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ARTICLE TYPE

About a fixed-point-type transformation to solve quadratic matrix equations using the Krasnoselskij method†

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Summary

In this paper, we study the simplest quadratic matrix equation: $Q(X) = X^2 + BX + C = 0$. We transform this equation into an equivalent fixed-point equation and based on it we construct the Krasnoselskij method. From this transformation, we can obtain iterative schemes more accurate than successive approximation method. Moreover, under suitable conditions, we establish different results for the existence and localization of a solution for this equation with the Krasnoselskij method. Finally, we see numerically that the predictor-corrector iterative scheme with the Krasnoselskij method as a predictor and the Newton method as corrector method, can improves the numerical application of the Newton method when approximating a solution of the quadratic matrix equation.

KEYWORDS:
Quadratic matrix equations, iterative methods, local convergence, semilocal convergence, error estimates.

1 | INTRODUCTION

Quadratic matrix equations arises in many areas of scientific computing and engineering applications. There is a great variety of quadratic matrix equations that have been studied due to their applicability. An important class of examples, arising in control theory, is algebraic Riccati equations, such as

$$XBX + XA + A^*X + C = 0,$$

where A, B, and C are given coefficient matrices. Theory of Riccati equations and numerical methods for their solution are well developed [13].

In this work, we focus on the simplest quadratic matrix equation:

$$Q(X) = X^2 + BX + C = 0, \qquad B, C \in \mathbb{R}^{m\times m}, \tag{1.1}$$

which we refer as (QME) throughout the work. Observe that, this type of quadratic matrix equation with B and C satisfying certain conditions, arise in noisy Wiener-Hopf problems for Markov chains. Notice that, the study can be extended to $\mathbb{C}^{m\times m}$. Although some Riccati equations are quadratic matrix equations, and vice versa, the two classes of equations require different techniques for analysis and solution in general. This equation (1.1) occurs in a variety of applications, for example, it may arise in the well known quadratic eigenvalue problem:

$$Q(\lambda)x = \lambda^2 Ax + \lambda Bx + Cx = 0, \text{ with } A, B, C \in \mathbb{C}^{m\times m},$$

which arises in the analysis of structural systems and vibration problems [16].

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Many studies have been made to solve quadratic matrix equation by methods which rely heavily on linear algebra and the theory of matrices, see [14, 15]. However, our main aim is to approximate a solution of equation (1.1) from the numerical point of view. A commonly used technique to approximate a solution of equation (1.1) is the application of iterative schemes. The most popular solver for these type of equations is the Newton method. As it is known, under some suitable conditions, the Newton method can achieve quadratic convergence speed. Davis ([5], [6]) considered the Newton method to the quadratic matrix (1.1) in detail.

In Bai et al.[3] for solving quadratic matrix equations, the authors transform equation (QME) into an equivalent fixed point equation, and based on it they construct a successive approximation method, these new methods are more accurate and effective than the known ones.

Notice that, if $X^* \in \mathbb{R}^{m\times m}$ is a solution of (QME), i.e.,

$$(X^*)^2 - BX^* - C = 0,$$

then, we have $(X^* - B)X^* = C$. Moreover, it follows that $(X^* - B)$ and X^* are both nonsingular matrices. In this case, following the transformation given in [3], for equation (1.1), we can construct the following fixed point equation:

$$X = T(X), \quad \text{where} \quad T(X) = (X - B)^{-1}C, \tag{1.2}$$

and $T : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$. Thus, $X^* \in \mathbb{R}^{m\times m}$ is a solution of (QME) if and only if it is a fixed-point of the matrix operator T, or equivalently, a zero of the matrix equation

$$F(X) = X - T(X) = 0,$$

with $F : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$. Moreover, as can be seen in [10], if we consider $T(X) = (X - B)^{-1}C$, the successive approximation method is stable.

Now, to define the operator T, it is important to do so in such a way that the successive approximation method is stable. It is known [1] that, in the application of iterative schemes, its algorithm is important to obtain a stable iterative scheme. It is known that there are several possibilities to express equation (1.1), such that, $X = T(X)$ with $T : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$. Obviously, it is necessary that a fixed matrix for operator T must be a solution of equation (1.1). For example, we can consider $T(X) = -(BX + C)^{1/2}$. This situation does not seem to be very favorable due to the instabilities that arise when approximating the square root or the p-th root of a matrix through iterative processes (see [1] and the references there given). On the other hand, if we consider $T(X) = B^{-1}(-X^2 - C)$, we can easily verify that the successive approximation method has a computationally unstable behavior.

