POSITIVE PERIODIC SOLUTIONS OF A SINGULAR MINKOWSKI-CURVATURE EQUATION

ZAITAO LIANG[1], SHENGJUN LI[2*]

Abstract. In this paper, we study the existence of positive periodic solutions of a singular Minkowski-curvature equation. The proof is based on the Mawhin's continuation theorem. Moreover, an example and the corresponding numerical simulations are given to illustrate our theoretical analysis. Some results in the literature are generalized and improved.

1. Introduction

Singular differential equations arise from different applied sciences, for example, in the study of the motion of particles under the influence of gravitational or electrostatic forces [12, 16, 32]. Nowadays, there is a wide range of nonlinear models involving singular terms. For instance, singular equations can be used to model the interaction between atomic particles [29, 30] in molecular dynamics, and in [16, 21], the singular term models the restoring force caused by a compressed perfect gas. During the past few decades, the existence of positive periodic solutions of different kinds of singular equations have been studied by many researchers. We just refer the reader to classical papers [8, 9, 10, 12, 13, 14, 15, 18, 22, 23, 25, 26, 31, 32, 34, 35] and the references therein.

In order to present our main results, we first mention the following two related results. In [34], Zhang studied the existence of positive periodic

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solutions of the following singular Liénard equation

$$u'' + f(u)u' + g(t, u) = 0, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $g : [0, T] \times (0, \infty) \to \mathbb{R}$ is an L^1-Carathéodory function, $g(t, u)$ is T-periodic with respect to its first variable and

$$\lim_{u \to 0^+} g(t, u) = -\infty \quad \text{uniformly in } t.$$

Assume that

$$\varphi(t) = \limsup_{u \to +\infty} \frac{g(t, u)}{u}$$

exists uniformly a.e. $t \in [0, T]$ and $\varphi \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, which means that for any $\varepsilon > 0$, there exists $g_\varepsilon \in L^1(0, T)$ such that

$$g(t, u) \leq (\varphi(t) + \varepsilon)u + g_\varepsilon(t), \tag{1.2}$$

for all $u > 0$ and a.e. $t \in [0, T]$. The following result was proved in [34] by using the coincidence degree theory [28].

Theorem 1.1. *Assume that the following conditions are satisfied:*

(H$_1$) *There exist constants M_1, M_2 with $0 < M_1 < M_2$ and $\tau \in [0, T]$ such that*

$$M_1 \leq u(\tau) \leq M_2,$$

if u is a positive T-periodic solution satisfying

$$\int_0^T g(t, u(t))dt = 0;$$

(H$_2$) *$\bar{g}(u) < 0$ for every $u \in (0, M_1)$ and $\bar{g}(u) > 0$ for every $u \in (M_2, +\infty)$, where $\bar{g}(u) = \frac{1}{T}\int_0^T g(t, u)\mathrm{d}t$;*

(H$_3$) *$g(t, u) = g_0(u) + g_1(t, u)$, where $g_0 \in C((0, +\infty), \mathbb{R})$ and $g_1 : [0, T] \times [0, +\infty) \to R$ is an L^1-Carathéodory function in the following sense*

(a$_1$) *$t \to g_1(t, u)$ is measurable for each $u > 0$,*

(a$_2$) *$u \to g_1(t, u)$ is continuous for a.e. $t \in [0, T]$,* (a$_3$) *for any* $b > 0$*, there exist* $h_b \in L^1((0,T);[0,\infty))$ *such that*

$$|g_1(t,u)| \le h_b(t),$$

for a.e. $t \in [0,T]$ *and all* $u \in [0,b]$*;*

(H$_4$) $\int_0^1 g_0(u)du = -\infty$*;*

(H$_5$) $\|\varphi^+\|_1 < \frac{\sqrt{3}}{T}$, $(\varphi^+(t) = \max(\varphi(t),0))$.

