F-Expansion Method and Its Application for Finding Exact Analytical Solutions of Fractional order RLW Equation

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Abstract: Exact nonlinear partial differential equation solutions are critical for describing new complex characteristics in a variety of fields of applied science. The aim of this research is to use the F-expansion method to find the generalized solitary wave solution of the regularized long wave (RLW) equation of fractional order. Fractional partial differential equations can also be transformed into ordinary differential equations using fractional complex transformation and the properties of the modified Riemann–Liouville fractional-order operator. Because of the chain rule and the derivative of composite functions, nonlinear fractional differential equations (NLFDEs) can be converted to ordinary differential equations. We have investigated various set of explicit solutions with some free parameters using this approach. The solitary wave solutions are derived from the moving wave solutions when the parameters are set to special values. Our findings show that this approach is a very active and straightforward way of formulating exact solutions to nonlinear evolution equations that arise in mathematical physics and engineering. It is anticipated that this research will provide insight and knowledge into the implementation of novel methods for solving wave equations.

Keywords: Exact soliton solution, F-expansion method, RLW equation, Maple 18.

1. Introduction

The fundamental properties of applied sciences are closely connected with the properties of nature and science that can be explained using nonlinear partial differential equations (NPDEs). NPDEs have recently been used to investigate the properties of a variety of real-world problems in fluid dynamics, population ecology, shallow-water wave propagation, plasma physics, solid-state physics, heat, and quantum mechanics, optical fibers and biology. In addition, their mathematical models have been published in the literature. Exact solutions can help people better understand the physical mechanisms underlying natural or social phenomena explained by nonlinear evolution equations (NEEs). Exploring exact solutions (especially solitary wave solutions) for the NEEs has thus been a hot and difficult topic in mathematical physics for a long time.

Several analytical methods for solving nonlinear partial differential equations have been given by trial equation method [1], extended trial equation method [2], Jacobi elliptic function method [3], Weierstrass elliptic function expansion method [4], F-expansion method [5-8], the first integral method [9], the extended fractional Riccati expansion method [10], the fractional complex transform [11], the Jacobi elliptic equation method [12], the modified extended tanh method [13], the $\exp(-\Phi(\xi))$ method [14,15], the generalized $\left(G'/G\right)$-expansion method [16–18], the exp $\left(-\psi\left(z\right)\right)$-expansion method [19–23], the $\left(m+1/G'\right)$-expansion method [24], the sine-Gordon expansion method [25–31]. The Jacobi elliptic function solutions of nonlinear partial differential equations are significant. The Jacobi elliptic F-expansion method, the improved Jacobi elliptic F-expansion method, and the generalized Jacobi elliptic F-expansion method are used to obtain Jacobi elliptic function-based solutions. Various exact or numerical solutions to FPDEs have been successfully developed using these methods.

In this paper a generalized fractional complex transform [32-34] used to convert FDE to ODE. The regularized long wave (RLW) equation is a crucial partial differential equation that explains dispersive wave action. Since they explain a wide range of important physical phenomena, such as shallow water waves and ion-acoustic plasma waves, the RLW equations are crucial in the study of nonlinear dispersive waves. This equation is of great importance in engineering sciences. By applying elliptical equations various type of travelling wave solutions to nonlinear partial differential equations can be obtained using F-expansion method. This method is a generalized form of Jacobi method [5-8]. The main uniqueness of this technique is that there is no need to calculate the Jacobi elliptic function. The solution of elliptic function is regarded as solution of corresponding Jacobi elliptic function. The suggested algorithm is quite useful for such type of complex problems and is user-friendly. Numerical results show accuracy and efficiency of method.

This paper consists of several sections. In Section 2, some preliminaries and notations are addressed. In Section 3, a chain rule and fractional complex transform is reviewed. In Section 4, a brief description of the F-expansion method is reviewed. With the aid of this method, we will retrieve several sets of exact solutions for the RLW Equation in Section 5. However, to the best of the authors' knowledge, this method has not been applied for equation (1) in previous studies. Finally, Section 6 concludes the paper.

2. Preliminaries and Notation

This section has some primary definition and properties of fractional calculus theory that allows you to be used similarly on this work. For the finite derivative in the closed interval $[a,b]$ fractional integral and derivatives are defined as below.

Definition 2.1 A real function is said to be in the space $C\theta, \theta \in R,$ If there exists a real number $(\rho > \theta)$: $h(x) = x^{\rho} h_1(x)$ where $h_1(x) = C(0,\infty)$ and it is said to be in the space $C_{\theta}^{n}\theta$ if $h^n \in C\theta, n \in N$

Definition 2.2 The Riemann-Liouville fractional integral operator of order $c \geq 0$ of a function $h \in C\theta, \theta \geq -1,$ is defined as

$$J^c(x) = \frac{1}{\Gamma(c)} \int_0^x (x-t)^{c-1} h(t)dt, c > 0, x > 0, \tag{1}$$

$$J^0(x) = h(x).$$

Properties of the operator J^c can be found in [35]; some of them are:

For $h \in C\theta, \theta \geq -1, c, d \geq 0$ and $e \geq -1$

$$\begin{aligned} &J^c J^d h(x) = J^{c+d} h(x), \\ &J^c J^d h(x) = J^d J^c h(x), \\ &J^c x^e = \frac{\Gamma(e+1)}{\Gamma(d+e+1)} x^{c+e}. \end{aligned} \tag{2}$$

When using fractional differential equations to model real-world phenomena, the Riemann—Liouville derivative has several disadvantages. As a result, we'll implement a relatively new technique known as the modified fractional differential operator, which was introduced by M. Caputo in his work on viscoelasticity theory [36].

