\nu)\theta+\nu\zeta$, the above inequality become

$$\frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]\leq D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta). \tag{3.9}$$

From (3.5) and (3.9), we get inequality (3.1).

Corollary 3.2. *If we choose $\beta=1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 3.1, it reduces to [20, Theorem 2.3], i.e.,*

$$\psi^n\left(\frac{\theta+\zeta}{2}\right)\leq\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^C D_{\theta+}^{\gamma}\psi(\zeta)+(-1)^{n\,C}D_{\zeta-}^{\gamma}\psi(\theta)\right]\leq\frac{\psi^n(\theta)+\psi^n(\zeta)}{2}.$$

Lemma 3.3. *Let $\psi\in L^1[\theta,\zeta]$, $\psi*K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$. For the differentiable function $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi:[\theta,\zeta]\to\mathbb{R}$ with $n-1<\gamma<n$, $0<\beta\leq1$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi\in L^1[\theta,\zeta]$ the following equality*

$$\begin{aligned}&\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]\\ &=\frac{\zeta-\theta}{2}\int_0^1\left[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\right]D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\nu\theta+(1-\nu)\zeta)d\nu,\end{aligned}$$

holds.

Proof. Consider

$$\begin{aligned}I&=\int_0^1\left[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\right]D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+1}\psi(\nu\theta+(1-\nu)\zeta)d\nu\\ &=\int_0^1\left[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\right]D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\nu\theta+(1-\nu)\zeta)d\nu\\ &+\int_0^1\left[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\right]D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\nu\theta+(1-\nu)\zeta)d\nu\\ &=I_1+I_2.\end{aligned} \tag{3.10}$$

Integrating I_1 by parts, we get

$$\begin{aligned}
I_1 &= (1-\nu)^{\beta(n-\gamma)} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} \Big|_0^1 \\
&+ \int_0^1 \beta(n-\gamma)(1-\nu)^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} d\nu \\
&- \nu^{\beta(n-\gamma)} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} \Big|_0^1 \\
&+ \int_0^1 \beta(n-\gamma)\nu^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} d\nu.
\end{aligned}$$

Substituting $x = \nu\theta + (1-\nu)\zeta$, we obtain

$$\begin{aligned}
I_1 &= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta) + D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta)}{\zeta-\theta} - \frac{\beta(n-\gamma)}{\zeta-\theta}\Big[\int_\zeta^\theta \Big(\frac{\theta-x}{\theta-\zeta}\Big)^{\beta(n-\gamma)-1} \\
&\times \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(x)}{\theta-\zeta} dx + \int_\zeta^\theta \Big(\frac{\zeta-x}{\zeta-\theta}\Big)^{\beta(n-\gamma)-1} \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(x)}{\theta-\zeta} dx\Big] \\
&= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)}{\zeta-\theta} - \frac{\beta(n-\gamma)}{(\zeta-\theta)^{\beta(n-\gamma)+1}} \int_\theta^\zeta (\zeta-x)^{\beta(n-\gamma)-1} D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(x)dx \\
&= \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)}{\zeta-\theta} - \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)+1}} D_{\theta^+}^{\gamma,\beta}\Phi(\zeta). \qquad (3.11)
\end{aligned}$$

Similarly, integrating I_2 by parts, we get

$$\begin{aligned}
I_2 &= (1-\nu)^{\beta(n-\gamma)} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} \Big|_0^1 \\
&+ \int_0^1 \beta(n-\gamma)(1-\nu)^{\beta(n-\gamma)-1} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} d\nu \\
&- \nu^{\beta(n-\gamma)} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta} \Big|_0^1 \\
&+ \int_0^1 \beta(n-\gamma)\nu^{\beta(n-\gamma)-1} \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(\nu\theta+(1-\nu)\zeta)}{\theta-\zeta}.
\end{aligned}$$

Substituting again $x = \nu\theta + (1-\nu)\zeta$, we get

$$I_2 = \frac{D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{\zeta-\theta} - \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)+1}} D_{\zeta^-}^{\gamma,\beta}\Phi(\theta). \qquad (3.12)$$

Using (3.11) and (3.12) in (3.10), we have

$$I = \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{\zeta-\theta} - \frac{\Gamma(\beta(n-\gamma)+1)}{(\zeta-\theta)^{\beta(n-\gamma)+1}} \left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right].$$

Thus by multiplying both sides with $\frac{\zeta-\theta}{2}$, we get the desired result.

