

# 1 Deviation of Mercury's spin axis from an exact Cassini 2 state induced by dissipation

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## 5 **Key Points:**

- 6 • Viscous and electromagnetic drag at the fluid core boundaries generate a deviation that  
7 does not exceed 0.1 arcsec
- 8 • In units of arcsec, the phase lag from tidal dissipation follows the empirical relation  $(80/Q)$ ,  
9 where  $Q$  is the quality factor
- 10 • The maximum phase lag allowed by observations gives a lower limit on the bulk man-  
11 tle viscosity of approximately  $10^{17}$  Pa s

**Abstract**

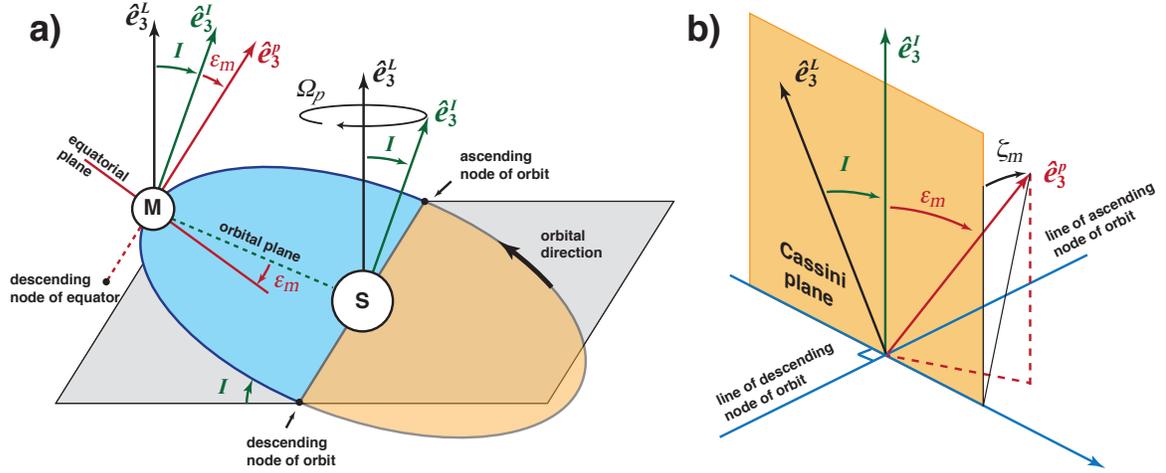
We compute predictions of the deviation of Mercury’s spin axis from an exact Cassini state caused by tidal dissipation, and viscous and electromagnetic (EM) friction at the core-mantle boundary (CMB) and inner core boundary (ICB). Viscous friction at the CMB generates a phase lead, viscous and EM friction at the ICB produce a phase lag; the magnitude of the deviation depends on the inner core size, kinematic viscosity and magnetic field strength, but cannot exceed an upper bound. For a small inner core, viscous friction at the CMB results in a maximum phase lead of 0.027 arcsec. For a large inner core (radius > 1000 km), EM friction at the ICB generates the largest phase lag, but it does not exceed 0.1 arcsec. Elastic deformations induced by the misaligned fluid and solid cores play a first order role in the phase lead/lag caused by viscous and EM coupling, and contribute to a perturbation in mantle obliquity on par with that caused by tidal deformations. Tidal dissipation results in a phase lag and its magnitude (in units of arcsec) is given by the empirical relation  $(80/Q)$ , where  $Q$  is the quality factor;  $Q=80$  results in a phase lag of  $\sim 1$  arcsec. A large inner core with a low viscosity of the order of  $10^{17}$  Pa s or lower can significantly affect  $Q$  and thus the resulting phase lag. The limited mantle phase lag suggested by observations ( $<10$  arcsec) implies a lower limit on the bulk mantle viscosity of approximately  $10^{17}$  Pa s.