Definition 2.3 For m to be the smallest integer that exceeds, c the Caputo time fractional derivative operator of order $c > 0$ and defined as

$$D_t^c h(x) = \frac{\partial^c v(x,t)}{\partial t^c} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-t)^{n-c-1} h(t)dt, -1 = \ n, n \in N, \\ \dfrac{\partial^c v(x,t)}{\partial t^c}, c = n. \end{cases} \tag{3}$$

3. Chain rule for fractional calculus and fractional complex transform

The authors used the chain rule to convert a fractional differential equation with alteration of the Riemann-Liouville derivative into its classical differential partner. The following relationship [11] was used by the authors to show that the chain rule is invalid.

$$D_t^{\alpha} v = \tau_t \frac{dv}{d\Omega} D_t^{\alpha}\Omega \text{ and } D_x^{\alpha} v = \tau_x \frac{dv}{d\Omega} D_x^{\alpha}\Omega.$$

To determine τ_z we consider a special case as $z = t^c$ and $v = z^m$, we have

$$\frac{\partial^c v}{\partial t^c} = \frac{\Gamma(1+nc)^{nc-c}}{\Gamma(1+nc-c)} = \tau.\frac{\partial v}{\partial z} = \tau n t^{nc-c}. \tag{4}$$

Thus τ_z is calculated as:

$$\tau_z = \frac{\Gamma(1+nc)}{\Gamma(1+nc-c)}. \tag{5}$$

Other fractional indexes $(\tau'_x, \tau'_y \tau'_z)$ can govern in similar means. Li and He [32-34] have developed a fractional complex transform method for transforming FDE so that all analytical methods for advanced calculus can be extended to fractional calculus.

$$v(x,t) = v(\Omega), v = \frac{\varpi t^c}{\Gamma(1+c)} + \frac{l x^d}{\Gamma(1+d)} + \frac{n x^e}{\Gamma(1+e)} + \Omega_0. \tag{6}$$

where l, ϖ and n are constants.

4. Analysis of F-expansion Method

Consider the following general nonlinear FPDE as:

$$Q(v, v_t, v_x, v_{xx}\, v_{xxx}, \ldots, D_t^c v, D_x^c v, D_{xx}^c v, \ldots) = 0,\ 0 < c \le 1. \tag{7}$$

where $D_t^c v, D_x^c v, D_{xx}^c v$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx respectively.

The following steps will demonstrate the nature of the F-expansion method:

Step 1: By transforming Eq. (7), look for solitary wave solutions (6).

Rewrite equation (7) in the following nonlinear ODE.

$$P(v, \varpi v', cv', dv', ev', ...) = 0, \tag{8}$$

where c, d, e and ϖ are constants and prime denotes the derivative with respect to χ.

Step 2: If necessary, integrate Eq. (8) term by term one or more times. As a consequence, the integration constant is defined (s). For convenience, the integration constant(s) can be set to zero.

Step 3: We assume that the wave solution can be expressed in the following form using the F-expansion process.

$$v(\chi) = \sum_{i=0}^{N} \alpha_i H^i(\chi). \tag{9}$$

where α_r real constants to be determined are, N is a positive integer to be determined. $F(\xi)$ satisfies the following auxiliary equation

$$[H'(\chi)]^2 = QH^4(\chi) + PH^2(\chi) + R. \tag{10}$$

where Q, P $P, Q,$ and R are constants. The last equation hence holds for

$$H''(\chi) = 2QH^3(\chi) + PH(\chi),$$

$$H'''(\chi) = 6QH^2H'(\chi) + PH'(\chi),$$

$$H^{(iv)}(\chi) = 24Q^2H^5(\chi) + 20QPH^3(\chi) + 12QRH(\chi) + Q^2H(\chi),$$

$$H^{v}(\chi) = 120Q^2H^4H'(\chi) + 60QPH^3H'(\chi) + 12QRH'(\chi) + P^2H'(\chi),$$
$$\vdots$$

Step 4: Make the decision N. This is generally done by balancing the highest order linear term(s) with the highest order nonlinear term(s) in Eq (8).

Step 5: putting Eq. (9) into Eq. (8) along with Eq. (10) yields an algebraic equation involving powers of $H^i H'$. Equating the coefficients of each power of $H^i H'$ to zero gives a system of algebraic equations for α_r, c, d, e and ϖ. Then, these constant are determined by use of computer algebra system for example Maple 18.

Eq. (11) [37-41] has 52 forms of exact solutions, as shown in Tables 1 and 2. In reality, these exact solutions can be used to create even more precise solutions (7).

Table 1: Relations between the coefficients (P, Q, R) and corresponding $H(\chi)$.

Case	Q	P	$\boldsymbol{R}$	$H(\chi)$
1	n^2	$-(1+n^2)$	1	$\mathrm{Sn}\xi$
2	n^2	$-(1+n^2)$	1	$\mathrm{cd}\xi = \mathrm{cn}\xi/\mathrm{dn}\xi$
3	$-n^2$	$2n^2-1$	$1-n^2$	$\mathrm{Cn}\xi$
4	-1	$2-n^2$	n^2-1	$\mathrm{Dn}\xi$
5	1	$-(1+n^2)$	n^2	$\mathrm{ns}\xi = (\mathrm{sn}\xi)^{-1}$
6	1	$-(1+n^2)$	n^2	$\mathrm{dc}\xi = \mathrm{dn}\xi/\mathrm{cn}\xi$
7	$1-n^2$	$2n^2-1$	$-n^2$	$\mathrm{nc}\xi = (\mathrm{cn}\xi)^{-1}$
8	n^2-1	$2-n^2$	-1	$\mathrm{nd}\xi = (\mathrm{dn}\xi)^{-1}$
9	$1-n^2$	$2-n^2$	1	$\mathrm{sc}\xi = \mathrm{sn}\xi/\mathrm{cn}\xi$
10	$-n^2(n^2-1)$	$2n^2-1$	1	$\mathrm{sd}\xi = \mathrm{sn}\xi/\mathrm{dn}\xi$
11	1	$2-n^2$	$1-n^2$	$\mathrm{cs}\xi = \mathrm{cn}\xi/\mathrm{sn}\xi$
12	1	$2n^2-1$	$-n^2(n^2-1)$	$\mathrm{ds}\xi = \mathrm{dn}\xi/\mathrm{sn}\xi$

