ONE DIMENSIONAL BROWNIAN MOTION WITH HOLDING AND JUMPING BOUNDARY

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Abstract. Let a particle start at some point in the unit interval $I := [0, 1]$ and undergo Brownian motion in I until it hits one of the end points. At this instant the particle stays put for a finite holding time with an exponential distribution and then jumps back to a point inside I with a probability density μ_0 or μ_1 parametrized by the boundary point it was at. The process starts afresh. The same evolution repeats independently each time. Many probabilistic aspects of this diffusion process are investigated in the paper [11]. The authors in the cited paper call this process diffusion with holding and jumping (DHJ). Our simple aim in this paper is to analyze the eigenvalues of a nonlocal boundary problem arising from this process. In particular we answer a question on the spectral gap of the DHJ process raised at the end of the paper in [11].

1. Introduction

Consider a diffusion equation on the unit interval $I := [0, 1]$

$$u_t(t,x) = L\,u(t,x) \qquad \text{where the generator is} \tag{1.1}$$

$$L := a(x)\,\frac{d^2}{dx^2} + b(x)\frac{d}{dx}\,. \tag{1.2}$$

Both $a(x) > 0$ and $b(x)$ are assumed to be continuous on I. The eigenvalue problem we would like to consider has the form

$$a(x)\,v''(x) + b(x)v'(x) + \lambda v(x) = 0 \tag{1.3}$$

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with boundary conditions

$$v(0) = \int_0^1 v(x)\, d\mu_0(x) - \sigma_0\, (Lv)(0), \tag{1.4}$$

$$v(1) = \int_0^1 v(x)\, d\mu_1(x) - \sigma_1\, (Lv)(1). \tag{1.5}$$

The boundary conditions above originate from a diffusion process considered in a famous paper of Feller [5]. As summarized in Theorem 8 of Feller's paper, the diffusion process starts from some point in I until it reaches one of the boundary points at some time t. Then there is a *finite sojourn time* T associated with each of the two boundary points 0 and 1. In other words, the particle is held for a duration of T which is assumed to be a random variable independent of the past and has an exponential distribution $\Pr(T > t) = \exp(-t/\sigma_i)$, $i = 0$ or 1. (As noted on p.3 in [5], $\sigma_i = 0$ and $\sigma_i = \infty$ correspond to instantaneous return and to absorbing barrier process respectively at the end point i.) At $t + T$, the particle jumps back to I according to a probability distribution μ_0 or μ_1 parametrized by the end points. The process starts afresh. Peng and Li call this type of process *diffusion with holding and jumping* (DHJ). Connection of DHJ to neuron science is discussed in [12]. Generalizations of DHJ to diffusion with sticky boundary conditions have appeared in many popular papers in probability, see for example [4].

The analysis with no holding at the boundary (i.e. $\sigma_i = 0$) was initiated in [1] and [2] in a general multi-dimensional setting. A question raised on the realness of eigenvalues of one-dimensional Brownian motion with jumping (BMJ) was answered in [8] and in [7]. There have been a lot of research on this type of process initiated by Feller's paper (see for example [3] and [4].) Peng and Li's paper [11] seems to be the only one in discussing the existence of non-real eigenvalues λ for the system in (1.3), (1.4) and (1.5). We continue the analysis using classical function theoretic techniques to derive sufficient conditions on the holding parameter σ_i's and on boundary measures μ that give all real eigenvalues. We give an affirmative answer on the principal eigenvalue question posed in Peng and Li's paper. Finally we give a numerical example showing the existence of a non-real principal eigenvalue for certain σ values.

For Brownian motion, the differential equation in (1.2) is simply

$$v''(x) + \lambda v(x) = 0. \tag{1.6}$$

The boundary conditions in (1.4) and (1.5) become:

$$v(0) = \int_0^1 v(x)\, d\mu_0(x) - \sigma_0\, v''(0) \tag{1.7}$$

$$v(1) = \int_0^1 v(x)\, d\mu_1(x) - \sigma_1\, v''(1). \tag{1.8}$$

This type of nonlocal boundary problem has an eigen-parameter involved. To compute the eigenvalues of (1.6), we let $\lambda = z^2$. A solution of the equation has the form

$$v(x) = c_1 e^{-izx} + c_2 e^{izx}.$$

The boundary conditions (1.7) and (1.8) correspond to the system:

$$c_1 + c_2 = \int_0^1 \left(c_1 e^{-izx} + c_2 e^{izx}\right) d\,\mu_0(x) + \sigma_0 z^2 \,(c_1 + c_2), \tag{1.9}$$

$$c_1 e^{-iz} + c_2 e^{iz} = \int_0^1 \left(c_1 e^{-izx} + c_2 e^{izx}\right) d\,\mu_1(x) + \sigma_1 z^2 \,\left(c_1 e^{-iz} + c_2 e^{iz}\right). \tag{1.10}$$

By grouping the coefficients of c_1 and c_2 in the equations above, we can find the values of z from the zeros of the determinant

$$\Delta := \begin{vmatrix} (1-\sigma_0 z^2) - \int_0^1 e^{-izx}\, d\mu_0 & (1-\sigma_0 z^2) - \int_0^1 e^{izx}\, d\mu_0 \\ (1-\sigma_1 z^2)e^{-iz} - \int_0^1 e^{-izx}\, d\mu_1 & (1-\sigma_1 z^2)e^{iz} - \int_0^1 e^{izx}\, d\mu_1 \end{vmatrix}. \tag{1.11}$$