Corollary 3.4. *If we take $\beta=1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 3.3, we obtain following [20, Lemma 2.2]*

$$\frac{\psi^n(\theta)+\psi^n(\zeta)}{2}-\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^C D^{\gamma}_{\theta+}\psi(\zeta)+(-1)^{n\,C}D^{\gamma}_{\zeta^-}\psi(\theta)\right]$$
$$=\frac{\zeta-\theta}{2}\int_0^1\left[(1-\nu)^{n-\gamma}-\nu^{n-\gamma}\right]\psi^{n+1}(\nu\theta+(1-\nu)\zeta)d\nu,$$

for Caputo fractional derivatives.

Theorem 3.5. *Let $\psi\in L^1[\theta,\zeta]$, $\psi * K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$ and $D^{\gamma+\beta(n-\gamma)}_{(\theta,\zeta)}\psi:[\theta,\zeta]\to\mathbb{R}$ be a differentiable function with $n-1<\gamma<n$ and $0<\beta\le 1$. If $D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi$ is convex on $[\theta,\zeta]$, then the following inequality is true*

$$\Big|\frac{D^{\gamma+\beta(n-\gamma)}_{\theta^+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta^-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D^{\gamma,\beta}_{\theta^+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta^-}\Phi(\theta)\right]\Big|$$
$$\le\frac{\zeta-\theta}{2(\beta(n-\gamma)+1)}\Big(1-\frac{1}{2^{\beta(n-\gamma)}}\Big)\Big(|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|\Big).$$

Proof. By using Lemma 3.3 and Definition 2.1 , we get

$$\Big|\frac{D^{\gamma+\beta(n-\gamma)}_{\theta^+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta^-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D^{\gamma,\beta}_{\theta^+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta^-}\Phi(\theta)\right]\Big|$$
$$\le\frac{\zeta-\theta}{2}\int_0^1|(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}|$$
$$\times\Big(\nu|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|+(1-\nu)|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\Big)d\nu$$
$$=\frac{\zeta-\theta}{2}\int_0^{\frac12}\Big[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\Big]$$
$$\times\Big(\nu|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|+(1-\nu)|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\Big)d\nu$$
$$+\int_{\frac12}^1\Big[\nu^{\beta(n-\gamma)}-(1-\nu)^{\beta(n-\gamma)}\Big]\Big(\nu|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|+(1-\nu)|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\Big)d\nu$$
$$=\frac{\zeta-\theta}{2}\Big[|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\int_0^{\frac12}\Big[(1-\nu)^{\beta(n-\gamma)+1}-(1-\nu)\nu^{\beta(n-\gamma)}\Big]d\nu$$
$$+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|\int_0^{\frac12}\Big[\nu(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)+1}\Big]d\nu$$
$$+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\int_{\frac12}^1\Big[(1-\nu)\nu^{\beta(n-\gamma)}-(1-\nu)^{\beta(n-\gamma)+1}\Big]d\nu$$
$$+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|\int_{\frac12}^1\Big[\nu^{\beta(n-\gamma)+1}-\nu(1-\nu)^{\beta(n-\gamma)}\Big]d\nu\Big]$$

$$= \frac{\zeta-\theta}{2}\Big[|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|\Big(\frac{1}{\beta(n-\gamma)+1}-\frac{1}{(\beta(n-\gamma)+1)2^{\beta(n-\gamma)}}\Big)$$
$$+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|\Big(\frac{1}{\beta(n-\gamma)+1}-\frac{1}{(\beta(n-\gamma)+1)2^{\beta(n-\gamma)}}\Big)\Big]$$
$$= \frac{\zeta-\theta}{2(\beta(n-\gamma)+1)}\Big(1-\frac{1}{2^{\beta(n-\gamma)}}\Big)\Big(|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\zeta)|+|D^{\gamma+\beta(n-\gamma)+1}_{(\theta,\zeta)}\psi(\theta)|\Big).$$

Hence the proof is complete.

