THE GROUND STATES FOR HARTREE-FOCK SYSTEMS WITH A GENERAL NONLINEARITY

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ABSTRACT. We consider the least energy solutions of Hatree-Fock system with the coupling term $\phi_{u,v}$ in the two equations, and the nonlinearity are general subcritical with a small perturbation. By Nehari's manifold approach, the existence of non-trivial ground state solutions is obtained. The asymptotic behaviors with respect to parameters λ and β are also studied.

1. INTRODUCTION AND MAIN RESULT

1.1. **Background.** In this paper, we consider the following Hatree-Fock system

$$\begin{cases} -\Delta u + u + \lambda \phi_{u,v} u = f(u) + \beta v, \\ -\Delta v + v + \lambda \phi_{u,v} v = g(v) + \beta u, \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where

$$\phi_{u,v}(x) := \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy \in D^{1,2}\left(\mathbb{R}^3\right),$$

and $\lambda > 0$, $0 < \beta < 1$. It can also be seen as a Schrödinger-Poisson type system. In the past ten years, Schrödinger-Poisson system has been used in various aspects owing to it's wide application in physics. From the perspective of physics, if we ignore the manetic effect, the equations could describes the system which has the interaction between the same charged particles. The nonlinear term f and g simulate the interaction between particles, and the coupling terms $\phi_{u,v} u$ and $\phi_{u,v} v$ are related to the interaction in the electric field. For more physical knowledge about the Schrödinger-Poisson system we could refer to [11, 24, 25, 27, 28] and the references therein. Recently, there has been an increasing attention towards systems like (1.1), and the existence of ground state solutions and the asymptotic behavior have been investigated when $\lambda = 0$ and $\beta \to 0$ in [15], more related content could refer to [18, 30, 31] and references therein.

1.2. **Related results.** Let us summarize some results related to Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-2} u, \\ -\Delta \phi = u^2, \end{cases} \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

In [22], D'Aprile and Mugnai obtained the existence of a nontrivial radial solution to the problem (1.2) for $p \in [4, 6)$. In [8], Ruiz obtained some nonexistence results for (1.2) and established the relation between the existence of the positive solutions to system (1.2) with the parameters

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$p \in (2,6)$ and $\lambda > 0$. For $p \in (4,6)$, Ruiz [9] investigated the existence of radial ground states to system (1.2) and obtained the different behavior of the solutions depending on p as $\lambda \to 0$. We also would like to cite some works in [4, 21], where system (1.2) was considered as $\lambda \to 0$. In [4,21], the authors concerned with the semi-classical states for system (1.2). Precisely, the authors studied the existence of radial positive solutions concentrating around a sphere. Recently, some works were focused on the existence of sign-changing solutions to the system (1.2). Subsequently, Ianni [13] obtained a similar result for $p \in [4,6)$. More recently, Wang and Zhou [32] considered the non-autonomous system, we also would like to mention [1,10,12,16,29] and the references therein.

The works discussed above mainly focused on the equation (1.2) with the very special nonlinearity $f(u) = |u|^{p-2}u$. In [4], Azzollini et al. concerned with the existence of a positive radial solution to the single equation of system (1.1) under the effect of a general nonlinear term, see also [2, 26]. Then, Zhang [7] obtained the existence of positive solutions to the equations involving a more general critical nonlinearity, where f is at critical growth.

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u), \\ -\Delta\phi = \lambda u^2, \end{cases} \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

When $\lambda = 0$, (1.1) leads to the local weakly coupled nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u = f\left(u\right) + \beta v, \\ -\Delta v + v = g\left(v\right) + \beta u, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

for $\beta \geq 0$, $N \geq 3$, which has been intensively studied in the past five years. Zou and Chen [30] obtained that, if f and g satisfy the critical condition, then there exists $\beta_0 \in (0,1)$ such that for any $0 < \beta < \beta_0$, (1.4) has a positive radially symmetric bound state $(u_\lambda, v_\lambda) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ when $N \geq 3$. In [31], for $f(u) = (1 + a(x))|u|^{p-1}u$ and $g(v) = (1 + b(x))|v|^{p-1}v$, suppose in addition that $a(x) + b(x) \geq 0$, then, for every $0 < \beta < 1$, (1.4) has a positive ground state solution. Furthermore, they also proved several properties of the ground states at infinity.

When $\lambda = 0$, $\beta = 0$ and f, g are special form, the system (1.1) becomes

$$\begin{cases} -\Delta u + u = |u|^{2q-2}u + \gamma|v|^q|u|^{q-2}u, \\ -\Delta v + \omega^2 v = |v|^{2q-2}v + \gamma|u|^q|v|^{q-2}v, \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

for $0 < \omega^2 \leq 1$, which has been intensively studied in the past fifteen years. Applying variation methods, the first work is considered by Lin and Wei [23] and also by Ambrosetti and Colorado [5], Maia, Montefusco, and Pellacci [17], Bartsch and Wang [20], Sirakov [6], then followed by an extensive literature presenting investigations of different aspects and variations of this problem. In fact this system is obtained when looking for solitary wave solutions of two coupled nonlinear Schrödinger equations which model, for instance, binary mixtures of Bose-Einstein condensates or propagation of wave packets in nonlinear optics.

Based on the study of (1.2) by Ruiz, Maia et al. which considered the Schrödinger equation in the simulation of the interaction force between nucleus and electron from the physical background. Due to the complexity of the system, it is necessary to consider the interaction between molecular orbitals and whether Pauli theorem is taken into account when using the model to approximate. Maia et al. [1] added a perturbation term containing parameter β to the right end of (1.2), and coupled the two equations into equations through $\phi_{u,v}$, then the following system is obtained,

$$\begin{cases} -\Delta u + u + \lambda \phi_{u,v} u = |u|^{2q-2}u + \beta |v|^q |u|^{q-2}u, \\ -\Delta v + v + \lambda \phi_{u,v} v = |v|^{2q-2}v + \beta |u|^q |v|^{q-2}v, \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.6}$$

the authors studied the existence of radial state solutions and the asymptotic behaviors with respect to parameter β. When $q \in (\frac{3}{2}, 3)$, $\lambda > 0$ and $\beta \geq 0$, system (1.6) has a radial ground state solution $(\mathfrak{u}_\beta, \mathfrak{v}_\beta) \neq (0,0)$. Moreover, if $\beta = 0$, the ground state solution is semitrivial, if β is sufficiently small or sufficiently large, the ground state solution has different property and asymptotic behaviors.