If we define

$$F(z) := \left((1-\sigma_0 z^2) - \int_0^1 e^{-izx}\, d\mu_0\right) \times \left((1-\sigma_1 z^2)e^{iz} - \int_0^1 e^{izx}\, d\mu_1\right), \tag{1.12}$$

then the determinant Δ is equal to $F(z) - F(-z)$. By noting that

$$e^{iz} = \cos(z) + i\sin(z),$$

we see that $F(z)$ can be decomposed into its real and imaginary parts as

$$F(z) = E(z) + i\,O(z),$$

where E is an even function while O is odd and both have all *real* coefficients. Hence $\Delta = 2i\,O(z)$. To analyze the roots of Δ and to clear up a point in the proof of Proposition 6.1 in [11], we first do a quick review on a topic in classical entire functions shown in Chapter VII of Levin's book [9].

An entire function $f(z)$ is said to be of *exponential type* if there are positive constants M and τ such that $|f(z)| \le M e^{\tau|z|}$ for all large $|z|$. The *indicator* function of $f(z)$ is

$$h_f(\theta) := \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r}.$$

The *defect* d_f ([9], p.319) of an entire function $f(z)$ of exponential type is defined by

$$d_f := \left(h_f(-\pi/2) - h_f(\pi/2)\right)/2.$$

If the defect of $f(z)$ is non-negative, it follows from the Lindelöf's maximum principle ([9], p.320) that

$$\left|\frac{f(z)}{\overline{f}(z)}\right| \le 1 \quad \text{for} \quad Im(z) > 0.$$

The function $\overline{f}(z)$ above is defined to be $\overline{f(\overline{z})}$. If $f(z) = P(z) + iQ(z)$ are its real and imaginary part decomposition (each with real Taylor coefficients), then $\overline{f}(z) = P(z) - iQ(z)$.

Definition 1.1. An entire function $f(z)$ of exponential type is said to belong to the class $\mathcal{P}$ if it has non-negative defect and has no roots in the open lower half plane $Im(z) < 0$.

Definition 1.2. An entire function $f(z)$ of exponential type is defined to be in the class $\overline{HB}$ (named after Hermite and Biehler) if $f(z)$ has no roots in the lower half plane and

$$|\frac{f(z)}{\overline{f}(z)}| \leq 1 \quad \text{when} \quad Im(z) > 0.$$

As a consequence of the definition, we see that the product of two $\overline{HB}$ functions also belongs to $\overline{HB}$. In addition, according to Lemma 1 in ([9], p.319) the two classes $\mathcal{P}$ and $\overline{HB}$ introduced above are identical.

The following statement is part of Theorem 3' in Levin's ([9], p.314):

Proposition 1.1. *If an entire function $f(z)$ of exponential type belongs to $\overline{HB}$, then its real and imaginary parts may be represented in the form*

$$P(z) = R(z)P_1(z), \quad \textit{and} \quad Q(z) = R(z)Q_1(z),$$

where the roots of the entire functions $R(z), P_1(z)$ and $Q_1(z)$ are all real and the roots of $P_1(z)$ and $Q_1(z)$ interlace.

2. Zeros of the determinant

It was shown in Lemma 5.3 in [11] that the eigenvalues $\lambda = z^2$ other than $\lambda = 0$ in (1.6) have positive real parts. We define the *principal eigenvalue* λ_1 as the one with smallest positive real part and the *spectral gap* as

$$\gamma_1 = \min\{\operatorname{Re}\lambda : 0 \neq \lambda \text{ is an eigenvalue of } (1.6)\}.$$

We'll show that when both σ's are large, all the eigenvalues of (1.6) are real regardless how the probability measures μ are defined at the end points. However when σ's get small, it is possible that some complex eigenvalues will appear for certain measures. In particular if $\sigma_0 = \sigma_1$ and μ_0 and μ_1 are point mass measure at $x = 1/2$, it was raised in the last part of [11] if the principal eigenvalue λ_1 is real while the second one is complex. We answer the question in the affirmative and pinpoint the spectral gap in this case. The example discussed by Peng and Li shows how eigenvalues switch from real to complex when σ gets smaller. However, they become real again when σ converges to the limit 0.

We first establish a result on existence of real roots of the determinant Δ defined in the previous section. The function F in (1.12) can be rephrased in a 'symmetric' form as:

$$F(z) := e^{iz/2}\{(1-\sigma_0 z^2) - \int_0^1 e^{-izt}\, d\mu_0(t)\} \times e^{iz/2}\{(1-\sigma_1 z^2) - \int_0^1 e^{-iz(1-t)}\, d\mu_1(t)\}. \tag{2.1}$$

Theorem 2.1. *Let*

$$f(z) = e^{iz/2}\left((1-\sigma z^2) - \int_0^1 e^{-izt}\, d\,\mu(t)\right). \tag{2.2}$$

1. *The function f belongs to $\overline{HB}$ if $\sigma > 2/\pi^2$ for a general probability measure μ.*
2. *For any $\sigma > 0$, f belongs to $\overline{HB}$ if the probability measure $d\mu = m(t)dt$ with $m(t)$ being a differentiable and non-increasing function on I .*