Corollary 3.6. *If we choose $\beta=1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 3.5, then we get [20, Theorem 2.4]*

$$\frac{\psi^n(\zeta)+\psi^n(\theta)}{2}-\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^CD^{\gamma}_{\theta+}\psi(\zeta)+(-1)^n{}^CD^{\gamma}_{\zeta-}\psi(\theta)\right]$$
$$\leq \frac{\zeta-\theta}{2(n-\gamma+1)}\Big(1-\frac{1}{2^{n-\gamma}}\Big)\Big(|\psi^{n+1}(\zeta)|+|\psi^{n+1}(\theta)|\Big).$$

Lemma 3.7. *Let $\psi\in L^1[\theta,\zeta]$, $\psi * K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$ and $D^{\gamma+\beta(n-\gamma)}_{(\theta,\zeta)}\psi$: $[\theta,\zeta]\to\mathbb{R}$ be twice differential mapping on (θ,ζ) with $n-1<\gamma<n$ and $0<\beta\leq 1$. If $D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi\in L^1[\theta,\zeta]$, then we have the following equality.*

$$\frac{D^{\gamma+\beta(n-\gamma)}_{\theta+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D^{\gamma,\beta}_{\theta+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta-}\Phi(\theta)\right]$$
$$=\frac{(\zeta-\theta)^2}{2}\int_0^1\frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\nu\theta+(1-\nu)\zeta)d\nu.$$

Proof. By using Lemma 3.3, we get

$$\frac{D^{\gamma+\beta(n-\gamma)}_{\theta+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}[D^{\gamma,\beta}_{\theta+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta-}\Phi(\theta)]$$
$$=\frac{\zeta-\theta}{2}\Big[\int_0^1\Big[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\Big]D^{\gamma+\beta(n-\gamma)+1}_{\theta+}\psi(\nu\theta+(1-\nu)\zeta)d\nu$$
$$+\int_0^1\Big[(1-\nu)^{\beta(n-\gamma)}-\nu^{\beta(n-\gamma)}\Big]D^{\gamma+\beta(n-\gamma)+1}_{\zeta-}\psi(\nu\theta+(1-\nu)\zeta)d\nu\Big].$$

Integrating by parts, we get

$$=\frac{\zeta-\theta}{2}\Big[\frac{D^{\gamma+\beta(n-\gamma)+1}_{\theta+}\psi(\zeta)-D^{\gamma+\beta(n-\gamma)+1}_{\theta+}\psi(\theta)+D^{\gamma+\beta(n-\gamma)+1}_{\zeta-}\psi(\zeta)-D^{\gamma+\beta(n-\gamma)+1}_{\zeta-}\psi(\theta)}{\beta(n-\gamma)+1}$$
$$-\frac{\zeta-\theta}{\beta(n-\gamma)+1}\int_0^1\Big[(1-\nu)^{\beta(n-\gamma)+1}+\nu^{\beta(n-\gamma)+1}\Big]D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\nu\theta+(1-\nu)\zeta)d\nu\Big]. \tag{3.13}$$

Since

$$D^{\gamma+\beta(n-\gamma)+1}_{\theta+}\psi(\zeta)-D^{\gamma+\beta(n-\gamma)+1}_{\theta+}\psi(\theta)=\int_\theta^\zeta D^{\gamma+\beta(n-\gamma)+2}_{\theta+}\psi(u)du.$$

Substituting $u = \nu\theta + (1-\nu)\zeta$, we get

$$\begin{aligned} &D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) \\ &= (\zeta-\theta)\int_0^1 D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu, \end{aligned} \tag{3.14}$$

and

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) = \int_\theta^\zeta D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(u)du.$$

Substituting again $u = \nu\theta + (1-\nu)\zeta$, we get

$$D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) = (\zeta-\theta)\int_0^1 D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu. \tag{3.15}$$

By adding (3.14) and (3.15), we obtain

$$\begin{aligned} &D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\zeta) - D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi(\theta) \\ &= (\zeta-\theta)\int_0^1 D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu. \end{aligned} \tag{3.16}$$

Using equation (3.16) into (3.13), we get the required result.