Indeed, when $\beta = 0$, the system (1.5) is uncoupled and it reduced to two equations of the same type, for such type of equations, well known results have been obtained about uniqueness of the positive solution. In our case, even for $\beta = 0$, the system remains coupled by $\phi_{u,v}$. Moreover, compared with equation (1.6), the system considered in this paper is more generally. This form could better describes the interaction between charged particles and which could be used to describe more cases in physics.

By the Nehari manifold and Ambrosetti–Rabinowitz condition, a voctorial ground state solution $(u \neq 0, v \neq 0)$ is obtained when $\lambda > 0$ and $0 < \beta < 1$, we also studied the asymptotic behaviors of the solutions.

1.3. **Main result.** Here we concerned with positive solutions of (1.1), and in the sequel we assume without loss of generality that $f(s) \equiv g(s) \equiv 0$ for all $s \leq 0$. Assume that $f(s)$ (the same as $g(s)$) satisfies the following properties throughout the paper:

- (f_1) $f : \mathbb{R} \to \mathbb{R}^+$ is continuous, and $\lim_{s \to 0} \frac{f(s)}{s} = 0$,
- (f_2) $\limsup_{s \to +\infty} \frac{f(s)}{s^{p-1}} < +\infty$, where $2 < p < 2^*$,
- (f_3) There exists $\mu > 4$ such that $f(t)t \geq \mu F(t)$, where $F(t) = \int_0^t f(s)ds$,
- (f_4) For every $t \in (0, +\infty)$, $f(t)/t^3$ is an increasing function of t.

The main result is as follows.

Theorem 1.1. *Suppose that f (the same as g) satisfies $(f_1) - (f_4)$, $\lambda > 0$ and $0 < \beta < 1$, the Hatree-Fock system (1.1) has a voctorial ground state solution.*

1.4. **Main difficulties and ideas.** If the problem is discussed in $\mathbb{R}^3$, we find the compactness condition cannot be satisfied. In order to deal with compactness issue, we will work in the radial setting and use the compact embedding of $H_r^1(\mathbb{R}^3)$ into $L^p\left(\mathbb{R}^3\right)$ for $p \in (2, 2^*)$. Then the functional will be restricted to $H_r := H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$ and the solutions will be found in H_r. The invariance of the functional under rotations and the Palais' Principle of Symmetric Criticality [19] makes natural this constraint. According to the variational method under the condions $(f_1) - (f_4)$ the energy functional $I_{\lambda,\beta}$ is well defined, and the minimum value of the energy functional is obtained in Nehari manifold $\mathcal{N}^{\lambda,\beta} := \{(u,v) \in H_r : \mathcal{J}_{\lambda,\beta}(u,v) = 0\}$. What's more, the ground state solution is proved to be positive by the strong maximum principle. At the end of the paper, because of the $I_{\lambda,\beta}$ has mountain pass geometry, the asymptotic behaviors are obtained when $\lambda \to 0$ and $\beta \to 0$.

The paper is organized as follows.

In Section 2, we present some preliminaries in order to prove our results. Particularly, we recall some results in [7] and [1], that will be used to rule out the semitrivial solution and briefly

descript the existence of solution for the limit problem.
In Section 3, the existence of ground state solutions is obtained. Moreover, the solution is vectorial.
In Section 4, the asymptotic behaviors of ground states with respect to parameters λ and β are also studied.

Notations.
- Unless otherwise stated, integrals will always be considered on the whole $\mathbb{R}^3$ with the Lebesgue measure.
- We denote with $\|\cdot\|$ the norm in $H^1(\mathbb{R}^3)$ and $\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2 \, dx\right)^{\frac{1}{2}}$.
- We denote with $\|\cdot\|_p$ the standard L^p- norm and $\|u\|_p := \left(\int_{\mathbb{R}^3} |u|^p \, dx\right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$.
- $H^1_r(\mathbb{R}^3)$ is the subspace of $H^1(\mathbb{R}^3)$ of radially symmetric functions.
- $\mathcal{D}^{1,2}(\mathbb{R}^3) := \left\{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\right\}$.

2. Preliminaries

Recalling the following well-known facts, λ_1 is the first eigenvalue of $-\triangle$, and $\mathcal{S}_p$ denotes the best constant of Soblolev embedding $H^1\left(\mathbb{R}^3\right) \hookrightarrow L^p\left(\mathbb{R}^3\right)$,

$$\mathcal{S}_p \left(\int_{\mathbb{R}^3} |u|^p \, dx\right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx, \quad \text{for all } u \in H^1\left(\mathbb{R}^3\right).$$

For $(u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, the Lax–Milgram theorem implies that there exists a unique $\phi_{u,v} \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$- \triangle \phi = 4\pi(u^2 + v^2),$$

with

$$\phi_{u,v}(x) := \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x - y|} dy \in \mathcal{D}^{1,2}\left(\mathbb{R}^3\right). \tag{2.1}$$

Assumption (f_2) implies f has (possibly) a subcritical growth at infinity. Moreover, for every $\varepsilon_1, \varepsilon_2 > 0$, there exists $\mathcal{C}_{\varepsilon_1}, \mathcal{C}_{\varepsilon_1} > 0$ such that

$$|f(t)| \leq \varepsilon_1 |t| + \mathcal{C}_{\varepsilon_1} |t|^{p-1},$$

and

$$|g(t)| \leq \varepsilon_2 |t| + \mathcal{C}_{\varepsilon_2} |t|^{p-1},$$

where $2 < p < 2^*$. Define $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$ and $\mathcal{C}_\varepsilon = max\{\mathcal{C}_{\varepsilon_1}, \mathcal{C}_{\varepsilon_2}\}$, then we have

$$|f(t)t + g(t)t| \leq \varepsilon |t|^2 + \mathcal{C}_\varepsilon |t|^p. \tag{2.2}$$

For the following system

$$\begin{cases} -\Delta u + u + \lambda\phi_{u,v} u = f(u) + \beta v, \\ -\Delta v + v + \lambda\phi_{u,v} v = g(v) + \beta u, \end{cases} \quad \text{in } \mathbb{R}^3,$$

we define the energy functional $I_{\lambda,\beta}: H_r \to \mathbb{R}$ by

$$
\begin{aligned}
I_{\lambda,\beta}(u,v) =& \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&+\frac{\lambda}{4}\int_{\mathbb{R}^3}(u^2+v^2)\phi_{u,v}dx-\int_{\mathbb{R}^3}\left(F(u)+G(v)\right)dx-\beta\int_{\mathbb{R}^3}uvdx.
\end{aligned}
$$

It is obvious to see that $I_{\lambda,\beta}$ is of class $\mathcal{C}^1$ in H_r, and easy to verify that $(u,v)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)$ is a solution of (1.1) if and only if $(u,v)\in H_r$ is a critical point of the functional $I_{\lambda,\beta}$.