In this work, following the ideas of Y.-H Gao in [9], we consider

$$T(X) = -(B + CX^{-1}), \tag{1.3}$$

which has a computationally stable behavior. Moreover, in this situation, as it is known [12], Krasnoselskij observed that the averaged mapping

$$T_\omega(X) = (1 - \omega)Id(X) + \omega T(X), \quad \omega \in [0, 1], \tag{1.4}$$

where Id is the identity in $\mathbb{R}^{m\times m}$, possesses the same fixed points as T, and has a better asymptotic behavior than T itself. Therefore, can be used as an iteration function to approximate fixed points of T. So, in our study, we consider the Krasnoselskij method, which is given by the following algorithm:

$$\begin{cases} X_0 \text{ given in } \mathbb{R}^{m\times m}, \\ X_{n+1} = T_\omega(X_n) = (1 - \omega)X_n - \omega(B + CX_n^{-1}), \ n \geqslant 0, \quad \omega \in [0, 1]. \end{cases} \tag{1.5}$$

Notice that, for $\omega = 1$, the successive approximation method [4] is obtained:

$$X_0 \text{ given in } \mathbb{R}^{m\times m}, \quad X_{n+1} = T(X_n), \quad n \geq 0. \tag{1.6}$$

In this work, from (1.3) and under suitable conditions, we prove the local convergence for the Krasnoselskij method given in (1.5). Besides, using the auxiliary matrix technique [8], we prove the existence of a solution of equation (1.1) and we locate domains of global convergence restricted to balls for the Krasnoselskij method. To separate a solution of equation (1.1) from other possible solutions we obtain a result of uniqueness of solution. Numerical results show that, we can localize from the Krasnoselskij method schemes more accurate, varying the parameter ω, than the successive approximation method. Furthermore, as we show in the numerical examples it is interesting to construct a predictor-corrector method [7, 11], Krasnoselskij-Newton, which improves accuracy and execution time when approximating a solution of equation (1.1).

The paper is organized as follows. In Section 2, we present different conditions for the existence and localization of solutions of equation (1.1) from the study of the global convergence restricted of the Krasnoselskij method given in (1.5). We also establish a global convergence restricted result in which we increase the value of the parameter ω that appears in (1.5). Finally, we illustrate some numerical results in Section 3 one of them related to the noisy Wiener-Hopf problem considered in ([10], [11]).

2 | ON THE APPROXIMATION OF A SOLUTION FOR (QME)

In what follows, we consider the local situation for the Krasnoselskij method given in (1.5). So, we suppose that there exists $X^* \in B(X^*, R)$ a fixed matrix of T given in (1.3). So, we look for conditions on R so that the Krasnoselskij method is convergent for any starting matrix X_0 in $B(X^*, R)$. Thus, we obtain a local convergence result.

Now, we give a basic technical result to achieve our goal.

Lemma 1. Let X^* be a solution of QME (1.1) and we suppose that there exists $(X^*)^{-1}$ with $\|(X^*)^{-1}\| \leq \beta^*$. For each $X \in B(X^*, R)$, with $R < 1/\beta^*$, then there exists X^{-1} with $\|X^{-1}\| \leq \dfrac{\beta^*}{1-\beta^* R}$.

Proof If we consider I the identity matrix in $\mathbb{R}^{m\times m}$, we have

$$\|(I - (X^*)^{-1}X\| \leq \|(X^*)^{-1}\|\|X - X^*\| < \beta^* R,$$

therefore, as $R < 1/\beta^*$, then by the perturbation lemma in matrix analysis, there exists X^{-1}, and $\|X^{-1}\| \leq \dfrac{\beta^*}{1-\beta^* R}$. ■

Theorem 1. Let X^* be a nonsingular solution of QME (1.1) such that

$$\|C\| = c \quad \text{and} \quad \|(X^*)^{-1}\| \leq \beta^*.$$

Assume that $X_0 \in \mathbb{R}^{m\times m}$ and there exists $R > 0$ such that $\|X_0 - X^*\| \leq R$. Then, if

$$\beta^* < \frac{\sqrt{R^2+4c} - R}{2c}, \tag{2.7}$$

the iterative sequence $\{X_n\}$, generated by the Krasnoselskij method (1.5) with X_0 as the initial guess and $\omega \in (0, 1]$, converges to X^*. Moreover, the sequence $\{X_n\} \subseteq B(X^*, R)$ and satisfies

$$\|X_{n+1} - X^*\| \leq q^*(\omega)\|X_n - X^*\|, n = 0, 1, 2, ...,$$

where

$$q^*(\omega) = 1 - \omega + \omega\frac{(\beta^*)^2 c}{1-\beta^* R}.$$

Proof Notice that, it is easy to check

$$\frac{\sqrt{R^2+4c} - R}{2c} < \frac{1}{R}.$$

Then, $\beta^* R < 1$ and from Lemma 1 we obtain that there exists X_0^{-1} and therefore $T(X_0)$, for T given in (1.3), is well defined. So, taking into account that as X^* is a solution of QME (1.1) then it is a fixed matrix of T, given in (1.3), we have

$$\begin{aligned} X_1 - X^* &= (1-\omega)X_0 + \omega T(X_0) - (1-\omega)X^* - \omega T(X^*) \\ &= (1-\omega)(X_0 - X^*) + \omega C((X^*)^{-1} - X_0^{-1}) \\ &= (1-\omega)(X_0 - X^*) + \omega C(X^*)^{-1}(X_0 - X^*)X_0^{-1}. \end{aligned} \tag{2.8}$$