Then equation (1.1) *has at least one positive* T*-periodic solution.*

In [33], Wang generalized the above result to the following equation

$$u''(t) + f(u(t))u'(t) + g(t,u(t-\sigma)) = 0, \tag{1.3}$$

where f and g satisfy the same conditions as in (1.1) with g is an L^2-Carathéodory function, and $0 \le \sigma \le T$ is a constant. By using the Mawhin's continuation theorem [17], Wang obtained the following result.

Theorem 1.2. *Assume that assumptions* (H$_1$)*,* (H$_2$) *and* (H$_4$) *are satisfied together with the following conditions:*

(H$_3'$) $g(t,u) = g_0(u) + g_1(t,u)$*, where* $g_0 \in C((0,+\infty),\mathbb{R})$ *and* $g_1 : [0,T] \times [0,+\infty) \to \mathbb{R}$ *is an* L^2*-Carathéodory function in the following sense*

(a$_1$) $t \to g_1(t,u)$ *is measurable for each* $u > 0$*,*

(a$_2$) $u \to g_1(t,u)$ *is continuous for a.e.* $t \in [0,T]$*,*

(a$_3'$) *for any* $b > 0$*, there exist* $h_b \in L^2((0,T);[0,\infty))$ *such that*

$$|g_1(t,u)| \le h_b(t),$$

for a.e. $t \in [0,T]$ *and all* $u \in [0,b]$*;*

(H$_5'$) $\|\varphi\|_\infty < (\frac{\pi}{T})^2$.

Then equation (1.3) *has at least one positive* T*-periodic solution.*

The aim of this paper is to generalize and improve the results in [33, 34] to the following singular Minkowski-curvature equation

$$\left(\frac{u'}{\sqrt{1-(u')^2}}\right)' + f(u)u' + g(t,u) = e(t), \tag{1.4}$$

where f and g satisfy the same conditions as in (1.1), $e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with zero mean value. The above equation is driven by a strongly nonlinear differential operator of ψ-Laplacian type

$$u \mapsto -(\psi(u'))', \text{ where } \psi(s) := \frac{s}{\sqrt{1-s^2}}.$$

As is well known, this is the one dimensional version of the partial differential operator

$$u \mapsto -\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big),$$

which in turn is usually meant as a mean-curvature operator in Lorentz-Minkowski spaces [3, 19]. Recently, there has been a significant interest in the study of the existence and multiplicity issues of the associated boundary value problems. See for instance [1, 2, 4, 5, 6, 11, 20, 24, 27] and the references therein. However, the study on the periodic problem of the Minkowski-curvature equation is considerably fewer [7]. In [7], by using the Mawhin's coincidence degree theory and the Poincaré-Birkhoff fixed point theorem, Boscaggin and Feltrin studied the existence, non-existence, multiplicity of positive periodic solutions, both harmonic and subharmonic to an indefinite Minkowski-curvature equation. To the best of our knowledge, the problem on the existence of periodic solutions of the singular Minkowski-curvature equation has not attracted much attention in the literature. The results of this paper will fill, at least partially, this gap.

Based on the Mawhin's continuation theorem [17], we obtain the following result.

Theorem 1.3. *Assume that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *hold. Then equation* (1.4) *has at least one positive* T*-periodic solution.*

Compared with the results in [33, 34], the novelties of our result are as follows. First, we dealt with a more meaningful and complex singular equation. Secondly, the conditions of our result are weakened with compare to Theorem 1.1 and Theorem 1.2, because we do not need the assumptions

(H_5) and (H_5'). Moreover, the existence results of the singular Minkowski-curvature equation presented in this paper are the first ones available in the literature.

The rest of this paper is organized as follows. In Section 2, we state some preliminary results. The proof of Theorem 1.3 will be presented in Section 3. In Section 4, an example and the corresponding numerical simulations (phase portrait and time series portrait of the positive periodic solution of the example) are given to illustrate our theoretical analysis.

2. Preliminaries

Let U and V be two real Banach spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively. A linear operator

$$L : D(L) \subset U \to V$$

is called a Fredholm operator of index zero if

(i) $\mathrm{Im}L$ is a closed subset of V,

(ii) $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < \infty$.

