AN INITIAL BOUNDARY VALUE PROBLEM FOR A PSEUDOPARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION

ANTONTSEV S.N.[1,2]

AITZHANOV S.E.[3,4,*]

ZHANUZAKOVA D.T.[3]

[1] *Lavrentyev Institute of Hydrodynamics of SB RAS, Russia*

[2] *CMAF-CIO, University of Lisbon, Portugal*

[3] *Al-Farabi Kazakh National University, Almaty, Kazakhstan*

[4] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*

E-mail addresses: anontsevsn@mail.ru, aitzhanov.serik81@gmail.com, dinara.zhan07@gmail.com.
Date: March 14, 2022.
*Corresponding author.

Abstract. An initial-boundary value problem for a quasilinear equation of pseudoparabolic type with a nonlinear boundary condition of the Neumann-Dirichlet type is investigated in this work. From a physical point of view, the initial-boundary value problem considered here is a mathematical model of quasi-stationary processes in semiconductors and magnets, which takes into account a wide variety of physical factors. Many approximate methods are suitable for finding eigenvalues and eigenfunctions in problems where the boundary conditions are linear with respect to the desired function and its derivatives. Among these methods, the Galerkin method leads to the simplest calculations. In this article, by the Galerkin method to prove the existence of a weak solution to the initial-boundary value problem for a pseudoparabolic equation in a bounded domain. On the basis of a priori estimates, we prove a local existence theoremand uniqueness for a weak generalized solution of the initial-boundary value problem for the quasilinear pseudoparabolic equation. A special place in the theory of nonlinear equations is occupied by the study of unbounded solutions, or, as they are called in another way, blow-up regimes. Nonlinear evolutionary problems admitting unbounded solutions are globally unsolvable. In the article, sufficient conditions for the blow-up of a solution in a finite time in a limited area with a nonlinear Neumann-Dirichlet boundary condition are obtained.

Keywords: Pseudoparabolic equations; nonlinear boundary conditions; Galerkin method; the existence of a solution; uniqueness of the solution; blow-up of the solution; asymptotic behavior of the solution.

MSC (2020): 35A01; 35B40; 35B44; 35S16.

1. Introduction

The first rigorous mathematical study of equations that are not equations of the Cauchy-Kovalevskaya type is the pioneering work of S.L. Sobolev [1]. The same work aroused great interest in the study of non-classical equations, called equations of the Sobolev or pseudoparabolic type. The study of problems for the pseudo-parabolic type began in the late 1970s. A large number of works are devoted to the study of nonlinear equations of pseudoparabolic type [3]– [38]. The modeling of physical processes leading to equations of the Sobolev type and, in particular, of the pseudoparabolic type, are devoted to the works [3], [4], [6], [7], [15]– [20], [24].

Questions of the asymptotic behavior of solutions of such problems at large times, as well as the theory of scattering and stability of solutions of the solitary wave type for one-dimensional and multidimensional equations of the Benjamin-Bon-Mahoney and Benjamin-Bon-Mahoney-Burgers, Rosenau-Burgers types were considered in [5]– [7]. Oskolkov A.P., Antontsev S.N., Kozhanov A.I., SveshnikovA.I., Korpusov M.O. and many other scientists have made a significant contribution to the study of the solvability of initial-boundary value problems for equations of pseudoparabolic type.

The systems of nonlinear equations of the Sobolev type (Kelvin-Voigt equations) describe flows of viscoelastic fluids. Investigations of the mathematical correctness of such equations are devoted to the works [15]– [17], [33]– [38].

In the work of M. O. Korpusov, A. G. Sveshnikov [22] a model equation is considered that describes the relaxation of an initial perturbation in a crystalline semiconductor in the case when its electrical conductivity depends nonlocally on the field.

$$\frac{\partial}{\partial t}(\Delta u - u) - \left(\int_\Omega dx \nabla u\right)^q \Delta u = 0,$$

$$u|_{\partial\Omega} = 0, \;\; u(x,0) = u_0(x).$$

For certain initial parameters, the effect of finite time "cooling down" is proved to occur. For other parameters, the first term of the longtime asymptoticsis found and the remainder of the asymptotic expansion is estimated.

In the work of [21] studied the mathematical model of wave processes in semiconductors in an external electric field, taking into account dissipation and the non-local connection of the current density with the strength of an electric field:

$$\frac{\partial}{\partial t}(\Delta u - u - |u|^q u) + \Delta u + u_{x_1} + uu_{x_1} - \left(\int_\Omega dx \nabla u\right)^q \Delta u = 0,$$

$$u|_{\partial\Omega} = 0, \;\; u(x,0) = u_0(x).$$

Sufficient conditions for the blow-up of a strong generalized solution are obtained. This article is devoted to the study of the problems of local and global solvability in time and the effect of the blow-up in a finite time of solutions of the initial-boundary value problem for nonlinear Sobolev-type equations.

1.1. **Formulation of a problem. Determining the solution.** We consider the quasilinear equation

$$\frac{\partial}{\partial t}(u - \chi\Delta u) - \left(a_0 + a_1 \|\nabla u\|_{2,\Omega}^{2q-2}\right) \Delta u = b(x,t)|u|^{p-2}u + f(x,t), \;\; (x,t) \in Q_T \tag{1.1}$$

with the nonlinear boundary

$$\left.\frac{\partial u}{\partial n} + k(x,t)|u|^{\sigma-2}u\right|_\Gamma = 0, \quad \Gamma = \partial\Omega \times (0,T), \tag{1.2}$$

and with the initial conditions

$$u(x,0) = u_0(x), \;\; x \in \Omega. \tag{1.3}$$

Here $Q_T = \{(x,t): \; x \in \Omega, \; \Omega \subset R^n, \; 0 < t < T\}$ is a cylinder, $\Omega \subset R^n, \; n \geq 3$ is a bounded domain, with a sufficiently smooth boundary $\partial\Omega$, so p, q, a_0, a_1 and σ are positive constants.

This problem is a mathematical model of wave processes in semiconductors in an external electric field, taking into account the dissipation and nonlocal connection of the current density with the electric field strength [19], [20], ([25], Chapter 7, p.516).

The functions $b(x,t)$, $f(x,t)$, $k(x,t)$ and $u_0(x)$ satisfy the following conditions:

$$\begin{aligned} &0 < b_0 \leq b(x,t) \leq b_1 < \infty, \;\; 0 < b_t(x,t) \leq b_1 < \infty, \;\; \forall (x,t) \in Q_T; \\ &0 < k_0 \leq k(x,t) \leq k_1 < \infty, \;\; 0 \leq \tfrac{k_t(x,t)}{k(x,t)} \leq K_1, \;\; \tfrac{|k_{tt}(x,t)|}{k(x,t)} \leq K_2, \;\; \forall (x,t) \in Q_T; \\ &\|f(x,t)\|_{2,\Omega}^2 \leq C_0, \quad \forall t \in [0,T], \;\; u_0(x) \in W_2^1(\Omega). \end{aligned} \tag{1.4}$$

Definition 1. A weak generalized solution to problem (1.1)-(1.3) is a function $u(x,t)$ from the space $W_2^1(0,T;\, W_2^1(\Omega))\bigcap L_\sigma(\Gamma)$, which satisfies the integral identity

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\frac{\partial u}{\partial t}v+\chi\nabla u_t\cdot\nabla v+\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)\nabla u\cdot\nabla v\frac{\partial v}{\partial x_i}\right)dxdt-\\
&-\int_0^T\int_\Omega b(x,t)|u|^{p-2}uvdxdt+\\
&+\int_0^T\int_\Gamma\left(\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^{\sigma-2}u+\chi k_t(x,t)|u|^{\sigma-2}u\right)vd\Gamma dt+\\
&+\chi(\sigma-1)\int_0^T\int_\Gamma k(x,t)|u|^{\sigma-2}u_tvd\Gamma dt=\int_0^T\int_\Omega fvdxdt,
\end{aligned}$$

for all $v(x,t)\in L_2(0,T;\, W_2^1(\Omega))$.

We will give an equivalent definition of this Definition 1 (see [25], [26]):

Definition 2. A weak generalized solution to problem (1.1)-(1.3) is a function $u(x,t)$ from the space $W_2^1(0,T;\, W_2^1(\Omega))\bigcap L_\sigma(\Gamma)$, that satisfies the integral identity

$$\begin{aligned}
&\int_0^T\int_\Omega\left(\frac{\partial u}{\partial t}v(x)\varphi(t)+\chi\varphi(t)\nabla u_t\cdot\nabla v+\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)\varphi(t)\nabla u\cdot\nabla v\right)dxdt-\\
&-\int_0^T\int_\Omega b(x,t)|u|^{p-2}uv(x)\varphi(t)dxdt+\\
&+\int_0^T\int_\Gamma\left(\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^{\sigma-2}u+\chi k_t(x,t)|u|^{\sigma-2}u\right)v(x)\varphi(t)d\Gamma dt+\\
&+\chi(\sigma-1)\int_0^T\int_\Gamma k(x,t)|u|^{\sigma-2}u_tv(x)\varphi(t)d\Gamma dt=\int_0^T\int_\Omega fv(x)\varphi(t)dxdt,
\end{aligned}\tag{1.5}$$

for all $\varphi(t)\in L_2(0,T),\;\; v(x)\in W_2^1(\Omega)$.

2. Galerkin's approximations

Let us choose in $H^1(\Omega)$ some system of functions $\{\Psi_j(x)\}$ forming a basis in this space. Such a system certainly exists, because the $H^1(\Omega)$ - is a separable space. We will seek an approximate solution to problem (1.1)-(1.3) in the form

$$\begin{aligned}
&\sum_{k=1}^m C'_{mk}(t)\int_\Omega\left[\Psi_k\Psi_j+\chi\sum_{i=1}^n\frac{\partial\Psi_k}{\partial x_i}\cdot\frac{\partial\Psi_j}{\partial x_i}\right]dx+\\
&+\chi(\sigma-1)\sum_{k=1}^m C'_{mk}(t)\int_\Gamma k(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_jd\Gamma+\\
&+\sum_{k=1}^m C_{mk}(t)\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Omega\sum_{i=1}^n\frac{\partial\Psi_k}{\partial x_i}\cdot\frac{\partial\Psi_j}{\partial x_i}dx+\\
&+\sum_{k=1}^m C_{mk}(t)\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_jd\Gamma+\\
&+\chi\sum_{k=1}^m C_{mk}(t)\int_\Gamma k_t(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_jd\Gamma-\\
&-\sum_{k=1}^m C_{mk}(t)\int_\Omega b(x,t)|u_m|^{p-2}\Psi_k\Psi_jdx=\int_\Omega f\cdot\Psi_jdx,\;\; j=1,...,m.
\end{aligned}\tag{2.1}$$

$$C_{mk}(0)=\int u_m(0)\Psi_k dx,\ u_{m0}=u_m(0)=\sum_{k=1}^m C_{mk}(0)\Psi_k=\sum_{k=1}^m \alpha_k\Psi_k, \tag{2.2}$$

moreover

$$u_{m0}\to u_0 \quad \text{strongly in} \ \ H^1(\Omega) \ \ \text{at} \ \ m\to\infty. \tag{2.3}$$

We introduce the notation $\vec{C}_m \equiv \{C_{1m}(t),...,C_{mm}(t)\}^T$,

$$a_{kj}=\int_\Omega \left[\Psi_k\Psi_j+\chi\left(\nabla\Psi_k,\nabla\Psi_j\right)\right]dx+\chi(\sigma-1)\int_\Gamma k(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_j d\Gamma,$$

$$b_{kj}=-\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Omega\left(\nabla\Psi_k,\nabla\Psi_j\right)dx-$$
$$-\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_j d\Gamma-\chi\sum_{k=1}^m C_{mk}(t)\int_\Gamma k_t(x,t)|u_m|^{\sigma-2}\Psi_k\Psi_j d\Gamma+$$
$$+\int_\Omega b\left(x,t\right)\left|u_m\right|^{p-2}\Psi_k\Psi_j dx+\int_\Omega f\cdot\Psi_j dx.$$

$$A_m\left(\vec{C}_m\right)\equiv\left\{a_{jk}\left(\vec{C}_m\right)\right\},\ \ \vec{G}_m\left(\vec{C}_m\right)\equiv\left\{b_{jk}\left(\vec{C}_m\right)\right\}\vec{C}_m.$$

Then the system of equations (2.1)-(2.2) takes the matrix form

$$A_m\vec{C}'_m=\vec{G}_m\left(\vec{C}_m\right),\ \ \vec{C}_m(0)=\vec{\alpha}. \tag{2.4}$$

The matrix A_m is invertible. In fact quadratic form

$$\sum_{k,j=1}^m a_{kj}\xi_k\xi_j=\int_\Omega|\eta|^2dx+\chi\int_\Omega|\nabla\eta|^2dx,\ \ \eta=\sum_{l=1}^m\xi_l\psi_l.$$

is equal to zero if and only if $\eta=0$. Considering the positivity of the matrix A_m, the problem (2.4) can be reduced to the following form

$$\vec{C}'_m=A_m^{-1}\vec{G}_m\left(\vec{C}_m\right),\ \ \vec{C}_m(0)=\vec{\alpha}. \tag{2.5}$$

According to Cauchy's theorem, the problem (2.5) has at least one solution $\vec{C}_m$ in some time interval $t\in(0,T_m),\ T_m>0$. At the next step, we obtain the a priori estimates which prove that the Cauchy problem (2.4) has the global solution in the interval $[0,T]$.