Proof. It is obvious that f is an entire function of exponential type. To calculate its defect, we look at the growth along the imaginary axis. For $z = iy$, $y < 0$,

$$f(iy) = e^{-y/2}\left(1 + \sigma y^2 - \int_0^1 e^{yt}\, d\mu(t)\right),$$

we see that

$$\overline{\lim_{y\to\infty}} \frac{\log|f(iy)|}{|y|} = \frac{1}{2}. \tag{2.3}$$

For $y > 0$,

$$f(iy) = (1+\sigma y^2)e^{-y/2} - e^{-y/2}\int_0^1 e^{yt}\, d\mu(t), \quad \text{and} \tag{2.4}$$

$$e^{-y/2}\int_0^1 e^{yt}\, d\mu(t) = \int_0^1 e^{y(t-1/2)}\, d\mu(t) \le e^{y/2}. \tag{2.5}$$

Since the first term on the right side of (2.4) goes to zero, we have

$$\overline{\lim_{y\to\infty}} \frac{\log|f(iy)|}{|y|} \le \frac{1}{2}. \tag{2.6}$$

Altogether, the defect d_f of f is non-positive.

To show that $f(z)$ does not have zeros in the lower half plane, we do some simple calculus here. Suppose $f(z) = 0$ for some $z = x + iy$ with $y < 0$, then its zeros in the lower half plane are computed from the equation:

$$\sigma z^2 = 1 - \int_0^1 e^{-ixt} e^{yt}\, d\,\mu(t).$$

By taking the modulus of both sides, we see that $\sigma|z|^2 \le 2$. So $x^2 \le 2/\sigma$.

By looking at the imaginary part of $e^{-iz/2}f(z) = 0$, we have

$$2\sigma xy = \int_0^1 \sin(xt)\, e^{yt}\, d\mu(t). \tag{2.7}$$

The equation is odd in x, we may perform an analysis only for $x > 0$. The right hand side is positive if $0 < x \le \pi$ while the left side is negative. For $f(z)$ to have zeros in the lower half plane, $x > \pi$ has to hold. Together with the inequality $x^2 \le 2/\sigma$, we conclude that $f(z)$ cannot vanish in the lower half-plane if $\sigma > 2/\pi^2$.

For the proof of the second statement, we write $\dfrac{1-\cos(xt)}{x}$ as an anti-derivative of $\sin(xt)$. With the measure $d\mu = m(t)dt$ after one integration by parts, (2.7) above becomes

$$2\sigma xy = \frac{1-\cos(x)}{x} e^{y}\, m(1) - \int_0^1 \frac{1-\cos(xt)}{x} e^{yt}\, (y\, m(t) + m'(t))\, dt. \tag{2.8}$$

For $y < 0$, the left side is negative while both terms on the right are non-negative since $y\, m(t) + m'(t) < 0$. In conclusion, f cannot have zeros in the lower half plane and is hence in $\overline{HB}$. □

Remark 2.1. $f(z)$ belongs to $\overline{HB}$ in the limiting case $\sigma = 0$. This was proved in [7]. Statement (2) in Theorem 2.1 also holds if the measure μ is piece-wise continuous and non-increasing on $[0, 1]$. A good example is $d\mu = \dfrac{1}{b}\, dt$ supported on $[0, b]$ for $0 < b < 1$. Interestingly it may not hold if μ is increasing. This is illustrated by μ being the constant function supported on $[1 - b, 1]$.

Definition 2.2. To simplify our language below, we shall say a probability measure μ is *non-increasing (non-decreasing)* if $d\mu = m(t)\, dt$ for some smooth function $m(t)$ that is non-increasing (non-decreasing) on $[0, 1]$.

Corollary 2.1. *Let $g(z) = e^{iz/2}(c - \int_0^1 e^{-izt}\, d\mu(t))$. The function belongs to $\overline{HB}$ for any real constant c whenever μ is non-increasing.*

Proof. The proof is almost identical to the one given above. By writing $z = x + iy$, the imaginary part of $e^{-z/2}g(z)$ is $\int_0^1 \sin(xt)e^{yt}\, d\mu(t)$. This integral is positive for $y < 0$. Hence g cannot have zeros in the lower half plane. □

Remark 2.2. The real part decomposition of the function g is

$$c\, \cos(z/2) - \int_0^1 \cos(z(1/2 - t))\, d\mu(t).$$

According to Proposition 1.1, this function has all real zeros for any real constant c when μ is non-increasing. Since the integrand is symmetric with to the mid-point 1/2, it also has all real zeros if μ is non-decreasing. Result of this type is well-known. See for example, problems 173-175 in Part V, Chapter 3 of [13].

Corollary 2.2. *If μ is non-increasing on I, then both*

$$C(z) := (1-\sigma z^2)\cos(z/2) - \int_0^1 \cos(z(t-1/2))\,d\mu \quad \textit{and} \tag{2.9}$$

$$S(z) := (1-\sigma z^2)\sin(z/2) - \int_0^1 \sin(z(t-1/2))\,d\mu \tag{2.10}$$

have all real zeros for any $\sigma > 0$.

In addition, the function $C(z)$ has all real zeros if μ is non-decreasing on I.