A continuous operator $N : \Omega \subset U \to V$ is said to be L-compact in $\bar{\Omega}$ if

(iii) $K_P(I - Q)N(\bar{\Omega})$ is a relative compact set of U,

(iv) $QN(\bar{\Omega})$ is a bounded set of V,

where $P : U \to \mathrm{Ker}L$, $Q : V \to V$ are continuous linear projectors satisfying

$$\mathrm{Im}P = \mathrm{Ker}L, \quad \mathrm{Ker}Q = \mathrm{Im}L,$$

and

$$K_P = L\big|^{-1}_{\mathrm{Ker}P \cap D(L)}.$$

Then we have the decompositions

$$U = \mathrm{Ker}L \oplus \mathrm{Ker}P, \quad V = \mathrm{Im}L \oplus \mathrm{Im}P.$$

Lemma 2.1. [17] *Let U, V be as above, Ω be an open and bounded set of U, $L : D(L) \subset U \to V$ be a Fredholm operator of index zero and the continuous* *operator* $N : \bar{\Omega} \subset U \to V$ *be* L*-compact on* $\bar{\Omega}$*. In addition, if the following conditions hold:*

(h_1) $Lv \neq \lambda Nv$, $\forall (v, \lambda) \in \partial\Omega \times (0, 1)$*;*

(h_2) $QNv \neq 0$, $\forall v \in \mathrm{Ker}L \cap \partial\Omega$*;*

(h_3) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$,

where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is any homeomorphism and* P, Q, K_P *are given above. Then* $Lv = Nv$ *has at least one solution in* $\bar{\Omega}$*.*

3. Proof of Theorem 1.3

Throughout this paper, we set

$$\bar{g}(u) = \frac{1}{T} \int_0^T g(t, u)\mathrm{d}t.$$

The equation (1.4) is equivalent to the following system

$$\begin{cases} u'(t) = \phi(v(t)), \\ v'(t) = -f(u(t))\phi(v(t)) - g(t, u(t)) + e(t), \end{cases} \tag{3.1}$$

where $\phi(v) = \frac{v}{\sqrt{1+v^2}}$.

Let

$$U = V = \{\omega = (u(t), v(t))^\top \in C(\mathbb{R}, \mathbb{R}^2),\ \omega(t) = \omega(t+T)\}$$

with the norm $\|\omega\| = \max\{\|u\|_\infty,\ \|v\|_\infty\}$, where

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)| \text{ and } \|v\|_\infty = \max_{t\in[0,T]} |v(t)|.$$

Obviously, U and V are Banach space. Let us define the operator

$$L : U \supset D(L) \to V,\ L\omega = \omega' = (u'(t), v'(t))^\top,$$

where

$$D(L) = \{\omega | \omega(t) = (u(t), v(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2), \omega(t) = \omega(t+T)\}.$$

One may easily see that $\mathrm{Ker}L = \mathbb{R}^2$ and

$$\mathrm{Im}L = \{u \in V, \int_0^T u(s)\mathrm{d}s = 0\}.$$

Therefore L is a Fredholm operator of index zero.

Let

$$X = \{\omega | \omega = (u(t), v(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^+ \times \mathbb{R}), \omega(t) = \omega(t+T)\}$$

and define a nonlinear operator $N : X \supset \bar{\Omega} \to V$ by

$$N\omega = (\phi(v), -f(u)\phi(v) - g(t,u) + e(t))^\top ,$$

where Ω is an open and bounded set with $\bar{\Omega} \subset X \subset U$. Then the system (3.1) can be written as

$$L\omega = N\omega \quad in\ \bar{\Omega}.$$

Let

$$P : U \to \mathrm{Ker}L, \quad P(u) = \frac{1}{T}\int_0^T u(s)\mathrm{d}s,$$
$$Q : V \to \mathrm{Im}Q, \quad Q(u) = \frac{1}{T}\int_0^T u(s)\mathrm{d}s.$$

Set

$$K_P = L|^{-1}_{\mathrm{Ker}P \cap D(L)},$$

we have

$$[K_P(u)](t) = \int_0^T G(t,s)u(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{s-T}{T}, & 0 \le t \le s, \\ \frac{s}{T}, & s \le t \le T. \end{cases}$$

For all Ω satisfying $\bar{\Omega} \subset (U \cap X) \subset U$, we get that $K_P(I-Q)N(\bar{\Omega})$ is a relative compact set of U, $QN(\bar{\Omega})$ is a bounded set of V. Thus N is L-compact in $\bar{\Omega}$.