In order to prove the main result Theorem 1.1, we find the minimizer of C^1-functional $I_{\lambda,\beta}$ under the constraint of

$$
\mathcal{N}^{\lambda,\beta}:=\left\{(u,v)\in H_r:\mathcal{J}_{\lambda,\beta}(u,v)=0\right\},
$$

where

$$
\begin{aligned}
\mathcal{J}_{\lambda,\beta}(u,v):=& \textstyle\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx+\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&+\lambda\textstyle\int_{\mathbb{R}^3}(u^2+v^2)\phi_{u,v}dx-\int_{\mathbb{R}^3}(f(u)u+g(v)v)dx-2\beta\int_{\mathbb{R}^3}uvdx.
\end{aligned}
\tag{2.3}
$$

Obviously, $\mathcal{N}^{\lambda,\beta}$ contains all nontrivial radial critical points of $I_{\lambda,\beta}$. Moreover, the following simple result assures us that any couple $(u,v)\in H_r\setminus\{0\}$ can be uniquely projected on $\mathcal{N}^{\lambda,\beta}$ via $\gamma_{u,v}(t)$ (which defined in the following) and gives us a further property of such a projection.

Lemma 2.1. *For any $(u,v)\in H_r\setminus\{0\}$, there exists a unique $t_{u,v}>0$ such that $\gamma_{u,v}(t_{u,v})\in\mathcal{N}^{\lambda,\beta}$ and*

$$
I_{\lambda,\beta}\left(\gamma_{u,v}\left(t_{u,v}\right)\right)=\max_{t>0}I_{\lambda,\beta}\left(\gamma_{u,v}\left(t\right)\right). \tag{2.4}
$$

Proof. Given $(u,v)\in H_r\setminus\{0\}$, we denote with $\gamma_{u,v}:[0,+\infty)\to H_r$ the curve

$$
\gamma_{u,v}:=(tu,tv). \tag{2.5}
$$

By the simple calculation we get

$$
\begin{aligned}
I_{\lambda,\beta}(\gamma_{u,v}(t)) =& \frac{t^2}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx+\frac{t^2}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&+\frac{\lambda t^4}{4}\int_{\mathbb{R}^3}(u^2+v^2)\phi_{u,v}dx-\int_{\mathbb{R}^3}(F(tu)+G(tv))dx-t^2\beta\int_{\mathbb{R}^3}uvdx,
\end{aligned}
$$

and

$$
\begin{aligned}
I'_{\lambda,\beta}(\gamma_{u,v}(t)) =& t\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx+t\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&+\lambda t^3\int_{\mathbb{R}^3}(u^2+v^2)\phi_{u,v}dx-\int_{\mathbb{R}^3}(uf(tu)+vg(tv))dx-2t\beta\int_{\mathbb{R}^3}uvdx.
\end{aligned}
$$

Let $(u,v) \in H_r \setminus \{0\}$, we obtain

$$\begin{aligned} I'_{\lambda,\beta}(\gamma_{u,v}(t)) = 0 &\Leftrightarrow (tu,tv) \in \mathcal{N}^{\lambda,\beta} \\ &\Leftrightarrow \|u\|^2 + \|v\|^2 - 2\beta \int_{R^3} uvdx \\ &= \frac{1}{t}\int_{R^3} (uf(tu) + vg(tv))\,dx - t^2\lambda \int_{R^3} (u^2+v^2)\phi_{u,v}dx \\ &= t^2 \left(\frac{1}{t^3}\int_{R^3} (uf(tu) + vg(tv))\,dx - \lambda \int_{R^3} (u^2+v^2)\phi_{u,v}dx \right). \end{aligned}$$

By (f_4), the right hand side is an increasing function of t, thus, there exists a unique $t = t_{u,v} > 0$, such that $I_{\lambda,\beta}(0) = 0$, $I_{\lambda,\beta}(\gamma_{u,v}(t_{u,v})) > 0$ for $t > 0$ small, $I_{\lambda,\beta}(\gamma_{u,v}(t_{u,v})) < 0$ for t large. Therefore $\max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t_{u,v}))$ is achieved at a unique $t = t_{u,v}$ such that $I'_{\lambda,\beta}(\gamma_{u,v}(t)) = 0$ and $\gamma_{u,v}(t_{u,v}) \in \mathcal{N}^{\lambda,\beta}$. We define

$$\begin{aligned} m &:= \inf_{(u,v)\in\mathcal{N}_{\lambda,\beta}} I_{\lambda,\beta}, \\ c &:= \inf_{(u,v)\in H_r\setminus\{o\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)), \end{aligned}$$

hence, by standard proof we obtain

$$m = c. \tag{2.6}$$

Lemma 2.2. *(see [18]). Let $q \in (2, 2^*)$, and $\{(u_n, v_n)\} \in H_r$ be such that $(u_n, v_n) \rightharpoonup (u, v)$ in H_r as $n \to +\infty$. We have , as $n \to +\infty$,*

$$\phi_{u_n,v_n} \to \phi_{u,v} \quad in \quad D_r^{1,2}(\mathbb{R}^3), \tag{2.7}$$

$$\int_{\mathbb{R}^3} (u_n^2 + v_n^2)\phi_{u_n,v_n} \to \int_{\mathbb{R}^3} (u^2+v^2)\phi_{u,v}. \tag{2.8}$$

Moreover, $\phi_{u,v} : H^1(\mathbb{R}^3) \mapsto D^{1,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets, and if (u,v) is radial function, so is $\phi_{u,v}$.

3. The proof of Theorem 1.1

Now we are ready to find the ground state solution of (1.1) by minimizing the functional $I_{\lambda,\beta}$ on $\mathcal{N}^{\lambda,\beta}$.

Proof of Theorem (1.1). We divide the proof in several steps.

Step 1. $\mathcal{N}^{\lambda,\beta}$ is bounded away from zero, i.e. $dist(\mathcal{N}^{\lambda,\beta}, 0) > 0$.