**Plain language summary:** As Mercury orbits the Sun, the plane of its orbit is slowly precessing about a fixed axis in space. This locks the spin axis of Mercury into its own precession at the same rate. This configuration is known as a Cassini state in which the spin axis is oriented in the same plane as that formed by the orbit normal and the fixed axis (the Cassini plane). Dissipation introduces a small deviation of Mercury’s spin axis from the Cassini plane. We compute predictions of this deviation. We show that viscous and electromagnetic friction at the boundaries of the fluid core result in a limited deviation which does not exceed 0.1 arcsec. Dissipation from tidal deformations produce a deviation that is inversely proportional to the mantle viscosity, a measure of how stiff the mantle is. Measurements of the orientations of Mercury’s spin axis in space limit the deviation away from the Cassini plane to a phase lag of approximately 10 arcsec, and our results show that this implies that the mantle viscosity cannot be much smaller than  $10^{17}$  Pa s.

**1 Introduction**

The spin axis of Mercury is in a Cassini state (Figure 1). The latter describes a configuration in which the planet’s spin axis and orbit normal remain coplanar to and precess about the normal to the Laplace plane [Colombo, 1966; Peale, 1969, 2006]. The precession is retrograde, and the latest estimate of its period is  $325,513 \pm 10,713$  years [Baland *et al.*, 2017]. Figure 2 shows the orientation in space of the spin axis reported in several recent studies, expressed at the J2000 epoch as is the usual convention. A visual inspection of Figure 2 reveals that, within measurement errors, Mercury’s spin axis aligns with the plane defined by the Laplace pole and orbit normal, a plane which we refer to as the Cassini plane, confirming that Mercury occupies a Cassini state.

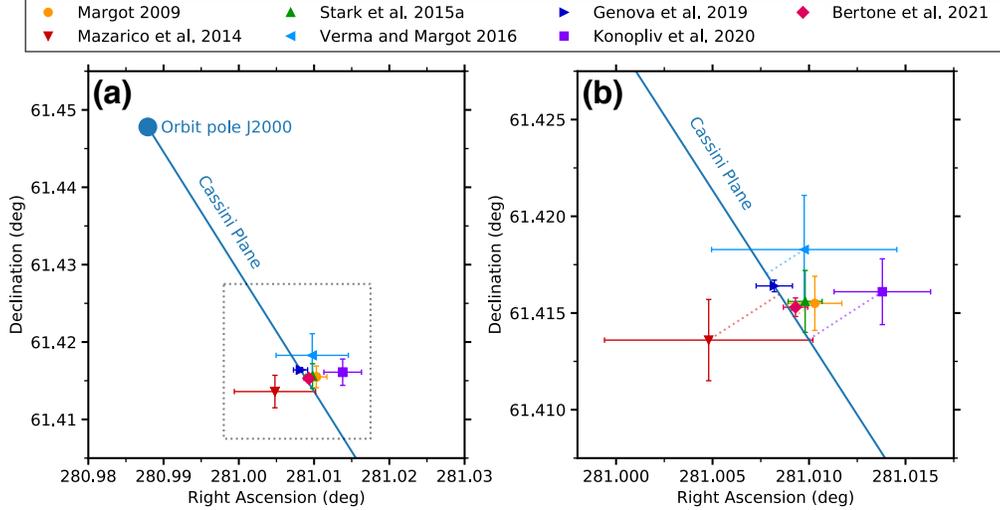
The retrograde precession of the Cassini plane implies that the line that depicts its location in Figure 2 is displaced toward the bottom-left as a function of time. Hence, a spin pole located to the top-right (bottom-left) with respect to this line is behind (ahead of) the expected



65 **Figure 1.** The Cassini state of Mercury. (a) The orbit of Mercury (M) around Sun (S) with re-  
 66 spect to the Laplace plane (grey shaded rectangle) and the Cassini state of Mercury. The normal to the  
 67 orbital plane ( $\hat{e}_3^I$ ) is offset from the normal to the Laplace plane ( $\hat{e}_3^L$ ) by an angle  $I = 8.5330^\circ$ . The  
 68 symmetry axis of the mantle  $\hat{e}_3^P$  (assumed to be exactly aligned with the mantle rotation vector in this  
 69 cartoon) is offset from  $\hat{e}_3^I$  by an obliquity angle of  $\varepsilon_m \approx 2$  arcmin. Both  $\hat{e}_3^I$  and  $\hat{e}_3^P$  precess about  $\hat{e}_3^L$  in  
 70 a retrograde direction at frequency  $\Omega_p = 2\pi/325,513 \text{ yr}^{-1}$ . The blue (orange) shaded region indicates  
 71 the portion of the orbit when Mercury is above (below) the Laplace plane. (b) In an ideal Cassini state,  
 72  $\hat{e}_3^P$  lies in the plane defined by  $\hat{e}_3^I$  and  $\hat{e}_3^L$  (the Cassini plane, orange shaded rectangle). Dissipation of  
 73 rotational energy displaces  $\hat{e}_3^P$  out of the Cassini plane by a phase-lag angle  $\zeta_m$ . In the complex notation  
 74 used in our study,  $\zeta_m = \text{Im}[\tilde{\varepsilon}_m]$ . Angles in both panels are not drawn to scale but exaggerated for the  
 75 purpose of illustration.