13	¼	$\frac{1-2n^2}{2}$	¼	$\mathrm{ns}\xi \pm \mathrm{cn}\xi$
14	$\frac{1-n^2}{4}$	$\frac{1+n^2}{2}$	$\frac{1-n^2}{4}$	$\mathrm{nc}\xi \pm \mathrm{sc}\xi$
15	¼	$\frac{n^2-2}{2}$	$\frac{n^2}{4}$	$\mathrm{ns}\xi \pm \mathrm{ds}\xi$
16	$\frac{n^2}{4}$	$\frac{n^2-2}{2}$	$\frac{n^2}{4}$	$\mathrm{sn}\xi \pm i\mathrm{cn}\xi$
17	$\frac{n^2}{4}$	$\frac{n^2-2}{2}$	$\frac{n^2}{4}$	$\sqrt{1-m^2}\mathrm{sd}\xi \pm \mathrm{cd}\xi$
18	¼	$\frac{1-n^2}{2}$	¼	$m\mathrm{cd}\xi \pm i\sqrt{1-m^2}\mathrm{nd}\xi$
19	¼	$\frac{1-2n^2}{2}$	¼	$m\mathrm{sn}\xi \pm i\mathrm{dc}\xi$
20	¼	$\frac{1-n^2}{2}$	¼	$\sqrt{1-m^2}\mathrm{sc}\xi \pm \mathrm{dc}\xi$
21	$\frac{n^2-1}{4}$	$\frac{n^2+1}{2}$	$\frac{n^2-1}{4}$	$m\mathrm{sd}\xi \pm \mathrm{nd}\xi$
22	$\frac{n^2}{4}$	$\frac{n^2-2}{2}$	¼	$\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}$
23	−¼	$\frac{n^2+1}{2}$	$\frac{(n^2-1)^2}{4}$	$m\mathrm{cn}\xi \pm \mathrm{dn}\xi$
24	$\frac{(n^2-1)^2}{4}$	$\frac{n^2+1}{2}$	¼	$\mathrm{ds}\xi \pm \mathrm{cs}\xi$
25	$\frac{n^4(1-n^2)}{2(n^2-2)}$	$\frac{2(1-n^2)}{n^2-2}$	$\frac{1-n^2}{2(n^2-2)}$	$\mathrm{dc}\xi \pm \sqrt{1-m^2}\mathrm{nc}\xi$

26	$Q>0$	$P<0$	$\dfrac{n^2P}{Q(1+n^2)^2}$	$\sqrt{\dfrac{-m^2Q}{(m^2+1)P}}\,\mathrm{sn}\left(\sqrt{\dfrac{-Q}{m^2+1}}\,\xi\right)$
27	$Q<0$	$P>0$	$\dfrac{n^4(n^2-1)Q^2}{Q(n^2-2)^2}$	$\sqrt{\dfrac{-Q}{(2-m^2)P}}\,\mathrm{dn}\left(\sqrt{\dfrac{Q}{2-m^2}}\,\xi\right)$
28	$Q<0$	$P>0$	$\dfrac{n^4(n^2-1)Q^2}{P(2n^2-1)^2}$	$\sqrt{\dfrac{-m^2Q}{(2m^2-1)P}}\,\mathrm{cn}\left(\sqrt{\dfrac{Q}{2m^2-1}}\,\xi\right)$
29	1	$2-4n^2$	1	$\dfrac{\mathrm{sn}\xi\ \mathrm{dn}\xi}{\mathrm{cn}\xi}$
30	n^4	2	1	$\dfrac{\mathrm{sn}\xi\ \mathrm{cn}\xi}{\mathrm{dn}\xi}$
31	1	n^2+2	$1-2n^2+4n^4$	$\dfrac{\mathrm{dn}\xi\ \mathrm{cn}\xi}{\mathrm{sn}\xi}$
32	$\dfrac{A^2(n-1)^2}{4}$	$\dfrac{n^2+1}{2}+3n$	$\dfrac{(n+1)^2}{4A^2}$	$\dfrac{\mathrm{cn}\xi\ \mathrm{dn}\xi}{A(1+\mathrm{sn}\xi)(1+m\mathrm{sn}\xi)}$
33	$\dfrac{A^2(n+1)^2}{4}$	$\dfrac{n^2+1}{2}-3n$	$\dfrac{(n+1)^2}{4A^2}$	$\dfrac{\mathrm{cn}\xi\ \mathrm{dn}\xi}{A(1+\mathrm{sn}\xi)(1-m\mathrm{sn}\xi)}$
34	$-\dfrac{4}{n}$	$6n-n^2-1$	$-2n^3+n^4+n^2$	$\dfrac{m\mathrm{cn}\xi\ \mathrm{dn}\xi}{m\mathrm{sn}^2\xi+1}$
35	$\dfrac{4}{n}$	$-6n-n^2-1$	$2n^3+n^4+n^2$	$\dfrac{m\mathrm{cn}\xi\ \mathrm{dn}\xi}{m\mathrm{sn}^2\xi-1}$
36	¼	$\dfrac{1-2m^2}{2}$	¼	$\dfrac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi}$
37	$\dfrac{1-n^2}{4}$	$\dfrac{1+n^2}{2}$	$\dfrac{1-n^2}{4}$	$\dfrac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$
38	$4n_1$	$2+6n_1-n^2$	$2+2n_1-n^2$	$\dfrac{m^2\mathrm{sn}\xi\ \mathrm{cn}\xi}{m_1-\mathrm{dn}^2\xi}$
39	$-4n_1$		$2-2n_1-n^2$	$-\dfrac{m^2\mathrm{sn}\xi\ \mathrm{cn}\xi}{m_1+\mathrm{dn}^2\xi}$

		$2-6n_1-n^2$		
40	$\frac{2-n^2-2n_1}{4}$	$\frac{n^2}{2}-1-3n_1$	$\frac{2-n^2-2n_1}{4}$	$\frac{m^2\operatorname{sn}\xi\operatorname{cn}\xi}{\operatorname{sn}^2\xi+(1+m_1)\operatorname{dn}\xi-1-m_1}$
41	$\frac{2-n^2+2n_1}{4}$	$\frac{n^2}{2}-1+3n_1$	$\frac{2-n^2+2n_1}{4}$	$\frac{m^2\operatorname{sn}\xi\operatorname{cn}\xi}{\operatorname{sn}^2\xi+(m_1-1)\operatorname{dn}\xi-1+m_1}$
42	$\frac{C^2n^4-(B^2+C^2)i^2+B^2}{4}$	$\frac{n^2+1}{2}$	$\frac{n^2-1}{4(C^2n^2+B^2)}$	$\frac{\sqrt{\frac{(B^2-C^2)}{(B^2-C^2m^2)}}+\operatorname{sn}\xi}{B\operatorname{cn}\xi+C\operatorname{dn}\xi}$
43	$\frac{C^2n^2+B^2}{4}$	$\frac{1}{2}-n^2$	$\frac{1}{4(C^2n^2+B^2)}$	$\frac{\sqrt{\frac{(B^2-C^2+C^2m^2)}{(B^2+C^2m^2)}}+\operatorname{cn}\xi}{B\operatorname{sn}\xi+C\operatorname{dn}\xi}$
44	$\frac{C^2+B^2}{4}$	$\frac{n^2}{2}-1$	$\frac{n^4}{4(C^2+B^2)}$	$\frac{\sqrt{\frac{(B^2+C^2-C^2m^2)}{(B^2+C^2)}}+\operatorname{dn}\xi}{B\operatorname{sn}\xi+C\operatorname{cn}\xi}$
45	$-(n^2+2n+1)B^2$	$2n^2+2$	$-\frac{2n+n^2+1}{B^2}$	$\frac{m\operatorname{sn}^2\xi-1}{B(m\operatorname{sn}^2\xi+1)}$
46	$-(n^2-2n+1)B^2$	$2n^2+2$	$-\frac{2n+n^2+1}{B^2}$	$\frac{m\operatorname{sn}^2\xi+1}{B(m\operatorname{sn}^2\xi-1)}$

