Water wave diffraction by a submerged prolate spheroid in ice-covered water

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Abstract

Using the multipoles method, we formulate the problems of diffraction (both surge and heave) of water waves by a submerged prolate spheroidal body in deep water with an ice-cover, with the ice-cover being modelled as an elastic plate of very small thickness. It investigates the linear hydrodynamic diffraction problem by prolate spheroidal body and obtained the analytical solution for the associated boundary value problem. The structural model is a spheroidal with its polar axis greater than its equatorial diameter, subjected to the action of incident wave. The hydrodynamic forces (Surge and heave exciting forces) are obtained and depicted graphically against the wave number for various parameters and also the flexural rigidity of the ice-cover to show the effect of the presence of ice-cover on these quantities. When the flexural rigidity is taken to be zero, the numerical results for the forces for water with free surface are recovered.

Keywords: Wave diffraction, Prolate spheroid, Spheroidal polar co-ordinate, Surge and heave exciting forces.

1. Introduction

The hydrodynamics of interaction of water waves, floating and submerged bodies is a well-known area of work since the last few decades due to its vast application in coastal engineering. Apparently it started with the exploration of the problems with circular cylindrical bodies. Water wave problem of fully submerged circular cylinder with single layer fluid bounded above by a free surface was studied and well established by many researchers(cf. Dean [1], Ursell [2], Garett [3], Black [4]). Then researchers showed their interest in spheroidal bodies, though the analytic studies on it is only limited to spheres from the starting years. Wang [5] applied the method of Havelock [6] to deal with diffraction and radiation problem of a submerged sphere in deep water. Also Linton [7] examined the problems on diffraction and radiation by a submerged sphere with free surface in a finite depth using the method of multipole. Rahman and Iakovlev [8] used the first order diffraction theory to deal with interaction of water waves with a submerged sphere in the ocean of finite depth.

Gradually the radius of work has been increased by dealing spheroidal bodies. The problems involving wave interaction with a spheroid submerged in multilayered or single layer fluid, with free surface or ice-covered surface is a familiar field of study for both normal and oblique incidences. Due to scientific activities in cold region and polar ocean interaction of ocean waves and sea ice is gained attention. In polar ocean it is most of the time covered by ice. The ice-cover is considered as a thin ice sheet of small thickness being consist of elements having elastic properties. Various researcher already worked on this geophysical topic. Fox and Squire [9] inquired the oblique reflection and penetration of ocean waves into shore fast sea ice. Linton and Chung [10] used the residue calculus technique to solve the scattering of water waves by the edge of a semi-infinite ice sheet in a finite depth ocean. Das and Mandal [11] formulated the problem of radiation of water waves by a submerged sphere in deep as well as in uniform finite depth water with an ice-cover. By using multipole expansion method Das and Thakur [12] researched on scattering of water wave in uniform finite depth water by a sphere beneath an ice-cover. Recently Das and Sahu [13] obtained the hydrodynamic forces by a submerged cylinder with an ice-covered water. Also marine engineers investigated the wave interaction with structures having elastic properties, such as Megafloat- a floating airport in Yakosuku Bay, Japan and super tankers in a sea way (cf. Andrianov and Hermans [14], Hermans ([15], [16]) and others). Definitely the work of Sir T.H.Havelock ([17], [18], [19]) on the wave resistance of a spheroid is a concrete direction to the researchers. The author determined the wave resistance of a submerged spheroid by replacing it with a distribution of sources and sinks or of doublets, the distribution being the image system for the solid in a uniform stream. After a few decades Wu and Eatock Taylor [20] investigated the hydrodynamic problem of submerged spheroid in regular waves based on linearized potential theory. Many researchers worked on both oblate and prolate spheroidal scattering and radiation problems by approaching different methods. By using the method to transform Green's function into the relevant harmonics Chatjigeorgiou and Miloh [21] calculated the hydrodynamics of non-axis symmetric oblate spheroids below a free surface. Chatjigeorgiou and Miloh [22] formulated free-surface hydrodynamics of a submerged prolate spheroid in finite water depth based on the method of multipole expansions. Using the method of multipole expansions constructed by employing Thorne's [23] formula Chatjigeorgiou [24] investigated the analytic solution for hydrodynamic diffraction by submerged prolate spheroids in infinite water by calculating the hydro dynamical forces. In this problem they transformed the spherical and polar coordinates of multipole potential terms into prolate spheroidal coordinates using suitable addition theorems. Chatjigeorgiou and Miloh [25] analyzed the Radiation and oblique diffraction by submerged prolate spheroid in water of finite depth. Here they used a different method. The problem is solved by using the ultimate image singularity system of external spheroidal harmonics distributed along the major axis of the spheroid between its two foci. Using the method of ultimate image singularities Anastasiou and Chatjigeorgiou [26] employed hydrodynamics of a submerged oblate spheroid in finite water depth. Instead of an oblate spheroid in this method a disc is considered. The whole process that has been performed for the spheroid, is performed for the disc as well. Anastasiou Chatjigeorgiou et. al. [27] worked on Miloh's image singularities for oblate spheroids. It is a method developed for the water wave diffraction and radiation problems. It is based on applying the method of the image singularity system for the case of spheroid. Though there are many published work on scattering and radiation by submerged spheroid with free surface, not much work is done for the same problem with an ice-cover.

Thus the present study is the extension of the work of Chatjigeorgiou [24] on hydrodynamics of spheroid by considering the scattering problem by a fully immersed prolate spheroid in water with an ice-cover under higher order boundary condition in infinite water depth. A submerged spheroid beneath an ice-covered ocean can also be considered to be a Work power device in polar oceans and considered as a subsurface storage tank as well as a fuel bladder in that polar oceans. Using the multipole expansion we calculated the hydrodynamic forces analytically and numerically. Also depicted the forces graphically. After replacing the ice-cover condition with the free surface condition it is observed that the hydrodynamical forces exactly coincides with the forces of Chatjigeorgiou [24]

2. Spheroidal Co-ordinate System

A prolate spheroid is a quadric surface obtained by rotating an ellipse about its major axis. It is elongated like a rugby ball. Specifically, here the prolate spheroid is submerged at a distance f (center of the spheroid) below the surface covered in a thin sheet of ice-cover. Considering the left handed rectangular cartesian co-ordinate system (x, y, z), placed on the undisturbed surface of ice-cover, xz-plane is the undisturbed surface of the ice-cover, with y-axis aiming vertically downward. Shifting the center of the spheroid to (x, y^*, z) by using $y = y^* + f$.