3. A priori estimates

3.1. **Local in time estimates in presence a source inside the domain Ω.** We multiply both sides of equality (2.1) by $C_{mj}(t)$ and sum over $j=\overline{1,m}$. As a result, we obtain the first energy equality

$$\begin{array}{l}\frac{dE}{dt}+a_0\int_\Omega|\nabla u_m|^2dx+a_1\left(\int_\Omega|\nabla u_m|^2dx\right)^q+\\ +\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u_m|^\sigma d\Gamma=\\ =\int_\Omega b(x,t)|u_m|^p dx-\frac{\chi}{\sigma}\int_\Gamma k_t(x,t)|u_m|^\sigma d\Gamma+\int_\Omega f\cdot u_m dx,\end{array} \tag{3.1}$$

where

$$E(t):=\frac{1}{2}\int_\Omega[|u_m|^2+\chi|\nabla u_m|^2]dx+\chi\frac{\sigma-1}{\sigma}\int_\Gamma k(x,t)|u_m|^\sigma d\Gamma.$$

Next, we will use the following statements.

Lemma 1. *([29], [30]. For any function $u(x) \in W_2^1(\Omega)$, the following inequalities hold*

$$\|u\|_{p,\Omega}^p \le C_0 \left(\|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \right)^{\frac{\theta p}{2}} \|u\|_{2,\Omega}^{(1-\theta)p} \le$$
$$\le C_0 \left(\|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \right)^{\frac{\theta p}{2} + \frac{(1-\theta)p}{2}} \le C_1 \left(\chi \|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \right)^{\frac{p}{2}},$$

where $C_1 = \max\left\{1;\ \frac{1}{\chi}\right\} \left(\frac{2(n-1)}{n-2}\right)^{\theta}$, $\theta = \frac{(p-2)n}{2p} < 1$, $2 < p < \frac{2n}{n-2}$, $n \ge 3$.

Lemma 2. *(First local estimate) Let conditions (1.4) be satisfied and $q > 1$, $2 < p < \frac{2n}{n-2}$, $n \ge 3$, $\sigma > 1$. Then there exists $T_0 > 0$ such that the function $u_m(x,t)$ the estimates are fair:*

$$E(t) \le C_4, \quad \text{for all } t \in [0,T],\ T < T_0,$$
$$\int_0^t \left(\int_\Omega |\nabla u_m(t)|^2 dx + \left(\int_\Omega |\nabla u_m(t)|^2 dx \right)^q \right) dt +$$
$$+ \int_0^t \left(a_0 + a_1 \|\nabla u_m\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k(x,t) |u_m|^\sigma d\Gamma dt \le C_5,$$

where the constants C_4, C_5 does not depend on $m \in N$ but depend on $0 < t < T_0$.

Proof. We estimate the right-hand side of identity (3.1) in the following way

$$\left| \int_\Omega b(x,t) |u_m|^p \right| \le b_0 \|u_m\|_{p,\Omega}^p \le b_0 C_1 \left(\chi \|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \right)^{\frac{p}{2}}, \tag{3.2}$$

$$\left| -\frac{\chi}{\sigma} \int_\Gamma k_t(x,t) |u_m|^\sigma d\Gamma \right| \le \frac{\chi}{\sigma} \int_\Gamma \frac{|k_t(x,t)|}{k(x,t)} k(x,t) |u_m|^\sigma d\Gamma \le K_1 \frac{\chi}{\sigma} \int_\Gamma k(x,t) |u_m|^\sigma d\Gamma, \tag{3.3}$$

$$\begin{aligned} &\left| \int_\Omega f \cdot u_m dx \right| \le \|f\|_{2,\Omega} \|u_m\|_{2,\Omega} \le \tfrac{1}{2} \|f\|_{2,\Omega}^2 + \tfrac{1}{2} \|u_m\|_{2,\Omega}^2 \le \\ &\le \tfrac{1}{2} \|f\|_{2,\Omega}^2 + \tfrac{1}{2} \left(\chi \|\nabla u_m\|_{2,\Omega}^2 + \|u_m\|_{2,\Omega}^2 \right). \end{aligned} \tag{3.4}$$

Substituting the obtained inequalities (3.2) and (3.4) into identity (3.1), we arrive at the inequality

$$\begin{aligned} &\tfrac{dE}{dt} + a_0 \int_\Omega |\nabla u_m|^2 dx + a_1 \left(\int_\Omega |\nabla u_m|^2 dx \right)^q + \left(a_0 + a_1 \|\nabla u_m\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k(x,t) |u_m|^\sigma d\Gamma \\ &\le C_2 E(t) + +C_3 E(t)^{\frac{p}{2}} + \tfrac{1}{2} \|f\|_{2,\Omega}^2, \end{aligned} \tag{3.5}$$

where $C_2 = \max\left\{\frac{K_1}{\sigma-1};\ 1\right\}$, $C_3 = b_0 C_1$. We introduce the function

$$z(t) = e^{-C_2 t} E(t).$$

Then (3.5) can be rewritten as

$$\frac{dz}{dt} \le C_3 e^{C_2 \frac{p-2}{2} t} [z(t)]^{\frac{p}{2}} + \frac{1}{2} e^{-C_3 t} \|f\|_{2,\Omega}^2.$$

Integrating the latest from 0 to t, we obtain

$$z(t) \le z(0) + C_3 \int_0^t e^{C_2 \frac{p-2}{2} s} [z(s)]^{\frac{p}{2}} ds + \frac{1}{2} \int_0^t e^{-C_2 s} \|f\|_{2,\Omega}^2 ds.$$

Using condition (1.4) we arrive at the nonlinear integral inequality

$$z(t) \leq z(0) + \frac{C_0}{2C_2} + C_3 \int_0^t e^{C_2 \frac{p-2}{2} s} [z(s)]^{\frac{p}{2}} ds.$$

Applying the Gronwall Bellman-Bihari lemma [39], we arrive at the estimate

$$z(t) \leq \frac{z(0) + \frac{C_0}{2C_2}}{\left[1 - \left(z(0) + \frac{C_0}{2C_2}\right)^{\frac{p-2}{2}} \frac{C_3}{C_2} \left(e^{C_2 \frac{p-2}{2} t} - 1\right)\right]^{\frac{2}{p-2}}},$$

if t satisfies the inequality

$$\frac{C_3}{C_2} \left(e^{C_2 \frac{p-2}{2} t} - 1\right) < \frac{1}{\left(z(0) + \frac{C_0}{2C_2}\right)^{\frac{p-2}{2}}}, \quad 0 \leq t < T,$$

that is

$$t < T_0 = \frac{2}{C_2(p-2)} \ln \left(1 + \frac{C_2}{C_3} \left(z(0) + \frac{C_0}{2C_2}\right)^{\frac{2-p}{2}}\right) := G(z(0)).$$

Then we arrive at final the inequality

$$E(t) \leq \frac{\left(E(0) + \frac{C_0}{2C_2}\right) e^{C_2 t}}{\left[1 - \left(E(0) + \frac{C_0}{2C_2}\right)^{\frac{p-2}{2}} \frac{C_3}{C_2} \left(e^{C_2 \frac{p-2}{2} t} - 1\right)\right]^{\frac{2}{p-2}}}. \tag{3.6}$$

From this estimate, we can conclude that there exists $T_0 > 0$ such that

$$\frac{1}{2} \left(\|u_m\|_{2,\Omega}^2 + \chi \|\nabla u_m\|_{2,\Omega}^2\right) + \chi \frac{\sigma - 1}{\sigma} \int_\Gamma k(x,t) \| u_m|^\sigma d\Gamma \leq C_4, \quad \text{for all } \ t \in [0,T], \ T < T_0, \tag{3.7}$$

where C_4 is a constant independent of $m \in N$ but depends on t.

Returning to (3.5) and taking into account (3.7), we obtain one more inequality:

$$\begin{aligned} &\int_0^T \left(\int_\Omega |\nabla u_m|^2 dx + \left(\int_\Omega |\nabla u_m|^2 dx\right)^q\right) dt + \\ &+ \int_0^T \left(a_0 + a_1 \|\nabla u_m\|_{2,\Omega}^{2q-2}\right) \int_\Gamma k(x,t) |u_m|^\sigma d\Gamma dt \leq C_5. \end{aligned} \tag{3.8}$$

□

Lemma 3. *(Second local estimate) Let conditions (1.4) and estimates (3.7), (3.8), as well as the inequalities of Lemma 1 are performed , then the functions $u_m(x,t)$ satisfy the estimate:*

$$\Lambda(t) \leq C_6, \ t \in (0, T_0),$$

where

$$\Lambda(t) := \int_0^t \left(\int_\Omega \left(|\partial_t u_m|^2 + \chi |\partial_t \nabla u_m|^2 \right) dx + \chi(\sigma - 1) \int_\Gamma k(x,t) |u_m|^{\sigma-2} |\partial_t u_m|^2 d\Gamma \right) dt,$$

and C_6 constant does not depend on $m \in N$.

Proof. Now we multiply equality (2.1) by $C'_{mj}(t)$ and sum over $j = \overline{1,m}$. Integrating obtained expression over τ from 0 to t, we arrive at the relations

$$\begin{array}{l}
\Lambda(t) + \frac{a_0}{2} \left\| \nabla u_m \right\|_{2,\Omega}^2 + \frac{a_1}{2q} \left\| \nabla u_m \right\|_{2,\Omega}^{2q} + = \\
= \int_\Omega b(x,t)|u_m|^p dx - \frac{1}{p} \int_0^t \int_\Omega b_\tau(x,t)|u_m|^p dx d\tau - \\
-\frac{1}{\sigma} \int_\Gamma \left(a_0 + a_1 \left\| \nabla u_m \right\|_{2,\Omega}^{2q-2} \right) k(x,t)|u_m|^\sigma d\Gamma + \\
+\frac{q-1}{\sigma} a_1 \int_0^t \left\| \nabla u_m \right\|_{2,\Omega}^{2q-4} \int_\Omega \nabla u_m \nabla \partial_\tau u_m dx \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma d\tau + \\
+\frac{1}{\sigma} \int_0^t \left(a_0 + a_1 \left\| \nabla u_m \right\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k_\tau(x,t)|u_m|^\sigma d\Gamma d\tau - \\
-\frac{\chi}{\sigma} \int_\Gamma k_t(x,t)|u_m|^\sigma d\Gamma + \frac{\chi}{\sigma} \int_0^t \int_\Gamma k_{\tau\tau}(x,t)|u_m|^\sigma d\Gamma d\tau + \\
+ \int_0^t \int_\Omega f\, \partial_\tau u_m dx d\tau + \frac{a_0}{2} \left\| \nabla u_m(x,0) \right\|_{2,\Omega}^2 + \frac{a_1}{2q} \left\| \nabla u_m(x,0) \right\|_{2,\Omega}^{2q} \\
- \int_\Omega b(x,0)|u_m(x,0)|^p dx + \\
+\frac{1}{\sigma} \int_\Gamma \left(a_0 + a_1 \left\| \nabla u_m(x,0) \right\|_{2,\Omega}^{2q-2} \right) k(x,0)|u_m(x,0)|^\sigma d\Gamma.
\end{array} \tag{3.9}$$