Weierstrass-elliptic function solutions for Eq. (4), where

$$D=\frac{1}{2}(-5P\pm\sqrt{9P^2-36QR})$$

and $$\psi'(\chi,g_2,g_3)=\frac{d}{d\chi}\psi(\chi,g_2,g_3)$$

Table B: Relations between the coefficients g_2, g_3 and corresponding $H(\chi)$.

Case	g_2	g_3	$H(\chi)$
47	$\frac{4}{3}(Q^2 - 3PR)$	$\frac{4Q(9PR - 2Q^2)}{27}$	$\sqrt{\frac{1}{P}\left(\phi(\xi; g_2, g_3) - \frac{1}{3}Q\right)}$
48	$\frac{4}{3}(Q^2 - 3PR)$	$\frac{4Q(9PR - 2Q^2)}{27}$	$\sqrt{\frac{3R}{3\phi(\xi; g_2, g_3) - Q}}$
49	$\frac{-(5DQ + 4Q^2 + 33PQR)}{12}$	$\frac{20Q^3 - 27PQR - 63PRD + 21Q^2D}{216}$	$\frac{\sqrt{12R\phi(\xi; g_2, g_3) + 2R(2Q + D)}}{12\phi(\xi; g_2, g_3) + D}$
50	$\frac{1}{12}Q^2 + PR$	$\frac{1}{216}Q(36PR - Q^2)$	$\frac{\sqrt{R}[6\phi(\xi; g_2, g_3) + Q]}{3\phi'(\xi; g_2, g_3)}$
51	$\frac{1}{12}Q^2 + PR$	$\frac{1}{216}Q(36PR - Q^2)$	$\frac{3\phi'(\xi; g_2, g_3)}{\sqrt{P}[6\phi(\xi; g_2, g_3) + Q]}$
52	$\frac{2Q^2}{9}$	$\frac{Q^3}{54}$	$\frac{Q\sqrt{-15Q/2P}3\wp(\xi; g_2, g_3)}{3\phi(\xi; g_2, g_3) + Q}, R = \frac{5Q^2}{36P}$

5. Numerical Applications

Consider the following fractional RLW Equation

$$\frac{\partial^\alpha V}{\partial t^\alpha} + \alpha V_x + 2VV_x + \beta V v_{xxt} = 0,\ 0 < \alpha < 1. \tag{11}$$

With the help of Eq. (7), the Eq. (11) transform to

$$\varpi V V_\Omega + \alpha l V_\Omega + 2lVV_\Omega + \beta l^2 \varpi V_{\Omega\Omega\Omega} = 0. \tag{12}$$

Integrating Eq. (12) once, gives

$$\varpi V + \alpha l V + l V^2 + \beta l^2 \varpi V = 0. \tag{13}$$

By using homogenous balancing principle, we have $N = 2$.

Therefore, trial solution is given by

$$W = \alpha_0 + \alpha_1 H / + \alpha_2 H^2. \tag{14}$$

Substitute Eq. (14) into (13) and using (10), we have

$$\varpi\alpha_0 + \varpi\alpha_1 H + \varpi\alpha_2 H^2 + \alpha l\alpha_0 + \alpha l\alpha_1 H + \alpha l\alpha_2 H^2 + l\alpha_0{}^2 + 2l\alpha_0\alpha_1 H + 2l\alpha_0\alpha_2 H^2 + l\alpha_1{}^2 H^2$$

$$2l\alpha_1 H^3\alpha_0 + l\alpha_2{}^2 H^4 + 2\beta\varpi l^2\alpha_1 QH^3 + \beta\varpi l^2\alpha_1 PH + 6\beta\varpi l^2\alpha_2 QH^4 +$$

$$4\beta\varpi l^2\alpha_2 PH^2 + 2\beta\varpi l^2\alpha_2 R = 0$$

On comparison of like powers of H we have system of algebraic equations

$$H^0: \quad \varpi\alpha_0 + \alpha l\alpha_0 + l\alpha_0{}^2 + 2\beta\varpi l^2\alpha_2 R = 0,$$

$$H^1: \quad \varpi\alpha_1 + \alpha l\alpha_0 + 2l\alpha_0\alpha_1 + \beta\varpi l^2\alpha_1 P = 0,$$

$$H^2: \quad \varpi\alpha_2 + \alpha l\alpha_2 + 2l\alpha_0\alpha_2 + l\alpha_1{}^2 + 4\beta\varpi l^2\alpha_2 P = 0,$$

$$H^3: \quad 2l\alpha_0\alpha_2 + 2\beta\varpi l^2\alpha_1 Q = 0,$$

$$H^4: \quad l\alpha_2{}^2 + 6\beta\varpi l^2\alpha_2 Q = 0.$$

Solving the above system for $\varpi, \alpha_0, \alpha_1, \alpha_2$ ω, a_0, a_1, a_2 with the help of MAPLE 18, yields

$$\left\{ \begin{array}{l} \varpi = \dfrac{\left(-1+4\sqrt{-3l^4QR\beta^2 + l^4P^2\beta^2}\right)dl}{48l^4QR\beta^2 - 16l^4P^2\beta^2 + 1}, \\ \alpha_0 = -\frac{1}{4}\beta\left(\dfrac{4l^2\left(-1+4\sqrt{-3l^4QR\beta^2 + l^4Q^2\beta^2}\right)Q\beta}{48l^4QR\beta^2 - 16l^4P^2\beta^2 + 1} + \dfrac{-1+4\sqrt{-3l^4QR\beta^2 + l^4P^2\beta^2}}{48l^4QR\beta^2 - 16l^4P^2\beta^2 + 1} + 1 \right), \\ \alpha_1 = 0, \alpha_2 = -\dfrac{3l^2Q\beta\left(-1+4\sqrt{-3l^4QR\beta^2 + l^4P^2\beta^2}\right)d}{48l^4QR\beta^2 - 16l^4P^2\beta^2 + 1} \end{array} \right\}. \tag{15}$$