Proof. The statements on $C(z)$ and $S(z)$ follow from the Theorem above and Proposition 1.1. The last statement follows from the fact that $\cos(z(t-1/2)$ is invariant when t is changed to $1-t$. □

Regarding cases when $\Delta(z)$ has all real roots, we have

Corollary 2.3. *If the measures μ_0 and μ_1 are respectively non-increasing and non-decreasing on I, then all the zeros of $\Delta(z)$ are real.*

Proof. This follows from the Theorem above and a factorization of F as shown in (2.1). □

Remark 2.3. In general, the function $f(z)$ would have complex roots in the lower half plane if μ is a point measure and σ is sufficiently small. The monotonic decreasing nature of $m(t)$ seems to be important. The smooth measure $d\mu = \frac{1}{630}(t(1-t))^4\,dt$ resembling the point mass measure at $t = 1/2$ gives a function $f(z)$ in (2.2) with roots in the lower half plane.

3. Location of principal eigenvalue

There are only a few cases that we can locate the principal eigenvalue of the system (1.6), (1.7) and (1.8) even if the eigenvalues are known to be real in advance. Its determination becomes difficult if there are non-real ones. As shown in Section 1, the eigenvalues are computed from the zeros of $\Delta(z)$ which is basically the imaginary part decomposition of $F(z)$ listed in (2.1). We re-write it as:

$$\left((1-\sigma_0 z^2)e^{iz/2} - \int_0^1 e^{-iz(t-1/2)}d\mu_0\right) \times \left((1-\sigma_1 z^2)e^{iz/2} - \int_0^1 e^{-iz(1/2-t)}d\mu_1\right). \tag{3.1}$$

Based on Theorem 2.1, F lies in $\overline{HB}$ if μ_0 is non-increasing while μ_1 is non-decreasing on $[0,1]$. In such a case $\Delta(z)$ has all real zeros. With further assumptions on σ_i's, we discuss a couple of examples that we can pin-point the location of the first positive zero and hence the principal eigenvalue.

We state a possibly well-known result on the monotonicity of a real analytic function.

Lemma 3.1. *Suppose $g(x)$ is a real analytic function defined on (a, b) with $\lim_{x\to a} g(x) = -\infty$ and $\lim_{x\to b} g(x) = \infty$. In addition assume that $g(x) - c = 0$ has only real zeros for each real c, then $g(x)$ is non-decreasing on (a, b).*

Proof. Suppose $g(x)$ has a local maximum $g(x_0) = w_0$ at some point x_0 in (a, b), then the integer value of the Cauchy integral

$$\frac{1}{2\pi i}\oint_C \frac{g'(z)}{g(z) - w}\, dz$$

is at least 2 when $w = w_0$ and C is a small circle on the complex plane surrounding the point x_0. Since $g(z) - w = 0$ does not have non-real roots for real w, the integral is 0 when the parameter w becomes slightly larger than w_0. This contradicts the fact that the Cauchy integral is a continuous function of w as it varies inside C. Hence $g(x)$ cannot have any local maximum. □

Theorem 3.2. *For the system (1.6), (1.7) and (1.8), assume that $\sigma_0 = \sigma_1$ and $\mu_0 = \mu_1$. In addition if the measure μ is monotonic, then the principal eigenvalue λ is equal to the minimum of $1/\sigma, 4\pi^2$ and z_0^2 where z_0 is the first root in $(\pi, 3\pi)$ where $(1 - \sigma z^2)\cos(z/2) - \int_0^1 \cos(z(1/2 - t))\, d\mu(t)$ has its first real zero.*

Proof. By expanding $F(z)$ in (3.1), we arrive at

$$(1 - \sigma z^2)^2 e^{iz} - 2(1 - \sigma z^2)e^{iz/2}\left(\int_0^1 \cos(z(1/2 - t)\, d\mu(t)\right) +$$
$$\left(\int_0^1 e^{iz(1/2-t)} d\mu\right) \times \left(\int_0^1 e^{-iz(1/2-t)} d\mu\right).$$

Since the last term above is always real, its imaginary decomposition part is determined by the sum of first two terms. So the imaginary part decomposition $O(z)$ is the function

$$2(1 - \sigma z^2)\sin(z/2)\left((1 - \sigma z^2)\cos(z/2) - \int_0^1 \cos(z(1/2 - t))\, d\mu(t)\right). \tag{3.2}$$

Its zeros are $1/\sqrt{\sigma}, 2\pi, 4\pi, \cdots$ and those of $C(z)$ which are all real according to Corollary 2.2. Our intention below is to show $C(z)$ has a zero in $(\pi, 3\pi)$.

Let us define the meromorphic functions:

$$Q_c(z) := \frac{\int_0^1 \cos(z(1/2 - t))\, d\mu}{\cos(z/2)}. \tag{3.3}$$

In the interval $(0,\pi)$, $Q_c(z) > 1$ because

$$\cos(z/2) - \int_0^1 \cos(z(1/2-t))\,d\mu = -2\int_0^1 \sin(\frac{zt}{2})\,\sin(\frac{z(1-t)}{2})\,d\mu < 0.$$

At $z=\pi$, $\int_0^1 \cos(\pi(1/2-t)\;d\mu = \int_0^1 \sin(\pi t)\,d\mu > 0$. Thus $Q_c(z)$ goes to $-\infty$ as z goes to π from the right.