Proof of Theorem 1.3. To apply Lemma 2.1, we consider the system

$$\begin{cases} u' = \lambda\phi(v(t)), \\ v' = -\lambda f(u(t))\phi(v(t)) - \lambda g(t, u(t)) + \lambda e(t), \end{cases} \tag{3.2}$$

where $\lambda \in (0,1)$. The most important work is to find an appropriate Ω such that all assumptions of Lemma 2.1 are satisfied.

Firstly, we claim that there exist positive constants R_0, R_1 and R_2 such that all possible positive T-periodic solutions of the system (3.2) satisfy

$$R_0 \leq u(t) \leq R_1, \quad | v(t) | \leq R_2, \quad \forall t \in [0, T].$$

Suppose that $(u(t), v(t))^\top$ is an arbitrary positive T-periodic solution of the system (3.2), then by the first equation of the system (3.2), we know that

$$|u'(t)| < 1, \text{ for all } t \in \mathbb{R}.$$

Integrating the second equation of the system (3.2) from 0 to T, we obtain

$$\begin{aligned}\int_0^T \lambda g(t, u(t))dt &= \int_0^T \big(- v'(t) - \lambda f(u(t))\phi(v(t)) + \lambda e(t)\big)dt \\ &= -\int_0^T f(u(t))u'(t)dt = 0.\end{aligned} \tag{3.3}$$

By (H_1) and (3.3), we know that there exist positive constants M_1, M_2 and $\tau \in [0, T]$ such that

$$M_1 \leq u(\tau) \leq M_2.$$

Therefore, for every $t \in [0, T]$, we have

$$\begin{aligned}|u(t)| &= \left| u(\tau) + \int_\tau^t u'(s)\mathrm{d}s \right| \\ &\leq |u(\tau)| + \int_0^T |u'(s)|\mathrm{d}s \\ &< |u(\tau)| + T \\ &\leq M_2 + T := R_1,\end{aligned} \tag{3.4}$$

Set

$$I_+ = \{t \in [0, T], g(t, u(t)) \geq 0\}.$$

By (1.2) and (3.3), we have

$$\begin{aligned}\int_0^T |g(t,u(t))|\mathrm{d}t &= 2\int_{I_+} g(t,u(t))\mathrm{d}t \\ &\le 2\int_{I_+}(\varphi(t)+\varepsilon)u(t)\mathrm{d}t + 2\int_{I_+} g_\varepsilon(t)\mathrm{d}t \\ &\le 2(\|\varphi\|_\infty+\varepsilon)\int_0^T |u(t)|\mathrm{d}t + 2\int_0^T |g_\varepsilon(t)|\mathrm{d}t \\ &\le 2TR_1(\|\varphi\|_\infty+\varepsilon) + 2\|g_\varepsilon\|_1.\end{aligned}$$

Since u is a T-periodic function, there exists $t_0 \in [0,T]$ such that $u'(t_0)=0$. It follows from the first equation of the system (3.2) that $v(t_0)=0$. Then we have

$$\begin{aligned}|v(t)| &= \left| v(t_0) + \int_{t_0}^t v'(s)\mathrm{d}s \right| \\ &\le \int_0^T |v'(s)|\mathrm{d}s \\ &\le \lambda\int_0^T |f(u(s))||\phi(v)|\mathrm{d}s + \lambda\int_0^T |g(s,u(s))|\mathrm{d}s + \lambda\int_0^T |e(s)|\mathrm{d}s \\ &\le \rho T + 2TR_1(\|\varphi\|_\infty+\varepsilon) + 2\|g_\varepsilon\|_1 + T\overline{|e|} := R_2, \end{aligned} \tag{3.5}$$

where $\rho = \max_{|u|\le R_1} |f(u)|$. Thus, by the first equation of the system (3.1), we have

$$|u'(t)| \le |\phi(v)| \le \frac{|v(t)|}{\sqrt{1+v^2(t)}} \le \frac{R_2}{\sqrt{1+R_2^2}} := R_3, \quad \forall t\in[0,T], \tag{3.6}$$

Obviously, $R_3 < 1$.