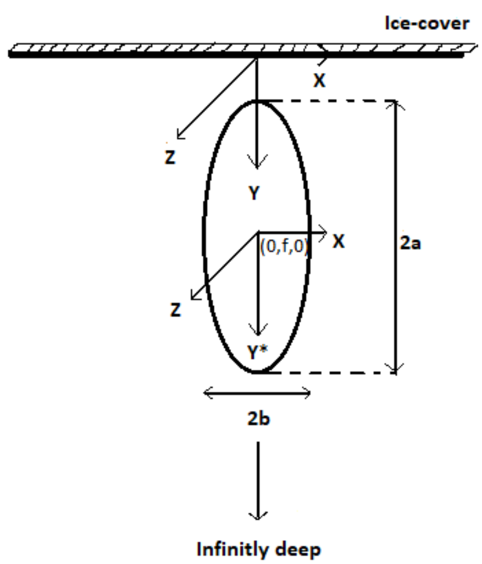

Fig.1: Schematic diagram of a prolate spheroid submerged at a distance f below ice-cover

The conversion formula from spheroidal to cartesian co-ordinates are (cf. Chatjigeorgiou [24])

$$x = c \sinh u \sin \nu \cos \psi \ ,$$

$$y^* = c\cosh u \cos\nu \ ,$$

$$z = c\sinh u \sin\nu \sin\psi \ ,$$

where $0 \le u < \infty$, $0 \le \nu \le \pi$ and $0 \le \psi < 2\pi$. Now the radial distance R is given by

$$R = \sqrt{x^2 + z^2} = c\sinh u \sin\nu \ .$$

The radial and the vertical co-ordinates from the center of the prolate spheroid are

$$R = c\sqrt{(1-\mu^2)(\xi^2 - 1)} \ ,$$

$$y^* = c\mu\xi \ ,$$

where $\xi = \cosh u$ and $\mu = \cos\nu$. For the case of oblate spheroid substituting ξ by iζ and c by $-ic$ in R and y^* such that

$$R = c\sqrt{(1-\mu^2)(\xi^2 + 1)} \ ,$$

$$y^* = c\mu\zeta \ ,$$

where $\zeta = \sinh u$ and $\mu = \cos\nu$. The transformation from oblate spheroid to cartesian co-ordinates is

$$x = c\cosh u \sin\nu \cos\psi \ ,$$

$$y^* = c\sinh u \cos\nu \ ,$$

$$z = c\cosh u \sin\nu \sin\psi \ ,$$

with $0 \le u < \infty$, $0 \le \nu \le \pi$ and $0 \le \psi < 2\pi$ and also (R, y^*) become

$$R = c\sqrt{(1-\mu^2)(\xi^2 + 1)} \ ,$$

$$y^* = c\mu\zeta \ ,$$

where $\zeta = \sinh u$ and $\mu = \cos\nu$. Eventually, on the center of the spheroid, a spherical co-ordinate system (r, θ, ψ) is described

$$x = r\sin\theta\cos\psi \ ,$$

$$y^* = r\cos\theta \ ,$$

$$z = r\sin\theta\sin\psi \ ,$$

where $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \psi < 2\pi$.

3. Mathematical Formulation

In the basis of linear potential flow theory the following assumptions, incompressible, irrotational flow, and non-viscous fluid is considered. A monochromatic incident wave with wave frequency ω is assumed. Considering Φ, the velocity potential (in spherical co-ordinate) ϕ, the complex potential, the small amplitude fluid motion is

$$\Phi(u,\nu,\psi,t) = Re\{\phi(u,\nu,\psi)e^{-\imath\omega t}\} \ . \tag{1}$$

In the content of linearized water wave theory ϕ satisfies the laplace equation,

$$\nabla^2\phi = 0 \ . \tag{2}$$

The linearized condition at the ice-cover modelled as thin elastic plate is

$$(D\frac{\partial^4}{\partial x^4} + 1 - \epsilon\lambda_0)\phi_y + \lambda_0\phi = 0 \ , \tag{3}$$

where $\lambda_0 = \frac{\omega^2}{g}$, where g is the acceleration due to gravity. Here $D = \frac{Eh_0^3}{12(1-\nu^2)\rho_1 g}$ is the flexural rigidity of the ice-cover, where E and ν are respectively Young's modulus and Poisson's ratio of the elastic material of the ice-cover, h_0 is the very small thickness of ice-cover, $\epsilon = \frac{\rho_0}{\rho_1}h_0$ with ρ_0 and ρ_1 densities of ice and water respectively and the bottom boundary condition for $y \to \infty$

$$\vec{\nabla}\phi \to 0 \ . \tag{4}$$

Also λ_1 is the unique real root of the dispersion equation $(Dk^4 + 1 - \epsilon\lambda_0)k - \lambda_0 = 0$. It is assumed that the total velocity potential ϕ is constructed by the combination of the incident and the diffraction component which will be denoted by ϕ_{inc}, ϕ_D respectively. The diffraction potential function ϕ_D satisfy the radiation condition

$$\lim_{R\to\infty}\sqrt{R}(\frac{\partial}{\partial R} - i\lambda_1)\phi_D = 0 \ .$$

On the surface of the body boundary condition is given by

$$(\frac{\partial\phi}{\partial u})_{u=r_0} = 0 \ ,$$

where r_0 is used to denote the boundary of the body in spheroidal co-ordinates. In fact r_0 is given by $r_0 = \tanh^{-1}(\frac{b}{a})$ where $a = c\cosh r_0$ and $b = c\sinh r_0$, denote respectively the semi major and semi minor axes of the spheroid.

4. Solution of the problem

The diffraction potential ϕ_D can be expressed in term of multipole potential function and it has the singularity at the center of spheroidal body within the fluid with ice-cover surface. It can be written as (cf. Chatjigeorgiou [24])

$$\phi_D(u,\nu,\psi) = \omega A e^{-\lambda_1 f}\sum_{n=0}^{\infty}\phi_D^m(u,\nu)\cos m\psi \ .$$

The expansion form of ϕ_D^m is given by

$$\phi_D^m = a^{m+2}F_m^m\phi_m^m + \sum_{n=m+1}^{\infty} a^{n+2}F_n^m\chi_n^m \ , \tag{5}$$

where, F_n^m is unknown expansion coefficients and χ_n^m is the wave free potential having singularity with ice-cover on water (cf. Dhillon and Mandal [28])

$$\chi_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{\lambda_1}{n-m}\frac{P_{n-1}^m(\cos\theta)}{r^n} + \frac{(-1)^{n+m}}{(n-m)!}\int_0^\infty g_1(k)k^n e^{-ky}J_m(kR)dk \ . \quad (6)$$