Corollary 3.8. *If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 3.7, we get the following equality for Caputo fractional derivatives*

$$\begin{aligned} &\frac{\psi^n(\theta)+\psi^n(\zeta)}{2} - \frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^C D_{\theta+}^{\gamma}\psi(\zeta) + (-1)^n {}^C D_{\zeta-}^{\gamma}\psi(\theta)\right] \\ &= \frac{(\zeta-\theta)^2}{2}\int_0^1 \frac{1-(1-\nu)^{n-\gamma+1}-\nu^{n-\gamma+1}}{n-\gamma+1}\psi^{n+2}(\nu\theta+(1-\nu)\zeta)d\nu. \end{aligned}$$

Lemma 3.9. *Let $\psi \in L^1[\theta,\zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta,\zeta]$, $n \in N$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi : [\theta,\zeta] \to \mathbb{R}$ is twice differentiable and measurable on $[\theta,\zeta]$, $n-1 < \gamma < n$ and $0 < \beta \leq 1$, then the equation*

$$\begin{aligned} &\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right] - D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right) \\ &= \frac{(\zeta-\theta)^2}{2}\int_0^1 m(\nu) D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu \end{aligned}$$

holds for $m(\nu) = \begin{cases} \nu - \frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}, & \nu \in [0,\frac{1}{2}); \\ 1-\nu - \frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}, & \nu \in [\frac{1}{2},1). \end{cases}$

Proof. Consider

$$\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\nu)D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu$$
$$=\frac{(\zeta-\theta)^2}{2}\Big[\int_0^{\frac{1}{2}}\nu\Big(D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)$$
$$+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)\Big)d\nu$$
$$+\int_{\frac{1}{2}}^1(1-\nu)\Big(D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)\Big)d\nu$$
$$-\int_0^1\left(\frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}\right)D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu\Big].$$

Let

$$I=\int_0^{\frac{1}{2}}\nu\Big(D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)\Big)d\nu$$
$$+\int_{\frac{1}{2}}^1(1-\nu)\Big(D_{\theta^+}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)\Big)d\nu$$
$$=I_1+I_2. \tag{3.17}$$

Integrating I_1 by parts, we get

$$I_1=\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi\Big(\frac{\theta+\zeta}{2}\Big)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi\Big(\frac{\theta+\zeta}{2}\Big)}{2(\theta-\zeta)}$$
$$-\frac{\Big[D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi\Big(\frac{\theta+\zeta}{2}\Big)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi\Big(\frac{\theta+\zeta}{2}\Big)-D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi(\zeta)-D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\theta)\Big]}{(\theta-\zeta)^2}. \tag{3.18}$$

Now integrating I_2 by parts, we get

$$I_2=-\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)+1}\psi\Big(\frac{\theta+\zeta}{2}\Big)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)+1}\psi\Big(\frac{\theta+\zeta}{2}\Big)}{2(\theta-\zeta)}$$
$$-\frac{\Big[D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi\Big(\frac{\theta+\zeta}{2}\Big)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi\Big(\frac{\theta+\zeta}{2}\Big)--D_{\theta^+}^{\gamma+\beta(n-\gamma)}\tilde{\psi}(\zeta)-D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi(\theta)\Big]}{(\theta-\zeta)^2}. \tag{3.19}$$

Substituting (3.18) and (3.19) to (3.17), we get

$$I=\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{(\zeta-\theta)^2}-\frac{2D_{\theta^+}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right)+2D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right)}{(\zeta-\theta)^2}.$$

Thus

$$\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\nu)D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)d\nu$$
$$=\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2}-D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right)$$

$$- \frac{(\zeta-\theta)^2}{2}\int_0^1 \left(\frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}\right)$$
$$\times D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\nu\theta+(1-\nu)\zeta)d\nu.$$

By using Lemma (3.7), we arrive at the desired result.