Then, by applying Lemma 1, we obtain

$$\|X_1 - X^*\| \leq \left[1 - \omega + \omega\frac{(\beta^*)^2 c}{1-\beta^* R}\right]\|X_0 - X^*\| = q^*(\omega)\|X_0 - X^*\|.$$

Now, taking into account that $q^*(0) = 1$ and $(q^*)'(\omega) = -1 + \dfrac{(\beta^*)^2 c}{1-\beta^* R}$, as from (2.7) it follows that $\dfrac{(\beta^*)^2 c}{1-\beta^* R} < 1$, then q is a decreasing real function. Therefore, $q^*(\omega) < 1$ for $\omega \in (0, 1]$, and then $\|X_1 - X^*\| < \|X_0 - X^*\| < R$.

Next, following a mathematical inductive procedure, taking into account (2.8), it follows that

$$\|X_{n+1} - X^*\| < q^*(\omega)^{n+1}\|X_0 - X^*\|. \tag{2.9}$$

Then, the iterative sequence $\{X_n\}$, generated by the Krasnoselskij method (1.5) with X_0 as the initial guess, converges to X^*. Moreover, the sequence $\{X_n\} \subseteq \overline{B(X^*, R)}$. ■

Note that the best a priori bound of the error, see (2.9), is obtained in the case $\omega = 1$, that is, for the Successive Approximation method. On the other hand, if $(\beta^*)^2 c < 1$, from condition (2.7), it follows that $R \in \left(0, \dfrac{1-(\beta^*)^2 c}{\beta^*}\right)$. In this case, we obtain that the local result that we have just proved gives us a ball, called the convergence ball, for which we obtain global convergence for the Krasnoselskij method. That is, this method converges to a solution of QME (1.1) for any initial guess X_0 considered in $B(X^*, R)$.

We follow the idea of locating domains of global convergence for the Krasnoselskij method restricted to balls. To dot hat, we prove a result of restricted global convergence, by using auxiliary matrices [8]. For this, we obtain previously some technical results.

Lemma 2. Let $\tilde{X} \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|\tilde{X}^{-1}\| \le \tilde{\beta}$. Then, for each $X \in \overline{B(\tilde{X}, R)}$, with $R < 1/\tilde{\beta}$, there exists $(\tilde{X} + t(X - \tilde{X}))^{-1}$, for $t \in [0, 1]$, with $\|(\tilde{X} + t(X - \tilde{X}))^{-1}\| \le f_R(t)$, where $f_R(t) = \dfrac{\tilde{\beta}}{1 - t\tilde{\beta}R}$.

Proof As in Lemma 1, we will apply the perturbation lemma in matrix analysis. For this, we consider

$$\|(I - (\tilde{X})^{-1}(\tilde{X} + t(X - \tilde{X}))\| \le \|\tilde{X}^{-1}\|\| - t(X - \tilde{X})\| \le t\tilde{\beta}R \le \tilde{\beta}R,$$

for $t \in [0, 1]$. Therefore, as $R < 1/\tilde{\beta}$, then there exists $(\tilde{X} + t(X - \tilde{X}))^{-1}$, and

$$\|(\tilde{X} + t(X - \tilde{X}))^{-1}\| \le f_R(t).$$

So, the lemma is proved. ■

From now, we will denote $F(X) = X - T(X)$ with T, given in (1.3). So, we can rewrite the Krasnoselskij method as

$$X_{n+1} = T_\omega(X_n) = X_n - \omega F(X_n), \quad n \geqslant 0. \tag{2.10}$$

It is easy to check that the Fréchet derivative $F'(X)$, for $X \in \mathbb{R}^{m\times m}$, is given by

$$F'(X)Y = Y + CX^{-1}YX^{-1}, \quad \text{for all } \ Y \in \mathbb{R}^{m\times m}.$$

Next, we analyze the behavior of the iterations $\{X_n\}$ given by the Krasnoselskij method.

Lemma 3. Let $\tilde{X} \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|\tilde{X}^{-1}\| \le \tilde{\beta}$ and $\|F(\tilde{X})\| \le \tilde{\eta}$. If $X_{n-1}, X_n \in B(\tilde{X}, R)$, with $R < 1/\tilde{\beta}$, then the following items are verified:

(i) there exists $(X_{n-1} + t(X_n - X_{n-1}))^{-1}$, for $t \in [0, 1]$, with $\|(X_{n-1} + t(X_n - X_{n-1}))^{-1}\| \le f_R(1)$.