Multiplying both sides of the second equation of the system (3.2) by $u'(t)$ and integrating from τ to t, we obtain

$$\begin{aligned}\int_\tau^t v'(s)u'(s)\mathrm{d}s =& -\lambda\int_\tau^t f(u(s))\phi(v)u'(s)\mathrm{d}s - \lambda\int_\tau^t g(s,u(s))u'(s)\mathrm{d}s \\ &+ \lambda\int_\tau^t e(s)u'(s)\mathrm{d}s \\ =& -\lambda\int_\tau^t f(u(s))\phi(v)u'(s)\mathrm{d}s - \lambda\int_\tau^t g_0(u(s))u'(s)\mathrm{d}s \\ &-\lambda\int_\tau^t g_1(s,u(s))u'(s)\mathrm{d}s + \lambda\int_\tau^t e(s)u'(s)\mathrm{d}s, \end{aligned} \tag{3.7}$$

By (3.6) and the condition (a_3), we have

$$\begin{aligned}\Big|\int_\tau^t g_1(s,u(s))u'(s)\mathrm{d}s\Big| &\le \int_0^T |g_1(s,u(s))||u'(s)|\mathrm{d}s\\ &\le \int_0^T h_{R_1}(s)|u'(s)|\mathrm{d}s\\ &\le R_3\|h_{R_1}\|_1\end{aligned}$$

and

$$\begin{aligned}\Big|\int_\tau^t v'(s)u'(s)ds\Big| &= \Big|\int_\tau^t \frac{\lambda v}{\sqrt{1+v^2}}v'(s)\mathrm{d}s\Big|\\ &= \big|\lambda(\sqrt{1+v^2(t)}-\sqrt{1+v^2(\tau)})\big|\\ &\le \lambda\sqrt{1+R_2^2}.\end{aligned}$$

It follows from (3.7) and the above two inequalities that

$$\begin{aligned}\lambda\Big|\int_{u(\tau)}^{u(t)} g_0(u)\mathrm{d}u\Big| =&\Big|-\int_\tau^t v'(s)u'(s)\mathrm{d}s-\lambda\int_\tau^t f(u(s))\phi(v)u'(s)\mathrm{d}s\\ &-\lambda\int_\tau^t g_1(s,u(s))u'(s)\mathrm{d}s+\lambda\int_\tau^t e(s)u'(s)\mathrm{d}s\Big|\\ \le&\Big|\int_\tau^t v'(s)u'(s)\mathrm{d}s\Big|+\lambda\int_0^T |f(u(s))||u'(s)|\mathrm{d}s\\ &+\lambda\int_\tau^t |g_1(s,u(s))||u'(s)|\mathrm{d}s+\lambda\int_0^T |e(s)||u'(s)|\mathrm{d}s\\ \le&\lambda\sqrt{1+R_2^2}+\lambda R_3\rho T+\lambda R_3\|h_{R_1}\|_1+\lambda R_3T\overline{|e|},\end{aligned}$$

which implies that

$$\Big|\int_{u(\tau)}^{u(t)} g_0(u)du\Big| \le 2\sqrt{1+R_2^2}+R_3\rho T+TR_3|\bar{e}|+R_3\|h_{R_1}\|_1<+\infty.$$