Corollary 3.10. *If we take $\beta = 1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Lemma 3.9 then the following equality for Caputo fractional derivatives*

$$\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^{C}D^{\gamma}_{\theta^+}\psi(\zeta)+(-1)^{n\,C}D^{\gamma}_{\zeta^-}\psi(\theta)\right]-\psi^{n}\left(\frac{\theta+\zeta}{2}\right)$$
$$=\frac{(\zeta-\theta)^2}{2}\int_0^1 m(\nu)\psi^{n+2}(\nu\theta+(1-\nu)\zeta)d\nu$$

holds, where $m(\nu)=\begin{cases}\nu-\frac{1-(1-\nu)^{n-\gamma+1}-\nu^{n-\gamma+1}}{n-\gamma+1}, & \nu\in[0,\frac{1}{2});\\ 1-\nu-\frac{1-(1-\nu)^{n-\gamma+1}-\nu^{n-\gamma+1}}{n-\gamma+1}, & \nu\in[\frac{1}{2},1).\end{cases}$

Theorem 3.11. *Let $\psi \in L^1[\theta,\zeta]$, $\psi * K_{(1-\beta)(n-\gamma)} \in AC^n[\theta,\zeta]$, $n \in N$ and $D^{\gamma+\beta(n-\gamma)}_{(\theta,\zeta)}\psi : [\theta,\zeta] \to \mathbb{R}$ be a twice differentiable function with $n-1<\gamma<n$ and $0<\beta\le 1$. If $|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi|$ is measurable, decreasing and geometric-arithmetically s-convex on $[\theta,\zeta]$ for some fixed $\gamma\in(0,\infty)$, $s\in(0,1]$, $0\le\theta<\zeta$, then the inequality*

$$\Big|\frac{D^{\gamma+\beta(n-\gamma)}_{\theta^+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta^-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D^{\gamma,\beta}_{\theta^+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta^-}\Phi(\theta)\right]\Big|$$
$$\le\frac{(\zeta-\theta)^2\left(|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|+|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|\right)}{2(\beta(n-\gamma)+1)}$$
$$\times\left(\frac{1}{s+1}-\frac{1}{\beta(n-\gamma)+s+2}\right),$$

holds.

Proof. By using Lemma 3.7, Lemma 2.5 and Definition 2.4, we have

$$\Big|\frac{D^{\gamma+\beta(n-\gamma)}_{\theta^+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta^-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D^{\gamma,\beta}_{\theta^+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta^-}\Phi(\theta)\right]\Big|$$
$$\le\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\int_0^1|1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}|$$
$$\times|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\nu\theta+(1-\nu)\zeta)|d\nu$$
$$\le\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\int_0^1|1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}||D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta^{\nu}\zeta^{1-\nu})|d\nu$$

$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\int_0^1 \Big(1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}\Big)\Big[\nu^s|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|$$
$$+(1-\nu)^s|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|\Big]d\nu$$
$$=\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\Big[\int_0^1\Big[\nu^s|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|+(1-\nu)^s|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|\Big]d\nu$$
$$-\int_0^1\Big[\nu^s(1-\nu)^{\beta(n-\gamma)+1}|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|+(1-\nu)^{\beta(n-\gamma)+s+1}|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|\Big]d\nu$$
$$-\int_0^1\Big[\nu^{\beta(n-\gamma)+s+1}|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|+\nu^{\beta(n-\gamma)+1}(1-\nu)^s|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|\Big]d\nu\Big].$$