54 Cassini state orientation, and corresponds to a phase lag (phase lead). We denote the offset from  
 55 the Cassini plane by an angle  $\zeta_m$ , defined positive for a phase lag (see Figure 1b). Table 1 gives  
 56 the spin pole orientations from the recent measurements that are plotted in Figure 2, as well  
 57 as their phase lag angles  $\zeta_m$ , calculated by the method described in Appendix A. For all spin  
 58 pole measurements, the  $1\sigma$  error on the phase lag is either larger than the phase lag itself, or  
 59 of similar magnitude. This confirms that, within measurement errors, Mercury's spin pole in-  
 60 deed occupies a Cassini state. The magnitude of the phase lags in Table 1 provides a quanti-  
 61 tative measure of the deviation from an exact Cassini state. For all spin pole measurements,  
 62 the phase lag is smaller than 10 arcsec; it is smaller than 1 arcsec for two of the most recent  
 63 measurements (those of *Genova et al.* [2019] and *Bertone et al.* [2021]). The only measurement  
 64 that suggest a phase lead ( $\zeta_m < 0$ ) is that from the study of *Mazarico et al.* [2014].

82 If the Mercury-Sun system were to be taken in isolation and if the dissipation of rotational  
 83 and orbital energy were negligibly small, Mercury would obey an exact Cassini state. However,  
 84 in reality, small deviations from an exact Cassini state are expected. First, the precession of  
 85 Mercury's pericentre at a period of 134,477 yr induced by gravitational forces from other plan-  
 86 ets generates a small nutation motion of approximately 0.85 arcsec of the spin axis with respect



**Figure 2.** a) Right ascension and declination angles of the spin pole of Mercury based on different studies. The location of the orbit pole and the orientation of the Cassini plane are taken from *Baland et al.* [2017]. b) Close up view near the spin pole locations. The horizontal and vertical lines indicate the  $1\sigma$  errors on right ascension and declination, respectively, and the dashed lines indicate the deviation to the Cassini plane. This deviation is the phase lag angle  $\zeta_m$ , positive (negative) for spin poles measurements located to the top-right (bottom-left) of the Cassini plane line.

to its position in the Cassini state [*Baland et al.*, 2017]. At epoch J2000, the phase of this nutation is such that the spin axis is displaced approximately perpendicular to (out-of) the Cassini plane, towards the top-right quadrant of Figure 2. As a result, the spin axis should not be aligned exactly with the Cassini plane, but it should lag behind it by an angle of  $\zeta_m \approx 0.85$  arcsec.

An additional deviation from the Cassini plane is expected from the dissipation of orbital and rotational energy which, even if small, is invariably present. Indeed, a Cassini state configuration is a state of minimum energy, and can only be attained as a result of dissipation of an earlier more energetic state. One source of dissipation is from tidal deformations that occur in response to the solar gravitational potential imposed on Mercury. Tidal deformations are never perfectly elastic, some of the energy being dissipated as heat within the planet. Tidal dissipation is characterized by a quality factor  $Q$ . As a reference, for  $Q \approx 100$ , a reasonable planetary value, tidal dissipation should induce a phase lag angle of  $\zeta_m \approx 1$  arcsec [*Baland et al.*, 2017]. A smaller  $Q$  would induce a larger  $\zeta_m$  and conversely, a larger  $Q$  would induce a smaller  $\zeta_m$ .

Taken together, the deviation away from the Cassini plane induced by the precession of the pericentre and tidal dissipation (based on  $Q \approx 100$ ) should lead to a phase lag of  $\zeta_m \approx 1.85$  arcsec. This is approximately equal to the error in  $\zeta_m$  derived from the spin pole orientation measurement of *Bertone et al.* [2021]. If we take this latter measurement as a benchmark, this implies that  $Q$  cannot be much smaller than 100.