For every $(u,v) \in \mathcal{N}^{\lambda,\beta}$, thanks to (2.2) and compact embedding of the radial functions we have

$$\begin{aligned} \|u\|^2 + \|v\|^2 &= \int_{\mathbb{R}^3} (f(u)u + g(v)v)dx - \lambda \int_{\mathbb{R}^3} (u^2+v^2)\phi_{u,v}dx + 2\beta \int_{\mathbb{R}^3} uvdx \\ &\le \varepsilon \int_{\mathbb{R}^3} (u^2+v^2)dx + \mathcal{C}_\varepsilon \int_{\mathbb{R}^3} (|u|^p + |v|^p)dx - \lambda \int_{\mathbb{R}^3} (u^2+v^2)\phi_{u,v}dx + 2\beta \int_{\mathbb{R}^3} uvdx \\ &\le \varepsilon(\|u\|_2^2 + \|v\|_2^2) + \mathcal{C}_\varepsilon(\|u\|_p^p + \|v\|_p^p) + 2\beta \|u\|_2 \|v\|_2, \end{aligned}$$

$$(1-\beta)(\|u\|^2 + \|v\|^2) \le \frac{\varepsilon}{\lambda_1}(\|u\|^2 + \|v\|^2) + \frac{\mathcal{C}_\varepsilon}{\mathcal{S}_p^p}(\|u\|^p + \|v\|^p),$$

then

$$((1-\beta)-\frac{\varepsilon}{\lambda_1})(\|u\|^2+\|v\|^2)\leq\frac{\mathcal{C}_\varepsilon}{\mathcal{S}_p^p}(\|u\|^p+\|v\|^p),$$

where λ_1 is the first eigenvalue, we get

$$\|u\|+\|v\|\geq\delta>0,$$

so that

$$dist(\mathcal{N}^{\lambda,\beta},0)>0,$$

which proves the claim.

Step 2. The functional $I_{\lambda,\beta}$ is coercive on $\mathcal{N}^{\lambda,\beta}$ and $m=\inf_{(u,v)\in\mathcal{N}^{\lambda,\beta}} I_{\lambda,\beta}>0$.
For every $(u,v)\in\mathcal{N}^{\lambda,\beta}$, by (f_3),

$$\begin{aligned}
I_{\lambda,\beta}(u,v) &= I_{\lambda,\beta}(u,v)-\frac{1}{\mu}\mathcal{J}_{\lambda,\beta}(u,v)\\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx+\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx\\
&\quad+\int_{\mathbb{R}^3}(f(u)u-\mu F(u))\,dx+\int_{\mathbb{R}^3}(g(v)v-\mu G(v))\,dx\\
&\quad+\left(\frac{1}{4}-\frac{1}{\mu}\right)\lambda\int_{\mathbb{R}^3}\phi\left(u^2+v^2\right)dx-(\frac{1}{2}-\frac{1}{\mu})2\beta\int_{\mathbb{R}^3}uvdx\\
&\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\|u\|^2+\|v\|^2-2\beta\int_{\mathbb{R}^3}uvdx\right),
\end{aligned}$$

as we can see, $0<\beta<1$ and $\frac{1}{4}-\frac{1}{\mu}>0$, then

$$I_{\lambda,\beta}(u,v)\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\|u\|^2+\|v\|^2\right),$$

which shows that $I_{\lambda,\beta}$ is coercive on $\mathcal{N}^{\lambda,\beta}$ and that

$$m=\inf_{(u,v)\in\mathcal{N}^{\lambda,\beta}} I_{\lambda,\beta}(u,v)\geq\inf_{(u,v)\in\mathcal{N}^{\lambda,\beta}}\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\|u\|^2+\|v\|^2\right)\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\delta^2>0.$$

by **Step1**.

Step 3. There exists $(u,v)\in\mathcal{N}^{\lambda,\beta}$ such that $I_{\lambda,\beta}(u,v)=m$.

Firstly, we prove that if $\{(u_n,v_n)\}_{n=1}^{\infty}$ is a minimizing sequence for $I_{\lambda,\beta}$ on $\mathcal{N}^{\lambda,\beta}$, then it is bounded. Hence, up to subsequence, it weakly converges to some (u,v) in H_r. As we know $m=\inf_{(u,v)\in\mathcal{N}^{\lambda,\beta}} I_{\lambda,\beta}$, there exists $\{(u_n,v_n)\}_{n=1}^{\infty}\subset\mathcal{N}^{\lambda,\beta}$ such that $I_{\lambda,\beta}(u_n,v_n)\to m>0$ and

$(u_n, v_n) \in \mathcal{N}^{\lambda,\beta}$, i.e. $I'_{\lambda,\beta}(u_n, v_n)(u_n, v_n) = 0$ for every $n \in \mathbb{R}$. Thus, by (f_3) we deduce that

$$\begin{aligned}
m + o_n(1) &= I_{\lambda,\beta}(u_n, v_n) - 0 \\
&= I_{\lambda,\beta}(u_n, v_n) - \frac{1}{\mu} I'_{\lambda,\beta}(u_n, v_n)(u_n, v_n) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + |u_n|^2\right) dx + \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + |v_n|^2\right) dx \\
&\quad + \left(\frac{1}{4} - \frac{1}{\mu}\right) \lambda \int_{\mathbb{R}^3} \phi\left(u_n^2 + v_n^2\right) dx - (\frac{1}{2} - \frac{1}{\mu}) 2\beta \int_{\mathbb{R}^3} u_n v_n dx \\
&\quad + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} f(u_n) u_n - F(u_n)\right) dx + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} g(v_n) v_n - G(v_n)\right) dx \\
&\geq (\frac{1}{2} - \frac{1}{\mu}) \left(\|u_n\|^2 + \|v_n\|^2 - 2\beta \int_{\mathbb{R}^3} u_n v_n dx\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \left(\|u_n\|^2 + \|v_n\|^2\right),
\end{aligned}$$

then

$$\|u_n\|^2 + \|v_n\|^2 \leq \frac{m + o_n(1)}{\left(\frac{1}{2} - \frac{1}{\mu}\right)},$$

$$sup\left(\|u_n\| + \|v_n\|\right) < +\infty,$$

the sequence $\{(u_n, v_n)\}$ is bounded.