We estimate the right-hand side of identity (3.9)

$$\left| \frac{1}{p} \int_\Omega b(x,t)|u_m|^p dx \right| \le \frac{b_0 C_1}{p} \left(\chi \left\| \nabla u \right\|_{2,\Omega}^2 + \left\| u \right\|_{2,\Omega}^2 \right)^{\frac{p}{2}} \le \frac{b_0 C_1}{p} \sqrt{C_4^p}. \tag{3.10}$$

$$\left| \frac{1}{\sigma} \left(a_0 + a_1 \left\| \nabla u_m \right\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma \right| \le \frac{C_4(a_0 + a_1 C_4^{q-1})}{\sigma}. \tag{3.11}$$

$$\begin{array}{l}
\left| a_1 \frac{q-1}{\sigma} \int_0^t \left\| \nabla u_m \right\|_{2,\Omega}^{2(q-2)} \int_\Omega \nabla u_m \nabla \partial_t u_m dx \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma d\tau \right| \le \\
\le a_1 \frac{q-1}{\sigma} \int_0^t \left\| \nabla u_m \right\|_{2,\Omega}^{2(q-2)} \left\| \nabla u_m \right\|_{2,\Omega} \left\| \nabla \partial_t u_m \right\|_{2,\Omega} \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma d\tau \le \\
\le a_1 \frac{q-1}{\sigma} C_4^{q-1} \int_0^t \left\| \nabla \partial_t u_m \right\|_{2,\Omega} d\tau \le \frac{\chi}{2} \int_0^t \left\| \nabla \partial_t u_m \right\|_{2,\Omega}^2 d\tau + a_1^2 \frac{(q-1)^2}{2\chi\sigma^2} C_4^{2q-2} T.
\end{array}$$

$$\begin{array}{l}
\left| -\frac{\chi}{\sigma} \int_\Gamma k_t(x,t)|u_m|^\sigma d\Gamma + \frac{\chi}{\sigma} \int_0^t \int_\Gamma k_{\tau\tau}(x,t)|u_m|^\sigma d\Gamma d\tau \right| \le \\
\le \frac{\chi}{\sigma} \int_\Gamma \frac{|k_t(x,t)|}{k(x,t)} k(x,t)|u_m|^\sigma d\Gamma + \\
+\frac{\chi}{\sigma} \int_0^t \int_\Gamma \frac{k_{\tau\tau}(x,\tau)}{k(x,\tau)} k(x,\tau)|u_m|^\sigma d\Gamma d\tau \le \frac{\chi}{\sigma} K_1 \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma + \\
+\frac{\chi}{\sigma} K_2 \int_0^t \int_\Gamma k(x,t)|u_m|^\sigma d\Gamma d\tau \le \frac{\chi}{\sigma} K_1 C_4 + \frac{\chi}{\sigma} K_2 C_4 T.
\end{array}$$

$$\begin{array}{l}
\left| \frac{1}{\sigma} \int_0^t \left(a_0 + a_1 \left\| \nabla u_m \right\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k_\tau(x,t)|u_m|^\sigma d\Gamma d\tau \right| \le \\
\le \frac{1}{\sigma} (a_0 + a_1 C_4^{q-1}) \int_0^t \int_\Gamma \frac{|k_\tau(x,t)|}{k(x,\tau)} k(x,\tau)|u_m|^\sigma d\Gamma d\tau \le \\
\le \frac{K_1}{\sigma} (a_0 + a_1 C_4^{q-1}) C_4 T.
\end{array}$$

$$\left| \frac{1}{p} \int_0^t \int_\Omega b_\tau(x,t)|u_m|^p dx d\tau \right| \le \frac{b_1}{p} \int_0^t \int_\Omega |u_m|^p dx d\tau \le \frac{b_1 C_1}{p} \sqrt{C_4^p} t \le \frac{b_1 C_1}{p} \sqrt{C_4^p} T.$$

$$\left| \int_0^t \int_\Omega f\, \partial_\tau u_m dx d\tau \right| \le \left\| f \right\|_{2,Q_t} \left\| \partial_\tau u_m \right\|_{2,Q_t} \le \frac{1}{2} \left\| f \right\|_{2,Q_t}^2 + \frac{1}{2} \left\| \partial_\tau u_m \right\|_{2,Q_t}^2. \tag{3.12}$$

Substituting the obtained inequalities into identity (3.9), we obtain

$$\tfrac{1}{2}\int_0^t\int_\Omega\left(|\partial_\tau u_m|^2+\chi|\partial_\tau\nabla u_m|^2\right)dxd\tau+\tfrac{a_0}{2}\left\|\nabla u_m\right\|_{2,\Omega}^2+\tfrac{a_1}{2q}\left\|\nabla u_m\right\|_{2,\Omega}^{2q}+$$
$$+\chi(\sigma-1)\int_0^t\int_\Gamma k(x,t)|u_m|^{\sigma-2}|\partial_\tau u_m|^2 d\Gamma d\tau\le C_5.$$

Thereby received the estimate

$$\int_0^T\left(\int_\Omega\left(|\partial_t u_m|^2+\chi|\partial_t\nabla u_m|^2\right)dx+\chi(\sigma-1)\int_\Gamma k(x,t)|u_m|^{\sigma-2}|\partial_t u_m|^2 d\Gamma\right)dt\le C_6. \quad (3.13)$$

□

3.2. **Global in time estimates in presence an absorption** $b<0$**.**

Lemma 4. *Suppose that in addition to (1.4) and $q>1, p>1, \sigma>1$ the term $b(x,t)$ satisfies the conditions*

$$0<b_0\le -b(x,t)\le b_1<\infty. \quad (3.14)$$

Then the Galerkin approximations for all $t\in[0,T]$ satisfy the estimates

$$sup_{t\in[0,T]}Y(u_m(t),t)+\int_0^T\Lambda\left(u_m(t),t\right)dt\le 2e^{2T}\left(\|f\|_{2,Q_T}^2+Y(u(0),0)+1\right), \quad (3.15)$$

where

$$Y(u_m(t),t):=\left(\frac{1}{2}\left(\int_\Omega[|u_m(t)|^2+\chi|\nabla u_m(t)|^2]dx\right)+\chi\frac{\sigma-1}{\sigma}\int_\Gamma k(x,t)|u_m(t)|^\sigma d\Gamma\right)\ge 0,$$

$$\Lambda(u_m(t),t):=a_0\int_\Omega|\nabla u_m|^2dx+a_1\left(\int_\Omega|\nabla u_m|^2dx\right)^q+\int_\Omega b_0|u_m|^p dx$$
$$+\left(a_0+a_1\left\|\nabla u_m\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u_m|^\sigma d\Gamma\ge 0.$$

Proof. Using (3.14), (1.4) and (3.1) we get the inequality

$$\begin{aligned}&\tfrac{d}{dt}Y(u_m(t),t)+\Lambda(u_m(t),t)\\&\le\left|\int_\Omega f\cdot u_m dx\right|\le\tfrac{1}{2}\int_\Omega f^2dx+Y(u_m(t),t).\end{aligned} \quad (3.16)$$

Integrating last one we arrive at estimate

$$\begin{aligned}&Y(u_m(t),t)+\int_0^t e^{-s}\Lambda(u_m(s),s)ds\\&\le e^T\left(Y(u_m(0),0)+\|f\|_{2,Q_T}^2\right)\le e^T\left(Y(u(0),0)+\|f\|_{2,Q_T}^2+1\right).\end{aligned} \quad (3.17)$$

From here we obtain

$$sup_{t\in[0,T]}Y(u_m(t),t)+\int_0^T\Lambda\left(u_m(t),t\right)dt\le 2e^{2T}\left(\|f\|_{2,Q_T}^2+Y(u(0),0)+1\right). \quad (3.18)$$

The estimate (3.15)is proved. □

Now let us prove the second global estimate under conditions (3.19).

3.3. **Global estimates in the presence a source $b > 0$.** First global estimate for $b > 0$. Now we assume that

$$p \leq \sigma, \;\; p < 2q, \;\; p \leq \frac{2(n-1)}{n-2}, \;\; 0 < k_0 \leq k(x,t). \tag{3.19}$$

Lemma 5. *(First global in time estimate in the presence a source inside the domain Ω, i.e. $b > 0$ and in the presence of an absorption on the boundary i.e.$k > 0$.) Assume that, in addition to (1.4), conditions (3.19) are satisfied. Then, for all m and any finite T, the following estimates*

$$E(t) \leq C_7, \quad \textit{for all } t \in [0,T], \tag{3.20}$$

$$\begin{aligned} &\int_0^t \left(\int_\Omega |\nabla u_m(t)|^2 dx + \left(\int_0^t |\nabla u_m(t)|^2 dx \right)^q \right) dt + \\ &+ \int_0^t \left(a_0 + a_1 \|\nabla u_m\|_{2,\Omega}^{2q-2} \right) \int_\Gamma k(x,t) |u_m|^\sigma d\Gamma dt \leq C_8, \end{aligned} \tag{3.21}$$

hold. Here the constants C_7, C_8 does not depend on $m \in N$ but depend on T.

Proof. First of all we derive a presentation for any function $u(x) \in W_2^1(\Omega) = H^1(\Omega)$.

According with the formula (7.12) page 71 from [2], any function $u(x) \in W_2^1(\Omega) = H^1(\Omega)$ may be presented by the formula

$$u(x) = u_0 + Q(\nabla u, x), \tag{3.22}$$

where

$$Q(\nabla u, x) = \int_\Omega \sum_{k=1}^n \frac{\varpi_k(y)}{r^{N-1}} \frac{\partial u(y)}{\partial y_k} dy, r = |x-y|,$$

$\varpi_k, \;\; k = 1,..,n$, are smooth functions and u_0 is a constant. Integrating both sides of the last expression over Γ we obtain

$$u_0 = \frac{1}{|\Gamma|} \left(\int_\Gamma u d\Gamma - \int_\Gamma Q(\nabla u, x) d\Gamma \right). \tag{3.23}$$

Respectively we arrive at the formula

$$u(x) = \frac{1}{|\Gamma|} \left(\int_\Gamma u d\Gamma - \int_\Gamma Q(\nabla u, x) d\Gamma \right) + Q(\nabla u, x). \tag{3.24}$$

We raise both sides of the last equality to the power of p and integrate over Ω. Then we arrive at the estimate

$$\int_\Omega |u(x)|^p dx \leq C \left(\left(\int_\Gamma |u| d\Gamma \right)^p + \left| \int_\Gamma Q(\nabla u, x) d\Gamma \right|^p + \int_\Omega |Q(\nabla u, x)|^p \, d\Omega \right), \tag{3.25}$$

where $C = C(|\Gamma|, |\Omega|, p, n)$. Using properties of integral operators and embedding theorem

$$\left| \int_\Gamma Q(\nabla u, x) d\Gamma \right|^p \leq C \left(\|\nabla u\|_{2,\Omega}^2 \right)^{\frac{p}{2}}, \;\; p \leq \frac{2(n-1)}{n-2}, \tag{3.26}$$

$$\int_\Omega \left| \int_\Omega Q(\nabla u, x) d\Omega \right|^p d\Omega \leq C \left(\|\nabla u\|_{2,\Omega}^2 \right)^{\frac{p}{2}}, \;\; p \leq \frac{2n}{n-2}, \tag{3.27}$$

we estimate

$$\int_{\Omega}|u(x)|^p dx \leq C\left(\left(\int_{\Gamma}|u|d\Gamma\right)^p+\left(\|\nabla u\|^2_{2,\Omega}\right)^{\frac{p}{2}}\right). \tag{3.28}$$

Then, combining (3.24)-(3.28) and using the inequalities

$$\left(\int_{\Gamma}|u|d\Gamma\right)^p \leq C\left(\int_{\Gamma}|u|^\sigma d\Gamma+1\right),\ \ C=C(k_0,\sigma,p),\ \ p\leq\sigma,$$

$$\left(\|\nabla u\|^2_{2,\Omega}\right)^{\frac{p}{2}} \leq \delta\left(\|\nabla u\|^2_{2,\Omega}\right)^q+C(\delta),\ \ \delta\in(0,1),\ \ p<2q$$

and taking into account (3.19), (1.4), we can rewrite (3.28) in the form

$$\int_{\Omega}b|u(x)|^p dx \leq C\left(\int_{\Gamma}k|u|^\sigma d\Gamma+1\right)+\delta\left(\|\nabla u\|^2_{2,\Omega}\right)^q+C(\delta),\ \ \delta\in(0,1),\ \ p<2q, \tag{3.29}$$

where $C=C(p,b_1,k_0,\sigma,q,\delta)$.

Choosing δ sufficiently small in comparison with a_1, similarly to (3.2)-(3.3), we arrive at the inequality

$$\begin{aligned}&\tfrac{dE}{dt}+a_0\textstyle\int_{\Omega}|\nabla u_m|^2dx+a_1\left(\int_{\Omega}|\nabla u_m|^2dx\right)^q+\left(a_0+a_1\|\nabla u_m\|^{2q-2}_{2,\Omega}\right)\int_{\Gamma}k(x,t)|u_m|^\sigma d\Gamma\leq\\&\leq C\left(E(t)+\|f\|^2_{2,\Omega}+1\right),\end{aligned} \tag{3.30}$$

Integration of the last inequality completes the proof of the lemma. ◻

Lemma 6. *(Second global estimate under conditions (3.19)) (Second global in time estimate in the presence a source inside the domain Ω, i.e. $b>0$ and in the presence of an absorption on the boundary i.e.$k>0$.)*

Assume that, in addition to (1.4), conditions (3.19) are satisfied. Then, for all m and any finite T, the following estimates

$$\Lambda(t)\leq C_9,\ \ t\in[0,T], \tag{3.31}$$

where

$$\Lambda(t):=\int_0^t\left(\int_{\Omega}\left(|\partial_t u_m|^2+\chi|\partial_t\nabla u_m|^2\right)dx+\chi(\sigma-1)\int_{\Gamma}k(x,t)|u_m|^{\sigma-2}|\partial_t u_m|^2d\Gamma\right)dt$$

hold. Here the constant C_9 does not depend on $m\in N$ but depend on T.

Proof. We will use the relations (3.9),(3.10)-(3.12) and inequality (3.29) to estimate $\int_{\Omega}b|u_m|^\sigma$. Taking into account Lemma 5, estimates (3.11)-(3.12) are carried out in a completely similar way, but for any fixed finite moment of time. Repeating the arguments of the Lemma 5 ends the proof.

◻

4. Passage to the limit as $m \to \infty$. Local and global existence theorems.

The estimates obtained in Lemmas 2,3 (local in time) and in Lemmas 4-6 (global in time) allow us to draw the following conclusions. Here and below $T < T_0$ in the case of local estimates and T is an arbitrary finite number in the case of global estimates

$$u_m \ \text{ limited in } \ L_\infty(0,T;\, H^1(\Omega)), \tag{4.1}$$

$$u_m' \ \text{ limited in } \ L_2(0,T;\, H^1(\Omega)), \tag{4.2}$$

$$k(x,t)|u_m|^{\sigma-2}u_m \ \text{ limited in } \ L_\infty(0,T;\, L_{\sigma'}(\Gamma)), \quad \sigma' = \frac{\sigma}{\sigma-1} > 1. \tag{4.3}$$

In addition, due to the conditions imposed on p:

$$|u_m|^{p-2}u_m \ \text{ limited in } \ L_\infty(0,T;\, L_{\frac{p}{p-1}}(\Omega)), \ 2 < p < \frac{2n}{n-2}, \ \ n \geq 3. \tag{4.4}$$

From(4.1) follows, that there exists a subsequence u_{m_k} of the sequence u_m, *-weakly converging to some element $u_m \in L_\infty(0,T;\, H^1(\Omega))$, that is

$u_{m_k} \ \ \to \ \ u$ *-weak in $L_\infty(0,T;\, H^1(\Omega))$.