Note that J_m is the mth order Bessel function of the first kind, P_n^m is the associated Legendre function of the first kind with order m, degree n and

$$g_1(k) = \frac{(Dk^4+1-\epsilon\lambda_0)k+\lambda_0}{D(k^4+k^3\lambda_1+k^2\lambda_1^2+k\lambda_1^3+\lambda_1^4+1-\epsilon\lambda_1)} \ .$$

Also the expansion of ϕ_n^m is given in spherical and polar co-ordinates (cf. Das and Mandal [11])

$$\phi_n^m(r,\theta) = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n+m}}{(n-m)!}\fint_0^\infty \frac{(Dk^4+1-\epsilon\lambda_0)k+\lambda_0}{(Dk^4+1-\epsilon\lambda_0)k-\lambda_0}k^n e^{-k(y+f)}J_m(kR)dk \ . \quad (7)$$

Here the first term harmonize to multipole singularity in an infinite medium and is connected with the second term to satisfy the ice-cover condition.

An incident wave potential of amplitude A is described by

$$\phi_{inc} = \omega A e^{-\lambda_1 y} + i\lambda_1\rho r_0\cos\psi = \omega A\sum_{m=0}^{\infty}\phi_{inc}^m(u,\nu)\cos m\psi,$$

where,

$$\phi_{inc}^m(R,y) = \frac{1}{\lambda_1}e^{-\lambda_1 y}\varepsilon_m i^m J_m(\lambda_1 R) \quad (8)$$

and $\varepsilon_m = 1$ for $m=0$, $\varepsilon_m = 2$ for $m \geq 1$.

5. Transformation

The incident wave potential and diffraction potential are in polar co-ordinates. Our primary object is to transform it into spheroidal Co-ordinate system for the application of the higher order boundary condition. For that reason taking the result (cf. Chatjigeorgiou [24])

$$J_m(kR)e^{-ky^*} = \sqrt{\frac{\pi}{2kc}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}I_{s+\frac{1}{2}}(kc)P_s^m(\mu)P_s^m(\xi) \ , \quad (9)$$

where $I_{s+\frac{1}{2}}$ is the modified bessel function of order $s+\frac{1}{2}$. Replacing y^* by $y-f$ in (9), it takes the form

$$J_m(kR)e^{-ky} = e^{-kf}\sqrt{\frac{\pi}{2kc}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}I_{s+\frac{1}{2}}P_s^m(\mu)P_s^m(\xi) \ , \quad (10)$$

(10) is used in the transformation of incident wave potential as well as diffraction potential. Also, in (6), (7) the term $\frac{P_n^m(\cos\theta)}{r^{n+1}}$ is in polar co-ordinate. So as per requirement the next

step is to convert $\frac{P_n^m(\cos\theta)}{r^{n+1}}$ into spheroidal co-ordinate. Starting with the well-known result (cf. Thorne [23])

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = \frac{2}{\pi(n-m)!}\int_0^\infty k^n \cos(k(y-f)-(n-m)\frac{\pi}{2})K_m(kR)dk, \tag{11}$$

K_m is the mth order of modified bessel function. Using $y^* = y - f$, (11) takes the form

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = \frac{e^{-i(n-m)\frac{\pi}{2}}}{\pi(n-m)!}\int_0^\infty k^n e^{iky^*}K_m(kR)dk + \frac{e^{i(n-m)\frac{\pi}{2}}}{\pi(n-m)!}\int_0^\infty k^n e^{-iky^*}K_m(kR)dk. \tag{12}$$

To evaluate the integration in (12) using the result for ascending series of bessel function

$$Y^{p+n} = \sum_{s=0}^{\infty}\frac{2^{p+n}(p+n+2s)\Gamma(p+n+2s)}{s!}J_{p+n+2s}(Y), \tag{13}$$

where $p+n$ is not a negative integer. If we put $y = ick$ and $p = \frac{1}{2}$ in (13) then after calculation we get

$$k^n = \frac{1}{(ic)^{(n+\frac{1}{2})}}k^{-\frac{1}{2}}\sum_{s=0}^{\infty}2^{n+\frac{1}{2}}\frac{(n+2s+\frac{1}{2})\Gamma(n+s+\frac{1}{2})}{s!}e^{i(n+2s+\frac{1}{2})\frac{\pi}{2}}I_{n+2s+\frac{1}{2}}(ck). \tag{14}$$

Then

$$\int_0^\infty k^n e^{\pm iky^*}K_m(kR)dk = \frac{1}{i^n c^{n+\frac{1}{2}}}\sum_{s=0}^{\infty}2^{n+\frac{1}{2}}\frac{(n+2s+\frac{1}{2})\Gamma(n+s+\frac{1}{2})}{s!}e^{i(n+2s)\frac{\pi}{2}}$$

$$\times\int_0^\infty e^{\pm iky^*}k^{-\frac{1}{2}}K_m(kR)I_{n+2s+\frac{1}{2}}(ck)dk. \tag{15}$$

Using the identities (cf. Cooke ([29], [30]))

$$(\frac{\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{-ky^*}k^{-\frac{1}{2}}J_m(kR)J_{n+\frac{1}{2}}(ck)dk = P_n^{-m}(\mu)q_n^m(\zeta) \tag{16}$$

and

$$(\frac{\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{-ky^*}k^{-\frac{1}{2}}Y_m(kR)J_{n+\frac{1}{2}}(ck)dk = -\frac{2}{\pi}Q_n^{-m}(\mu)q_n^m(\zeta) \tag{17}$$

where Y_m is the mth order Bessel function of the second kind and Q_n^m, q_n^m are the associate Legendre function of the second kind with real and complex arguments respectively,

$$q_n^m(\zeta) = e^{i(n+1)\frac{\pi}{2}-im\pi}Q_n^m(i\zeta). \tag{18}$$

After applying the relations of Hankel and Bessel functions

$$H_\nu^1(y) = J_\nu(y) + iY_\nu(y),$$

$$H_\nu^2(y) = J_\nu(y) - iY_\nu(y)$$

(16), (17) take the form,

$$(\frac{\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{-ky^*}k^{-\frac{1}{2}}H_m^1(kR)J_{n+\frac{1}{2}}(ck)dk = (P_n^{-m}(\mu) - \frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta) \tag{19}$$

and

$$(\frac{\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{-ky^*}k^{-\frac{1}{2}}H_m^2(kR)J_{n+\frac{1}{2}}(ck)dk=(P_n^{-m}(\mu)+\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta) \qquad (20)$$

where $H_m^{1,2}$ are the Hankel functions of order m of first and second kind respectively. In equation (19), (20) there is J_m instead of I_m. Therefore substituting c by ic in (19), it becomes

$$(\frac{i\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{-iky^*}k^{-\frac{1}{2}}H_m^1(ikR)J_{n+\frac{1}{2}}(ick)dk=(P_n^{-m}(\mu)-\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta) \qquad (21)$$

and substituting c by $-ic$ in (20) it becomes

$$(\frac{-i\pi c}{2})^{\frac{1}{2}}\int_0^\infty e^{iky^*}k^{-\frac{1}{2}}H_m^2(-ikR)J_{n+\frac{1}{2}}(-ick)dk=(P_n^{-m}(\mu)+\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta). \qquad (22)$$