(ii) $\|F(X_n)\| \le \left[|1 - \dfrac{1}{\omega}| + c f_R(1)^2\right] \|X_n - X_{n-1}\|,$

(iii) $\|X_{n+1} - \tilde{X}\| \le \left[|1 - \omega| + \omega\tilde{\beta}c f_R(1)\right] \|X_n - \tilde{X}\| + \omega\tilde{\eta}.$

Proof To prove item (i), we consider

$$\|(I - \tilde{X}^{-1}(X_{n-1} + t(X_n - X_{n-1}))\| \le \|\tilde{X}^{-1}\|\|t(\tilde{X} - X_n) + (1-t)(\tilde{X} - X_{n-1})\| \le \tilde{\beta}R,$$

for $t \in [0, 1]$. Therefore, as $R < 1/\tilde{\beta}$, then, by the perturbation lemma in matrix analysis, there exists $(X_{n-1} + t(X_n - X_{n-1}))^{-1}$, and

$$\|(X_{n-1} + t(X_n - X_{n-1}))^{-1}\| \le f_R(1).$$

Next, to prove (ii), we consider

$$\begin{aligned}\omega F(X_n) &= \omega F(X_{n-1}) + \omega(F(X_n) - F(X_{n-1}))\\ &= -(X_n - X_{n-1}) + \omega \int_{X_{n-1}}^{X_n} F'(Z)\, dZ\\ &= \omega \int_0^1 (F'(X_{n-1} + t(X_n - X_{n-1})) - \frac{1}{\omega} Id)(X_n - X_{n-1})\, dt,\end{aligned}$$

however,

$$(F'(X_{n-1} + t(X_n - X_{n-1})) - \frac{1}{\omega} Id)(X_n - X_{n-1}) =$$
$$(1 - \frac{1}{\omega})(X_n - X_{n-1}) + C(X_{n-1} + t(X_n - X_{n-1}))^{-1}(X_n - X_{n-1})(X_{n-1} + t(X_n - X_{n-1}))^{-1}.$$

Now, taking into account the previous expressions and the item (i), then the item (ii) follows easily.

Finally, to prove item (iii), notice that

$$\begin{aligned}X_{n+1} - \tilde{X} &= X_n - \omega(F(X_n) - F(\tilde{X})) - \omega F(\tilde{X}) - \tilde{X}\\ &= X_n - \tilde{X} - \omega \int_{\tilde{X}}^{X_n} F'(Z)\, dZ - \omega F(\tilde{X})\\ &= \int_0^1 \left(Id - \omega F'(\tilde{X} + t(X_n - \tilde{X}))\right)(X_n - \tilde{X})\, dt - \omega F(\tilde{X}).\end{aligned}$$

On the other hand,

$$(Id - \omega F'(\tilde{X} + t(X_n - \tilde{X})))(X_n - \tilde{X}) =$$
$$(1 - \omega)(X_n - \tilde{X}) - \omega C(\tilde{X} + t(X_n - \tilde{X}))^{-1}(X_n - \tilde{X})(\tilde{X} + t(X_n - \tilde{X}))^{-1}.$$

Then, from Lemma 2, we have

$$\begin{aligned}\|X_{n+1} - \tilde{X}\| &\leq \int_0^1 \|Id - \omega F'(\tilde{X} + t(X_n - \tilde{X}))\| \|X_n - \tilde{X}\|\, dt + \omega \|F(\tilde{X})\|\\ &= \int_0^1 (|1 - \omega| + \omega c f_R(t)^2)\, dt \|X_n - \tilde{X}\| + \omega \|F(\tilde{X})\|\\ &\leq \left[|1 - \omega| + \omega \tilde{\beta} c f_R(1)\right] \|X_n - \tilde{X}\| + \omega\, \tilde{\eta},\end{aligned}$$

what it proves (iii). ■

Theorem 2. Let $\tilde{X} \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|\tilde{X}^{-1}\| \leq \tilde{\beta}$ and $\|F(\tilde{X})\| \leq \tilde{\eta}$. Assume that $X_0 \in \mathbb{R}^{m\times m}$ and there exists $R > 0$ such that $\|X_0 - \tilde{X}\| < R$. Then, if

$$\tilde{\beta} < \frac{1}{R + \sqrt{c}} \quad \text{and} \quad \tilde{\eta} \leq R(1 - \tilde{\beta} c f_R(1)), \tag{2.11}$$

the iterative sequence $\{X_n\}$, generated by the Krasnoselskij method (1.5) with X_0 as the initial guess, converges to X^* a solution of QME (1.1). Moreover, the $X_n, X^* \in \overline{B(\tilde{X}, R)}$ and satisfies