Substituting solution set (15) into Eq. (13), yields

$$V(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{-3l^4QR\beta^2+l^4Q^2\beta^2}\right)P\beta}{48l^4QR\beta^2-16l^4P^2\beta^2+1}+\frac{-1+4\sqrt{-3l^4QR\beta^2+l^4P^2\beta^2}}{48l^4QR\beta^2-16l^4P^2\beta^2+1}+1\right)$$
$$-\frac{3l^2Q\beta\left(-1+4\sqrt{-3l^4QR\beta^2+l^4P^2\beta^2}\right)dH(\Omega)^2}{48l^4QR\beta^2-16l^4P^2\beta^2+1}. \tag{16}$$

Combining Table A and solution of Eq. (16), more generalized Jacobian-elliptic function can be obtained for solution of Eq. (16). The following exact solutions are obtained after that.

Case 1: When $Q=n^2, P=-(1+n^2), R=1, H(\chi)=zi\chi,$ then this is the exact solution of the given equation:

$$V_1(\Omega)=-\tfrac{1}{4}d\left(\begin{array}{l}\dfrac{4l^2\left(-1+4\sqrt{-3l^4n^2\beta^2+l^4\left(-n^2-1\right)^2\beta^2}\right)\left(-n^2-1\right)\beta}{48l^4n^2\beta^2-16l^4\left(-n^2-1\right)^2\beta^2+1}+\\ \dfrac{-1+4\sqrt{-3l^4n^2\beta^2+k^4\left(-n^2-1\right)^2\beta^2}}{48l^4n^2\beta^2-16l^4\left(-n^2-1\right)^2\beta^2+1}+1\end{array}\right).$$
$$-\frac{3l^2n^2\beta\left(-1+4\sqrt{-3l^4n^2\beta^2+l^4\left(-n^2-1\right)^2\beta^2}\right)d\sin(\Omega)^2}{48l^4n^2\beta^2-16l^4\left(-n^2-1\right)^2\beta^2+1} \tag{17}$$

Case 2: When $Q=n^2, P=-\left(1+n^2\right), R=1, H(\chi)=\phi d\chi,$ then this is the exact solution of the given equation:

$$V_2(\Omega) = -\tfrac{1}{4} d \left(\frac{4l^2\left(-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-n^2-1\right)^2\beta^2}\right)\left(-n^2-1\right)\beta}{48l^4n^2\beta^2 - 16l^4\left(-n^2-1\right)^2\beta^2 + 1} + \frac{-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-n^2-1\right)^2\beta^2}}{48l^4n^2\beta^2 - 16kl^4\left(-n^2-1\right)^2\beta^2 + 1} + 1 \right)$$

$$-\frac{3l^2n^2\beta\left(-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-n^2-1\right)^2\beta^2}\right)d\phi^2 d^2\Omega^2}{48l^4n^2\beta^2 - 16l^4\left(-n^2-1\right)^2\beta^2 + 1}.$$

$$-\frac{3l^2n^2\beta\left(-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-n^2-1\right)^2\beta^2}\right)d\sin(\Omega)^2}{48l^4n^2\beta^2 - 16l^4\left(-n^2-1\right)^2\beta^2 + 1}$$

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Case 3: When $Q = -n^2, P = 2n^2 - 1, R = 1 - m^2, H(\chi) = \phi i \chi,$ then this is the exact solution of the given equation:

$$V_3(\Omega) = -\tfrac{1}{4} d \left(\frac{4l^2\left(-1+4\sqrt{-3l^4n^2\left(-n^2+1\right)\beta^2 + l^4\left(2n^2-1\right)^2\beta^2}\right)\left(2n^2-1\right)\beta}{48l^4n^2\left(-n^2+1\right)\beta^2 - 16l^4\left(2n^2-1\right)^2\beta^2 + 1} + \frac{-1+4\sqrt{-3l^4n^2\left(-n^2+1\right)\beta^2 + l^4\left(2n^2-1\right)^2\beta^2}}{48l^4n^2\left(-n^2+1\right)\beta^2 - 16l^4\left(2n^2-1\right)^2\beta^2 + 1} + 1 \right) \tag{19}$$

$$-\frac{3l^2m^2\beta\left(-1+4\sqrt{-3l^4n^2\left(-n^2+1\right)\beta^2 + l^4\left(2n^2-1\right)^2\beta^2}\right)d\phi^2 i^2\eta^2}{48l^4n^2\left(-n^2+1\right)\beta^2 - 16l^4\left(2n^2-1\right)^2\beta^2 + 1}.$$

Case 4: When $Q = -1, P = 2 - n^2, R = n^2 - 1, H(\chi) = di\chi,$ then this is the exact solution of the given equation:

$$V_4(\Omega) = -\tfrac{1}{4} d \left(\frac{4l^2\left(-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2 + l^4\left(-2n^2+1\right)^2\beta^2}\right)\left(-2n^2+1\right)\beta}{-48l^4\left(n^2-1\right)b^2 - 16l^4\left(-2n^2+1\right)^2\beta^2+1} + \frac{-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2 + l^4\left(-2m^2+1\right)^2\beta^2}}{-48l^4\left(n^2-1\right)\beta^2 - 16l^4\left(-2n^2+1\right)^2\beta^2+1} + 1 \right)$$

$$+ \frac{3l^2\beta\left(-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2 + l^4\left(-2n^2+1\right)^2\beta^2}\right)dd^2i^2\Omega^2}{-48l^4\left(n^2-1\right)\beta^2 - 16l^4\left(-2n^2+1\right)^2\beta^2+1}. \tag{20}$$