Using the identities

$$H_m^1(ikR)=\frac{2}{\pi}(i)^{-m}K_m(kR),$$

$$H_m^2(-ikR)=\frac{2}{\pi}(i)^{m+1}K_m(kR),$$

and

$$J_\nu(iy)=i^\nu I_\nu(y)$$

(21), (22) becomes

$$\sqrt{(\frac{2c}{\pi})}\int_0^\infty e^{-iky^*}k^{-\frac{1}{2}}K_m(ikR)I_{n+\frac{1}{2}}(ck)dk=e^{-i(n-m)\frac{\pi}{2}}(P_n^{-m}(\mu)-\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta)$$

and

$$\sqrt{(\frac{2c}{\pi})}\int_0^\infty e^{iky^*}k^{-\frac{1}{2}}K_m(ikR)I_{n+\frac{1}{2}}(ck)dk=e^{i(n-m)\frac{\pi}{2}}(P_n^{-m}(\mu)+\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta).$$

Combining we get

$$\sqrt{(\frac{2c}{\pi})}\int_0^\infty e^{\mp iky^*}k^{-\frac{1}{2}}K_m(ikR)I_{n+\frac{1}{2}}(ck)dk=e^{\mp i(n-m)\frac{\pi}{2}}(P_n^{-m}(\mu)\mp\frac{2i}{\pi}Q_n^{-m}(\mu))q_n^m(\zeta. \qquad (23))$$

Using $n+2s$ instead of n and from (23), (14) and (18) we get

$$\frac{P_n^m(\cos\theta)}{r^{n+1}}=\frac{(-1)^m(\frac{2}{c})^{n+1}}{(n-m)!\sqrt{\pi}}e^{i(n+1)\frac{\pi}{2}}\sum_{s=0}^\infty(-1)^s\frac{(n+2s+\frac{1}{2})\Gamma(n+s+\frac{1}{2})}{s!}Q_{n+2s}^m(i\zeta)P_{n+2s}^{-m}(\mu). \qquad (24)$$

Here we are applying the identity (cf. Gradshteyn and Ryzhik [31])

$$P_n^{-m}(x)=(-1)^m\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}P_n^m(x)$$

(24) is expressed in oblate spheroidal co-ordinate system. Replacing c by ic and $i\zeta$ by ξ, we get the desired expression of $\frac{P_n^m(\cos\theta)}{r^{n+1}}$ in prolate spheroidal co-ordinate which is

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = \frac{(-1)^m(\frac{2}{c})^{n+1}}{(n-m)!\sqrt{\pi}} \sum_{s=m}^{\infty} \frac{(-1)^s\Gamma(n+s-m+\frac{1}{2})\Gamma(n+2s-3m+1)}{\Gamma(n+2s-m+1)\Gamma(s-m+1)}$$

$$\times(n+2s-2m+\frac{1}{2})P_{n+2s-2m}^m(\mu)Q_{n+2s-2m}^m(\xi). \tag{25}$$

Applying (10), (25)the terms in ϕ_m^m and χ_n^m in ϕ_D^m can be calculated. Now

$$\phi_m^m = \frac{(-1)^m(\frac{2}{c})^{m+1}}{\sqrt{\pi}} \sum_{s=m}^{\infty}(-1)^s \frac{\Gamma(s+\frac{1}{2})\Gamma(2s-2m+1)(2s-m+\frac{1}{2})}{\Gamma(2s+1)\Gamma(s-m+1)} P_{2s-m}^m(\mu)Q_{2s-m}^m(\xi)$$

$$+\sqrt{\frac{\pi}{2c}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}J(\lambda_1 f,m,s)P_s^m(\mu)P_s^m(\xi), \tag{26}$$

where

$$J(\lambda_1 f,m,s) = ⨍_0^\infty \frac{(Dk^4+1-\epsilon\lambda_0)k+\lambda_0}{(Dk^4+1-\epsilon\lambda_0)k-\lambda_0} k^{m-\frac{1}{2}}e^{-2kf}I_{s+\frac{1}{2}}(kc)dk. \tag{27}$$

In the second term of the expression of ϕ_m^m equation (12) has been used. (26)is the complete form of ϕ_m^m in the spheroidal co-ordinate system. Also the wave free potential χ_n^m is

$$\chi_n^m = \frac{(-1)^m(\frac{2}{c})^{n+1}}{(n-m)!\sqrt{\pi}} \sum_{s=m}^{\infty}(-1)^s \frac{\Gamma(n+s-m+\frac{1}{2})\Gamma(n+2s-3m+1)(n+2s-2m+\frac{1}{2})}{\Gamma(n+2s-m+1)\Gamma(s-m+1)}$$

$$\times P_{n+2s-2m}^m(\mu)Q_{n+2s-2m}^m(\xi)$$

$$+\frac{\lambda_1(-1)^m(\frac{2}{c})^n}{(n-m)!\sqrt{\pi}} \sum_{s=m}^{\infty}(-1)^s \frac{\Gamma(n+s-m-\frac{1}{2})\Gamma(n+2s-3m)(n+2s-2m-\frac{1}{2})}{\Gamma(n+2s-m)\Gamma(s-m+1)}$$

$$\times P_{n+2s-2m-1}^m(\mu)Q_{n+2s-2m-1}^m(\xi)$$

$$+\frac{(-1)^{m+n}}{(n-m)!}\sqrt{\frac{\pi}{2c}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}A_{ns}P_s^m(\mu)P_s^m(\xi), \tag{28}$$

where

$$A_{ns} = \int_0^\infty e^{-kf}k^{n-\frac{1}{2}}g_1(k)I_{s+\frac{1}{2}}(kc)dk. \tag{29}$$