$$\|X_{n+1} - X_n\| \leq \tilde{q}(\omega)\|X_n - X_{n-1}\|, n = 0, 1, 2, ...,$$
$$\|X^* - X_n\| \leq \frac{\omega \tilde{q}(\omega)^n}{1 - \tilde{q}(\omega)} \|F(X_0)\|, n = 0, 1, 2, ...,$$

where

$$\tilde{q}(\omega) = 1 - \omega + \omega c f_R(1)^2.$$

Proof Firstly, notice that

$$\frac{1}{R} > \frac{1}{R+\sqrt{c}} > \tilde{\beta}.$$

Therefore, $R < \dfrac{1}{\tilde{\beta}}$. Taking into account Lemma 2 for $t = 1$, there exists X_0^{-1} and then $X_1 = T_\omega(X_0)$ is well defined. Now, as item (iii) in Lemma 3, we have

$$\|X_1 - \tilde{X}\| \leq \left(|1-\omega| + \omega\tilde{\beta}c f_R(1)\right)\|X_0 - \tilde{X}\| + \omega\tilde{\eta},$$

so, from (2.11), it follows that $X_1 \in B(\tilde{X}, R)$.

Moreover, X_2 is well defined and taking into account that $X_0, X_1 \in B(\tilde{X}, R)$, from Lemma 3 we obtain

$$\|F(X_1)\| \leq \left(|1-\frac{1}{\omega}| + c f_R(1)^2\right)\|X_1 - X_0\|.$$

Therefore

$$\|X_2 - X_1\| \leq \tilde{q}(\omega)\|X_1 - X_0\|.$$

However, $\tilde{q}(0) = 1$ and $\tilde{q}'(\omega) = -(1 - c f_R(1)^2)$. But, from (2.11) we have that $\tilde{\beta}\sqrt{c} < 1$ and then $c f_R(1)^2 < 1$. So, it follows that $\tilde{q}$ is a strictly decreasing real function. Then, $\tilde{q}(\omega)) < 1$, for all $\omega \in (0, 1]$, and therefore $\|X_2 - X_1\| < \|X_1 - X_0\|$.

Now, applying a mathematical inductive procedure, the sequence $\{X_n\}$, given by the Krasnoselskij method is well defined, belongs to $B(\tilde{X}, R)$ and the real sequence $\{\|X_{n+1} - X_n\|\}$ is a strictly decreasing real sequence of positive real numbers. So, $\{X_n\}$ converges and, by the continuity of F and item (ii) of Lemma 3, it follows that $\{X_n\}$ converges to X^* a solution of QME (1.1). Moreover,

$$\begin{aligned}
\|X_{m+n} - X_n\| &\leq \|X_{m+n} - X_{m+n-1}\| + \cdots + \|X_{n+1} - X_n\| \\
&\leq \sum_{j=0}^{m-1} \|X_{n+j+1} - X_{n+j}\| \\
&\leq \sum_{j=0}^{m-1} \tilde{q}(\omega)^j \|X_{n+1} - X_n\| \\
&= \tilde{q}(\omega)^n \frac{1-\tilde{q}(\omega)^m}{1-\tilde{q}(\omega)} |\|X_1 - X_0\| \\
&= \omega\tilde{q}(\omega)^n \frac{1-\tilde{q}(\omega)^m}{1-\tilde{q}(\omega)} \|F(X_0)\|,
\end{aligned}$$

so, it is easy to check that

$$\|X^* - X_n\| \leq \frac{\omega\tilde{q}(\omega)^n}{1-\tilde{q}(\omega)}\|F(X_0)\|.$$

■

Firstly, notice that this results prove that there exists a solution of equation (QME) in $\overline{B(\tilde{X}, R)}$. Secondly, notice that if $\tilde{\beta}^2 c < 1$, we can ensure the global convergence of the Krasnoselskij method in $B(\tilde{X}, R)$ with $R \in \left(0, \dfrac{1}{\tilde{\beta}} - \sqrt{c}\right)$.

On the one hand, considering $\tilde{X} = X^*$, therefore admitted the existence of X^*, using decomposition (2.8), for any $n \geqslant 1$, instead of item (iii) of Lemma 3, we obtain as a consequence Theorem 1.

On the other hand, taking $\tilde{X} = X_0 \in \mathbb{R}^{m\times m}$, we can obtain the following semilocal convergence result.