At $z = 3\pi$, the numerator of $Q_c(z)$ is equal to $\int_0^1 \cos(3\pi(1/2-t)\;d\mu = -\int_0^1 \sin(3\pi t)d\mu$. With the property that μ is either increasing or decreasing on $[0,1]$, we draw the conclusion that $\int_0^1 \sin(3\pi t)\,d\mu > 0$ by splitting it into a sum of three integrals over $(0,1/3),(1/3,2/3)$ and $(2/3,1)$. The middle integral is smaller than either the first or the third one. Thus the quotient $Q_c(z)$ in (3.3) goes to ∞ as z goes to 3π from the left.

From the remark made after Corollary 2.1, we see that $Q_c(z) = c$ has only real zeros for any real c. Lemma 3.1 implies that $Q_c(z)$ is monotonically increasing in $(\pi,3\pi)$. The same conclusion applies to $Q_c(z)$ over the intervals $(3\pi,5\pi),(5\pi,7\pi),\cdots$.

Finally, the parabola $1-\sigma z^2$ will intersect $Q_c(z)$ in $(\pi,3\pi)$ at only one point z_0. This is the first positive root of $(1-\sigma z^2)\cos(z/2) - \int_0^1 \cos(z(1/2-t))\,d\mu(t) = 0$.

The eigenvalues of the system (1.6), (1.7) and (1.8) are determined by the roots of the equation (3.2), which consist of zeros of $(1-\sigma z^2)\sin(z/2)$ and those of $C(z)$. The candidates for the first positive roots are the $1/\sqrt{\sigma}, 2\pi$ and z_0. Hence the principal eigenvalue $\lambda = z^2$ is the minimum of $1/\sigma, 4\pi^2$ and z_0^2.

We remark that if an extra assumption is made on μ, then a better bound on the principal eigenvalue can be found. For example if $d\mu = m(t)dt$ with $m'(t) \geq 0$ and $m''(t) \leq 0$, then

$$\begin{aligned} Q_c(2\pi) &= -\int_0^1 \cos(2\pi(1/2-t))\,m(t)dt \\ &= -\frac{1}{2\pi}\int_0^1 \sin(2\pi t)m'(t)dt. \end{aligned}$$

Since $m'(t)$ is non-increasing, then $Q_c(2\pi) \leq 0$ Now $Q_c(z)$ is monotonically increasing in $(\pi,3\pi)$ and $Q_c(2\pi) \leq 0$, then z_0 is always larger than the minimum of $1/\sqrt{\sigma}$ and 2π. $\square$

Incidentally we note that the roots of $O(z)$ in (3.2) are identical if $d\mu$ is dt, $2tdt$ or $2(1-t)dt$. For all these cases, $O(z) = 2(1-\sigma z^2)\left((1-\sigma z^2)\cos(z/2) - 2\sin(z/2)/z\right)$. From an inverse eigenvalue problem point of view, it seems that the knowledge of the entire set of eigenvalues still cannot determine the single measure μ even when σ is given..

We finish this section with another case of real eigenvalues where the holding rates σ_i's are different but both end-point measures are equal to the Lebesgue measure. The case here is to illustrate the difficulty of locating the first positive root even if is known a priori that all the roots are real.

Proposition 3.1. *For the Brownian motion described by* (1.6), (1.7) *and* (1.8) *with* $\sigma_0 < \sigma_1$ *and both measures at the end points being the Lebesgue measure on* $[0,1]$*, all the eigenvalues are real.*

Let ρ *be the first positive root of* $x\cot(x/2) = \dfrac{1}{1-\sigma_0 x^2} + \dfrac{1}{1-\sigma_1 x^2}$.

Then the principal eigenvalue is $4\pi^2$ *if* $2\pi \leq 1/\sqrt{\sigma_1}$*. On the other hand, if* $2\pi > 1/\sqrt{\sigma_1}$*, then the principal eigenvalue is* ρ^2 *with* $1/\sqrt{\sigma_1} < \rho < 2\pi$.

Proof. We will just give a sketch of proof here. By replacing each of the measures by Lebesgue measure in (3.1), one gets

$$F = \left((1-\sigma_)z^2)e^{iz/2} - 2\frac{\sin(z/2}{z}\right) \times \left((1-\sigma_1 z^2)e^{iz/2} - 2\frac{\sin(z/2}{z}\right).$$

After expanding the product above, the odd part decomposition is

$$O(z) = 2\sin(z/2)\left((1-\sigma_0 z^2)(1-\sigma_1 z^2)\cos(z/2) - \frac{\sin(z/2)}{z}((1-\sigma_0 z^2)+(1-\sigma_1 z^2)\right).$$

The case $\sigma_0 = \sigma_1$ has been settled in the remark at the end of Theorem 3.2. We may assume that $\sigma_0 < \sigma_1$. The first positive zero x_0 of $O(z)$ is determined by the minimum of 2π and the first positive zero ρ of $x\cot(x/2) = R(x)$, where

$$R(x) := \frac{1}{1-\sigma_0 x^2} + \frac{1}{1-\sigma_1 x^2}.$$

$x\cot(x/2)$ decreases from 2 to $-\infty$ in $(0,2\pi)$, from ∞ to $-\infty$ in $(2\pi,4\pi), \cdots$ etc. $R(x)$ increases from 2 to ∞ in the interval $(0, 1/\sqrt{\sigma_1})$ and from $-\infty$ to ∞ in $(1/\sqrt{\sigma_1},\ 1/\sqrt{\sigma_0})$.