The desired expression of diffraction potential is

$$\phi_D^m = a^{m+2}F_m^m\{\frac{(\frac{2}{c})^{m+1}}{\sqrt{\pi}} \sum_{s=m}^{\infty}(-1)^{s+m} \frac{\Gamma(s+\frac{1}{2})\Gamma(2s-2m+1)(2s-m+\frac{1}{2})}{\Gamma(2s+1)\Gamma(s-m+1)} P_{2s-m}^m(\mu)$$

$$\times Q_{2s-m}^m(\xi)$$

$$+\sqrt{\frac{\pi}{2c}}\sum_{s=m}^{\infty}(-1)^{s-m}\frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)}J(\lambda_1 f,m,s)P_s^m(\mu)P_s^m(\xi)\} + \sum_{n=m+1}^{\infty} a^{n+2}F_n^m$$

$$\times\{\frac{(\frac{2}{c})^{n+1}}{(n-m)!\sqrt{\pi}} \sum_{s=m}^{\infty}(-1)^{s+m} \frac{\Gamma(n+s-m+\frac{1}{2})\Gamma(n+2s-3m+1)(n+2s-2m+\frac{1}{2})}{\Gamma(n+2s-m+1)\Gamma(s-m+1)}$$

$$\times P^m_{n+2s-2m}(\mu) Q^m_{n+2s-2m}(\xi)$$
$$+\frac{\lambda_1(\frac{2}{c})^n}{(n-m)!\sqrt{\pi}} \sum_{s=m}^{\infty} (-1)^{s+m} \frac{\Gamma(n+s-m-\frac{1}{2})\Gamma(n+2s-3m)(n+2s-2m-\frac{1}{2})}{\Gamma(n+2s-m)\Gamma(s-m+1)}$$
$$\times P^m_{n+2s-2m-1}(\mu) Q^m_{n+2s-2m-1}(\xi)$$
$$+\frac{1}{(n-m)!}\sqrt{\frac{\pi}{2c}} \sum_{s=m}^{\infty} (-1)^{s+n} \frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)} A_{ns} P^m_s(\mu) P^m_s(\xi)\}. \tag{30}$$

Applying (10) in the component of incident wave potential, it becomes,

$$\phi^m_{inc} = \frac{1}{\lambda_1} e^{-\lambda_1 f} \varepsilon_m \imath^m \sqrt{\frac{\pi}{2\lambda_1 c}} \sum_{s=m}^{\infty} (-1)^{s-m} \frac{(2s+1)\Gamma(s-m+1)}{\Gamma(s+m+1)} I_{s+\frac{1}{2}}(\lambda_1 c) P^m_s(\mu) P^m_s(\xi), \tag{31}$$

which is the desired expression in spheroidal co-ordinate.

6. Expansion coefficient

To find the solution of the problem unknown coefficient F^m_n in equation (30) is determined from the surface body boundary condition $(\frac{\partial \phi}{\partial u})_{u=r_0} = 0$ and it transforms to

$$\frac{\partial \hat{\phi}^m_D(u,\vartheta)}{\partial u} = e^{-\lambda_1 f} \frac{\partial \hat{\phi}^m_{inc}(u,\vartheta)}{\partial u}.$$

It is applied for $u = r_0$ for $0 \leq \vartheta \leq \pi$. Taking $\frac{dP^m_s(\xi)}{du} = P'^m_s(\xi_0)\sinh r_0$ and $\frac{dQ^m_s(\xi)}{du} = Q'^m_s(\xi_0)\sinh r_0$, where $\xi_0 = \cosh r_0$. To simplify this the orthogonality relation of associated Legendre function of the first kind is employed.

$$\int_{-1}^{1} P^m_n(\mu) P^m_s(\mu) d\mu = \delta_{ns} \frac{(n+m)!}{(n+\frac{1}{2})(n-m)!}$$

where δ_{ns} is Kronecker's delta. After step by step calculations the required system of equation involving F^m_n takes the form:

$$F^m_m C^m_{ms} + \sum_{n=m+1}^{\infty} F^m_n C^m_{ns} = B^m_s. \tag{32}$$

To find the solution, equation (32) has to be truncated for a finite number of modes M. Here the indices vary like $m = 0, 1, 2, ..., M$ and $n, s = m, m+1, ..., M$. For different values of m, the elements C^m_{ns}, C^m_{ms} and B^m_s frame different complex matrices. When $m = 0$ the elements C^m_{ns}, C^m_{ms} and B^m_s formulate $N \times N$, $N \times 1$ and $N \times 1$ complex matrices, where $N = M + 1$. The expressions of B^m_s, C^m_{ms} and C^m_{ns} are

$$B^m_s = -\frac{2(-1)^{s+m}}{\lambda_1} \epsilon_m \imath^m \sqrt{\frac{\pi}{2\lambda_1 c}} I_{s+\frac{1}{2}}(\lambda_1 c) P'^m_s(\xi_0)\sinh r_0,$$

$$C^m_{ms} = a\{\frac{(-1)^{s+m}(\frac{2a}{c})^{m+1}}{\pi^{\frac{1}{2}}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-m+1)} Q'^m_{2s-m}(\xi_0)\sinh r_0$$

$$+2(-1)^{s+m}a^{m+1}\sqrt{\frac{\pi}{2c}}J(\lambda_1 f,m,s)P_s'^m(\xi_0)\sinh r_0\},$$

$$C_{ns}^m = a\{\frac{(-1)^{s+m}(\frac{2a}{c})^{n+1}}{(n-m)!\pi^{\frac{1}{2}}}\frac{\Gamma(n+s-m+\frac{1}{2})}{\Gamma(s-m+1)}Q'^m_{n+2s-2m}(\xi_0)\sinh r_0+\frac{\lambda_1(\frac{2a}{c})^n a(-1)^{s+m}}{(n-m)!\pi^{\frac{1}{2}}}$$
$$\times\frac{\Gamma(n+s-m-\frac{1}{2})}{\Gamma(s-m+1)}Q'^m_{n+2s-2m-1}(\xi_0)\sinh r_0+2(-1)^{n+s}a^{n+1}\sqrt{\frac{\pi}{2c}}A_{n,s}P_s'^m(\xi_0)\sinh r_0\}.$$

7. Linear exciting forces

The exciting forces are obtained by the linear hydrodynamics pressure on the wetted surface of the body is(cf. Chatjigeorgiou [24])

$$F_r = -i\omega\rho\int\int_{S_0}(\phi_{inc}+\phi_D)n_r dS. \tag{33}$$

where S_0 denotes the wetted surface of the spheroid, ρ is the water density, $n_r(r=x,y,z)$ is the normal out of the body surface with

$$n_x = \frac{a\sin\vartheta\cos\psi}{(b^2\cos^2\vartheta+a^2\sin^2\vartheta)^{\frac{1}{2}}}, \tag{34}$$

$$n_y = \frac{b\cos\vartheta}{(b^2\cos^2\vartheta+a^2\sin^2\vartheta)^{\frac{1}{2}}}, \tag{35}$$

$$n_z = \frac{a\sin\vartheta\sin\psi}{(b^2\cos^2\vartheta+a^2\sin^2\vartheta)^{\frac{1}{2}}}. \tag{36}$$