Since $I_{\lambda,\beta}$ is coercive on $\mathcal{N}^{\lambda,\beta}$, the sequence $\{(u_n, v_n)\}$ is bounded, due to the compact embedding and up to subsequences, there exists $\left\{\left(u_{n_j}, v_{n_j}\right)\right\} \subset H_r$, such that

$$\begin{aligned}
\left(u_{n_j}, v_{n_j}\right) &\rightharpoonup (u, v) \quad \text{in} \quad H_r\left(\mathbb{R}^3\right), \\
\left(u_{n_j}, v_{n_j}\right) &\rightarrow (u, v) \quad \text{in} \quad L^p\left(\mathbb{R}^3\right).
\end{aligned}$$

As we know, there exists

$$\begin{aligned}
\omega_1 \in L^p\left(\mathbb{R}^3\right) \quad s.t. \quad \left|u_{n_j(x)}\right| \leq \omega_1, |u(x)| \leq \omega_1, \\
\omega_2 \in L^p\left(\mathbb{R}^3\right) \quad s.t. \quad \left|v_{n_j(x)}\right| \leq \omega_2, |v(x)| \leq \omega_2,
\end{aligned}$$

then, we deduce

$$F\left(u_{n_j}(x)\right) \leq \frac{\varepsilon}{2} \omega_1^2 + \frac{C_\varepsilon}{P} \omega_1^p,$$

because F is continuous and $u_{n_j(x)} \rightarrow u(x)$ in L^p , we have

$$F\left(u_{n_j}(x)\right) \rightarrow F(u(x)) \quad a.e. \ \text{ in } L^p\left(\mathbb{R}^3\right),$$

using dominated convergence theorem, we get

$$\int_{\mathbb{R}^3} F\left(u_{n_j}(x)\right) dx \rightarrow \int_{\mathbb{R}^3} F(u(x))\, dx \ \text{ as } \ j \rightarrow +\infty, \tag{3.1}$$

$$\int_{\mathbb{R}^3} f\left(u_{n_j}\right) u_{n_j} dx \rightarrow \int_{\mathbb{R}^3} f(u)\, u dx \ \text{ as } \ j \rightarrow +\infty. \tag{3.2}$$

Similarly, the conclusions above holds for $G\left(v_{n_j}\right)$,

$$\int_{\mathbb{R}^3} G\left(v_{n_j}(x)\right) dx \to \int_{\mathbb{R}^3} G(v(x))\, dx \quad \text{as} \quad j \to +\infty, \tag{3.3}$$

$$\int_{\mathbb{R}^3} g\left(v_{n_j}\right) v_{n_j} dx \to \int_{\mathbb{R}^3} g(v)\, v dx \quad \text{as} \quad j \to +\infty. \tag{3.4}$$

Eventually passing to this suitable subsequence,

$$\|u\| \leq \liminf_{j\to\infty} \left\|u_{n_j}\right\| \qquad \text{and} \qquad \|v\| \leq \liminf_{j\to\infty} \left\|v_{n_j}\right\|. \tag{3.5}$$

Thus, for one hand, by (2.8) in 2.2, (3.1), (3.3) and (3.5) we deduce

$$\begin{aligned}
I_{\lambda,\beta}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx + \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&\quad + \frac{\lambda}{4}\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx - \int_{\mathbb{R}^3}(F(u)+G(v))\,dx - \beta\int_{\mathbb{R}^3} uvdx \\
&\leq \liminf_{j\to\infty} \frac{1}{2}\left(\left\|u_{n_j}\right\|^2+\left\|v_{n_j}\right\|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\left(u_{n_j}^2+v_{n_j}^2\right)\phi_{u_j,v_j}dx \\
&\quad - \int_{\mathbb{R}^3}\left(F(u_{n_j})+G(v_{n_j})\right)dx - \beta\int_{\mathbb{R}^3} u_{n_j}v_{n_j}dx \\
&\leq \liminf_{j\to\infty} I_{\lambda,\beta}(u_{n_j},v_{n_j}).
\end{aligned}$$

as our previous proof $I_{\lambda,\beta}\left(u_{n_j},v_{n_j}\right)\to m$ when $j\to\infty$, we deduce

$$I_{\lambda,\beta}(u,v) \leq m.$$

For another hand, because of the constraint condition $\mathcal{J}_{\lambda,\beta}\left(u_{n_j},v_{n_j}\right)=0$ for every $j\in\mathbb{R}$, thanks to (3.2), (3.4)and (3.5),

$$\left\|u_{n_j}\right\|^2+\left\|v_{n_j}\right\|^2 = \int_{\mathbb{R}^3} f(u_{n_j})u_{n_j}+g(v_{n_j})v_{n_j}dx - \lambda\int_{\mathbb{R}^3}\left(u_{n_j}^2+v_{n_j}^2\right)\phi_{u_{n_j},v_{n_j}}dx + 2\beta\int_{\mathbb{R}^3} u_{n_j}v_{n_j}dx,$$

$$\|u\|^2+\|v\|^2 \leq \int_{\mathbb{R}^3}(f(u)u+g(v)v)\,dx - \lambda\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx + 2\beta\int_{\mathbb{R}^3} uvdx,$$

$$I'_{\lambda,\beta}(u,v)(u,v) \leq 0.$$

using the derivation process in **Step1** we get

$$\overline{\delta} \leq \left((1-\beta)-\frac{\varepsilon}{\lambda_1}\right)\left(\|u\|^2+\|v\|^2\right) \leq \mathcal{C}_\varepsilon\left(\|u\|^p+\|v\|^p\right),$$

where $\overline{\delta}=\left((1-\beta)-\frac{\varepsilon}{\lambda_1}\right)\delta$, then

$$\|u\|^p+\|v\|^p \geq \left(\frac{\overline{\delta}}{\mathcal{C}_\varepsilon}\right)^{\frac{1}{p}} > 0,$$

and

$$\inf\left(\|u\|^p+\|v\|^p\right) > 0 \quad \text{for every} \quad (u,v)\in\mathcal{N}^{\lambda,\beta}.$$

Noting that $\left(u_{n_j}, v_{n_j}\right) \in \mathcal{N}^{\lambda,\beta}$, and $\left(u_{n_j}, v_{n_j}\right) \to (u,v)$, then

$$\left\|u_{n_j}\right\|^p + \left\|v_{n_j}\right\|^p \geq \left(\frac{\overline{\delta}}{\mathcal{C}_\varepsilon}\right)^{\frac{1}{p}},$$

$$\|u\|^p + \|v\|^p \geq \left(\frac{\overline{\delta}}{\mathcal{C}_\varepsilon}\right)^{\frac{1}{p}} > 0,$$

we obtain

$$(u,v) \not\equiv (0,0).$$

According to Lemma 2.1, for every $(u,v) \in H_r$, there exists a unique $t_{u,v} > 0$, such that $I_{\lambda,\beta}\left(\gamma_{u,v}\left(t_{u,v}\right)\right) = \max_{t>0} I_{\lambda,\beta}\left(\gamma_{u,v}(t)\right)$ where $(t_{u,v}u, t_{u,v}v) \in \mathcal{N}^{\lambda,\beta}$. From [1], we know that, in paiticular $(u,v) \in \mathcal{N}^{\lambda,\beta}$ if and only if $t_{u,v} = 1$ and then

$$0 < m = \inf_{(u,v)\in\mathcal{N}^{\lambda,\beta}} I_{\lambda,\beta}(u,v) = \inf_{(u,v)\in H_r\setminus\{0\}} I_{\lambda,\beta}\left(\gamma_{u,v}\left(t_{u,v}\right)\right) = \inf_{(u,v)\in H_r\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}\left(\gamma_{u,v}(t)\right)$$

i.e.