Similarly, it follows from (4.2)-(4.4) that there exists a sequence $\{u_{m_k}\} \subset \{u_m\}$, such that

$u'_{m_k} \ \ \to \ \ u'$ limited in $L_2(0,T;\, H^1(\Omega))$.

By the Sobolev theorem $W_2^1(Q_T) \in L_m(Q_T)$, $m \leq \frac{2(n+1)}{n-1}$. This embedding is compact if $m < \frac{2(n+1)}{n-1}$. By the Rellich-Kondrashov theorem, the embedding of $W_2^1(Q_T)$ in $L_2(Q_T)$ is compact. This means that the sequence $u_{m_;}$ can be chosen so that $u_{m_k} \ \ \to \ \ u$ in the norm $L_2(Q_T)$, and hence converging almost everywhere (see [40], Theorem 16.1, p. 123.)

From (4.3) it follows that $k(x,t)|u_m|^{\sigma-2}u_m \in L_\infty(0,T;\, L_{\sigma'}(\Gamma))$, $\sigma' = \frac{\sigma}{\sigma-1} > 1$ converges almost everywhere in $(0,T)$.

The boundedness of $\sqrt{k(x,t)|u_m|^{\sigma-2}u_m}$ in $L_2(0,T;\; L_{\sigma'}(\Gamma))$, $\sigma' = \frac{\sigma}{\sigma-1} > 1$ implies the weak convergence in this space of the subsequence $k(x,t)|u_{m_k}|^{\sigma-2}u_{m_k}$ of some function $\chi(x,t)$.

By Lemma 1.3, proved in [31], it follows that $\chi(x,t) = k(x,t)|u|^{\sigma-2}u$.

Remark 1. *Convergence of norms $\|\nabla u_i(t)\|^2_{2,\Omega}$ and strong convergence ∇u_i. We consider the sequence of functions*

$$I_i(t) := \|\nabla u_i(t)\|^2_{2,\Omega}, \ t \in [0,T]. \tag{4.5}$$

According to Lemmas 2, 3 and estimates (3.20),(3.21) (in the local case) and Lemmas 5, 6 and estimate (3.31)(in the global case) this sequence is uniformly bounded and moreover

$$\int_0^T |\frac{dI_i(t)}{dt}|^2(t)dt \leq 4 \sup_{t\in[0,T]} \|\nabla u_i(t)\|_{2,\Omega}^2 \|\nabla u_{it}\|_{2,Q_T}^2 \leq C. \tag{4.6}$$

Hence, it follows that

$$\|\nabla u_i(t) - \nabla u_i(\tau)\|_{2,\Omega} \leq \int_\tau^t \|\nabla u_{is}\|_{2,\Omega} ds \leq C|t-\tau|^{1/2}. \tag{4.7}$$

Thus, the sequence $I_i(t)$ is compact in the space $C^\alpha[0,T]$ with any $0<\alpha<1/2$. Therefore, we can single out a subsequence $I_{i_m}(t) = \|\nabla u_{i_m}(t)\|_{2,\Omega}^2$, converging in $C^\alpha[0,T]$. We show that the corresponding subsequence $\nabla u_{i_m}(x,t)$ ((henceforth, we retain the previous notation $\nabla u_i(x,t)$) that converges weakly in $L_2(Q_T)$ to $\nabla u(x,t)$ will also converge strongly. We will use the formula

$$\int_{Q_T} |\nabla u - \nabla u_i|^2 dxdt = \|\nabla u_i\|_{2,Q_T}^2 - \|\nabla u\|_{2,Q_T}^2 \tag{4.8}$$

$$+2\int_{Q_T} (\nabla u - \nabla u_i)\nabla u dxdt.$$

Here

$$\|\nabla u_i\|_{2,Q_T}^2 - \|\nabla u\|_{2,Q_T}^2 \to 0,$$

by virtue of the convergence of norms and

$$2\int_{Q_T} (\nabla u - \nabla u_i)\nabla u dxdt \to 0,$$

by virtue of the definition of a weak solution.

The above considerations make it possible to go to the limit in (2.1). But first, we multiply each of equalities (2.1) by $d_j(t) \in C[0,T]$ and sum both sides of the resulting equality over $j=\overline{1,m}$. Then we integrate over t from 0 to T, we get

$$\begin{gathered}\int_0^T \int_\Omega \left(\frac{\partial u_m}{\partial t}\mu + \chi\sum_{i=1}^n \frac{\partial^2 u_m}{\partial t \partial x_i}\frac{\partial \mu}{\partial x_i} + \left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2q-2}\right)\sum_{i=1}^n \frac{\partial u_m}{\partial x_i}\frac{\partial \mu}{\partial x_i}\right) dxdt- \\ -\int_0^T\int_\Omega b(x,t)|u_m|^{p-2}u_m\mu dxdt+ \\ +\int_0^T\int_\Gamma \left(\left(a_0 + a_1\|\nabla u_m\|_{2,\Omega}^{2q-2}\right)k(x,t)|u_m|^{\sigma-2}u_m + \chi k_t(x,t)|u_m|^{\sigma-2}u_m\right)\mu d\Gamma dt+ \\ +\chi(\sigma-1)\int_0^T\int_\Gamma k(x,t)|u_m|^{\sigma-2}u_{mt}\mu d\Gamma dt = \int_0^T\int_\Omega f\mu dxdt,\end{gathered} \tag{4.9}$$

where $\mu(x,t) = \sum_{j=1}^m d_j(t)\Psi_j(x)$.

Taking into account the inclusions and convergence obtained we can pass in to the limit in (4.9) as $m\to\infty$ and obtain (1.5) for $\varphi(t)v(x) = \mu(x,t)$. Since the set of all functions $\mu(x,t)$ is dense in $W_2^1(0,T;W_2^1(\Omega))$, then the limit relation holds for all $v(x,t) \in L_2(0,T;W_2^1(\Omega))\bigcap L_\sigma(\Gamma)$.

Then we formulate the following theorems.

Theorem 1. *(Local existence) Let conditions (1.4) be satisfied, and also* $2 < p < \frac{2n}{n-2}$, $n \geq 3$, $q > 1$, $2 < \sigma < \frac{2(n-1)}{n-2}$. *Then on the interval* $(0,T)$, $T < T_0$, *there exists a weak generalized solution* $u(x,t)$ *of problem (1.1)-(1.3), and the following inclusions take place:*

$$u \in L_\infty(0,T;\, H^1(\Omega)),\;\; u_t \in L_2(0,T;\, H^1(\Omega)),$$

$$|u|^{\sigma-2}u \in L_\infty(0,T;\, L_{\sigma'}(\Gamma)),\quad \sigma' = \frac{\sigma}{\sigma-1} > 1,$$

$$|u|^{p-2}u \in L_\infty(0,T;\, L_{\frac{p}{p-1}}(\Omega)).$$

Theorem 2. *(Global existence) Let the conditions of Lemma (4) (or of Lemma (5)) and, accordingly, estimates (3.14), (3.15) (or (3.20), (3.21)) be satisfied. Then the solution* $u(x,t)$ *to the problem (1.1)-(1.3) exists on any finite time interval* $T < \infty$.

5. The uniqueness of a weak generalized solution.

Theorem 3. *Let us assume that*

$$2 \leq \sigma \leq 2 + \frac{2}{n-2},\;\; 2 \leq p \leq 2 + \frac{2}{n-2},\;\; n \geq 3, q > 2.$$

Then the weak generalized solution $u \in W_2^1(0,T;\, W_2^1(\Omega)) \cap L_\sigma(\Gamma)$ *to problem (1.1)-(1.3) is unique on the interval* $(0,T)$.

Proof. Suppose that problem (1.1)-(1.3) has two solutions: $u_1(x,t)$ and $u_2(x,t)$. Then their difference $u(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the condition $u(x,0) = 0$. Take $\varphi(t) = 1$ in equality (1.5) . Then we will have equality for almost all t :

$$\begin{aligned}
&\int_0^t \int_\Omega \left(\frac{\partial u(x,\tau)}{\partial \tau} v(x,\tau) + \chi \sum_{i=1}^n \frac{\partial^2 u}{\partial \tau \partial x_i} \frac{\partial v}{\partial x_i} \right) dx d\tau + \\
&+ a_0 \int_0^t \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx d\tau + \\
&+ a_1 \int_0^t \left(\|\nabla u_1\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_1}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \|\nabla u_2\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_2}{\partial x_i} \frac{\partial v}{\partial x_i} dx \right) d\tau - \\
&- \int_0^t \int_\Omega b(x,\tau) \left(|u_1|^{p-2} u_1 - |u_1|^{p-2} u_1 \right) v dx d\tau - \\
&+ a_0 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2} u_1 - |u_2|^{\sigma-2} u_2 \right) v d\Gamma d\tau + \\
&+ a_1 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2} u_1 \|\nabla u_1\|_{2,\Omega}^{2q-2} - |u_2|^{\sigma-2} u_2 \|\nabla u_2\|_{2,\Omega}^{2q-2} \right) v d\Gamma d\tau + \\
&+ \chi \int_0^t \int_\Gamma k_\tau(x,\tau) \left(|u_1|^{\sigma-2} u_1 - |u_2|^{\sigma-2} u_2 \right) v d\Gamma d\tau + \\
&+ \chi(\sigma-1) \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2} \partial_\tau u_1 - |u_1|^{\sigma-2} \partial_\tau u_1 \right) v d\Gamma d\tau = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_\Omega \left(uv + \chi \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \right) dx+ \\
&+a_0 \int_0^t \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \right) dx d\tau + \\
&+a_1 \int_0^t \left(\|\nabla u_1\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_1}{\partial x_i}\frac{\partial v}{\partial x_i} dx - \|\nabla u_2\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_2}{\partial x_i}\frac{\partial v}{\partial x_i} dx \right) d\tau - \\
&- \int_0^t \int_\Omega b(x,\tau) \left(|u_1|^{p-2}u_1 - |u_1|^{p-2}u_1 \right) v dx d\tau - \\
&+a_0 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2 \right) v d\Gamma d\tau + \\
&+a_1 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2}u_1 \|\nabla u_1\|_{2,\Omega}^{2q-2} - |u_2|^{\sigma-2}u_2 \|\nabla u_2\|_{2,\Omega}^{2q-2} \right) v d\Gamma d\tau + \\
&+\chi \int_\Gamma k(x,t) \left(|u_1|^{\sigma-2}u_1 - |u_1|^{\sigma-2}u_1 \right) v d\Gamma = 0.
\end{aligned}$$

We now put $v(x) = u(x,t)$. Then, we get

$$\begin{aligned}
&\int_\Omega \left(|u(x,t)|^2 + \chi |\nabla u(x,t)|^2 \right) dx + \chi \int_\Gamma k(x,t) \left(|u_1|^{\sigma-2}u_1 - |u_1|^{\sigma-2}u_1 \right) u(x,t) d\Gamma + \\
&+a_0 \int_0^t \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u(x,t)}{\partial x_i} \right) dx d\tau + \\
&+a_1 \int_0^t \left(\|\nabla u_1\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_1}{\partial x_i}\frac{\partial u(x,t)}{\partial x_i} dx - \|\nabla u_2\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_2}{\partial x_i}\frac{\partial u(x,t)}{\partial x_i} dx \right) d\tau - \\
&- \int_0^t \int_\Omega b(x,\tau) \left(|u_1|^{p-2}u_1 - |u_1|^{p-2}u_1 \right) u(x,t) dx d\tau - \\
&+a_0 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2 \right) u(x,t) d\Gamma d\tau + \\
&+a_1 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2}u_1 \|\nabla u_1\|_{2,\Omega}^{2q-2} - |u_2|^{\sigma-2}u_2 \|\nabla u_2\|_{2,\Omega}^{2q-2} \right) u(x,t) d\Gamma d\tau = 0.
\end{aligned} \tag{5.1}$$

Using the following inequalities

$||u_1|^q u_1 - |u_2|^q u_2| \le (q+1)\left(|u_1|^q + |u_2|^q\right)|u_1 - u_2|$ at $q > 0$,

$|\left(|u_1|^q u_1 - |u_2|^q u_2\right)(u_1 - u_2)| \ge |u_1 - u_2|^{q+2}$ at $q > 0$,