Therefore, the surge and heave exciting forces will be

$$F_x = -i\omega\rho ab\int_0^{2\pi}\{\int_0^{\pi}(\phi_{inc}+\phi_D)P_1^1(\mu)\sin\vartheta d\vartheta\}\cos\psi d\psi, \tag{37}$$

$$F_y = -i\omega\rho b^2\int_0^{2\pi}\{\int_0^{\pi}(\phi_{inc}+\phi_D)P_1^0(\mu)\sin\vartheta d\vartheta\}d\psi, \tag{38}$$

where $P_1^1(\mu) = -\sin\vartheta$ and $P_1^0(\mu) = -\cos\vartheta$. The sway force F_z becomes zero.

$$\frac{F_x}{\frac{4}{3}\rho\pi\omega^2 a^2 bAe^{-\lambda_1 f}} = i\frac{3}{4}\{a^2F_1^1\int_0^{\pi}\hat{\phi}_1^1P_1^1(\mu)\sin\vartheta d\vartheta+\sum_{n=2}^{\infty}a^{n+1}F_n^1\int_0^{\pi}\hat{\chi}_n^1P_1^1(\mu)\sin\vartheta d\vartheta\}, \tag{39}$$

$$\frac{F_y}{\frac{4}{3}\rho\pi\omega^2 a^2 bAe^{-\lambda_1 f}} = -i\frac{3b}{2a}\{aF_0^0\int_0^{\pi}\hat{\phi}_0^0P_1^0(\mu)\sin\vartheta d\vartheta+\sum_{n=1}^{\infty}a^{n+1}F_n^0\int_0^{\pi}\hat{\chi}_n^0P_1^0(\mu)\sin\vartheta d\vartheta\}, \tag{40}$$

where

$$\int_0^{\pi}\hat{\phi}_1^1P_1^1(\mu)\sin\vartheta d\vartheta = (\frac{2}{c})^2\frac{1}{2}Q_1^1(\xi_0)+2\sqrt{\frac{\pi}{2c}}J(\lambda_1 f,1,1)P_1^1(\xi_0)+2\pi i\sqrt{\frac{\pi}{2c}}K_0(\lambda_1)^{\frac{1}{2}}e^{-2\lambda_1 f}$$
$$\times P_1^1(\xi_0)I_{\frac{3}{2}}(\lambda_1 c), \tag{41}$$

$$\int_0^\pi \hat{\phi}_0^0 P_1^0(\mu)\sin\vartheta d\vartheta = -2\sqrt{\frac{\pi}{2c}}J(\lambda_1 f,0,1)P_1^0(\xi_0) - 2\pi i\sqrt{\frac{\pi}{2c}}K_0(\lambda_1)^{-\frac{1}{2}}e^{-2\lambda_1 f}P_1^0(\xi_0)I_{\frac{3}{2}}(\lambda_1 c), \tag{42}$$

(where, $K_0 = \frac{(D(\lambda_1)^4+1-\epsilon\lambda_0)\lambda_1+\lambda_0}{5D(\lambda_1)^4+1-\epsilon\lambda_0}$)

$$\begin{aligned}\int_0^\pi \hat{\chi}_n^1 P_1^1(\mu)\sin\vartheta d\vartheta &= \frac{(\frac{2}{c})^{n+1}}{(n-1)!\sqrt{\pi}}\frac{(\frac{1}{2}+n)\Gamma(\frac{1}{2}+n)\Gamma n}{\Gamma(n+2)}Q_n^1(\xi_0)\int_0^\pi P_n^1(\mu)P_1^1(\mu)\sin\vartheta d\mu \\ &+\frac{(\frac{2}{c})^n\lambda_1}{(n-1)!\sqrt{\pi}}\frac{(n-\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}Q_{n-1}^1(\xi_0)\int_0^\pi P_{n-1}^1(\mu)P_1^1(\mu)\sin\vartheta d\mu \\ &+\frac{2}{(n-1)!}\sqrt{\frac{\pi}{2c}}(-1)^{n+1}A_{n1}P_1^1(\xi_0),\end{aligned} \tag{43}$$

and

$$\begin{aligned}&\int_0^\pi \hat{\chi}_n^0 P_1^0(\mu)\sin\vartheta d\vartheta = \frac{(\frac{2}{c})^{n+1}}{n!\sqrt{\pi}}(\frac{1}{2}+n)\Gamma(\frac{1}{2}+n)Q_n^0(\xi_0)\int_0^\pi P_n^0(\mu)P_1^0(\mu)\sin\vartheta d\mu \\ &+\frac{(\frac{2}{c})^n\lambda_1}{n!\sqrt{\pi}}(n-\frac{1}{2})\Gamma(n-\frac{1}{2})Q_{n-1}^0(\xi_0)\int_0^\pi P_{n-1}^0(\mu)P_1^0(\mu)\sin\vartheta d\mu + \frac{2}{n!}\sqrt{\frac{\pi}{2c}}(-1)^{n+1}A_{n1}P_1^0(\xi_0).\end{aligned} \tag{44}$$

8. Numerical results

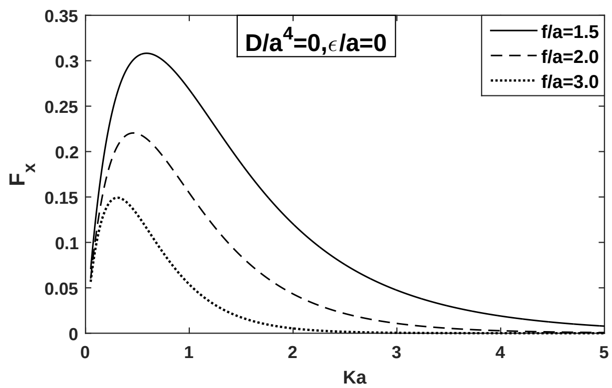

Fig.2:Magnitudes of surge exciting force on prolate spheroid (b/a=0.4) with various immersions