Case 5: When $Q=1, P=-\left(1+n^2\right), R=n^2, H(\Omega)=iz\Omega,$ then this is the exact solution of the given equation:

$$V_5(\Omega) = -\tfrac{1}{4} d \left(\frac{4k^2\left(-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-2n^2-1\right)^2\beta^2}\right)\left(-2n^2-1\right)\beta}{48l^4n^2\beta^2 - 16l^4\left(-2n^2-1\right)^2\beta^2+1} + \frac{-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-2n^2-1\right)^2\beta^2}}{48l^4n^2\beta^2 - 16l^4\left(-2n^2-1\right)^2\beta^2+1} + 1 \right)$$

$$- \frac{3l^2\beta\left(-1+4\sqrt{-3l^4n^2\beta^2 + l^4\left(-2n^2-1\right)^2\beta^2}\right)di^2z^2\Omega^2}{48l^4n^2\beta^2 - 16kl\left(-2n^2-1\right)^2\beta^2+1}. \tag{21}$$

Case6: When $Q=1, P=-\left(1+n^2\right), R=n^2, H(\chi)=d\phi\chi,$ then this is the exact solution of the given equation:

$$V_6(\Omega)=-\tfrac{1}{4}d\left(\frac{\frac{4l^2\left(-1+4\sqrt{-3kl^4n^2\beta^2+l^4\left(-2n^2-1\right)^2\beta^2}\right)\left(-2n^2-1\right)\beta}{48l^4n^2\beta^2-16l^4\left(-2n^2-1\right)^2\beta^2+1}+}{\frac{-1+4\sqrt{-3l^4n^2\beta^2+l^4\left(-2n^2-1\right)^2\beta^2}}{48l^4n^2\beta^2-16l^4\left(-2n^2-1\right)^2\beta^2+1}+1}\right)$$
$$-\frac{3k^2\beta\left(-1+4\sqrt{-3l^4n^2\beta^2+l^4\left(-2n^2-1\right)^2\beta^2}\right)dd^2\phi^2\Omega^2}{48k^4n^2\beta^2-16l^4\left(-2n^2-1\right)^2\beta^2+1}. \tag{22}$$

Case 7: When $Q=1-n^2, P=2n^2-1, R=-n^2, H(\chi)=iz\chi,$ then this is the exact solution of the given equation:

$$V_7(\Omega)=-\tfrac{1}{4}d\left(\frac{\frac{4l^2\left(-1+4\sqrt{-3l^4(1-n)^2n^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}\right)\left(2n^2-1\right)\beta}{48l^4(1-n)^2n^2\beta^2-16l^4\left(2n^2-1\right)^2\beta^2+1}+}{\frac{-1+4\sqrt{-3l^4(1-n)^2n^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}}{48l^4(1-n)^2n^2\beta^2-16l^4\left(2n^2-1\right)^2\beta^2+1}+1}\right)$$
$$-\frac{3l^2(1-n)^2\beta\left(-1+4\sqrt{-3l^4\left(-n^2+1\right)n^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}\right)di^2\phi^2\Omega^2}{48l^4(1-n)^2n^2\beta^2-16l^4\left(2n^2-1\right)^2\beta^2+1}. \tag{23}$$

Case 8: When $Q=n^2-1, P=2-n^2, R=-1, H(\chi)=id\chi,$ then this is the exact solution of the given equation:

$$V_8(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2+l^4\left(-n^2+2\right)^2\beta^2}\right)\left(-n^2+2\right)\beta}{-48l^4\left(n^2-1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+\frac{-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2+l^4\left(-n^2+2\right)^2\beta^2}}{-48l^4\left(n^2-1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+1\right)$$
$$-\frac{3l^2\left(n^2-1\right)\beta\left(-1+4\sqrt{3l^4\left(n^2-1\right)\beta^2+l^4\left(-n^2+2\right)^2\beta^2}\right)di^2d^2\Omega^2}{-48l^4\left(n^2-1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}. \tag{24}$$

Case 9: When $Q=1-n^2, P=2-n^2, R=1, H(\chi)=z\phi\chi,$ then this is the exact solution of the given equation:

$$V_9(\Omega)=-\tfrac{1}{4}d\left(\frac{4k^2\left(-1+4\sqrt{-3l^4(1-n)^2\beta^2+l^4\left(-n^2+2\right)^2\beta^2}\right)\left(-n^2+2\right)\beta}{48l^4(1-n)^2\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+\frac{-1+4\sqrt{-3l^4(1-n)^2\beta^2+l^4\left(-n^2+2\right)^2\beta^2}}{48l^4(1-n)^2\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+1\right)$$
$$-\frac{3l^2(1-n)^2\beta\left(-1+4\sqrt{3l^4(1-n)^2\beta^2+l^4\left(-n^2+2\right)^2\beta^2}\right)dz^2\phi^2\Omega^2}{48l^4(1-n)^2\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}. \tag{25}$$

Case 10: When $Q=-n^2\left(1-n^2\right), P=2n^2-1, R=1, H(\chi)=zd\chi,$ then this is the exact solution of the given equation:

$$V_{10}(\Omega)=-\tfrac{1}{4}d\left(\begin{array}{l}\dfrac{4l^2\left(-1+4\sqrt{3l^4n^2(1-n)^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}\right)\left(2n^2-1\right)\beta}{-48l^4n^2(1-n)^2\beta^2-16l^4\left(2n^2-1\right)^2\beta^2+1}+\\ \dfrac{-1+4\sqrt{3l^4n^2(1-n)^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}}{-48l^4n^2(1-n)^2b^2-16l^4\left(2n^2-1\right)^2\beta^2+1}+1\end{array}\right) \tag{26}$$

$$+\frac{3l^2n^2(1-n)^2\beta\left(-1+4\sqrt{3l^4n^2(1-n)^2\beta^2+l^4\left(2n^2-1\right)^2\beta^2}\right)dz^2d^2\Omega^2}{-48l^4n^2(1-n)^2\beta^2-16l^4\left(2n^2-1\right)^2\beta^2+1}.$$

Case 11: When $Q=1, P=2-n^2, R=1-n^2, H(\chi)=\phi z\chi,$ then this is the exact solution of the given equation:

$$V_{11}(\Omega)=-\tfrac{1}{4}d\left(\begin{array}{l}\dfrac{4l^2\left(-1+4\sqrt{-3l^4\left(-n^2+1\right)\beta^2+l^4\left(-n+2\right)^2\beta^2}\right)\left(-n^2+2\right)\beta}{48l^4\left(-n^2+1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+\\ \dfrac{-1+4\sqrt{3l^4\left(-n^2+1\right)\beta^2+l^4\left(-n^2+2\right)^2\beta^2}}{48l^4\left(-n^2+1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}+1\end{array}\right) \tag{27}$$

$$-\frac{3l^2\beta\left(-1+4\sqrt{3l^4\left(-n^2+1\right)\beta^2+l^4\left(-n^2+2\right)^2\beta^2}\right)d\phi^2z^2\Omega^2}{48l^4m^2\left(-n^2+1\right)\beta^2-16l^4\left(-n^2+2\right)^2\beta^2+1}.$$