$$I_{\lambda,\beta}(u,v) \geq m,$$

then

$$I_{\lambda,\beta}(u,v) = m.$$

Step 4. Let $(u,v) \in \mathcal{N}^{\lambda,\beta}$ be such that $I_{\lambda,\beta}(u,v) = m$, then $I'_{\lambda,\beta}(u,v) = 0$.
Firstly, we should prove (u,v) is a regular point of $\mathcal{N}^{\lambda,\beta}$, i.e. $\mathcal{J}_{\lambda,\beta}(u^*,v^*) \neq 0$, we consider the following formula

$$\begin{aligned}\mathcal{J}_{\lambda,\beta}(u,v) &= I'_{\lambda,\beta}(u,v)(u,v)\\ &= \int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2\, dx + \int_{\mathbb{R}^3} |\nabla v|^2 + |v|^2\, dx\\ &\quad + \lambda\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx - \int_{\mathbb{R}^3}\left(f(u)u + g(v)v\right)dx - 2\beta\int_{\mathbb{R}^3} uvdx.\end{aligned}$$

$$\begin{aligned}\mathcal{J}'_{\lambda,\beta}(u,v)(u,v) = {}& 2\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |u|^2\right)dx + 2\int_{\mathbb{R}^3}\left(|\nabla v|^2 + |v|^2\right)dx + 2\lambda\int (u^2+v^2)\phi_{u,v}dx\\ & - \int_{\mathbb{R}^3}\left(f'(u)u^2 + f(u)u\right)dx - \int_{\mathbb{R}^3}\left(g'(v)v^2 + g(v)v\right)dx - 4\beta\int_{\mathbb{R}^3} uvdx.\end{aligned}$$

because $(u,v) \in \mathcal{N}^{\lambda,\beta}$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |u|^2\right)dx + \int_{\mathbb{R}^3}\left(|\nabla v|^2 + |v|^2\right)dx\\ &= \int_{\mathbb{R}^3}\left(f(u)u + g(v)v\right)dx - \lambda\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx + 2\beta\int_{\mathbb{R}^3} uvdx,\end{aligned}$$

then, we deduce

$$\begin{aligned}\mathcal{J}'_{\lambda,\beta}(u,v)(u,v) &= 2\int_{\mathbb{R}^3}(f(u)u+g(v)v)\,dx \\ &\quad -\int_{\mathbb{R}^3}\left(f'(u)u^2+g'(v)v^2\right)dx-\int_{\mathbb{R}^3}(f(u)u+g(v)v)\,dx \\ &= \int_{\mathbb{R}^3}f(u)udx-\int f'(u)u^2dx+\int_{\mathbb{R}^3}\left(g(v)v+g'(v)v^2\right)dx \\ &= \int_{\mathbb{R}^3}u\left[f(u)-f'(u)u\right]dx+\int_{\mathbb{R}^3}v\left[g(v)-g'(v)v\right]dx,\end{aligned}$$

noting that $f(u)$(the same as $g(v)$)satisfies the assumption (f_4) for every $(u,v)\in\mathcal{N}^{\lambda,\beta}$, and $(u,v)\not\equiv(0,0)$

$$3f(u)-f'(u)u<0 \quad\text{and}\quad 3g(v)-g'(v)v<0,$$

from (f_1) we know $f:\mathbb{R}\to\mathbb{R}^+$, the same as g, then

$$f(u)-f'(u)u<0 \quad\text{and}\quad g(v)-g'(v)v<0,$$

i.e.

$$\mathcal{J}'_{\lambda,\beta}(u,v)(u,v)<0.$$

Then, thanks to the lagrange multiplier rule we know that, for some $l\in\mathbb{R}$,

$$I'_{\lambda,\beta}(u,v)+l\mathcal{J}'(u,v)=0. \tag{3.6}$$

In order to obtain $I'_{\lambda,\beta}(u,v)=0$, we only need to prove $l=0$. By (3.6),it is obvious that

$$I'_{\lambda,\beta}(u,v)(u,v)+l\mathcal{J}'(u,v)(u,v)=0,$$

as we know $(u,v)\in\mathcal{N}^{\lambda,\beta}$, then $I'_{\lambda,\beta}(u,v)(u,v)=0$, and $\mathcal{J}'_{\lambda,\beta}(u,v)(u,v)<0$, which deduce $l=0$.

Next, we prove that the ground state solution is vectorial.

Lemma 3.1. *Assume that (u,v) is a nontrivial solution of (1.1), then $u>0$ and $v>0$.*

Proof. It is easily seen from above that both $u\not\equiv0$ and $v\not\equiv0$. Indeed, if $u\not\equiv0$, means $v=0$, then the equation only has zero solution and if $u\not\equiv0$, the situation is similar, which contradicts the results we proved earlier.

Assume by contradiction that $\left\{x\in\mathbb{R}^3:u(x)<0\right\}$ is not empty. By a standard regularity theory, we see that $u,v\in W^{2,q}_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ for any $q\in(2,2^*)$, which implies

$$-\Delta u+u+\lambda\phi_{u,v}u=f(u)+\beta v,\quad \text{a.e. } x\in\mathbb{R}^3,$$

and $u,v\in C^{1,\alpha}_{\mathrm{loc}}\left(\mathbb{R}^3\right)$, $0<\alpha<1$, therefore, $\Delta u\in C\left(\mathbb{R}^3\right)$.

Let $u(x_1)=\inf_{\mathbb{R}^3}u(x)<0$. We claim that $\Delta u(x_1)\geq0$. Indeed, if $\Delta u(x_1)<0$, there exists $r>0$ such that

$$u(x)<0,\quad \Delta u(x)<0,\quad \forall x\in B(x_1,r),$$

which implies from the maximum principle(cf. [14]) that $\inf_{B(x_1,r)}u(x)\geq\inf_{\partial B(x_1,r)}u(x)$. Note that $u(x_1)=\inf_{\mathbb{R}^3}u(x)$, we obtain that $u\equiv$ const in $B(x,r)$ from the strong maximum principle(cf. [14]), so $\Delta u(x_1)=0$, which is a contradiction.