By using the definition of the beta function, we get

$$\Big|\frac{D^{\gamma+\beta(n-\gamma)}_{\theta^+}\Phi(\zeta)+D^{\gamma+\beta(n-\gamma)}_{\zeta^-}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\Big[D^{\gamma,\beta}_{\theta^+}\Phi(\zeta)+D^{\gamma,\beta}_{\zeta^-}\Phi(\theta)\Big]\Big|$$
$$\leq\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\Big[\frac{|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|}{s+1}+\frac{|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|}{s+1}$$
$$-\frac{|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|}{\beta(n-\gamma)+s+2}-|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|B(s+1,\beta(n-\gamma)+2)$$
$$-|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|B(s+1,\beta(n-\gamma)+2)-\frac{|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|}{\beta(n-\gamma)+s+2}\Big]$$
$$\leq\frac{(\zeta-\theta)^2\Big(|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\zeta)|+|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi(\theta)|\Big)}{2(\beta(n-\gamma)+1)}$$
$$\times\left(\frac{1}{s+1}-\frac{1}{\beta(n-\gamma)+s+2}\right).$$

Which completes the proof of the result.

Corollary 3.12. *If we take $\beta=1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 3.11 then the following result for Caputo fractional derivatives holds:*

$$\Big|\frac{\psi^n(\theta)+\psi^n(\zeta)}{2}-\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\Big[{}^CD^{\gamma}_{\theta+}\psi(\zeta)+{}^CD^{\gamma}_{\zeta-}\psi(\theta)\Big]\Big|$$
$$\leq\frac{(\zeta-\theta)^2\Big(|\psi^{n+2}(\theta)|+|\psi^{n+2}(\zeta)|\Big)}{2(n-\gamma+1)}\left(\frac{1}{s+1}-\frac{1}{n-\gamma+s+2}\right).$$

Theorem 3.13. *Let $\psi\in L^1[\theta,\zeta]$, $\psi*K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$. Consider $D^{\gamma+\beta(n-\gamma)}_{(\theta,\zeta)}\psi:[\theta,\zeta]\to\mathbb{R}$ be twice differentiable function with $n-1<\gamma<n$ and $0<\beta\leq 1$. If $|D^{\gamma+\beta(n-\gamma)+2}_{(\theta,\zeta)}\psi|^q$ is measurable, decreasing and geometric arithmetically* *s-convex on* $[\theta, \zeta]$ *for some fixed* $\gamma \in (0, \infty)$, $s \in (0, 1]$, $0 \leq \theta < \zeta$, *then the inequality*

$$\left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \right|$$
$$\leq \frac{(\zeta-\theta)^2 \max\left(1-2^{1-\beta(n-\gamma)}, 2^{1-\beta(n-\gamma)}-1\right)}{2(\beta(n-\gamma)+1)}$$
$$\times \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q}{s+1} \right)^{\frac{1}{q}},$$

is true.

Proof. We shall prove this theorem in two cases:

Case 1: Let $\gamma \in (0, 1)$ and $\beta(n-\gamma) \in [0, 1]$, then by using Lemma 3.7, Holder's inequality, Lemma 2.5, Definition 2.4 and Lemma 2.6, we obtain

$$\left| \frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta) + D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2} - \frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta) + D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)] \right|$$
$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)|^q d\nu \right)^{\frac{1}{q}}$$
$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta^\nu\zeta^{1-\nu})|^q d\nu \right)^{\frac{1}{q}}$$
$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\int_0^1 |1-(1-\nu)^{\beta(n-\gamma)+1} - \nu^{\beta(n-\gamma)+1}|^p d\nu \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left[\nu^s|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + (1-\nu)^s|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q\right] d\nu \right)^{\frac{1}{q}}$$
$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[(1-\nu)^{\beta(n-\gamma)} + \nu^{\beta(n-\gamma)} - 1\right]^p d\nu \right)^{\frac{1}{p}}$$
$$\leq \frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)} \left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q + |D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1} \right)^{\frac{1}{q}}$$

$$\times\left(\int_0^1 (2^{1-\beta(n-\gamma)}-1)^p d\nu\right)^{\frac{1}{p}}$$

$$=\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(2^{1-\beta(n-\gamma)}-1\right). \tag{3.20}$$