$||u_1|^q - |u_2|^q| \ge |u_1 - u_2|^q$ at $q > 0$,

$\left| \int_\Gamma k(x,\tau)\left(|u_1|^{\sigma-2}u_1 - |u_2|^{\sigma-2}u_2\right) u d\Gamma \right| \ge \int_\Gamma k(x,\tau)|u|^\sigma d\Gamma$,

we get estimates

$$\begin{aligned}
&\left| a_1 \int_0^t \int_\Gamma k(x,\tau) \left(|u_1(\tau)|^{\sigma-2}u_1(\tau) \|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} - |u_2(\tau)|^{\sigma-2}u_2(\tau) \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \right) u(t) d\Gamma d\tau \right| \le \\
&\le a_1 \left| \int_0^t \int_\Gamma k(x,\tau) \left(|u_1(\tau)|^{\sigma-2}u_1(\tau) - |u_2(\tau)|^{\sigma-2}u_2(\tau) \right) \|\nabla u_1\|_{2,\Omega}^{2q-2} u(t) d\Gamma d\tau \right| + \\
&+a_1 \left| \int_0^t \int_\Gamma k(x,\tau) \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} - \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \right) |u_2(\tau)|^{\sigma-2}u_2(\tau) u(t) d\Gamma d\tau \right|,
\end{aligned}$$

$$\int_0^t \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_1(\tau)}{\partial x_i}\frac{\partial u(t)}{\partial x_i}dx - \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u_2(\tau)}{\partial x_i}\frac{\partial u(t)}{\partial x_i}dx \right) d\tau =$$
$$= \int_0^t \|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} \int_\Omega \sum_{i=1}^n \frac{\partial u(\tau)}{\partial x_i}\frac{\partial u(t)}{\partial x_i}dx d\tau +$$
$$+ \int_0^t \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} - \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \right) \int_\Omega \sum_{i=1}^n \frac{\partial u_2(\tau)}{\partial x_i}\frac{\partial u(t)}{\partial x_i}dx d\tau \le$$
$$\le C_1 \|\nabla u(t)\|_{2,\Omega} \int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau + C_2 \|\nabla u(t)\|_{2,\Omega} \int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau,$$

$$\int_0^t \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} - \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \right) \int_\Omega \sum_{i=1}^n \frac{\partial u_2(\tau)}{\partial x_i}\frac{\partial u(t)}{\partial x_i}dx d\tau \le$$
$$\le \int_0^t \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-4} + \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-4} \right) \int_\Omega \left(|\nabla u_1(\tau)|^2 - |\nabla u_2(\tau)|^2 \right) dx \, \|\nabla u_2(\tau)\|_{2,\Omega} \|\nabla u(t)\|_{2,\Omega} d\tau \le$$
$$\le C_2' \|\nabla u(t)\|_{2,\Omega} \int_0^t \|\nabla u(\tau)\|_{2,\Omega} \|\nabla u_1(\tau) + \nabla u_2(\tau)\|_{2,\Omega} d\tau \le C_2 \|\nabla u(t)\|_{2,\Omega} \int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau.$$

Then (5.1) can be written as

$$\begin{aligned}
&\textstyle\int_\Omega \left(|u(x,t)|^2 + \chi|\nabla u(x,t)|^2\right) dx + \chi \int_\Gamma k(x,t)|u(x,t)|^\sigma d\Gamma \le \\
&\le a_0 \|\nabla u(t)\|_{2,\Omega} \textstyle\int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau + \\
&+ a_1 C_1 \|\nabla u(t)\|_{2,\Omega} \textstyle\int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau + a_1 C_2 \|\nabla u(t)\|_{2,\Omega} \int_0^t \|\nabla u(\tau)\|_{2,\Omega} d\tau + \\
&+ b_1(p-1) \textstyle\int_0^t \int_\Omega \left(|u_1|^{p-2} + |u_2|^{p-2}\right) u(\tau)u(t) dx d\tau + \\
&+ a_0(\sigma-1) \textstyle\int_0^t \int_\Gamma k(x,\tau) \left(|u_1|^{\sigma-2} + |u_2|^{\sigma-2}\right) u(\tau)u(t) d\Gamma d\tau + \\
&+ a_1 \left| \textstyle\int_0^t \int_\Gamma k(x,\tau) \left(|u_1(\tau)|^{\sigma-2}u_1(\tau) - |u_2(\tau)|^{\sigma-2}u_2(\tau)\right) \|\nabla u_1\|_{2,\Omega}^{2q-2} u(t) d\Gamma d\tau \right| + \\
&+ a_1 \left| \textstyle\int_0^t \int_\Gamma k(x,\tau) \left(\|\nabla u_1(\tau)\|_{2,\Omega}^{2q-2} - \|\nabla u_2(\tau)\|_{2,\Omega}^{2q-2} \right) |u_2(\tau)|^{\sigma-2} u_2(\tau) u(t) d\Gamma d\tau \right|.
\end{aligned} \tag{5.2}$$

We denote by $y^2(t) = \int_\Omega \left(|u(x,t)|^2 + \chi|\nabla u(x,t)|^2\right) dx$.

We estimate the right-hand side of inequality (5.2), using the Holder and Minkowski inequality

$$\begin{aligned}
&\left| \textstyle\int_\Omega b(x,\tau) \left(|u_1(\tau)|^{p-2}u_1(\tau) - |u_2(\tau)|^{p-2}u_2(\tau)\right) u(t) dx \right| \le \\
&\le b_1(p-1) \textstyle\int_\Omega \left(|u_1(\tau)|^{p-2} + |u_2(\tau)|^{p-2}\right) u(\tau)u(t) dx \le \\
&\le b_1(p-1) \left(\textstyle\int_\Omega \left(|u_1|^{p-2} + |u_2|^{p-2}\right)^2 u^2(\tau) dx \right)^{\frac{1}{2}} \left(\int_\Omega u^2(t) dx \right)^{\frac{1}{2}} \le \\
&\le b_1(p-1) \left(\textstyle\int_\Omega \left(|u_1|^{p-2} + |u_2|^{p-2}\right)^{\frac{2r}{r-2}} dx \right)^{\frac{r-2}{2r}} \left(\int_\Omega u^r(\tau) dx \right)^{\frac{1}{r}} \left(\int_\Omega u^2(t) dx \right)^{\frac{1}{2}} \le \\
&\le b_1(p-1) \left(\left(\textstyle\int_\Omega |u_1|^{\frac{2r(p-2)}{r-2}} dx \right)^{\frac{r-2}{2r}} + \left(\int_\Omega |u_2|^{\frac{2r(p-2)}{r-2}} dx \right)^{\frac{r-2}{2r}} \right) \times \\
&\times \left(\textstyle\int_\Omega u^r(\tau) dx \right)^{\frac{1}{r}} \left(\int_\Omega u^2(t) dx \right)^{\frac{1}{2}}.
\end{aligned}$$

We put $r = \frac{2n}{n-2}$, $p \le 2 + \frac{2}{n-2}$, $n \ge 3$. Then by the Sobolev embedding theorem $H^1(\Omega) \subset L_r(\Omega)$ and $H^1(\Omega) \subset L_{2r(p-2)/(r-2)}(\Omega)$.

In this case, taking into account the smoothness class of the solutions $u_1(x,t)$ and $u_2(x,t)$, we obtain the estimate

$$\begin{aligned}&\left|\int_0^t\int_\Omega b(x,\tau)\left(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\right)u(t)dxd\tau\right|\le C_3\left\|u(t)\right\|_{2,\Omega}\int_0^t\left\|u(\tau)\right\|_{r,\Omega}d\tau\le\\&\le C_3y(t)\int_0^t y(\tau)d\tau,\ C_3=C_3\left(b_1,p,|\Omega|,\sup_{t\in(0,T)}\int_\Omega|\nabla u_i(t)|^2dx\right),i=1,2.\end{aligned}\tag{5.3}$$

Now we estimate $\int_\Gamma k(x,\tau)\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)u(\tau)u(t)d\Gamma$ the term

$$\begin{aligned}&\chi a_0(\sigma-1)\int_\Gamma k(x,\tau)\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)u(\tau)u(t)d\Gamma\le\\&\le\chi a_0(\sigma-1)K_1\int_\Gamma\left(|u_1(\tau)|^{\sigma-2}+|u_2(\tau)|^{\sigma-2}\right)u(\tau)u(t)dx\le\\&\le\chi a_0(\sigma-1)K_1\left(\int_\Gamma\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)^2u^2(\tau)dx\right)^{\frac12}\left(\int_\Gamma u^2(t)dx\right)^{\frac12}\le\\&\le\chi a_0(\sigma-1)K_1\left(\int_\Gamma\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)^{\frac{2r}{r-2}}dx\right)^{\frac{r-2}{2r}}\left(\int_\Gamma u^r(\tau)dx\right)^{\frac1r}\left(\int_\Gamma u^2(t)dx\right)^{\frac12}\le\\&\le\chi a_0(\sigma-1)K_1\left(\left(\int_\Gamma|u_1|^{\frac{2r(\sigma-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}+\left(\int_\Gamma|u_2|^{\frac{2r(p-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}\right)\times\\&\times\left(\int_\Gamma u^r(\tau)dx\right)^{\frac1r}\left(\int_\Gamma u^2(t)dx\right)^{\frac12}.\end{aligned}$$

We put $r=\frac{2(n-1)}{n-2},\ \sigma\le2+\frac{2}{n-2},\quad n\ge3.$ Using the theorem on traces of functions from the Sobolev class, we obtain

$$\begin{aligned}&\chi a_0(\sigma-1)\int_0^t\int_\Gamma k(x,\tau)\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)u(\tau)u(t)d\Gamma d\tau\le C_4\left\|u(t)\right\|_{2,\Omega}\int_0^t\left\|u(\tau)\right\|_{r,\Omega}d\tau\le\\&\le C_4y(t)\int_0^t y(\tau)d\tau.\end{aligned}$$

Similarly, the following terms are evaluated

$$\begin{aligned}&a_1\left|\int_\Gamma k(x,\tau)\left(|u_1(\tau)|^{\sigma-2}u_1(\tau)-|u_2(\tau)|^{\sigma-2}u_2(\tau)\right)\left\|\nabla u_1\right\|_{2,\Omega}^{2q-2}u(t)d\Gamma\right|\le\\&\le a_1(\sigma-1)K_1C_5\int_\Gamma\left(|u_1|^{\sigma-2}+|u_2|^{\sigma-2}\right)u(\tau)u(t)d\Gamma\le\\&\le a_1(\sigma-1)K_1C_5'\left(\left(\int_\Gamma|u_1|^{\frac{2r(\sigma-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}+\left(\int_\Gamma|u_2|^{\frac{2r(p-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}\right)\times\\&\times\left(\int_\Gamma u^r(\tau)dx\right)^{\frac1r}\left(\int_\Gamma u^2(t)dx\right)^{\frac12},\end{aligned}$$

$$\begin{aligned}&a_1\left|\int_0^t\int_\Gamma k(x,\tau)\left(|u_1(\tau)|^{\sigma-2}u_1(\tau)-|u_2(\tau)|^{\sigma-2}u_2(\tau)\right)\left\|\nabla u_1\right\|_{2,\Omega}^{2q-2}u(t)d\Gamma d\tau\right|\le\\&\le C_5\left\|u(t)\right\|_{2,\Omega}\int_0^t\left\|u(\tau)\right\|_{r,\Omega}d\tau\le C_5y(t)\int_0^t y(\tau)d\tau,\end{aligned}$$

$$\begin{aligned}&a_1\left|\int_0^t\int_\Gamma k(x,\tau)\left(\left\|\nabla u_1(\tau)\right\|_{2,\Omega}^{2q-2}-\left\|\nabla u_2(\tau)\right\|_{2,\Omega}^{2q-2}\right)|u_2(\tau)|^{\sigma-2}u_2(\tau)u(t)d\Gamma d\tau\right|\le\\&\le a_1K_1\int_0^t\left(\left\|\nabla u_1(\tau)\right\|_{2,\Omega}^{2q-2}-\left\|\nabla u_2(\tau)\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma|u_2(\tau)|^{\sigma-1}u(t)d\Gamma d\tau\le\\&\le a_1K_1\int_0^t\left(\left\|\nabla u_1(\tau)\right\|_{2,\Omega}^{2q-3}+\left\|\nabla u_2(\tau)\right\|_{2,\Omega}^{2q-3}\right)\int_\Omega\left(|\nabla u_1(\tau)|^2-|\nabla u_2(\tau)|^2\right)dx\left\|u_2(\tau)\right\|_{\sigma,\Gamma}\left\|u(t)\right\|_{\sigma,\Gamma}d\tau\le\\&\le a_1K_1C_2'\left\|u(t)\right\|_{\sigma,\Gamma}\int_0^t\left\|u_2(\tau)\right\|_{\sigma,\Gamma}\left\|\nabla u_1(\tau)+\nabla u_2(\tau)\right\|_{2,\Omega}d\tau\le C_6\left\|u(t)\right\|_{\sigma,\Gamma}\int_0^t\left\|u_2(\tau)\right\|_{\sigma,\Gamma}d\tau\le\\&\le C_6y(t)\int_0^t y(\tau)d\tau\end{aligned}$$

where $\sigma\le2+\frac{2}{n-2},\quad n\ge3.$

By virtue of (5.3), we obtain

$$\begin{aligned}&y^2(t)+\chi\int_\Gamma k(x,t)|u(x,t)|^\sigma d\Gamma\le\\&\le a_0\left\|\nabla u(t)\right\|_{2,\Omega}\int_0^t\left\|\nabla u(\tau)\right\|_{2,\Omega}d\tau+\\&+a_1C_1\left\|\nabla u(t)\right\|_{2,\Omega}\int_0^t\left\|\nabla u(\tau)\right\|_{2,\Omega}d\tau+a_1C_2\left\|\nabla u(t)\right\|_{2,\Omega}\int_0^t\left\|\nabla u(\tau)\right\|_{2,\Omega}d\tau+\\&+(C_3+C_4+C_5+C_6)y(t)\int_0^t y(\tau)d\tau,\end{aligned}$$

or

$$\begin{aligned} &y^2(t)+\chi\int_\Gamma k(x,t)|u(x,t)|^\sigma d\Gamma \le \\ &\le (a_0+a_1C_1+a_1C_2+C_3+C_4+C_5+C_6)y(t)\int_0^t y(\tau)d\tau. \end{aligned} \tag{5.4}$$

From (5.4) follows the inequality

$$y(t)\le C_7\int_0^t y(\tau)d\tau,$$

which, by the Gronwall lemma, implies $\int_\Omega \left(\chi|\nabla u|^2+|u|^2\right)dx=0$ almost everywhere on the time interval $(0,T)$.This means the uniqueness of the weak generalized solutions.