Thus $\Delta u(x_1)\geq0$. Note that $f(s)\equiv0$ for $s\leq0$, and if we have $f(u(x_1))=0$, then $-\Delta u(x_1)+u(x_1)+\lambda\phi_{u(x_1),v(x_1)}u(x_1)=\beta v(x_1)\leq u(x_1)<\beta u(x_1)$, that is, $v(x_1)<\inf_{\mathbb{R}^3}u(x)<$

0 , so $\inf_{\mathbb{R}^3} v(x) < \inf_{\mathbb{R}^3} u(x)$. Similarly, we can obtain that $\inf_{\mathbb{R}^3} u(x) < \inf_{\mathbb{R}^3} v(x)$, which is a contradiction. Hence, $\left\{x \in \mathbb{R}^3 : u(x) < 0\right\} = \emptyset$, and $u \geq 0$. Similarly, $v \geq 0$. Then by the strong maximum principle, we see that $u, v > 0$.

4. Asymptotic Behavior of Ground State Solutions

As it is usual for elliptic equations, the solutions satisfy a suitable identity called Pohozaev identity. It can be obtained, at least formally, by the relation

$$\frac{d}{dt} I_{\lambda,\beta}\left(u_t, v_t\right)\bigg|_{t=1} = 0 \quad \text{where} \quad u_t(x) := u(x/t).$$

Next, Pohozaev identity is given.

Lemma 4.1. *If (u, v, ϕ) is a solution of (1.1) then it satisfies the Pohozaev identity*

$$\begin{aligned} &\tfrac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \tfrac{3}{2}\left(\|u\|_2^2 + \|v\|_2^2\right) + \tfrac{5}{4}\lambda \textstyle\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx \\ &= 3\textstyle\int_{\mathbb{R}^3}\left(F(u)+G(v)\right)dx + 3\beta\textstyle\int_{\mathbb{R}^3} uvdx. \end{aligned} \tag{4.1}$$

Proof. The proof is similar to reference [18].

Next, we discuss the asymptotic behaviors of the system (1.1).
Assume that $\lambda > 0$, $0 < \beta < 1$. Let $(u_{\lambda,\beta}, v_{\lambda,\beta})$ be a solution of

$$\begin{cases} -\Delta u + u + \lambda\phi_{u,v}u = f\left(u\right) + \beta v, \\ -\Delta v + v + \lambda\phi_{u,v}v = g\left(v\right) + \beta u, \end{cases} \quad \text{in } \mathbb{R}^3.$$

Then one may prove that $(u_{\lambda,\beta}, v_{\lambda,\beta})$ satisfied the Nehari manifold (3.1) and Pohozaev identity (4.1). Firstly, we consider the energy functional, and write (u, v) for $(u_{\lambda,\beta}, v_{\lambda,\beta})$. For one thing, thanks to Pohozaev identity,

$$\begin{aligned} I_{\lambda,\beta}(u,v) =& \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |u|^2\right)dx + \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2 + |v|^2\right)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx \\ &- \frac{1}{6}\int_{\mathbb{R}^3}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right)dx - \frac{1}{2}\int_{\mathbb{R}^3}\left(|u|_2^2 + |v|_2^2\right)dx + \frac{5}{12}\lambda\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx \\ =& \frac{1}{3}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |u|^2\right)dx + \frac{1}{3}\int_{\mathbb{R}^3}\left(|\nabla v|^2 + |v|^2\right)dx - \frac{1}{6}\lambda\int_{\mathbb{R}^3}\left(u^2+v^2\right)\phi_{u,v}dx \\ &- \frac{1}{3}\int_{\mathbb{R}^3}\left(|u|^2 + |v|^2\right)dx \\ \leq& \frac{1}{3}\left(\|u\|^2 + \|v\|^2\right). \end{aligned}$$

For another thing,

$$\begin{aligned}
I_{\lambda,\beta}(u,v) &= I_{\lambda,\beta}(u,v) - \frac{1}{\mu}\mathcal{J}_{\lambda,\beta}(u,v) \\
&= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|\nabla u|^2+|u|^2\right)dx + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|\nabla v|^2+|v|^2\right)dx \\
&\quad + \int_{\mathbb{R}^3}\left(f(u)u-\mu F(u)\right)dx + \int_{\mathbb{R}^3}\left(g(v)v-\mu G(v)\right)dx \\
&\quad + \left(\frac{1}{4}-\frac{1}{\mu}\right)\lambda\int_{\mathbb{R}^3}\phi\left(u^2+v^2\right)dx - (\frac{1}{2}-\frac{1}{\mu})2\beta\int_{\mathbb{R}^3}uvdx \\
&\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\left(\|u\|^2+\|v\|^2-2\beta\int_{\mathbb{R}^3}uvdx\right) \\
&\geq C\left(\|u\|^2+\|v\|^2\right).
\end{aligned}$$

Obviously, if the uniform boundness of $I_{\lambda,\beta}(u,v)$ is obtained, then $\|u\|^2+\|v\|^2$ will has uniformly bounded estimates independent of parameters λ and β. Next, we will discuss the asymptotic behavior of the solution under three asymptotic conditions of the parameter, which ties in find convergent subsequences under asymptotic behavior.

4.1. **The case fixed λ, $\beta \to 0$.** In this condition, the limit equation is

$$\begin{cases} -\Delta u + u + \lambda\phi_{u,v}u = f\left(u\right), \\ -\Delta v + v + \lambda\phi_{u,v}v = g\left(v\right), \end{cases} \quad \text{in } \mathbb{R}^3. \tag{4.2}$$

Suppose the ground state energy of the first equation is minimal and we define the ground state energy is $\mathfrak{n}_1$ and satisfied $\mathfrak{n}_1 = I_{\lambda,0}(\mathfrak{m}_1)$. I_{λ,β_n} is the ground state energy of the original equation in this case. Indeed, since for any $u \in H_r^1(R^3)$ it holds

$$I_{\lambda,0}(u) = I_{\lambda,\beta}(u,0),$$

then

$$0 < \mathfrak{n}_1 = I_{\lambda,0}(\mathfrak{m}_1) = \inf_{u\in H_r^1(R^3)\setminus\{0\}}\max_{t>0} I_{\lambda,\beta}(\zeta_u(t),0) \geq \inf_{u\in H_r^1(R^3)\setminus\{0\}}\max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) = I_{\lambda,\beta_n}.$$

Next, the lower bound considered. Let $\mathfrak{g}$ be the ground state of

$$-\triangle u + u = f(u) \ \text{ in } \mathbb{R}^3,$$

then

$$I_{\lambda,\beta_n} \geq I_{0,\beta_n}(\mathfrak{g},0) = I_{0,0}(\mathfrak{g}) > 0, \tag{4.3}$$

thus, $I_{\lambda,\beta_n}(u,v)$ is bounded, then (u,v) has uniformly bounded estimates independent of parameters λ and β.