Case 2: Let $\gamma\in[1,\infty)$ and $\beta(n-\gamma)\in[1,\infty)$. By using Lemma 3.7, Holder's inequality, Lemma 2.5, Definition 2.4 and Lemma 2.6, we obtain

$$\Big|\frac{D_{\theta^+}^{\gamma+\beta(n-\gamma)}\Phi(\zeta)+D_{\zeta^-}^{\gamma+\beta(n-\gamma)}\Phi(\theta)}{2}-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]\Big|$$

$$\le\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^1 (1-2^{1-\beta(n-\gamma)})^p d\nu\right)^{\frac{1}{p}}$$

$$=\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(1-2^{1-\beta(n-\gamma)}\right). \tag{3.21}$$

Now from (3.20) and (3.21), we obtain the required result.

Corollary 3.14. *If we take $\beta=1$ ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 3.13, we get the following inequality for Caputo fractional derivatives*

$$\left|\frac{\psi^n(\theta)+\psi^n(\zeta)}{2}-\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}[{}^CD_{\theta^+}^{\gamma}\psi(\zeta)+(-1)^{n\,C}D_{\zeta^-}^{\gamma}\psi(\theta)]\right|$$

$$\le\frac{(\zeta-\theta)^2\max\left(1-2^{1-n+\gamma},2^{1-n+\gamma}-1\right)}{2(n-\gamma+1)}\left(\frac{|\psi^{n+2}(\theta)|^q+|\psi^{n+2}(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Theorem 3.15. *Let $\psi\in L^1[\theta,\zeta]$, $\psi*K_{(1-\beta)(n-\gamma)}\in AC^n[\theta,\zeta]$, $n\in N$ and $D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi:[0,\zeta]\to\mathbb{R}$ be differentiable function with $n-1<\gamma<n$ and $0<\beta\le1$. If $|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi|^q$ is measurable for $1<q<\infty$, decreasing and geometric arithmetically s-convex on $[0,\zeta]$ for some fixed $\gamma\in(0,\infty)$, $s\in(0,1]$, $0\le\theta<\zeta$, then the following fractional inequality holds:*

$$\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]-D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right)$$

$$\le\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(\frac{(\beta(n-\gamma)+1)2^{-p-1}+\left(\beta(n-\gamma)+0.5\right)^{p+1}-(\beta(n-\gamma))^{p+1}}{p+1}\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 3.9, Lemma 2.5, Holder's inequality and Definition 2.4, we get

$$
\begin{aligned}
&\left|\frac{\Gamma(\beta(n-\gamma)+1)}{2(\zeta-\theta)^{\beta(n-\gamma)}}\left[D_{\theta^+}^{\gamma,\beta}\Phi(\zeta)+D_{\zeta^-}^{\gamma,\beta}\Phi(\theta)\right]-D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)}\psi\left(\frac{\theta+\zeta}{2}\right)\right| \\
&\leq \frac{(\zeta-\theta)^2}{2}\int_0^1 |m(\nu)||D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\nu\theta+(1-\nu)\zeta)|d\nu \\
&\leq \frac{(\zeta-\theta)^2}{2}\int_0^1 |m(\nu)||D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta^\nu\zeta^{1-\nu})|d\nu \\
&\leq \frac{(\zeta-\theta)^2}{2}\left(\int_0^1 |m(\nu)|^p d\nu\right)^{\frac{1}{p}}\left(\int_0^1\left|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta^\nu\zeta^{1-\nu})\right|^q d\nu\right)^{\frac{1}{q}} \\
&\leq \frac{(\zeta-\theta)^2}{2}\left(\int_0^1 |m(\nu)|^p d\nu\right)^{\frac{1}{p}}\Big(\int_0^1\Big[\nu^s|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q \\
&+(1-\nu)^s|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q\Big]d\nu\Big)^{\frac{1}{q}} \\
&=\frac{(\zeta-\theta)^2}{2}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}} \\
&\times\Big[\int_0^{\frac{1}{2}}\left|\nu-\frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}\right|^p d\nu \\
&+\int_{\frac{1}{2}}^1\left|(1-\nu)-\frac{1-(1-\nu)^{\beta(n-\gamma)+1}-\nu^{\beta(n-\gamma)+1}}{\beta(n-\gamma)+1}\right|^p d\nu\Big]^{\frac{1}{p}} \\
&=\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}} \\
&\times\Big[\int_0^{\frac{1}{2}}\left|\beta(n-\gamma)\nu-1+(1-\nu)^{\beta(n-\gamma)+1}+\nu^{\beta(n-\gamma)+1}\right|^p d\nu \\
&+\int_{\frac{1}{2}}^1\left|\beta(n-\gamma)+1-\beta(n-\gamma)\nu-\nu-1+(1-\nu)^{\beta(n-\gamma)+1}+\nu^{\beta(n-\gamma)+1}\right|^p d\nu\Big]^{\frac{1}{p}} \\
&\leq\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}} \\
&\times\left(\int_0^{\frac{1}{2}}((\beta(n-\gamma)+1)\nu)^p\,d\nu+\int_{\frac{1}{2}}^1(\beta(n-\gamma)-\nu+1)^p\,d\nu\right)^{\frac{1}{p}} \\
&\leq\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}
$$