□

6. Blow up of the solution in a finite time

We obtain conditions, under which the solution u of the problem (1.1)-(1.3) blows up in a finite moment of time.We say that the solution $u(x,t)$ blows up at moment time T_1 if some norm $\|u(\cdot,t)\|_\Omega$ of the solution tends to infinity when $t\to T_1$. We introduce the notation

$$\Phi(t)=\frac{1}{2}\|u\|^2_{2,\Omega}+\frac{\chi}{2}\|\nabla u\|^2_{2,\Omega}+\chi\frac{\sigma-1}{\sigma}\int_\Gamma k(x,t)|u|^\sigma d\Gamma,$$

$$J(t)=\|u_t\|^2_{2,\Omega}+\chi\|\nabla u_t\|^2_{2,\Omega}+\chi(\sigma-1)\int_\Gamma k(x,t)|u|^{\sigma-2}|\partial_t u|^2 d\Gamma.$$

Theorem 4. *Let the conditions (1.4) is fulfilled ,* $2<p<\frac{2n}{n-2}$, $n\ge 3$, $q>1$, $2<\sigma<\frac{2(n-1)}{n-2}$, *and also*

$$\begin{aligned} &\int_\Omega b(x,0)|u_0|^p dx-a_0\int_\Omega|\nabla u_0|^2dx-a_1\left(\int_\Omega|\nabla u_0|^2dx\right)^q- \\ &-\left(a_0+a_1\|\nabla u_0\|^{2q-2}_{2,\Omega}\right)\int_\Gamma k(x,0)|u_0|^\sigma d\Gamma-\frac{\chi}{\sigma}\int_\Gamma k_t(x,0)|u_0|^\sigma d\Gamma>0, \end{aligned}$$

$$|\Phi'(0)|^2>\frac{\beta\Phi^2(0)}{\alpha-1}+\frac{16p}{4\alpha-p-2}\Phi^{\frac{p+2}{2}}(0)+\frac{2Cp}{2\alpha-1}\Phi(0),$$

where $\Phi(0)=\frac{1}{2}\|u_0\|^2_{2,\Omega}+\frac{\chi}{2}\|\nabla u_0\|^2_{2,\Omega}+\chi\frac{\sigma-1}{\sigma}\int_\Gamma k(x,0)|u_0|^\sigma d\Gamma$,

$$\begin{aligned} \Phi'(0)=&\int_\Omega b(x,0)|u_0|^p dx-a_0\int_\Omega|\nabla u_0|^2dx-a_1\left(\int_\Omega|\nabla u_0|^2dx\right)^q- \\ &-\left(a_0+a_1\|\nabla u_0\|^{2q-2}_{2,\Omega}\right)\int_\Gamma k(x,0)|u_0|^\sigma d\Gamma-\frac{\chi}{\sigma}\int_\Gamma k_t(x,0)|u_0|^\sigma d\Gamma. \end{aligned}$$

Then the solution to the initial-boundary value problem (1.1) - (1.3) blows up in some finite time T_1*, where*

$$T_1=\frac{\Phi^{1-\alpha}(0)}{A}>0,$$

$$A^2=|\Psi'(0)|^2-\beta(\alpha-1)|\Psi(0)|^2-\frac{16p(\alpha-1)^2}{4\alpha-p-2}\Psi^{\frac{4\alpha-p-2}{2(\alpha-1)}}(0)-\frac{2Cp(\alpha-1)^2}{2\alpha-1}\Psi^{\frac{2\alpha-1}{\alpha-1}}(0).$$

Proof. We multiply equation (1.1) sequentially by the functions $u(x,t)$, $u_t(x,t)$ and integrate over the domain Ω. Then we obtain the following relations

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega[|u|^2+\chi|\nabla u|^2]dx+\chi\tfrac{\sigma-1}{\sigma}\tfrac{d}{dt}\int_\Gamma k(x,t)|u|^\sigma d\Gamma+\\&+a_0\int_\Omega|\nabla u|^2dx+a_1\left(\int_\Omega|\nabla u|^2dx\right)^q+\\&+\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u|^\sigma d\Gamma+\tfrac{\chi}{\sigma}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma=\\&=\int_\Omega b(x,t)|u|^pdx,\end{aligned}$$

$$\begin{aligned}&\int_\Omega\left(|\partial_t u|^2+\chi|\partial_t\nabla u|^2\right)dx+\tfrac{d}{dt}\left(\tfrac{a_0}{2}\|\nabla u\|_{2,\Omega}^2+\tfrac{a_1}{2q}\|\nabla u\|_{2,\Omega}^{2q}\right)\\&+\chi(\sigma-1)\int_\Gamma k(x,t)|u|^{\sigma-2}|\partial_t u|^2d\Gamma=\\&=\tfrac{d}{dt}\int_\Omega b(x,t)|u|^pdx-\tfrac{1}{p}\int_\Omega b_t(x,t)|u|^pdx-\\&-\tfrac{1}{\sigma}\tfrac{d}{dt}\int_\Gamma\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^\sigma d\Gamma+\\&+\tfrac{q-1}{\sigma}a_1\|\nabla u\|_{2,\Omega}^{2q-4}\int_\Omega\nabla u\nabla\partial_t u dx\int_\Gamma k(x,t)|u|^\sigma d\Gamma+\\&+\tfrac{1}{\sigma}\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma-\\&-\tfrac{\chi}{\sigma}\tfrac{d}{dt}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma+\tfrac{\chi}{\sigma}\int_\Gamma k_{tt}(x,t)|u|^\sigma d\Gamma.\end{aligned}$$

By virtue of this notation we have

$$\begin{aligned}&\tfrac{d\Phi(t)}{dt}=-a_0\int_\Omega|\nabla u|^2dx-a_1\left(\int_\Omega|\nabla u|^2dx\right)^q-\\&-\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u|^\sigma d\Gamma-\tfrac{\chi}{\sigma}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma+\\&+\int_\Omega b(x,t)|u|^pdx,\end{aligned}\tag{6.1}$$

$$\begin{aligned}&J(t)=-\tfrac{d}{dt}\left(\tfrac{a_0}{2}\|\nabla u\|_{2,\Omega}^2+\tfrac{a_1}{2q}\|\nabla u\|_{2,\Omega}^{2q}\right)+\\&+\tfrac{1}{p}\tfrac{d}{dt}\int_\Omega b(x,t)|u|^pdx-\tfrac{1}{p}\int_\Omega b_t(x,t)|u|^pdx-\\&-\tfrac{1}{\sigma}\tfrac{d}{dt}\int_\Gamma\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^\sigma d\Gamma+\\&+\tfrac{q-1}{\sigma}a_1\|\nabla u\|_{2,\Omega}^{2q-4}\int_\Omega\nabla u\nabla\partial_t u dx\int_\Gamma k(x,t)|u|^\sigma d\Gamma+\\&+\tfrac{1}{\sigma}\left(a_0+a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma-\\&-\tfrac{\chi}{\sigma}\tfrac{d}{dt}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma+\tfrac{\chi}{\sigma}\int_\Gamma k_{tt}(x,t)|u|^\sigma d\Gamma.\end{aligned}\tag{6.2}$$

Using the notation for Φ we derive

$$\begin{aligned}&|\Phi'(t)|\le\|u\|_{2,\Omega}\|u_t\|_{2,\Omega}+\chi\|\nabla u\|_{2,\Omega}\|\nabla u_t\|_{2,\Omega}+\\&+\chi(\sigma-1)\left(\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right)^{\frac12}\left(\int_\Gamma k(x,t)|u|^{\sigma-2}|u_t|^2d\Gamma\right)^{\frac12}+\\&+\chi\tfrac{\sigma-1}{\sigma}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma.\end{aligned}\tag{6.3}$$

Further, note that

$$\begin{aligned}&\Phi(t)J(t)=\left(\tfrac12\|u\|_{2,\Omega}^2+\tfrac{\chi}{2}\|\nabla u\|_{2,\Omega}^2+\chi\tfrac{\sigma-1}{\sigma}\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right)\times\\&\times\left(\|u_t\|_{2,\Omega}^2+\chi\|\nabla u_t\|_{2,\Omega}^2+\chi(\sigma-1)\int_\Gamma k(x,t)|u|^{\sigma-2}|\partial_t u|^2d\Gamma\right)\ge\\&\ge\tfrac{1}{\sigma}\left(\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2+\chi(\sigma-1)\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right)\times\\&\times\left(\|u_t\|_{2,\Omega}^2+\chi\|\nabla u_t\|_{2,\Omega}^2+\chi(\sigma-1)\int_\Gamma k(x,t)|u|^{\sigma-2}|\partial_t u|^2d\Gamma\right).\end{aligned}\tag{6.4}$$

From (6.3), (6.4) and the inequality

$$(ab+cd+ef)^2\le(a^2+c^2+e^2)(b^2+d^2+f^2),$$

follows next inequality

$$\begin{aligned}&|\Phi'(t)|^2 \leq \left(\|u\|_{2,\Omega}^2 + \chi \|\nabla u\|_{2,\Omega}^2 + \chi(\sigma-1)\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right) \times \\ &\times \left(\|u_t\|_{2,\Omega}^2 + \chi \|\nabla u_t\|_{2,\Omega}^2 + \chi(\sigma-1)\int_\Gamma k(x,t)|u|^{\sigma-2}|u_t|^2 d\Gamma\right).\end{aligned}$$

That is

$$|\Phi'(t)|^2 \leq \sigma\Phi(t)J(t) \leq \sigma\Phi(t)J(t) + \chi\frac{\sigma-1}{\sigma}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma. \tag{6.5}$$

After elementary transformations (6.2) can be written in the following form

$$\begin{aligned}&J(t) = \tfrac{1}{p}\tfrac{d}{dt}\left(\int_\Omega b(x,t)|u|^p dx - a_0\int_\Omega |\nabla u|^2 dx - a_1\left(\int_\Omega |\nabla u|^2 dx\right)^q - \right.\\ &\left. - \left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,t)|u|^\sigma d\Gamma - \tfrac{\chi}{\sigma}\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma\right) - \\ &-\tfrac{a_0(p-2)}{2p}\tfrac{d}{dt}\int_\Omega |\nabla u|^2 dx - \tfrac{a_1(p-2q)}{2pq}\tfrac{d}{dt}\left(\int_\Omega |\nabla u|^2 dx\right)^q + \\ &+\tfrac{\sigma-p}{p}\int_\Gamma\left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^{\sigma-2}uu_t d\Gamma + \\ &-\tfrac{\chi(p-1)}{p}\int_\Gamma k_t(x,t)|u|^{\sigma-2}uu_t d\Gamma - \\ &-\tfrac{1}{p}\int_\Omega b_t(x,t)|u|^p dx + \\ &+\tfrac{q-1}{p}a_1\|\nabla u\|_{2,\Omega}^{2q-4}\int_\Omega \nabla u\nabla\partial_t u dx\int_\Gamma k(x,t)|u|^\sigma d\Gamma + \\ &+\tfrac{1}{p}\left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma - \\ &+\tfrac{\chi}{\sigma p}\int_\Gamma k_{tt}(x,t)|u|^\sigma d\Gamma.\end{aligned} \tag{6.6}$$

We estimate the right side of (6.6) by the following way

$$\begin{aligned}&\left|-\tfrac{a_0(p-2)}{2p}\tfrac{d}{dt}\int_\Omega |\nabla u|^2 dx\right| \leq \tfrac{a_0(p-2)}{p}\|\nabla u\|_{2,\Omega}\|\nabla u_t\|_{2,\Omega} \leq \\ &\leq \varepsilon_1\|\nabla u_t\|_{2,\Omega}^2 + \tfrac{a_0^2(p-2)^2}{4p^2\varepsilon_1}\|\nabla u\|_{2,\Omega}^2 \leq \varepsilon_1 J(t) + \tfrac{a_0^2(p-2)^2}{4p^2\varepsilon_1}\Phi(t).\end{aligned}$$

$$\begin{aligned}&\left|\tfrac{a_1(p-2q)}{2pq}\tfrac{d}{dt}\left(\int_\Omega |\nabla u|^2 dx\right)^q\right| \leq \tfrac{a_1(p-2q)}{2p}\left(\int_\Omega |\nabla u|^2 dx\right)^{q-1}\|\nabla u\|_{2,\Omega}\|\nabla u_t\|_{2,\Omega} \leq \\ &\leq \varepsilon_2\|\nabla u_t\|_{2,\Omega}^2 + \tfrac{a_1^2(p-2q)^2}{16p^2\varepsilon_2}\|\nabla u\|_{2,\Omega}^{4q-2} \leq \varepsilon_2 J(t) + \tfrac{a_1^2(p-2q)^2}{16p^2\varepsilon_2}\Phi^{2q-1}(t) \leq \\ &\leq \varepsilon_2 J(t) + \Phi^{\frac{p}{2}}(t) + C_2(\varepsilon_2),\end{aligned}$$

where $C_2(\varepsilon_2) = \left(\frac{a_1^2(p-2q)^2}{16p^2\varepsilon_2}\right)^{\frac{p}{p-4q+2}}\frac{(p-4q+2)(4q-2)^{\frac{4q-2}{p-4q+2}}}{p^{\frac{p}{p-4q+2}}}$.