If $1/\sqrt{\sigma_1} \geq 2\pi$, then $x\cot(x/2)$ will not intersect $R(x)$ in $(0,2\pi)$. So 2π is the first positive root of $O(z)$. If $1/\sqrt{\sigma_1} < 2\pi$, then the first point of intersection of the two curve occurs inside the interval $(1/\sqrt{\sigma_1},\ 2\pi)$. □

3.1. **Peng and Li's question.** When both σ_i's are the same σ and the μ_i's are both point measure $\delta_{1/2}$, Peng and Li in [11] raise the problem of showing that the principal eigenvalue is real and is equal to the minimum of $1/\sigma$ and $4\pi^2$. In the limiting case when both $\sigma_i = 0$, i.e. there is no holding at the boundary points, it was proved in [1] for a general point measure δ_p, $p \in (0,1)$ that the principal eigenvalue is $4\pi^2$..

With both μ's being $\delta_{1/2}$,

$$O(z) = 2\sin(z/2)\,(1-\sigma z^2)\,\big((1-\sigma z^2)\,\cos(z/2)-1\big)\,. \tag{3.4}$$

So the zeros of $O(z)$ are those of

$$h(z) := (1-\sigma z^2)\cos(z/2) - 1 = 0$$

plus $z = 1/\sqrt{\sigma}$ and $2n\pi$ for $n = 1, 2, \cdots$. For the rest of this subsection, we perform an analysis on the zeros of $h(z) = 0$. According to Theorem 2.1, $h(z)$ being the real part of $e^{iz/2}((1-\sigma z^2) - e^{-iz/2})$ has all real zeros whenever $\sigma > 2/\pi^2$.

For large σ, the parabola $1-\sigma x^2$ intersects various branches of $\sec(x/2)$ at two different points in each of the intervals $(\pi, 3\pi)$, $(5\pi, 7\pi), \cdots$ in the lower half plane. The roots of $h(x) = 0$ together with the double root at the origin are all real. Since $\cos(x/2)$ is negative in each of these intervals, all the real roots satisfy the inequality $1-\sigma x^2 < 0$. So all the positive real roots of $h(x)$ are larger than $1/\sqrt{\sigma}$.

Inside the first interval $(\pi, 3\pi)$, as σ gets small, the parabola becomes first tangent to $\sec(x/2)$ at some point x_1 which is a double root of $h(x) = 0$. As σ gets smaller, all the real roots in this interval disappear since the x values are less than $1/\sqrt{\sigma}$. However each root of $h(z)$ is continuous (actually locally analytic) in terms of σ, the double root x_1 on the real axis in $(\pi, 3\pi)$ will split into a pair of conjugate complex roots on the complex plane. We need to approximate the location of x_1 in order to compute the spectral gap of this nonlocal boundary value problem. Our aim in the following is to show $x^2 - y^2 > 4\pi^2$, i.e. Re $z^2 > 4\pi^2$ for all the non-real roots $z = x + iy$ of $h(z) = 0$.

All the double roots can be computed from the system of equations:

$$\begin{aligned} h(x) &= (1-\sigma x^2)\,\cos(x/2) - 1 = 0, \\ h'(x) &= -2\sigma x\,\cos(x/2) - \frac{1-\sigma x^2}{2}\,\sin(x/2) = 0. \end{aligned}$$

By eliminating the parameter σ, we arrive at

$$\phi(x) := 4\cos(x/2) - 4\cos^2(x/2) - x\,\sin(x/2) \tag{3.5}$$

whose roots are the double roots of $h(x)$. Since

$$\begin{aligned} \phi(x) &= 4\cos(\frac{x}{2})\left(1-\cos(\frac{x}{2})\right) - x\,\sin(\frac{x}{2}) \quad \text{and} \\ \phi'(x) &= -3\sin(\frac{x}{2})\left(1-\cos(\frac{x}{2})\right) + \cos(x/2)\left(\sin(\frac{x}{2}) - \frac{x}{2}\right), \end{aligned}$$

we see easily that $\phi(0) = 0$, $\phi(x) < 0$ in $(\pi, 2\pi)$. In addition $\phi'(x) < 0$ in $(0, \pi)$ and $\phi'(x) > 0$ in $(\pi, 3\pi)$. Now $\phi(7\pi/3) = -2\sqrt{3} - 3 + 7\pi/6 < 0$. From these results, we conclude that the first positive root x_1 of $\phi(x)$ is larger than $7\pi/3 = 7.33038$. Numerically $x_1 = 7.74873$. This is the place where $1-\sigma x^2$ is tangent to $\sec(x/2)$ in $(\pi, 3\pi)$. The corresponding value of σ^* is 0.03906.

We note that $\lambda = x_1^2$ is a real eigenvalue of multiplicity 2 corresponding to $\sigma = \sigma^*$ in the system (1.6) and (1.7). The double roots have to split into a conjugate pair on the complex plane as σ decreases further. Each trajectory can be parametrized implicitly as an analytic function of σ for $0 < \sigma < \sigma^*$. As σ goes to zero, the limiting equation is $h(z) = \cos(z/2) - 1$. So the conjugate pair of trajectories will converge to the point $(4\pi, 0)$ on the real axis. Similarly in the interval $(5\pi, 7\pi)$, the two real roots on the real line will converge to a double root and then split into a conjugate complex pair as σ gets smaller further.