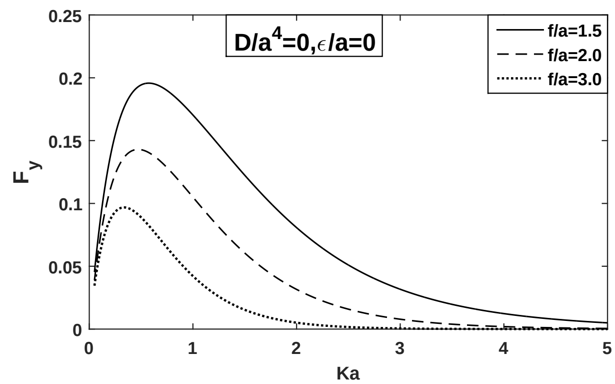

Fig.3:Magnitude of heave exciting force on a prolate spheroid (b/a=0.4) for various immersions

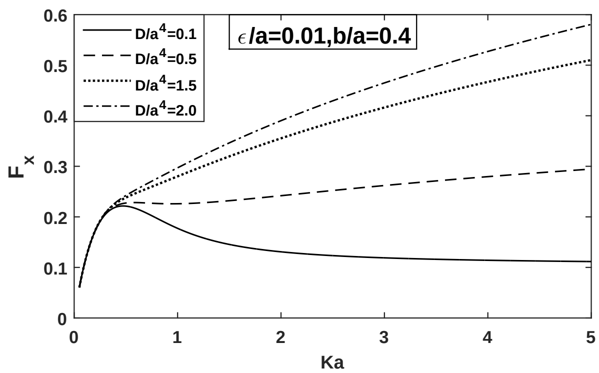

Fig.4:Magnitude of surge exciting force on prolate spheroid with fixed immersion f/a=2.0

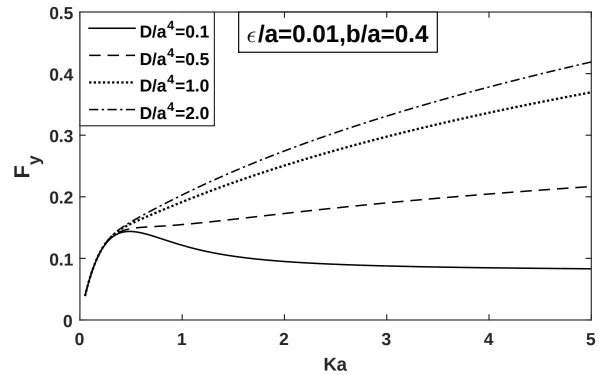

Fig.5:Magnitude of heave exciting force on prolate spheroid with fixed immersion f/a=2.0

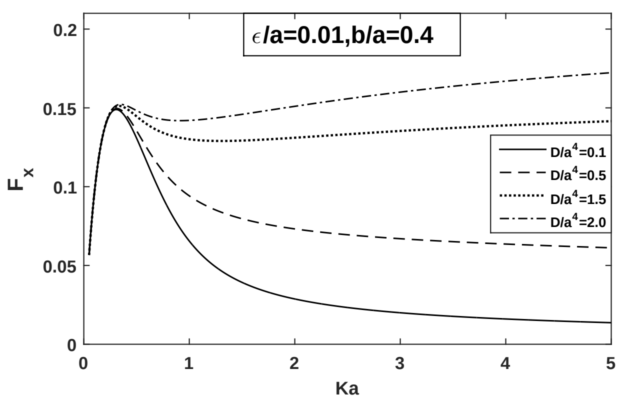

Fig.6:Magnitudes of surge exciting force on prolate spheroid with fixed immersion f/a=3.0

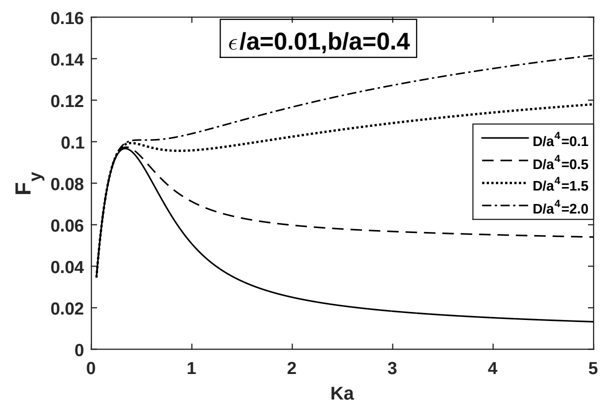

Fig.7:Magnitudes of heave exciting forces on prolate spheroid with fixed immersions f/a=3.0

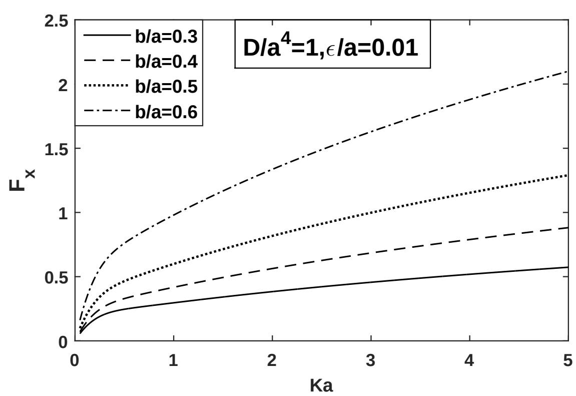

Fig.8:Magnitudes of surge exciting force on prolate spheroid with fixed immersion f/a=1.5

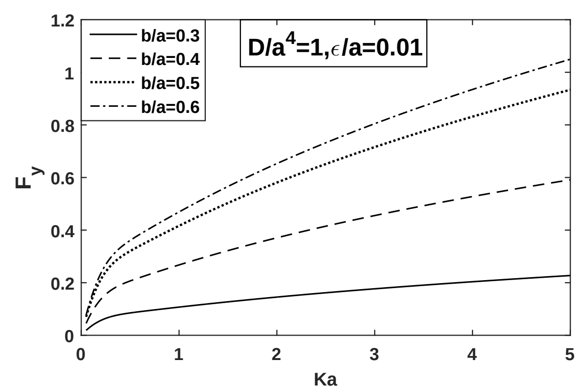

Fig.9:Magnitudes of heave exciting force on prolate spheroid with fixed immersion f/a=1.5

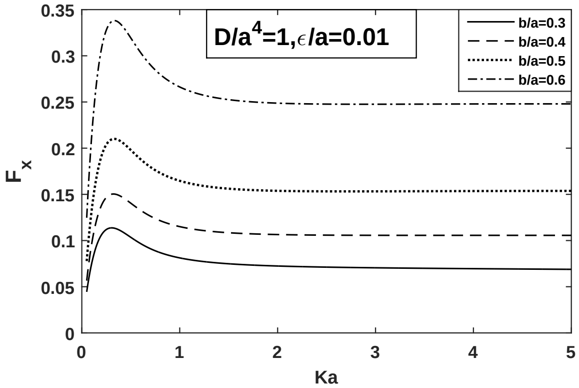

Fig.10:Magnitudes of surge exciting forces on prolate spheroid with fixed immersions f/a=3.0