$$\begin{aligned}&\left|\tfrac{\sigma-p}{p}\int_\Gamma\left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)k(x,t)|u|^{\sigma-2}uu_t d\Gamma\right| \leq \\ &\leq \tfrac{\sigma-p}{p}\left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)\left(\int_\Gamma k(x,t)|u|^{\sigma-2}|u_t|^2 d\Gamma\right)^{\frac{1}{2}}\left(\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right)^{\frac{1}{2}} \leq \\ &\leq \varepsilon_3\int_\Gamma k(x,t)|u|^{\sigma-2}|u_t|^2 d\Gamma + \tfrac{(\sigma-p)^2}{4p^2\varepsilon_3}\left(a_0 + a_1\|\nabla u\|_{2,\Omega}^{2q-2}\right)^2\int_\Gamma k(x,t)|u|^\sigma d\Gamma \leq \\ &\leq \varepsilon_3 J(t) + \tfrac{(\sigma-p)^2 a_0^2}{2p^2\varepsilon_3}\Phi(t) + \Phi^{\frac{p}{2}}(t) + C_3(\varepsilon_3),\end{aligned}$$

where $C_3(\varepsilon_3) = \left(\frac{(\sigma-p)^2 a_1^2}{2p^2\varepsilon_3}\right)^{\frac{p}{p-4q+2}}\frac{(p-4q+2)(4q-2)^{\frac{4q-2}{p-4q+2}}}{p^{\frac{p}{p-4q+2}}}$.

$$\begin{aligned}&\left|\tfrac{\chi(p-1)}{p}\int_\Gamma k_t(x,t)|u|^{\sigma-2}uu_t d\Gamma\right| \leq \tfrac{\chi(p-1)}{p}\int_\Gamma \tfrac{|k_t(x,t)|}{k(x,t)}k(x,t)|u|^{\sigma-1}|u_t| d\Gamma \leq \\ &\leq \varepsilon_4 J(t) + \tfrac{\chi^2(p-1)^2}{4p^2\varepsilon_4}K_1^2\Phi(t).\end{aligned}$$

$$\left|\int_\Omega b_t(x,t)|u|^p dx\right| \le b_1\left\|u\right\|_{p,\Omega}^p \le b_1C_1\left(\chi\left\|\nabla u\right\|_{2,\Omega}^2+\left\|u\right\|_{2,\Omega}^2\right)^{\frac{p}{2}} \le b_1C_1\Phi^{\frac{p}{2}}(t).$$

$$\begin{array}{l}\left|\frac{q-1}{p}a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-4}\int_\Omega \nabla u\nabla\partial_t u dx\int_\Gamma k(x,t)|u|^\sigma d\Gamma\right| \le \\ \le \varepsilon_5 J(t)+\frac{(q-1)^2}{4p^2\varepsilon_5}a_1^2\Phi^{2q-1}(t) \le \varepsilon_5 J(t)+\Phi^{\frac{p}{2}}(t)+C_5(\varepsilon_5),\end{array}$$

where $C_5(\varepsilon_5)=\left(\frac{(q-1)^2}{4p^2\varepsilon_5}a_1^2\right)^{\frac{p}{p-4q+2}}\frac{(p-4q+2)(4q-2)^{\frac{4q-2}{p-4q+2}}}{p^{\frac{p}{p-4q+2}}}$.

$$\begin{array}{l}\frac{1}{p}\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)\left|\int_\Gamma k_t(x,t)|u|^\sigma d\Gamma\right| \le \\ \le \frac{1}{p}\left(a_0+a_1\left\|\nabla u\right\|_{2,\Omega}^{2q-2}\right)\int_\Gamma \frac{|k_t(x,t)|}{k(x,t)}k(x,t)|u|^\sigma d\Gamma \le \\ \le \frac{K_1a_0}{p}\Phi(t)+\Phi^{\frac{p}{2}}(t)+C_6,\end{array}$$

where $C_6=\left(\frac{K_1a_1}{p}\right)^{\frac{p}{p-2q+2}}\frac{(p-2q+2)(2q-2)^{\frac{2q-2}{p-2q+2}}}{p^{\frac{p}{p-2q+2}}}$.

$$\left|\frac{\chi}{\sigma p}\int_\Gamma k_{tt}(x,t)|u|^\sigma d\Gamma\right| \le \frac{\chi}{\sigma p}K_2\int_\Gamma k(x,t)|u|^\sigma d\Gamma \le \frac{\chi}{\sigma p}K_2\Phi(t).$$

Substituting the obtained estimates in (6.6)

$$\begin{array}{l}(1-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5)J(t) \le \frac{1}{p}\Phi''(t)+ \\ +\left(\frac{a_0^2(p-2)^2}{4p^2\varepsilon_1}+\frac{(\sigma-p)^2a_0^2}{2p^2\varepsilon_3}+\frac{\chi^2(p-1)^2}{4p^2\varepsilon_4}K_1^2+\frac{K_1a_0}{p}+\frac{\chi}{\sigma p}K_2\right)\Phi(t)+ \\ +(4+b_1C_1)\Phi^{\frac{p}{2}}(t)+C_2(\varepsilon_2)+C_3(\varepsilon_3)+C_5(\varepsilon_5)+C_6.\end{array} \tag{6.7}$$

We multiply inequality (6.7) by $\Phi(t)$ and, taking into account (6.5) reduces to the ordinary differential inequality

$$\Phi''(t)\Phi(t)-\alpha|\Phi'(t)|^2+\beta\Phi^2(t)+4p\Phi^{\frac{p+2}{2}}(t)+Cp\Phi(t)\ge 0, \tag{6.8}$$

where $\alpha=\frac{p(1-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5)}{\sigma}$, $\varepsilon_i=\frac{p-\sigma}{10p}$, $i=1,2,...,5$;

$$\beta=p\left(\frac{a_0^2(p-2)^2}{4p^2\varepsilon_1}+\frac{(\sigma-p)^2a_0^2}{2p^2\varepsilon_3}+\frac{\chi^2(p-1)^2}{4p^2\varepsilon_4}K_1^2+\frac{K_1a_0}{p}+\frac{\chi}{\sigma p}K_2+K_1\chi\frac{\sigma-1}{\sigma}\right).$$

We require that the coefficients satisfy the condition $p>\sigma$. In this case ε can be chosen, so that $\alpha>1$.

We introduce a new function

$$\Psi(t)=\Phi^{1-\alpha}(t),\ \alpha>1. \tag{6.9}$$

Dividing both sides of inequality (6.8) by $\Phi^{1+\alpha}(t)\ge 0$, we obtain the inequality

$$\frac{\Phi''(t)}{\Phi^\alpha(t)}-\alpha\frac{|\Phi'(t)|^2}{\Phi^{1+\alpha}(t)}+\beta\Phi^{1-\alpha}(t)+4p\Phi^{\frac{p}{2}-\alpha}(t)+Cp\Phi^{-\alpha}(t)\ge 0,$$

which for the function $\Psi(t)$ takes the form

$$\frac{\Psi''(t)}{1-\alpha}+\beta\Psi(t)+4p\Psi^{\frac{p-2\alpha}{2(1-\alpha)}}(t)+Cp\Psi^{\frac{\alpha}{\alpha-1}}(t)\geq 0. \tag{6.10}$$

Now we make another assumption about the initial data:

$$\begin{array}{l}\Phi'(0)=-a_0\int_\Omega|\nabla u_0|^2dx-a_1\left(\int_\Omega|\nabla u_0|^2dx\right)^q-\\ -\left(a_0+a_1\|\nabla u_0\|_{2,\Omega}^{2q-2}\right)\int_\Gamma k(x,0)|u_0|^\sigma d\Gamma-\frac{\chi}{\sigma}\int_\Gamma k_t(x,0)|u_0|^\sigma d\Gamma+\\ +\int_\Omega b(x,0)|u_0|^p dx>0.\end{array} \tag{6.11}$$

Hence, there exists a time instant $t_1>0$, such that

$$\Phi'(t)\geq 0 \text{ for all } t\in[0,t_1]. \tag{6.12}$$

From relation (6.9) we have the expression

$$\Psi'(t)=(1-\alpha)\Phi^{-\alpha}(t)\Phi'(t), \tag{6.13}$$

from which, by virtue of (6.12) taking into account $\alpha>1$ we obtain

$$\Psi'(t)\leq 0 \text{ for all } t\in[0,t_1]. \tag{6.14}$$

Then, multiplying (6.10) by $\Psi'(t)$, we obtain the following inequality

$$\tfrac{\Psi''(t)\Psi'(t)}{1-\alpha}+\beta\Psi(t)\Psi'(t)+4p\Psi^{\frac{p-2\alpha}{2(1-\alpha)}}(t)\Psi'(t)+Cp\Psi^{\frac{\alpha}{\alpha-1}}(t)\Psi'(t)\leq 0 \text{ for all } t\in[0,t_1],$$

from which we get

$$\Psi''(t)\Psi'(t)\geq\beta(\alpha-1)\Psi(t)\Psi'(t)+4p(\alpha-1)\Psi^{\frac{p-2\alpha}{2(1-\alpha)}}(t)\Psi'(t)+Cp(\alpha-1)\Psi^{\frac{\alpha}{\alpha-1}}(t)\Psi'(t) \text{ for all } t\in[0,t_1].$$

Hence it follows that

$$\frac{1}{2}\frac{d}{dt}|\Psi'(t)|^2\geq\frac{\beta(\alpha-1)}{2}\frac{d}{dt}|\Psi(t)|^2+\frac{8p(\alpha-1)^2}{4\alpha-p-2}\frac{d}{dt}\Psi^{\frac{4\alpha-p-2}{2(\alpha-1)}}(t)+\frac{Cp(\alpha-1)^2}{2\alpha-1}\frac{d}{dt}\Psi^{\frac{2\alpha-1}{\alpha-1}}(t).$$

Integrating the last inequality, we have

$$\begin{array}{l}|\Psi'(t)|^2\geq A^2+\beta(\alpha-1)|\Psi(t)|^2+\frac{16p(\alpha-1)^2}{4\alpha-p-2}\Psi^{\frac{4\alpha-p-2}{2(\alpha-1)}}(t)+\\ +\frac{2Cp(\alpha-1)^2}{2\alpha-1}\Psi^{\frac{2\alpha-1}{\alpha-1}}(t)\geq A^2,\end{array} \tag{6.15}$$

where

$$A^2=|\Psi'(0)|^2-\beta(\alpha-1)|\Psi(0)|^2-\frac{16p(\alpha-1)^2}{4\alpha-p-2}\Psi^{\frac{4\alpha-p-2}{2(\alpha-1)}}(0)-\frac{2Cp(\alpha-1)^2}{2\alpha-1}\Psi^{\frac{2\alpha-1}{\alpha-1}}(0).$$

We require the fulfillment of one more condition on the initial data

$$\begin{array}{l}A^2=(\alpha-1)^2|\Phi'(0)|^2\Phi^{-2\alpha}(0)-\\ -\beta(\alpha-1)\Phi^{2-2\alpha}(0)-\frac{16p(\alpha-1)^2}{4\alpha-p-2}\Phi^{\frac{p+2-4\alpha}{2}}(0)-\frac{2Cp(\alpha-1)^2}{2\alpha-1}\Phi^{1-2\alpha}(0)>0.\end{array}$$

$$|\Phi'(0)|^2>\frac{\beta\Phi^2(0)}{\alpha-1}+\frac{16p}{4\alpha-p-2}\Phi^{\frac{p+2}{2}}(0)+\frac{2Cp}{2\alpha-1}\Phi(0). \tag{6.16}$$

Then from inequality (6.15), we obtain

$$|\Psi'(t)| \geq A > 0 \text{ for all } t \in [0, t_1]. \tag{6.17}$$

Hence it follows that

$-\Psi'(t) \geq A > 0 \quad \Rightarrow \quad \Psi'(t) \leq -A < 0$ for all $t \in [0, t_1]$.