We show how the trajectories are obtained since we have to prove that all these roots satisfy the inequality $\mathrm{Re}(z^2) > 4\pi^2$. With $z = x + iy$, the equation $h(z) = 0$ written as

$$\sigma z^2 = 1 - \frac{1}{\cos(z/2)}$$

can be decomposed into its real and imaginary parts as follows:

$$\sigma(x^2 - y^2) = 1 - \frac{\cos(x/2)\cosh(y/2)}{\cosh(y/2)^2 - \sin(x/2)^2},$$
$$\sigma\, 2\, x\, y = -\frac{\sin(x/2)\sinh(y/2)}{\cosh(y/2)^2 - \sin(x/2)^2}.$$

By eliminating the parameter σ, these two equations yield

$$-\frac{x^2 - y^2}{2xy}\sin(\frac{x}{2})\sinh(\frac{y}{2}) = \cosh^2(\frac{y}{2}) - \sin^2(\frac{x}{2}) - \cos(\frac{x}{2})\cosh(\frac{y}{2}).$$

Let us define

$$H(x, y) := \cosh^2(\frac{y}{2}) - \sin^2(\frac{x}{2}) - \cos(\frac{x}{2})\cosh(\frac{y}{2}) + \frac{x^2 - y^2}{2xy}\sin(\frac{x}{2})\sinh(\frac{y}{2}). \tag{3.6}$$

All the non-real zeros $z = x + iy$ of $h(z) = 0$ lie on the contours defined by $H(x, y) = 0$. As illustrated in Figure 1, their heights of the contours get larger as they move out to infinity.

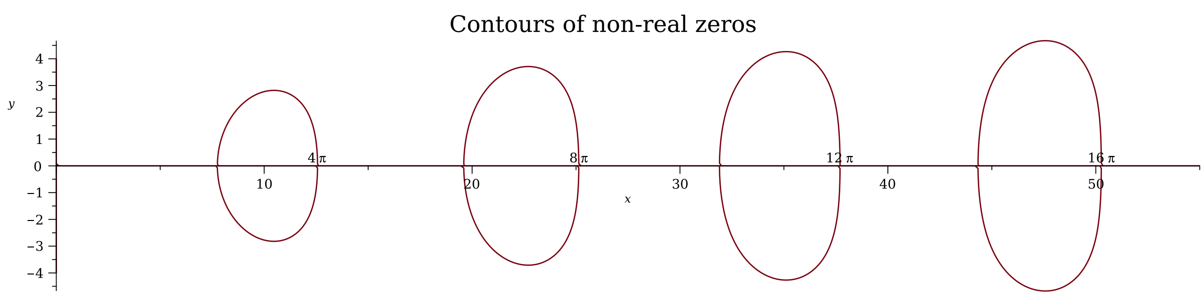

FIGURE 1. Level curves of H(x,y) = 0

The graph is symmetric with respect to both axes and we'll only consider the part in the first quadrant. Let $\alpha := 3/7$. We intend to show that all the points (x, y) on the contours lie within the angular sector $y < \alpha x$ with $x > \pi/\alpha$. Consequently $x^2 - y^2 > \dfrac{40}{9}\pi^2 > 4\pi^2$. We showed earlier that the trajectory of the roots of $h(z) = 0$ come out from the point $x_1 > 7\pi/3$ on the real axis. It has to come down to the x axis at 4π as σ decreases to 0. To show that the graph cannot cross the vertical line $x = 7\pi/3$, we inspect the value

$$\begin{aligned} H(7\pi/3, y) &= \cosh^2(\frac{y}{2}) - \frac{1}{4} + \frac{\sqrt{3}}{2}\cosh(\frac{y}{2}) + \frac{7\pi}{12y}\sinh(\frac{y}{2}) - \frac{3y}{28\pi}\sinh(\frac{y}{2}) \\ &= \left(\frac{\sqrt{3}}{2}\cosh(\frac{y}{2}) - \frac{1}{4}\right) + \left(\cosh^2(\frac{y}{2}) - \frac{3y}{28\pi}\sinh(\frac{y}{2})\right) + \frac{7\pi}{12y}\sinh(\frac{y}{2}). \end{aligned}$$

With the terms arranged as shown, each of the groups is positive. Thus $H(7\pi/3, y) > 0$ for all $y \geq 0$. In other words, no points of the vertical line $(7\pi/3, y)$ can lie on the contours.

We now prove that the contours lie below the radial line $y = \alpha x$. The proof is a routine calculus problem.

$$H(x, \alpha x) = \cosh^2(\frac{\alpha x}{2}) - \sin^2(\frac{x}{2}) - \cos(\frac{x}{2})\cosh(\frac{\alpha x}{2}) + \frac{1-\alpha^2}{2\alpha}\sin(\frac{x}{2})\sinh(\frac{\alpha x}{2}).$$

The sum of the last three terms is bounded above by $1 + \cosh(\alpha x/2) + \sinh(\alpha x/2)$ which is $1 + e^{\alpha x/2}$. It is a simple task to show that $\cosh^2(t) > 1 + e^t$ for $t > 3/2$. We'll give a proof at the end of this section. So $H(x, \alpha x) > 0$ if $\alpha x/2 > 3/2$ or $x > 7$. In other words, all the points $z = x + iy$ on the contours lie within the angular sector bounded by $y = \pm 3x/7$ if $x > 7$. This is definitely true since we have shown that $x > 7\pi/3$ for all points on the contours. The following result answers the question raised at the end of the paper by Peng and Li [11].