Corollary 1. Let $X_0 \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|X_0^{-1}\| \leq \beta_0$ and $\|F(X_0)\| \leq \eta_0$. If

$$\beta_0 < \frac{1}{R+\sqrt{c}} \quad \text{and} \quad \eta_0 \leq R\left(1 - \frac{c\beta_0^2}{1-\beta_0 R}\right), \tag{2.12}$$

then, the iterative sequence $\{X_n\}$, generated by the Krasnoselskij method (1.5) with X_0 as the initial guess, converges to X^* a solution of equation given in (1.1). Moreover, the sequence $\{X_n\} \subseteq B(X_0, R)$ and satisfies

$$\|X_{n+1} - X_n\| \leq q_0(\omega)\|X_n - X_{n-1}\|, n = 0, 1, 2, ...,$$

$$\|X^* - X_n\| \leq \frac{\omega q_0(\omega)^n}{1-q_0(\omega)}\|F(X_0)\|, n = 0, 1, 2, ...,$$

where

$$q_0(\omega) = 1 - \omega + \omega \frac{c\beta_0^2}{(1-\beta_0 R)^2}.$$

■

Once a solution of equation (QME) is located by Theorem 2 in $B(X_0, R)$, we now separate it from other possible solutions by the following result of uniqueness of solution.

Theorem 3. Under the hypotheses of Theorem 2, the solution X^* of (1.1) is unique in $B(X_0, \frac{2}{\tilde{\beta}} - R)$.

Proof
Assume that Y^* is another solution of (1.1) in $B(X_0, r)$. Thus,

$$0 = F(X^*) - F(Y^*) = \int_{Y^*}^{X^*} F'(Z)dZ = \int_0^1 F'(Y^* + \tau(X^* - Y^*)d\tau(X^* - Y^*)$$

Now, if the operator $Q(X) = \int_0^1 F'(Y^* + \tau(X^* - Y^*)d\tau(X)$ is invertible, then $Y^* = X^*$. We can deduce the last from the perturbation lemma in matrix analysis, pr0vided that $\|Id - Q\| < 1$.

Observe that

$$(Id - Q)(X) = -\int_0^1 \left(-C(Y^* + \tau(X^* - Y^*)^{-1}X(Y^* + \tau(X^* - Y^*)^{-1}\right) d\tau.$$

Now, taking into account that

$$\|Id - \widetilde{X}^{-1}(Y^* + \tau(X^* - Y^*)\| < \tilde{\beta} \int_0^1 ((1-\tau)r + \tau R)d\tau < 1,$$

there exists $(Y^* + \tau(X^* - Y^*)^{-1}$ with $\|(Y^* + \tau(X^* - Y^*)^{-1}\| \leq f_R(1)$. Then, $\|Id - Q\| \leq c f_R(1)^2 < 1$ and therefore $Y^* = X^*$. So, the proof is complete. ■

2.1 | A modification of the Krasnoselskij method

Next, we propose a modification of the Krasnoselskij method to increase the size of the set of values for the parameter w. To do that, we consider $\tilde{X} \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|\tilde{X}^{-1}\| \leq \tilde{\beta}$ and $\|F(\tilde{X})\| \leq \tilde{\eta}$. The convergence conditions on the parameters $\tilde{\beta}$ and $\tilde{\eta}$ arise from the fact that the sequence $\{\|X_{n+1} - X_n\|\}$ is strictly decreasing and that $X_n \in B(\tilde{X}, R)\ \forall n \geq 0$.

Theorem 4. Let $\tilde{X} \in \mathbb{R}^{m\times m}$ a nonsingular matrix such that $\|\tilde{X}^{-1}\| \leq \tilde{\beta}$ and $\|F(\tilde{X})\| \leq \tilde{\eta}$. Assume that $X_0 \in \mathbb{R}^{m\times m}$ and there exists $R > 0$ such that $\|X_0 - \tilde{X}\| < R$. If

$$\tilde{\beta} < \frac{1}{R + \sqrt{c}} \quad \text{and} \quad \tilde{\eta} \leq R\left(1 - c f_R(1)\right), \tag{2.13}$$

then, the iterative sequence $\{X_n\}$, generated by the Krasnoselskij method (1.5) with X_0 as the initial guess, converges to X^* a solution of QME (1.1), for all $w \in \left(1, \frac{2}{1 + c f_R(1)^2}\right)$. Moreover, the sequence $\{X_n\} \subseteq B(\tilde{X}, R)$ and satisfies

$$\|X_{n+1} - X_n\| \leq \tilde{q}(\omega)\|X_n - X_{n-1}\|, n = 0, 1, 2, ...,$$

where

$$\tilde{q}(\omega) = \omega - 1 + \omega c f_R(1)^2.$$

Proof Analogously to the proof of Theorem 2 we prove that the sequence $\{X_n\}$, given by the Krasnoselskij method is well defined, belongs to $B(\tilde{X}, R)$ and the real sequence $\{\|X_{n+1} - X_n\|\}$ is a strictly decreasing real sequence of positive real numbers when $w > 1$.