106 Another source of dissipation is viscous and electromagnetic (EM) drag at the core-mantle  
 107 boundary (CMB) and inner core boundary (ICB) of Mercury. If the core of Mercury were fully  
 108 solidified, the orientation of the spin (and symmetry) axis depicted in Figures 1 and 2 would  
 109 characterize that of the entire planet. However, the electrically conducting core must be par-  
 110 tially fluid, as motions within it are required to sustain Mercury’s internally generated mag-  
 111 netic field [Anderson *et al.*, 2011, 2012; Johnson *et al.*, 2012]. The observed amplitude of Mer-  
 112 cury’s 88-day libration provides additional support for a partially fluid core [e.g. Margot *et al.*,  
 113 2007, 2012]. Just like the Earth, the central region of Mercury’s core may be solid, although  
 114 the size of this solid inner core, if it exists, is not well constrained [e.g. Steinbrügge *et al.*, 2021].  
 115 The measurements shown in Figure 2 reflect then the orientation of the spin (and symmetry)  
 116 axis of Mercury’s outer solid shell comprised of its mantle and crust. We do not have direct mea-  
 117 surements of the orientation of the spin axis of the fluid core nor, if present, that of the solid  
 118 inner core. However, we expect that their spin axes also obey a Cassini state, though with dif-  
 119 ferent obliquity angles than that of the mantle [e.g. Peale *et al.*, 2014, 2016; Dumberry, 2021].  
 120 The differentially rotating mantle, fluid core and inner core imply viscous drag at the CMB and  
 121 at the ICB. The shearing of the magnetic field threading the ICB also leads to EM drag, a pro-  
 122 cess that also occurs at the CMB if the lowermost region of the mantle is electrically conduct-  
 123 ing.

124 Dissipation from viscous and EM drag at the CMB and ICB drains some of Mercury’s ro-  
 125 tational energy and, consequently, induces a deviation of the spin pole away from the Cassini  
 126 plane. The magnitude of these internal sources of dissipation, and hence the resulting  $\zeta_m$ , de-  
 127 pend on parameters that are not well known, including the viscosity of the fluid core, the elec-  
 128 trical conductivity of both the inner and fluid cores, and the strength of the magnetic field in-  
 129 side the core. However, predictions can be computed based on a range of model parameters.  
 130 The calculations presented in Peale *et al.* [2014] suggest that viscous and/or EM coupling may  
 131 amount to a phase lag of 0.05 arcsec. Clearly, the total dissipation from the combined effects  
 132 of tidal deformations and viscous and EM friction at the boundaries of the fluid core must be  
 133 limited, as otherwise the spin pole would deviate from the Cassini plane by a greater angle than  
 134 the upper bound of a few arcsec suggested by measurements.

135 The main objective in this work is to compute estimates of the dissipation and phase lag  
 136 angle  $\zeta_m$  induced by tidal deformation and by viscous and EM drag at the CMB and ICB of  
 137 Mercury. A model to compute the Cassini state of Mercury comprising a fluid core and solid  
 138 inner core is presented in detail in Dumberry [2021] (referred to hereafter by D21). This model  
 139 includes viscous and EM coupling at the ICB and CMB. The focus in D21 was on the effects  
 140 that viscous and EM coupling have on the obliquity angle, in other words on the component  
 141 of the spin pole orientation contained in the Cassini plane. The present work can be thought  
 142 of as the second part of D21, focused here on the component of the spin pole out of the Cassini  
 143 plane. We provide an update on the predictions of  $\zeta_m$  made by Peale *et al.* [2014] due to vis-  
 144 cous drag (which dominates EM drag) at the CMB and complemented by the inclusion of EM  
 145 drag (which dominates viscous drag) at the ICB. The model developed in D21 did not include  
 146 viscoelastic deformations induced by tidal forces and by the differential rotation of Mercury’s  
 147 interior regions. We modify here the model in D21 to include these effects. A connection be-  
 148 tween  $\zeta_m$  and the tidal quality factor  $Q$  is presented in Baland *et al.* [2017]; our model is con-  
 149 sistent with their results, and we make an additional effort to relate  $Q$  to the viscosities of the  
 150 mantle and inner core.