Theorem 3.3. *For the Brownian motion described by* (1.6), (1.7) *and* (1.8) *with holding rates at the end points equal to the same σ and $\mu_0 = \mu_1 = \delta_{1/2}$, then the principal eigenvalue is min* $\{1/\sigma, 4\pi^2\}$.

Proof. The roots of (3.4) provide the eigenvalues of the system. Since $\lambda = z^2$, the real eigenvalues are $1/\sigma$, $4\pi^2, 16\pi^2, \cdots$ regardless what the holding rate σ is. If z is a real root of $h(z) = 0$, we showed then $1 - \sigma z^2 > 0$, i.e. $1/\sigma$ provides a lower bound for all other real eigenvalues.

Let $z = x+iy$ be a non-real root of $h(z) = 0$ when $\sigma < \sigma^*$. We showed above that $x > 7\pi/3$ and $y < 3x/7$. So Re $(z^2) = x^2 - y^2 > x^2(1 - 9/49)$, i.e. Re $\lambda =$ Re $(z^2) > \frac{49\pi^2}{9}\frac{40}{49} = \frac{40\pi^2}{9} > 4\pi^2$.

Hence the principal eigenvalue is real and is the minimum of the two numbers $1/\sigma$ and $4\pi^2$. $\square$

We conclude this section with a numerical example showing the existence of a non-real principal eigenvalue. In (3.1), if $\sigma_0 = \sigma_1$ and the boundary measures are $\mu_0 = \delta_p$ and $\mu_1 = \delta_{1-p}$ respectively, $F(z)$ is equal to

$$\begin{aligned}&\Big((1-\sigma z^2)e^{iz/2} - e^{-i(p-1/2)z}\Big)^2\\ &= \big((1-\sigma z^2)(\cos(z/2)+i\sin(z/2)) - \cos((p-1/2)z) + i\sin((p-1/2)z)\big)^2\\ &= \big(\big[(1-\sigma z^2)\cos(z/2) - \cos((p-1/2)z)\big] + i\,\big[(1-\sigma z^2)\sin(z/2)+\sin((p-1/2)z)\big]\big)^2.\end{aligned}$$

So the imaginary part $O(z)$ is equal to

$$2\left[(1-\sigma z^2)\cos(z/2) - \cos((p-1/2)z)\right] \times \left[(1-\sigma z^2)\sin(z/2)+\sin((p-1/2)z)\right.$$

We take $\sigma = 0.01$ and $p = 2/3$, it can be shown that all the zeros of the factor

$$(1-0.01z^2)\cos(z/2) - \cos(z/6)$$

are real and that the first positive zero is at 3π. The second factor has the form

$$(1-0.01z^2)\sin(z/2) + \sin(z/6).$$

It has a pair of complex zeros $\zeta = 7.66688 \pm i\,2.49563$ and all the others are real. Numerically the real part of $\zeta^2 = 52.55300$ is smaller than the square of the other real roots. Also this number is smaller than $9\pi^2 = 88.82643$. Thus ζ^2 is the principal eigenvalue here.

We note that in [7], it was shown for the case of $\sigma = 0$ (Brownian motion with jumps only), this symmetric jump of jumping from 0 to 2/3 and from 1 to 1/3 gives the largest principal eigenvalue among all possible probability measures μ_0 and μ_1.

Remark 3.1. The inequality $\cosh(t)^2 > 1 + e^t$ is implied by $x^2 - 4x - 2 > 0$ with $x = e^t$. We have to show that $e^t > 2 + \sqrt{6}$ for $t > 3/2$. If we replace e^t by a lower bound $1 + t + \cdots + t^5/5!$, its value at $t = 3/2$ is $5711/1280 = 4.46172$. An upper bound of $2 + \sqrt{6}$ is $2 + 49/20 = 4.45$. So $\cosh(t)^2 > 1 + e^t$ for $t = 3/2$ and beyond.

4. Conclusion

Though the Brownian model studied here is pretty straight forward, we have shown in general the computation of the spectral gap is not a simple problem. In particular we use the classical Hermite-Biehler Theorem to help us analyze the distribution of the eigenvalues. Slight generalization of the simple Brownian motion model $v'' + \lambda v = 0$, such as $(1 + x)^4 v''(x) + z^2 v(x) = 0$ or Bessel-like diffusion $v''(x) + n/(1+x)v'(x) + z^2 v(x) = 0$ also have similar eigenvalue distributions. For a diffusion equation of the form $v''(x) + cv'(x) + z^2 v(x) = 0$ with no holding conditions, we note that the paper [10] seems to be the first one in pointing out the existence of a non-real principal eigenvalue under the jump boundary conditions $v(0) = v(1/2) = v(1)$. Our results show that if the boundary conditions (1.4) and (1.5) yield both real and non-real roots in the parameter z, then it is a real pain to sort out the minimal value of real part of $\lambda = z^2$.

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