From expression (6.13)

$$\Psi'(t) = (1-\alpha)\Phi^{-\alpha}(t)\Phi'(t) \leq -A \quad \Rightarrow \quad \Phi'(t) \geq \frac{A}{\alpha-1}\Phi^{\alpha}(t) > 0,$$

whence it follows that, under the initial condition (6.11) the value $\Phi'(t)$ remains greater than zero for the entire interval of existence of a solution to the original problem. Thus, from (6.17) it follows that

$$\Psi'(t) \leq -A < 0 \text{ for all } t \in [0, t_1].$$

Integrating the last inequality, we obtain

$$\Psi(t) \leq \Psi(0) - At,$$

$$\Phi^{1-\alpha}(t) \leq \Phi^{1-\alpha}(0) - At \quad \Rightarrow \quad \Phi(t) \geq \frac{1}{[\Phi^{1-\alpha}(0) - At]^{\frac{1}{\alpha-1}}}.$$

Hence, in a finite time $T_0 \in (0, T_1]$, where

$$T_1 = \frac{\Phi^{1-\alpha}(0)}{A} > 0,$$

The function $\Psi(t)$ becomes zero. Which means

$$\lim_{t \to T_0} \Phi(t) = +\infty.$$

□

7. Exponential decay of the solution in time

In this section we consider the quasilinear equation

$$\frac{\partial}{\partial t}(u - \chi\Delta u) - \left(a_0 + a_1 \|\nabla u\|_{2,\Omega}^{2q-2}\right)\Delta u + u + |u|^{p-2}u = f(x,t), \tag{7.1}$$

in the cylinder $Q_T = \{(x,t): \; x \in \Omega, \; \Omega \subset R^n, \; 0 < t < T\}$ with nonlinear boundary

$$\left.\frac{\partial u}{\partial n} + |u|^{\sigma-2}u\right|_{\Gamma} = 0, \quad \Gamma = \partial\Omega \times (0,T), \tag{7.2}$$

and with the initial condition

$$u(x,0) = u_0(x). \tag{7.3}$$

Here $\Omega \subset R^n$, $n \geq 3$ a bounded domain, with a sufficiently smooth boundary $\partial\Omega$, a_0, a_1 and σ are positive constants. We are interested in the asymptotic behavior of the solution when time tends to infinity.

We multiply equation (7.1) by the function $u(x,t)$ and integrate over the domain Ω, we obtain

$$\begin{aligned}&\rho'(t) + a_0 \int_\Omega |\nabla u|^2 dx + a_1 \left(\int_\Omega |\nabla u|^2 dx\right)^q + \\ &+ \left(a_0 + a_1 \|\nabla u\|_{2,\Omega}^{2q-2}\right) \int_\Gamma |u|^\sigma d\Gamma + \int_\Omega |u|^2 dx + \int_\Omega |u|^p dx = \int_\Omega f u dx.\end{aligned} \tag{7.4}$$

where

$$\rho(t) = \frac{1}{2}\left(\int_\Omega [|u|^2 + \chi|\nabla u|^2] dx + 2\chi \frac{\sigma-1}{\sigma} \int_\Gamma |u|^\sigma d\Gamma\right).$$

We estimate the right-hand side (7.4)

$$\left|\int_\Omega f u dx\right| \le \tfrac{1}{2}\|u\|_{2,\Omega}^2 + \tfrac{1}{2}\int_\Omega |f|^2 dx.$$

Substituting the estimates obtained in (7.4), we obtain

$$\rho'(t) + Q \le \frac{1}{2}\int_\Omega |f|^2 dx \tag{7.5}$$

where

$$Q = a_0 \int_\Omega |\nabla u|^2 dx + a_1 \left(\int_\Omega |\nabla u|^2 dx\right)^q + \left(a_0 + a_1 \|\nabla u\|_{2,\Omega}^{2q-2}\right) \int_\Gamma |u|^\sigma d\Gamma + \frac{1}{2}\int_\Omega |u|^2 dx + \int_\Omega |u|^p dx.$$

It easy verify that

$$\alpha\rho \le Q, \;\; \alpha = min(a_0, \frac{a_0}{\chi}, \frac{a_0\sigma}{\chi(\sigma-1)}). \tag{7.6}$$

Relation (7.5) and (7.6) lead us to the Cauchy problem for ordinary differential inequality

$$\frac{d}{dt}\rho(t) + \alpha\rho(t) \le \frac{1}{2}\|f(t)\|_{2,\Omega}^2, \;\; \rho(0) = \rho_0. \tag{7.7}$$

Let us assume that

$$\frac{1}{2}\|f(t)\|_{2,\Omega}^2 \le C_f exp(-\mu t), \;\; \mu \ge \alpha, \;\; C_f = const. \tag{7.8}$$

An analysis of solutions to problem (7.7) allows us to make the following assertion.

Theorem 5. *Let conditions (7.8) hold and $u \in W_2^1(\Omega) \bigcap L_\sigma(\Gamma)$. Then the solution to problem (7.1)-(7.3) satisfies estimates*

$$\rho(t) \le \exp(-\alpha t)\left(\rho(0) + \frac{C_f}{\mu - \alpha}\right), \quad \mu > \alpha,$$

$$\rho(t) \le \exp(-\alpha t)\left(\rho(0) + tC_f\right), \quad \mu = \alpha.$$

8. Conclusion

In conclusion, we note the following. In equation (1.1), the Laplace operator can be replaced by a more general second-order elliptic operator $A \equiv div(a(x,t)\nabla u) + b(x,t)u$. The boundary condition (1.2) can be replaced by a non-local condition of the form

$$\frac{\partial u}{\partial n} + \int_0^t k(x,\tau)|u|^{\sigma-2} u d\tau\Big|_\Gamma = 0, \quad \Gamma = \partial\Omega \times (0,T).$$

The technique used in this paper can be applied to many other equations of hyperbolic and pseudohyperbolic types.

Acknowledgments. The first author was supported by the Lavrenty'ev Institute of Hydrodynamics of the Siberian Branch RAS (project no. III.22.4.2, Analysis of mathematical models of continua with singularities, discontinuities and intrinsic inhomogeneities), Novosibirsk, Russia and also partially supported by the Portuguese Foundation for Science and Technology, Portugal, under the project: UIDB/04561/2020. The work of the second and third authors was financially supported by the Ministry of Education and Science of the Republic of Kazakhstan (grant No, AP13068505).

Conflicts of interests. This work does not have any conflicts of interests.

References

[1] S.L. Sobolev, *On a new problem of mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat., 18 (1954), no 1, p. 3–50. (in Russian) 1

[2] S.L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, M.:Nauka, 1988. (in Russian) 3.3

[3] G.I. Barenblatt , Yu.P. Zheltov, I.N. Kochina, *On the basic concepts of the theory of filtration in fractured media*, Appl. math. and mechan. 24 (1960), no. 5. P. 58-73. (in Russian) 1

[4] T. W. Ting, *Parabolic and pseudoparabolic partial differential equations*, J. Math. Soc. Japan. 14 (1969), p. 1-26. 1

[5] T.B. Benjamin, *Lectures on nonlinear wave motion* , Ltd. Appl. Math. Vol. Amer. Math. Soc: Providence; RL, 15 (1974), p. 3 - 7 . 1

[6] T.B. Benjamin, J.L. Bona, J.J. Mahony, *Model equations for long waves in nonlinear dispersive systems* , Pliilos. Trans. Roy. Soc. London A, 272 (1972), no. 1220, p. 47 - 78 . 1

[7] J.P. Albert, J.L. Bona, J.-C. Saut, *Model equations for waves in stratified fluids* , Proc. Roy. Soc. London A, 453 (1997), no. 1961, p. 1233-1260. 1

[8] R. E. Showalter, *Existence and representation theorems for a semilinear Sobolev equation in Banach space* , SIAM J. Math. Anal., 3(1972), no. 3, p. 527-543.

[9] R.E. Showalter *The Sobolev type equations I* , Appl. Anal. 5 (1975), no. 1, p. 15-22.

[10] R.E. Showalter *The Sobolev type equations* , Appl. Anal. 5 (1975), no. 2, p. 81-99.

[11] R.E. Showalter, T. W. Ting, *Pseudoparabolic partial differential equations* , SIAM J. Math. Anal. 1 (1970), no. 1, p. 1–26.

[12] U. Stefanelli, *On a class of doubly nonlinear nonlocal evolution equations* , Differ. Integr. Equations, 15 (2002), no. 8, p. 897–922.

[13] G. A. Sviridyuk, *Variety of solutions of one singular parabolic equation* , Sov. Math. Dokl. 289 (1986), no. 6, p. 1315–1318.

[14] S. I. Pohozhaev, *On an approach to nonlinear equations*, Sov. Math. Dokl., 20 (1979), 912–916.

[15] A. P. Oskolkov, *Initial-boundary value problems for equations of motion of Kelvin–Voight fluids and Oldroyd fluids*, Proc. Steklov Inst. Math., 179 (1989), p. 137–182. 1

[16] A. P. Oskolkov, *To the stability theory for the solutions of the semilinear dissipative Sobolev type equations*, J. Math. Sci., 77 (1995), no.3, p. 3225–3231.

[17] A. P. Oskolkov, *Nonlocal problems for some class nonlinear operator equations arising in the theory Sobolev type equations*, J. Soviet Math., 64 (1993), no. 1, p. 724–736. 1

[18] S.L. Gabov, A.G. Sveshnikov, *Linear problems of the theory of non-stationary internal waves*, M.: Nauka, 1990. (in Russian)

[19] M. O. Korpusov, A. G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudoparabolic type in problems of mathematical physics*, Comput. Math. Math. Phys., 43:12 (2003), p. 1765–1797. 1.1

[20] M. O. Korpusov, A. G. Sveshnikov, *Three-dimensional nonlinear evolutionary pseudoparabolic equations in mathematical physics II*, Comput. Math. Math. Phys., 44:11 (2004), p. 1942–1948. 1, 1.1
[21] M. O. Korpusov, A. G. Sveshnikov, *On blowup of a solution to a Sobolev-type equation with a nonlocal source*, Siberian Math. J. 46 (2005), no. 3. p.443–452. 1
[22] M. O. Korpusov, A. G. Sveshnikov, *Finite-time relaxation of the solution of a nonlinear pseudoparabolic equation*, Comput. Math. Math. Phys. 51 (2011), no.3, p.377–403. 1
[23] I. A. Shishmarev, *On a nonlinear Sobolev type equation* , Differential Equations, 41 (2005), p. 146–149.
[24] M. O. Korpusov, A. G. Sveshnikov, *On the solvability of strongly nonlinear pseudoparabolic equation with double nonlinearity*, Comput. Math. Math. Phys., 43:7 (2003), p. 944–961. 1
[25] A. B. Alshin, M. O. Korpusov, A. G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations, Series in nonlinear analisys and applications*, 15, De Gruyter, 2011, 648 p. 1.1, 1.1
[26] P.A. Makarov, *Blow-Up of the Solution of the Initial Boundary-Value Problem for the Generalized Boussinesq Equation with Nonlinear Boundary Condition*, Math. Notes, 92:4 (2012), p. 519–531. 1.1
[27] A.S. Berdyshev, S.E. Aitzhanov, G.O. Zhumagul, *Solvability of Pseudoparabolic Equations With Non-Linear Boundary Condition*, Lobachevskii J. Math., 41:9 (2020), p. 1772—1783.
[28] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations*, Berlin: Degruyter, 1995, 560 p.
[29] O.A. Ladyzhenskaia, V.A. Solonnikov, N.N. Uraltseva, *Linear and Quasi-linear Equations of Parabolic Type Translations of Mathematical Monographs*, 23. Providence, RI: American Mathematical Society, 1968. 1
[30] S. N. Antontsev, J. I. Diaz, and S. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, 2002. 1
[31] J.-L. Lions, *Some methods for solving non-linear boundary value problems*, M.: Nauka, 1972, 588 p. (in Russian) 4
[32] S.N. Antontsev, E. Ozturk, *Well-posedness and long time behavior for p- Laplacian equation with nonlinear boundary conditions*, J. Math.Anal.Appl., 472 (2019), p. 1604-1630.
[33] S. N. Antontsev and K. Khompysh, *Kelvin-Voight equation with p-Laplacian and damping term: Existence, uniqueness and blow-up*, J. Math. Anal. Appl., 446 (2017), no. 2, 1255–1273. 1
[34] S. N. Antontsev and K. Khompysh, *Generalized Kelvin-Voigt equations with p-Laplacian and source/absorption terms*, J. Math. Anal. Appl., 456 (2017), no. 1, 99–116.
[35] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, *Existence and large time behavior for generalized Kelvin-Voigt equations governing nonhomogeneous and incompressible fluids.* 2019 J. Phys.: Conf. Ser. 1268 012008.
[36] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, *Kelvin -Voigt equationa with anisotropic diffusion, relaxation and damping:Blow-up and large time behavior*, Asymptotic Analysis, 121 (2021), no. 2, p. 125-157.
[37] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, *Generalized Kelvin-Voigt equations for nonhomogeneous and incompressible fluids*, Communications in Mathematical Sciences, 17 (2019), no. 7, p.1915-1948.
[38] S.N. Antontsev, H.B. de Oliveira, Kh. Khompysh, *Kelvin-Voigt equations perturbed by anisotropic relaxation, diffusion and damping*, J. Math.Anal.Appl. 473 (2019), p. 1122-1154. 1
[39] B.P. Demidovich, *Lectures on the mathematical theory of stability*, M: Nauka, 1967. (in Russian) 3.1
[40] J.-L. Lions, *Inhomogeneous boundary value problems and their applications*, M: Mir, 1971. 372 p. (in Russian) 4