Study	Right ascension (°)	Declination (°)	Phase lag (arcsec)
<i>Margot et al.</i> [2012]	$281.0103 \pm 0.0015$	$61.4155 \pm 0.0013$	$2.50 \pm 2.83$
<i>Mazarico et al.</i> [2014]	$281.0048 \pm 0.0054$	$61.41436 \pm 0.0021$	$-7.76 \pm 9.16$
<i>Stark et al.</i> [2015a]	$281.00980 \pm 0.00088$	$61.4156 \pm 0.0016$	$1.79 \pm 2.23$
<i>Verma and Margot</i> [2016]	$281.00975 \pm 0.0048$	$61.41828 \pm 0.0028$	$4.55 \pm 8.44$
<i>Genova et al.</i> [2019]	$281.0082 \pm 0.0009$	$61.4164 \pm 0.0003$	$0.00364 \pm 1.52$
<i>Konopliv et al.</i> [2020]	$281.0138 \pm 0.0025$	$61.4161 \pm 0.0017$	$8.90 \pm 4.49$
<i>Bertone et al.</i> [2021]	$281.0093 \pm 0.00063$	$61.4153 \pm 0.00048$	$0.645 \pm 1.15$

**Table 1.** Right ascension, declination and phase lag angle with respect to the Cassini plane at J2000 for recent measurements of Mercury’s spin pole orientation. The phase lag angles give the distance to the Cassini plane of the central value of each of the spin pole orientation measurements. See Appendix A for details of the calculations of the phase lags and their estimated errors.

As Table 1 illustrates, current measurements of the spin pole orientation are not sufficiently precise to determine the phase lag with high accuracy. Hence, we do not have a specific observational target that we aim to match. Our study is instead an exploration of the different dissipative mechanisms and the phase lag they produce. It is likely that the accuracy of the spin pole orientation will improve with future observations, such as that from the upcoming Bepi-Colombo satellite mission [*Cicalò et al.*, 2016]. Predictions of the phase lag by a combination of tidal dissipation and viscous and EM drag at the CMB and ICB may provide an opportunity to further constrain the internal dissipation taking place within Mercury, and in turn, the physical parameters associated with these processes.

## 2 Theory

The rotational model of Mercury that we use and the way we construct interior models of Mercury are presented in detail in D21. For convenience we briefly mention some of their salient features below. We modify the rotational model of D21 to take into account viscoelastic deformations. These modifications are presented in Appendix B.

### 2.1 Interior structure

Mercury (mass  $M$ ) is modelled as a simple four layer planet comprised of an inner core, fluid core, mantle, and crust, each with a uniform density. The outer spherical mean radii of each of these layers, are denoted by  $r_s$ ,  $r_f$ ,  $r_m$ , and  $R$ , and their densities by  $\rho_s$ ,  $\rho_f$ ,  $\rho_m$ , and  $\rho_c$ , respectively. The inner core radius  $r_s$  corresponds to the ICB radius, the fluid core radius  $r_f$  to the CMB radius, and  $R$  to the planetary radius of Mercury.

For the crust, we assume a density of  $\rho_c = 2974 \text{ kg m}^{-3}$  and a thickness of  $h = R - r_m = 26 \text{ km}$  [*Sori*, 2018]. Individual interior models are constructed for each choice of ICB radius, ensuring that they are consistent with  $M$  and chosen values of the moments of inertia of the whole planet  $C$  and that of the combined mantle and crust  $C_m$ . The latter two are determined from the observed obliquity  $\varepsilon_m$  and the observed amplitude of the 88-day longitudi-

180 nal librations. We use here the same choices of  $C$  and  $C_m$  as in D21:  $C/MR^2 = 0.3455$  and  
 181  $C_m/MR^2 = 0.1475$ . Two possible end-member scenarios for how the densities of the solid ( $\rho_s$ )  
 182 and fluid ( $\rho_f$ ) cores may evolve with inner core growth were considered in D21. In the first,  $\rho_s$   
 183 is held constant and  $\rho_f$  is adjusted with inner core size to match  $M$ . This captures a Fe-S core  
 184 composition with little or no S being incorporated into the inner core as it crystallizes. In the  
 185 second scenario, it is the density contrast at the ICB which is set to a constant, capturing a Fe-  
 186 Si core composition in which Si is expected to partition into the solid core. Specific solutions  
 187 of the rotational model depend on which of these scenarios is used, but their qualitative behaviour  
 188 are equivalent. Numerical results are computed here according to the first scenario, with  $\rho_s =$   
 189  $8,800 \text{ kg m}^{-3}$ .