Since $u(\tau)>M_1$, we can deduce from (H_4) and the above inequality that there exists a constant $R_4>0$ such that

$$u(t)\ge R_4,\quad \forall t\in[\tau,T].$$

Analogously, we can prove that there exists a constant $R_5>0$ such that

$$u(t)\ge R_5,\quad \forall t\in[0,\tau].$$

Set $R_0 = \min\{R_4, R_5\}$, by the above two inequalities, we have

$$u(t) \geq R_0, \quad \forall t \in [0, T]. \tag{3.8}$$

Now, we define the set

$$\Omega = \{\omega : \omega(t) = (u(t), v(t))^\top \in X, r_0 < u(t) < r_1, |v(t)| < r_2, t \in [0, T]\},$$

where r_0, r_1 and r_2 are constants which are independent of λ and satisfy

$$0 < r_0 < \min\{R_0, M_1\}, \quad r_1 > \max\{R_1, M_2\} \quad \text{and} \quad r_2 > R_2.$$

By (3.4), (3.5) and (3.8), we konw that the condition (h_1) of Lemma 2.1 is satisfied.

Next, we consider the condition (h_2). In fact, if $\omega \in \partial\Omega \cap \mathrm{Ker}L$, then

$$\omega = (r_0, \pm r_2)^\top \text{ or } (r_1, \pm r_2)^\top,$$

in this case , we get

$$QN\omega = (\phi(\pm r_2), -\bar{g}(r_0))^\top,$$

or

$$QN\omega = (\phi(\pm r_2, -\bar{g}(r_1))^\top.$$

By condition (H_2), we know that

$$\bar{g}(r_0) \neq 0, \quad \bar{g}(r_1) \neq 0,$$

which means that

$$QN\omega \neq (0, 0)^\top.$$

That implies the condition (h_2) is established.

Finally, in order to verify the condition (h_3), we define

$$J : \mathrm{Im}L \to \mathrm{Ker}L, \; J(\omega) = \omega.$$

If $\omega \in \Omega \cap \ker L$, then

$$\omega = (b_1, b_2)^\top,$$

where b_1, b_2 are constant with $r_0 < b_1 < r_1$, $|b_2| < r_2$. We have

$$JQN\omega = (\phi(b_2), -\bar{g}(b_1))^\top.$$

Note that $\phi(v)$ is a continuous strictly increasing function, combining with the condition (H_2), one can easy to calculate that

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, \mathbf{0}\} \neq 0.$$

Therefore, the condition (h_3) is satisfied.

Up to now, all assumptions of Lemma 2.1 are satisfied. By Lemma 2.1, we prove that the equation (1.4) has at least one positive T-periodic solution.

4. An example and numerical simulations

In this section, an example and the corresponding numerical simulations (phase portrait and time series portrait of the positive periodic solution of the example) are given to illustrate our theoretical analysis.

Consider the following singular equation

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + (u^\alpha + 1)u' + u(1+\sin t) - \frac{1}{u^\gamma} = \cos t, \tag{4.1}$$

where $\alpha > 0$ and $\gamma > 0$ are constants.

Corollary 4.1. *Assume that $\alpha > 0$ and $\gamma \geq 1$. Then equation* (4.1) *has at least one positive 2π-periodic solution.*

Proof. Equation (4.1) can be regarded as a problem of the form (1.4), where

$$f(u) = u^\alpha + 1, \quad g(t,u) = u(1+\sin t) - \frac{1}{u^\gamma}, \quad e(t) = \cos t.$$

By direct computations, we can obtain

$$\begin{aligned} \bar{g}(u) &= \frac{1}{2\pi}\int_0^{2\pi} g(t,u)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} [u(1+\sin t) - \frac{1}{u^\gamma}]\mathrm{d}t \\ &= u - \frac{1}{u^\gamma}, \quad \forall u \in \mathbb{R}^+. \end{aligned}$$