Case 12: When $Q=1, P=2n^2-1, R=-n^2\left(1-n^2\right), H(\chi)=dz\chi,$ then this is the exact solution of the given equation:

$$V_{12}(\Omega)=-\tfrac{1}{4}d\left(\frac{\frac{4l^2\left(-1+4\sqrt{3l^4n^2(-n^2+1)\beta^2+l^4(2n^2-1)^2b^2}\right)(2n^2-1)\beta}{-48l^4n^2(-n^2+1)\beta^2-16l^4(2n^2-1)^2\beta^2+1}+}{\frac{-1+4\sqrt{3l^4n^2(-n^2+1)\beta^2+l^4(2n^2-1)^2\beta^2}}{-48l^4n^2(-n^2+1)\beta^2-16l^4(2n^2-1)^2\beta^2+1}+1}\right)$$
$$-\frac{3k^2b\left(-1+4\sqrt{3l^4n^2(-n^2+1)\beta^2+l^4(2n^2-1)^2\beta^2}\right)dd^2z^2\Omega^2}{-48l^4n^2(-n^2+1)\beta^2-16l^4(2n^2-1)^2\beta^2+1}. \quad (28)$$

Case 13: When $Q=1/4, P=\left(1-2n^2\right)/2, R=1/4, H(\chi)=iz\chi\pm\phi i\chi,$ then this is the exact solution of the given equation:

$$V_{13}(\Omega)=-\tfrac{1}{4}d\left(\frac{\frac{4k^2\left(-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2}\right)\left(-n^2+\frac{1}{2}\right)\beta}{3l^4\beta^2-16l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2+1}+}{\frac{-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2}}{3l^4\beta^2-16l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2+1}+1}\right)$$
$$-\frac{3}{4}\frac{l^2\beta\left(-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2}\right)d\left(iz\Omega\pm\phi i\Omega\right)^2}{3l^4\beta^2-16l^4\left(-n^2+\frac{1}{2}\right)^2\beta^2+1}. \quad (29)$$

Case 14: When $Q=\left(1-n^2\right)/4, P=\left(1+n^2\right)/2, R=\left(1-n^2\right)/4, H(\chi)=i\phi\chi\pm z\phi\chi,$ then this is the exact solution of the given equation:

$$V_{14}(\Omega)=-\tfrac{1}{4}d\left(\frac{\left(4l^2\left(-1+4\sqrt{-3l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)\beta\right)}{\left(48l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1\right)}+\frac{-1+4\sqrt{-3l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}}{48l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1}\right)-\frac{\left(3l^2\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)\beta\left(-1+4\sqrt{-3l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)d\left(ic\Omega\pm z\phi\Omega\right)^2\right)}{\left(48l^4\left(-\tfrac{1}{4}n^2+\tfrac{1}{4}\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1\right)}. \tag{30}$$

Case 15: When $Q=1/4, P=\left(n^2-2\right)/2, R=n^2/4, H(\chi)iz\chi\pm dz\chi,$ then this is the exact solution of the given equation:

$$V_{15}(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^2\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2-1\right)\beta}{3l^4n^2\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+\frac{-1+4\sqrt{-\tfrac{3}{16}l^4n^2\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}}{3l^4n^2\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+1\right)-\tfrac{3}{4}\frac{l^2b\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^2\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)d\left(iz\Omega\pm d\phi\Omega\right)^2}{3l^4n^2\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}. \tag{31}$$

Case 16: When $Q=n^2/4, P=\left(n^2-2\right)/2, R=n^2/4, H(\chi)=zi\chi\pm r\phi i\chi,$ then this is the exact solution of the given equation:

$$V_{16}\left(\Omega\right)=-\tfrac{1}{4}d\left(\frac{\dfrac{4l^2\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2-1\right)\beta}{3l^4n^4\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+}{\dfrac{-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}}{3l^4n^4\beta^2-16k^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+1}\right)$$
$$-\tfrac{3}{4}\frac{l^2n^2\beta\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)d\left(zi\Omega\pm r\phi i\Omega\right)^2}{3l^4n^4\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}. \tag{32}$$

Case 17: When $Q={n^2}/{4}, p=\left(n^2-2\right)/2, R={n^2}/{4}, H\left(\chi\right)=\sqrt{1-n^2}zd\chi\pm\phi d\chi,$ then this is the exact solution of the given equation:

$$V_{17}\left(\Omega\right)=-\tfrac{1}{4}d\left(\frac{\dfrac{4l^2\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2-1\right)\beta}{3l^4n^4\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+}{\dfrac{-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}}{3l^4n^4\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}+1}\right)$$
$$-\tfrac{3}{4}\frac{l^2n^2\beta\left(-1+4\sqrt{-\tfrac{3}{16}l^4n^4\beta^2+l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2}\right)d\left(\sqrt{-n^2+1}zd\Omega\pm\phi d\Omega\right)^2}{3l^4n^4\beta^2-16l^4\left(\tfrac{1}{2}n^2-1\right)^2\beta^2+1}. \tag{33}$$

Case 18: When $Q=1/4, P=\left(1-n^2\right)/2, R=1/4, H\left(\chi\right)=n\phi d\chi\pm r\sqrt{1-n^2}id\chi,$ then this is the exact solution of the given equation:

$$V_{18}(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{-\frac{3}{16}l^4n^2\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)\left(-\frac{1}{2}n^2+\frac{1}{2}\right)\beta}{3l^4n^2\beta^2-16l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2+1}+\frac{-1+4\sqrt{-\frac{3}{16}l^4n^2\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}}{3l^4n^2\beta^2-16l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1}+1\right)$$

$$-\tfrac{3}{4}\frac{1}{3l^4n^2\beta^2-16l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1} \tag{34}$$

$$\left(l^2n^2\beta\left(-1+4\sqrt{-\frac{3}{16}l^4n^2\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)d\left(n\phi d\Omega\pm r\sqrt{-n^2+1}id\Omega\right)^2\right).$$