We suppose that $X_0 \in B(\tilde{X}, R)$. If $\tilde{\beta}R < 1$ then there exists X_0^{-1} and then $X_1 = T_\omega(X_0)$ is well defined. Moreover, taking into account the condition on $\tilde{\eta}$, given in (2.13), it follows that

$$\begin{aligned} \|X_1 - \tilde{X}\| &\leq \left(|1-\omega| + \omega\tilde{\beta}c f_R(1)\right)\|X_0 - \tilde{X}\| + \omega\tilde{\eta} \\ &< \left(\omega - 1 + \omega\tilde{\beta}c f_R(1)\right) R + \omega\tilde{\eta} \leq R. \end{aligned}$$

Taking into account that $X_1 \in B(\tilde{X}, R)$, there exists X_1^{-1} and then $X_2 = T_\omega(X_1)$ is well defined. And, we have

$$\|X_2 - X_1\| = \|wF(X_1)\| \leq w\left(|1 - \frac{1}{\omega}| + c f_R(1)^2\right)\|X_1 - X_0\| \leq \tilde{q}(\omega)\|X_1 - X_0\|.$$

Now, since $\tilde{q}(1) < 1$ and $\tilde{q}$ is a strictly an increasing real function, it follows that $\tilde{q}(\omega)) < 1$, for $\omega \in \left(1, \dfrac{2}{1+c f_R(1)^2}\right)$. Therefore $\|X_2 - X_1\| < \|X_1 - X_0\|$.

Now, applying a mathematical inductive procedure, we prove that $\{X_n\}$ converges to X^* a solution of QME (1.1). ■

Note that, we have established a semilocal convergence result in which we increase the value of the parameter ω, from $(0, 1]$ to $\left(0, \dfrac{2}{1+c f_R(1)^2}\right)$.

3 | NUMERICAL EXPERIMENTS

In order to analyze numerically the application of the Krasnoselskij method (K) to approximate a solution of equation (1.1), we consider two numerical examples related to the noisy Wiener-Hopf problem considered in (see [10], [11] and the references there given). In both numerical examples, we compare the accuracy and the execution time with the Newton (NM) and the predictor-corrector (KN) methods. The methods were implemented in Mathematica Version 10.0 in both examples.

Firstly, we take the (QME) with matrices:

$$B = -\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 100 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & -1 \end{pmatrix} \in \mathbb{R}^{100\times 100}. \tag{3.14}$$

We show in Table 1 the iterations (n) and residuals $RES = \|X_n^2 + BX_n + C\|_F$ of the Krasnoselskij method apply to equation (1.1) with matrices given in (3.14) for different values of the parameter ω, $\omega = 0.3, 0.6, 0.8, 0.9, 1$. We denote by $\|M\|_F$ the Frobenius norm of a matrix M, which is defined as $\|M\|_F^2 = trace(M^T M)$.

TABLE 1 Iterations and residuals of the Krasnoselskij method apply to (3.14) with $X_0 = (100 + \sqrt{10004})I_{100}$, and stopping criteria $RES < 10^{-9}$.

ω	n	RES
0.3	89	0.117835×10^{-10}
0.6	34	0.213112×10^{-10}
0.8	19	0.375782×10^{-10}
0.9	17	0.115693×10^{-10}
1	23	0.185747×10^{-10}

Observe that, although the a priori error bound optimal is reached for $\omega = 1$, see Theorem 2, a better numerical behavior than for the successive approximation method can appear for different values of ω, in this case for $\omega = 0.8, 0.9$.

Notice that, taking $T(X) = B + CX^{-1}$, the Krasnoselskij method has a reduced operational cost. Then, as in [11], we can think of the use of a predictor-corrector method. Thus, using the Krasnoselskij method as a predictor, we can increase the speed of convergence using the Newton method [2] as a corrector. Notice that, the Newton method at each iteration step needs to solve the Sylvester equation: $X_{n+1}X_n + CX_n^{-1}X_{n+1} = -BX_n$. Then, the hybrid predictor-corrector method may have less operational cost than directly applying the Newton method. So, we can consider the iterative process

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\text{Given an initial guess } X_0 \in \mathbb{R}^{m\times m},\\ X_{n+1} = X_n - F(X_n), \quad n = 0, 1, \ldots, N_0 - 1,\end{array}\right.\\ \left\{\begin{array}{l}Y_0 = X_{N_0},\\ Y_{n+1} = Y_n - [F'(Y_n)]^{-1}F(Y_n), \quad n \geqslant 0,\end{array}\right.\end{array}\right.$$

to approximate a solution of equation (1.1), where $F : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, with $F(X) = X + B + CX^{-1}$.