Obviously, if we choose two constants M_1, M_2 with $M_1 \in (0,1)$ and $M_2 \in (1,\infty)$, then we have

$$\bar{g}(u) < 0, \quad \forall u \in (0, M_1)$$

and

$$\bar{g}(u) > 0, \quad \forall u \in (M_2, +\infty).$$

Therefore, if

$$\int_0^{2\pi} g(t, u(t))dt = 0,$$

then there exists $\tau \in [0, 2\pi]$ such that

$$M_1 \leq u(\tau) \leq M_2.$$

Define

$$g_0(u) = -\frac{1}{u^\gamma}, \quad g_1(t, u) = u(1 + \sin t), \quad h_b(t) = b(1 + t),$$

then

$$| g_1(t, u) | \leq b \mid 1 + \sin t \mid \leq b(1 + t) = h_b(t),$$

for a.e. $t \in [0, 2\pi]$ and all $u \in [0, b]$.

Moreover, we have

$$\int_0^1 g_0(u)\mathrm{d}u = \int_0^1 -\frac{1}{u^\gamma}\mathrm{d}u = -\infty.$$

and

$$\varphi(t) = \limsup_{u \to +\infty} \frac{g(t, u)}{u} = 1 + \sin t.$$

Up to now, all assumptions of Theorem 1.3 are satisfied. By Theorem 1.3, we know that equation (4.1) has at least one positive 2π-periodic solution. □

Remark 4.2. *In fact, by direct computations, we have*

$$\|\varphi^+\|_1 = 2\pi > \frac{\sqrt{3}}{T} = \frac{\sqrt{3}}{2\pi},$$

which contradicts the condition (H_5)*, and*

$$\|\varphi\|_\infty = 2 > (\frac{\pi}{T})^2 = \frac{1}{4},$$

which contradicts the condition (H_5')*.*

Finally, applying Matlab software, we obtain the time series portrait and phase portrait of the 2π-periodic solution of the equation (4.1) with $\alpha = 0.18$ and $\gamma = 1.23$, which is show in FIGURE 1 and 2.

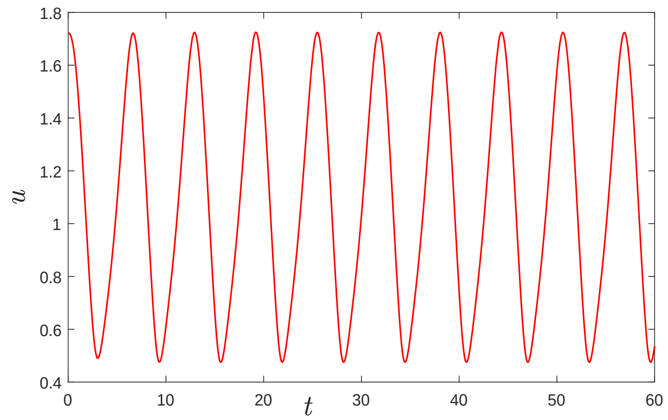

FIGURE 1. Time series portrait of the 2π-periodic solution of the equation (4.1) with the initial condition $u(0) = 1.722,\ u'(0) = 0$.

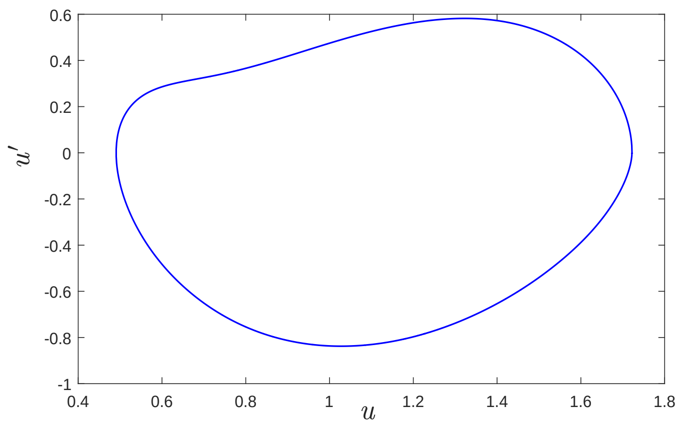

FIGURE 2. Phase portrait of the 2π-periodic solution of the equation (4.1) with the initial condition $u(0) = 1.722,\ u'(0) = 0$.

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Declarations

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