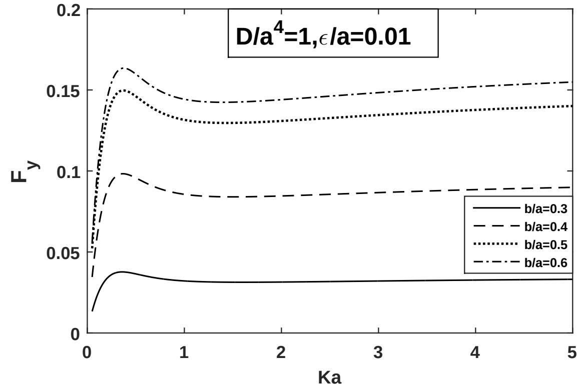

Fig.11:Magnitudes of heave exciting force on prolate spheroid with fixed immersions f/a=3.0

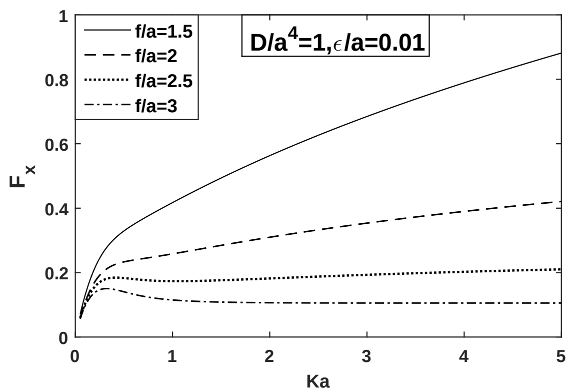

Fig.12:Magitudes of surge exciting force on prolate spheriod (b/a=0.4)with various immersions

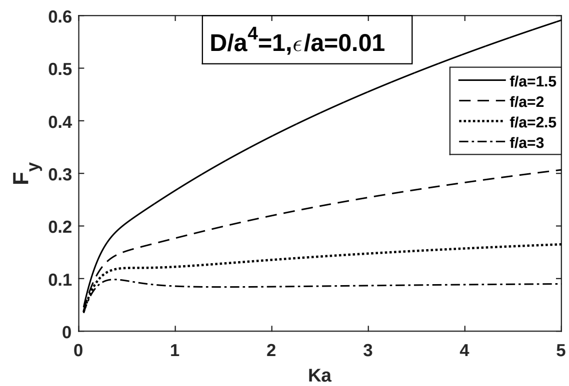

Fig.13:Magnitudes of heave exciting force on prolate spheriod (b/a=0.4) with various immersions

The hydrodynamic forces (Surge and heave) are plotted in Fig.2-13 when the prolate spheroid submerged in water with an ice-cover. In Fig.2 and 3, the different curves correspond to different values of $\frac{f}{a}$ e.g. $\frac{f}{a} = 1.5, 2.0, 3.0$ with $\frac{D}{a^4} = 0$,$\frac{\epsilon}{a} = 0$ and $\frac{b}{a} = 0.4$. Fig.2 and Fig.3 show the surge exciting force and heave exciting force respectively for the prolate spheroid. It is observed that forces increase when the spheroid is close to the ice-cover ($\frac{f}{a} = 1.5$) for both the cases. All the curves for both cases first increases with increasing Ka, then obtained a moderate value and after that decrease with increasing Ka. Also it is noted that for $\frac{D}{a^4} = 0$ and $\frac{\epsilon}{a} = 0$, then the ice-cover becomes a free surface and for this case the curves in Fig.2 and 3 exactly coincide with the corresponding curves for a forces exciting on the prolate spheroid submerged in water with free surface (cf. figures 4 and 5 in I.K.Chatjigeorgiou [24]). Also it is observed that the surge exciting forces in Fig.2 is greater than for heave (Fig.3) for low to moderate values of Ka.

Fig.4, 5, 6, 7 depict the exciting forces (Surge and heave) plotted against Ka for different values flexural rigidity $\frac{D}{a^4}$ of the ice-cover, e.g. $\frac{D}{a^4} = 0.1, 0.5, 1.5, 2.0$ and in all the cases $\frac{\epsilon}{a}$ is 0.01 and $\frac{b}{a} = 0.4$ but $\frac{f}{a}$ is 2 for Fig.4 and 5 and $\frac{f}{a}$ is 3 for Fig.6 and 7. Here it is observed that cases in the figures increase with $\frac{D}{a^4}$ increases for all the cases. But the curves for surge exciting forces is greater than the exciting forces for heave prolate spheroid. It is also noted that when spheroid is closed to the ice-cover ($\frac{f}{a} = 2$) the deviation of the forces is somewhat greater than the case of spheroid is deeply submerged ($\frac{f}{a} = 3$).

The Fig.8, 9, 10, 11 depict the exciting forces (both surge and heave) plotted against Ka for different values of $\frac{b}{a}$, e.g. $\frac{b}{a} = 0.3, 0.4, 0.5, 0.6$ and in all the cases $\frac{D}{a^4} = 0$ and $\frac{\epsilon}{a} = 0.01$ but $\frac{f}{a} = 1.5$ for figures 8 and 9 and $\frac{f}{a} = 3.0$ for figures 10 and 11. It is observed that in all the cases the forces increase with $\frac{b}{a}$ increases. These also lead to somewhat similar results as in the figs.4-7 and display the same characteristics but figures for surge is greater than that obtained for heave. Fig.12-13 depict the surge and heave exciting forces respectively. It is observed that when the spheroid is closed to the ice-cover surface ($\frac{f}{a} = 1.5$), large amount of forces are generated.

9. Conclusion

The problem of wave diffraction by a prolate spheroid submerged in water beneath the free surface is extended here when the free surface is replaced by a thin ice-cover modelled as a thin elastic plate. Numerical results for the exciting forces for surge and heave spheroid are obtained. The method of multipoles has been shown to be an extremely powerful method for solving the problem involving submerged spheroid and the numerical results for exciting force are obtained for different values of various parameters. When the ice-cover is replaced by a free surface (by making $\frac{D}{a^4} = 0, \frac{\epsilon}{a} = 0$) curves for exciting forces exactly coincide with the curves for the case of deep water with free surface.

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