Fix any $\lambda > 0$, and let $\beta_n \in (0,1)$, such that $\beta_n \to 0$ as $n \to +\infty$. Let $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})$ be any positive radial ground state of (1.1) with $\beta = \beta_n$. By the proof above, we see that $\{(u_{\lambda,\beta_n}, v_{\lambda,\beta_n})\}_{n\in\mathbb{N}}$ is bounded in H_r. Then passing to a subsequence, we may assume that $(u_{\lambda,\beta_n}, v_{\lambda,\beta_n}) \to (u_{\lambda,0}, v_{\lambda,0})$ weakly in H_r, and so $(u_{\lambda,0}, v_{\lambda,0})$ satisfies (4.2). We claim that $(u_{\lambda,0}, v_{\lambda,0}) \not\equiv (0,0)$. Assume by controdiction that $(u_{\lambda,0}, v_{\lambda,0}) \equiv (0,0)$, then $I_{\lambda,0} = 0$, which controdict to (4.3), so $(u_{\lambda,0}, v_{\lambda,0}) \not\equiv (0,0)$, but $(u_{\lambda,0}, v_{\lambda,0})$ would be the semitrivial.

4.2. **The case fixed β, $\lambda \to 0$.** In this condition, the limit equation is

$$\begin{cases} -\Delta u + u = f(u) + \beta v, \\ -\Delta v + v = g(v) + \beta u, \end{cases} \quad \text{in } \mathbb{R}^3. \tag{4.4}$$

Suppose the ground state energy of the first equation is maximum and we define the ground state energy is $\mathfrak{n}_2$ and satisfied $\mathfrak{n}_2 = I_{0,\beta}(\mathfrak{m}_2)$. $I_{\lambda_n,\beta}$ is the ground state energy of the original equation in this case. Indeed, since for any $u \in H_r^1(R^3)$ it holds

$$I_{0,\beta}(u) = I_{\lambda,\beta}(u, 0),$$

then

$$0 < \mathfrak{n}_2 = I_{\lambda,0}(\mathfrak{m}_2) = \inf_{(u,v)\in H_r\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\varsigma_u(t), 0) \le \inf_{(u,v)\in H_r\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) = I_{\lambda_n,\beta}. \tag{4.5}$$

Next, we consider to get the desired upper bound. Let $\mathfrak{s}$ be the ground state of

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u), \\ -\Delta \phi = \lambda u^2, \end{cases} \quad \text{in } \mathbb{R}^3,$$

thanks to (2.7), let $u = \mathfrak{s}$, $v = 0$, such that

$$\begin{aligned} I_{\lambda_n,\beta} &= \inf_{(u,v)\in H_r\setminus\{0\}} \max_{t>0} I_{\lambda_n,\beta}(tu, tv) \le \inf_{(u,v)\in H_r\setminus\{0\}} \max_{t>0} I_{\lambda_n,\beta}(t\mathfrak{s}, 0) \\ &\le \max_{t>0} I_{\lambda_n,\beta}(t\mathfrak{s}, 0) \le \max_{t>0} I_{1,\beta}(t\mathfrak{s}, 0) = I_{1,\beta}(\mathfrak{s}, 0). \end{aligned}$$

thus, $I_{\lambda_n,\beta}(u, v)$ is bounded, then (u, v) has uniformly bounded estimates independent of parameters λ and β.

Fix any $0 < \beta < 1$, and let $\lambda_n > 0$, such that $\lambda_n \to 0$ as $n \to +\infty$. Define $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})$ be any positive radial ground state of (1.1) with $\lambda = \lambda_n$. By the proof above, we see that $\{(u_{\lambda_n,\beta}, v_{\lambda_n,\beta})\}_{n\in\mathbb{N}}$ is bounded in H_r. Then passing to a subsequence, we may assume that $(u_{\lambda_n,\beta}, v_{\lambda_n,\beta}) \to (u_{0,\beta}, v_{0,\beta})$ weakly in H_r, and so $(u_{0,\beta}, v_{0,\beta})$ satisfies (4.4). By counter evidence $(u_{0,\beta}, v_{0,\beta}) \not\equiv (0, 0)$. Assume $(u_{0,\beta}, v_{0,\beta}) \equiv (0, 0)$, then $I_{0,\beta} = 0$, which controdict to the (4.5), so $(u_{0,\beta}, v_{0,\beta}) \not\equiv (0, 0)$. Furthermore $(u_{0,\beta}, v_{0,\beta})$ would not be the semitrivial, we assume without loss of generality that $u_{0,\beta} > 0$ and $(u_{0,\beta}, 0)$ is solution to the system above, substitute the equation, obviously, $u_{0,\beta} = 0$, which controdict to the $u_{0,\beta} > 0$, thus the solution of the system would be vectorial.

4.3. **The case $\lambda \to 0$, $\beta \to 0$.** In this condition, the limit equation is

$$\begin{cases} -\Delta u + u = f(u), \\ -\Delta v + v = g(v), \end{cases} \quad \text{in } \mathbb{R}^3. \tag{4.6}$$

Suppose the ground state energy of the first equation is minimal and we define the ground state energy is $\mathfrak{n}_3$ and satisfied $\mathfrak{n}_3 = I_{0,0}(\mathfrak{m}_3)$. I_{λ_n,β_n} is the ground state energy of the original equation in this case. Let $\mathfrak{g}$ be the ground state of

$$-\triangle u + u = f(u) \quad \text{in } R^3,$$

then

$$I_{\lambda_n,\beta_n} \ge I_{0,0}(\mathfrak{g}, 0) = I_{0,0}(\mathfrak{g}) > 0, \tag{4.7}$$

thus, $I_{\lambda,\beta_n}(u, v)$ is bounded, then (u, v) has uniformly bounded estimates independent of parameters λ and β. In this condition, let $\lambda_n > 0$ and $0 < \beta_n < 1$ such that $\lambda_n \to 0, \beta_n \to 0$ as $n \to +\infty$. Set $(u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n})$ be any positive radial ground state of (1.1) with $\lambda = \lambda_n$, $\beta = \beta_n$.

We see that $\{(u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n})\}_{n\in\mathbb{N}}$ is bounded in H_r. Then passing to the subsequence, we may assume that $(u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n}) \to (u_{0,0}, v_{0,0})$ weakly in H_r, and so $(u_{0,0}, v_{0,0})$ satisfies (4.6), then the system has a positive radially symmetric ground state solution, and from (4.7), the energy functional has positive lower bound, this ensures that the solution of the equation is nontrivial.

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