Case 19: When $Q=1/4, P=\left(1-2n^2\right)/2, R=1/4, H(\chi)=nzi\chi\pm id\phi\chi,$ then this is the exact solution of the given equation:

$$V_{19}(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)\left(-\frac{1}{2}n^2+\frac{1}{2}\right)\beta}{3l^4\beta^2-16l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1}+\frac{-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}}{3l^4\beta^2-16l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1}+1\right)$$

$$-\tfrac{3}{4}\frac{1}{3k^4b^2-16l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1} \tag{35}$$

$$\left(l^2\beta\left(-1+4\sqrt{-\frac{3}{16}l^4\beta^2+l^4\left(-\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)d\left(nzi\Omega\pm rd\phi\Omega\right)^2\right).$$

Case 20: When $Q=1/4, P=\left(1-n^2\right)/2, R=1/4, H(\chi)=\sqrt{1-n^2}z\phi\chi\pm d\phi\chi,$ then this is the exact solution of the given equation:

$$V_{20}(\Omega)=-\tfrac{1}{4}d\left(\frac{4l^2\left(-1+4\sqrt{-\tfrac{3}{16}l^4\beta^2+l^4\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)\beta}{3l^4\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1}+\frac{-1+4\sqrt{-\tfrac{3}{16}l^4\beta^2+l^4\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}}{3l^4\beta^2-16l^4\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1}+1\right)$$

$$-\tfrac{3}{4}\frac{1}{3l^4\beta^2-16l^4\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1} \tag{36}$$

$$\left(l^2\beta\left(-1+4\sqrt{-\tfrac{3}{16}l^4\beta^2+l^4\left(-\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)d\left(\sqrt{-n^2+1}z\phi\Omega\pm d\phi\Omega\right)^2\right).$$

Case 21: When $Q=\left(n^2-1\right)/4, P=\left(n^2+1\right)/2, R=\left(n^2-1\right)/4, H(\chi)=nzd\chi\pm id\chi,$ then this is the exact solution of the given equation:

$$V_{21}(\Omega)=-\tfrac{1}{4}d\left(\frac{\left(4l^2\left(-1+4\sqrt{-3l^4\left(\tfrac{1}{4}n^2-\tfrac{1}{4}\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)\beta\right)}{\left(48l^4\left(\tfrac{1}{4}n^2-\tfrac{1}{4}\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1\right)}+\frac{-1+4\sqrt{-3l^4\left(\tfrac{1}{4}n^2-\tfrac{1}{4}\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}}{48l^4\left(\tfrac{1}{4}n^2-\tfrac{1}{4}\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1}+1\right)- \tag{37}$$

$$16l^4-\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1.$$

Case 22: When $Q=n^2/4, P=\left(n^2-2\right)/2, R=1/4, H(\chi)=\dfrac{zi\chi}{1\pm di\chi},$ then this is the exact solution of the given equation:

$$V_{22}(\Omega)=-\tfrac{1}{4}d\left(\begin{array}{l}\dfrac{4l^2\left(-1+4\sqrt{-\frac{3}{16}l^4n^2\beta^2+l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2}\right)\left(\frac{1}{2}n^2-1\right)\beta}{3l^4n^2\beta^2-16l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2+1}+\\ \dfrac{-1+4\sqrt{-\frac{3}{16}l^4n^2\beta^2+l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2}}{3l^4n^2\beta^2-16k^4\left(\frac{1}{2}n^2-1\right)^2\beta^2+1}+1\end{array}\right) \tag{38}$$

$$-\tfrac{3}{4}\frac{l^2n^2\beta\left(-1+4\sqrt{-\frac{3}{16}l^4n^4\beta^2+l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2}\right)dz^2i^2d^2\Omega^2}{\left(3l^4n^2\beta^2-16l^4\left(\frac{1}{2}n^2-1\right)^2\beta^2+1\right)\left(1\pm id\Omega\right)^2}.$$

Case 23: When $Q=-\frac{1}{4}, P=\left(n^2+1\right)/2, R=\left(1-n^2\right)^2/4, H(\chi)=n\phi i\chi\pm di\chi,$ then this is the exact solution of the given equation:

$$V_{23}(\Omega)=-\tfrac{1}{4}d\left(\begin{array}{l}\dfrac{\left(4l^2\left(-1+4\sqrt{\frac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)\left(\frac{1}{2}n^2+\frac{1}{2}\right)\beta\right)}{\left(-3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1\right)}\\ \qquad+\dfrac{-1+4\sqrt{\frac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}}{-3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1}+1\end{array}\right) \tag{39}$$

$$+\tfrac{3}{4}\frac{\left(l^2\beta\left(-1+4\sqrt{\frac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2}\right)d\left(n\phi i\Omega\pm di\Omega\right)^2\right)}{\left(-3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\frac{1}{2}n^2+\frac{1}{2}\right)^2\beta^2+1\right)}.$$

Case 24: When $Q=\dfrac{\left(1-n^2\right)^2}{4}, P=\left(n^2+1\right)/2, R=\frac{1}{4}, H(\chi)=dz\chi\pm\phi z\chi,$ then this is the exact solution of the given equation:

$$V_{24}\left(\Omega\right)=-\tfrac{1}{4}d\left(\frac{\left(4l^2\left(-1+4\sqrt{-\tfrac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)\beta\right)}{\left(3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1\right)} + \frac{-1+4\sqrt{-\tfrac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+k^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}}{3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1}+1\right) \tag{40}$$
$$-\tfrac{3}{4}\frac{\left(l^2\left(-n^2+1\right)^2\beta\left(-1+4\sqrt{-\tfrac{3}{16}l^4\left(-n^2+1\right)^2\beta^2+l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2}\right)d\left(dz\Omega\pm\phi z\Omega\right)^2\right)}{\left(3l^4\left(-n^2+1\right)^2\beta^2-16l^4\left(\tfrac{1}{2}n^2+\tfrac{1}{2}\right)^2\beta^2+1\right)}.$$

In the same manner with the aid of Table 1 and 2, other exact solutions of Eq. (16) can be obtained.

6. Conclusion

The fractional kind generalized solitary solutions of the nonlinear regularized long wave RLW equation are obtained using the F-expansion method. The results of this study show that this approach has exact solutions for all forms of functions, including Weierstrass-elliptic and Jacobian-elliptic functions. These solutions are validated by substituting them back into the physical model, and they are found to be superior to other existing methods. Furthermore, this approach has major advantages in terms of ease of use, precision, and computational performance. The obtained results are strengthened by addition of MAPLE 18. In future, this method will be used in applied sciences for further modification in results.

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