$$\times\left((\beta(n-\gamma)+1)\int_0^{\frac{1}{2}}\nu^p d\nu+\int_{\frac{1}{2}}^1(\beta(n-\gamma)-\nu+1)^p d\nu\right)^{\frac{1}{p}}$$

$$=\frac{(\zeta-\theta)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\theta)|^q+|D_{(\theta,\zeta)}^{\gamma+\beta(n-\gamma)+2}\psi(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(\frac{(\beta(n-\gamma)+1)2^{-p-1}+(\beta(n-\gamma)+0.5)^{p+1}-(\beta(n-\gamma))^{p+1}}{p+1}\right)^{\frac{1}{p}}.$$

Which completes the proof of the result.

Corollary 3.16. *If we take $\beta=1$ and ψ is symmetric about $\frac{\theta+\zeta}{2}$ in Theorem 3.15, then the inequality*

$$\frac{\Gamma(n-\gamma+1)}{2(\zeta-\theta)^{n-\gamma}}\left[{}^C D_{\theta+}^{\gamma}\psi(\zeta)+(-1)^{n}{}^C D_{\zeta-}^{\gamma}\psi(\theta)\right]-\psi^n\left(\frac{\theta+\zeta}{2}\right)$$

$$\leq\frac{(\zeta-\theta)^2}{2(n-\gamma+1)}\left(\frac{|\psi^{n+2}(\theta)|^q+|\psi^{n+2}(\zeta)|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left(\frac{(n-\gamma+1)2^{-p-1}+(n-\gamma+0.5)^{p+1}-(n-\gamma)^{p+1}}{p+1}\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, holds for Caputo fractional derivatives.

4. Concluding remarks

In this paper, we established some new Hermite Hadamard-type inequalities for Hilfer fractional derivative by using convexity theory and Holder's inequality. We discuss the special case of our general results. The results of this paper may stimulate further research for researcher working in this field.

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Muhammad Samraiz-Department of Mathematics, University of Sargodha, Sargodha, Pakistan
E-mail address: muhammad.samraiz@uos.edu.pk

Zahida Perveen -Department of Mathematics, University of Sargodha, Pakistan
E-mail address: zahidaperveen1995@gmail.com

Sajid Iqbal-Department of Mathematics, Ripha International University, Faisalabad Campus, Satyana Road, 38000 Faisalabad, Pakistan
E-mail address: sajid_uos2000@yahoo.com

Saima Naheed-Department of Mathematics, University of Sargodha, Pakistan
E-mail address: saimasamraiz@gmail.com; saima.naheed@uos.edu.pk

Thabet Abdeljawad-Department of Mathematics and General Sciences, Prince Sultan University, Riyadh 12345, Saudi Arabia, Department of Medical Research, China Medical University, Taichung 40402, Taiwan, Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan
E-mail address: tabdeljawad@psu.edu.sa