So, we denote by $\{Z_n\}$ the hybrid method:

$$Z_n = \left\{\begin{array}{l}X_n, \ n = 0, 1, \ldots, N_0 - 1,\\ Y_n, \ n \geqslant N_0.\end{array}\right.$$

To approximate a solution of equation (1.1), in the following Table 2 we apply the Krasnoselskij methodd (K), the Newton method (NM) and the predictor-corrector method (KN). Thus, we consider two and three iterations to predict with the Krasnoselskij method and then iterate with the Newton method to increase the accuracy of the solution until the stopping criterion $RES < 10^{-10}$ is reached.

TABLE 2 Numerical results with $X_0 = (100 + \sqrt{10004})I_{100}$, and stopping criteria $RES = \|X_k^2 + BX_k + C\|_F < 10^{-9}$.

	$n(w = 1)$	$RES(w = 1)$	iter$(w = 0.9)$	$RES(w = 0.9)$
K	23	0.185747×10^{-10}	17	0.115693×10^{-10}
NM	5	0.551624×10^{-14}		
KN	$3 + 3$	0.718181×10^{-14}	$3 + 3$	0.41076×10^{-13}
KN	$2 + 3$	$0.7158738 \times 10^{-10}$	$2 + 4$	0.414911×10^{-13}

As we show in Table 2, the use of the hybrid method can make sense in certain occasions due to the reduction in the operational cost.

Secondly, we apply the Krasnoselskij, the Newton and the predictor-corrector (KN) methods to the following noisy Wiener-Hopf problem. Thus, we consider the simplest quadratic matrix equation:

$$X^2 - VX + Q = 0,$$

and

$$X^2 + VX + Q = 0,$$

with Q-matrix solutions Γ_+ and Γ_-, respectively.

In particular, we consider the following matrices:

$$V = \begin{pmatrix} aI_{10} & 0 \\ 0 & bI_{10} \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & -1 \end{pmatrix} \in \mathbb{R}^{20\times 20}.$$

We analyze the following cases for a and b:

(a) $a = 1$, $b = -1$, so Γ_+ and Γ_- are both singular Q-matrices.

(b) $a = 2$, $b = -1$, so Γ_+ is a singular Q-matrix and Γ_- is a non-singular Q-matrix.

We apply for each case, the Newton (NM), the Krasnoselskij and the predictor-corrector (KN) methods, to approximate solutions of Γ_+ and Γ_-, denoted by $\tilde{\Gamma}_+$ and $\tilde{\Gamma}_-$, respectively. In Tables 3–4, we report the computational results of each experiment in terms of iteration number denoted by $n_\pm$, residuals $RES_\pm := \| \tilde{\Gamma}_\pm^2 \mp V\tilde{\Gamma}_\pm + Q\|_\infty$ and the computing time in seconds $t_\pm$ used until convergence to a Q-matrix such that $RES_\pm < 10^{-10}$ for (a) and (b). We take, $X_0 = \max_{1\leq i\leq 20}(\pm v_i + \sqrt{v_i^2 - 4q_i})/2I_{20}$ just like in [11] and, $K_0 = 8$ when apply the predictor-corrector method in both cases. In our implementations all iterations are run in Mathematica 10.

TABLE 3 Numerical results of experiment (a) under stopping criterion $RES_{\pm} < 10^{-10}$.

	$n_{\pm}$	$RES_{\pm}$	$t_{\pm}$
NM	35	0.673475×10^{-16}	14.5
K	77	0.754063×10^{-11}	1.5
KN	$8 + 4$	0.108633×10^{-16}	2.2

TABLE 4 Numerical results of experiment (b) under stopping criterion $RES_{\pm} < 10^{-10}$.

	n_{+}	RES_{+}	t_{+}	k_{-}	RES_{-}	t_{-}
NM	27	0.294812×10^{-18}	11.5	14	0.638975×10^{-13}	6.6
K	49	0.587327×10^{-11}	1.1	43	0.678975×10^{-11}	1.
KN	$8 + 3$	0.520372×10^{-20}	1.7	$8 + 3$	0.723316×10^{-14}	1.2

Observe that, in all cases analyzed the method predictor-corrector is more accurate than the Newton method. The number of iterations that the Newton method uses is considerably reduced by the predictor-corrector method. Moreover, the Krasnoselskij and the predictor-corrector methods require the use of smaller execution time than the Newton method. That is what indicates the best computational efficiency of these methods compared to the Newton method.

4 | CONCLUSIONS

From a fixed-point-type transformation of quadratic matrix equation, we consider the stable iterative scheme of Krasnoselskij. Using this scheme we carried out a qualitative study of equation given in (1.1). We obtain domains of existence of solutions that allow us to locate and separate them. The numerical examples confirm that varying the parameter ω that appear in the Krasnoselskij method, we can improve numerically the successive approximation method. Even, the predictor-corrector iterative scheme with the Krasnoselskij method as a predictor and the Newton method as corrector method, respectively, it can improves the numerical application of the Newton method when approximating a solution of equation given in (1.1).

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