190 Each layer is triaxial in shape. We define the polar geometrical ellipticity of each layer  
 191 as the difference between the mean equatorial and polar radii, divided by the mean spherical  
 192 radius. Likewise, we define the equatorial geometrical ellipticity of each layer as the difference  
 193 between the maximum and minimum equatorial radii, divided by the mean spherical radius.  
 194 The polar and equatorial geometrical ellipticities are denoted by  $\epsilon_i$  and  $\xi_i$  respectively, with the  
 195 subscript  $i = s, f, m$ , and  $r$  denoting the ICB, CMB, crust-mantle boundary, and surface, re-  
 196 spectively. The polar and equatorial flattenings at the surface are taken from *Perry et al.* [2015]  
 197 and their values are given in Table 1 of D21. We assume that the shapes of the ICB and CMB  
 198 coincide with equipotential surfaces at hydrostatic equilibrium, and the flattenings at all in-  
 199 terior boundaries are specified such that they match the observed degree 2 spherical harmonic  
 200 coefficients of gravity  $J_2$  and  $C_{22}$  (their numerical values are given in Table 1 of D21).

201 With the densities and ellipticities of each interior regions known, one can compute the  
 202 moments of inertia of the fluid core ( $C_f > B_f > A_f$ ) and solid inner core ( $C_s > B_s > A_s$ ).  
 203 The rotational model involves the mean equatorial moments of inertia  $\bar{A}, \bar{A}_f, \bar{A}_s$  of the whole  
 204 planet, fluid core and solid inner core and the dynamical ellipticities  $e, e_f, e_s, \gamma$  and  $\gamma_s$ . These  
 205 are defined and computed according to Equations 2 and 3 of D21.

## 206 2.2 Rotational model

207 Mercury rotates in a 3:2 spin-orbit resonance. Its sidereal frequency  $\Omega_o = 2\pi/58.64623$   
 208  $\text{day}^{-1}$  is 1.5 times its orbital frequency (or, mean motion)  $n = 2\pi/87.96935 \text{ day}^{-1}$  [*Stark et al.*,  
 209 2015b]. Mercury’s rotation is also characterized by a Cassini state. The latter defines a con-  
 210 figuration in which the orientations of the normal to the orbital plane (or, orbital pole,  $\hat{\mathbf{e}}_3^I$ ) and  
 211 the symmetry axis ( $\hat{\mathbf{e}}_3^P$ ) are both coplanar with, and precess about, the normal to the Laplace  
 212 plane (or, Laplace pole,  $\hat{\mathbf{e}}_3^L$ ). The rotation vector of Mercury  $\boldsymbol{\Omega}$  is not exactly aligned with the  
 213 symmetry axis  $\hat{\mathbf{e}}_3^P$  in the Cassini state equilibrium, but the offset between the two is small, ap-  
 214 proximately 0.015 arcsec (see Equation 5a below). The Cassini state of Mercury is illustrated  
 215 in Figure 1. The orientation of the Laplace pole varies on long timescales, but it is convenient  
 216 here to assume that it is invariant in inertial space. The precession of  $\hat{\mathbf{e}}_3^I$  and  $\hat{\mathbf{e}}_3^P$  about the Laplace  
 217 normal is retrograde with frequency  $\Omega_p = 2\pi/325,513 \text{ yr}^{-1}$  [*Baland et al.*, 2017].

218 Since Mercury has a fluid core and (possibly) a solid inner core,  $\hat{\mathbf{e}}_3^P$  and  $\boldsymbol{\Omega}$  characterize  
 219 the symmetry and rotation axes of the solid shell of Mercury comprised of its mantle and crust.  
 220 Three additional orientation vectors are required to fully describe the Cassini state: the rota-  
 221 tion vectors of the fluid core ( $\boldsymbol{\Omega}_f$ ) and inner core ( $\boldsymbol{\Omega}_s$ ) and the symmetry axis of the inner core

222  $(\hat{e}_3^s)$  (see Figure 2 of D21); these also precess in the retrograde direction with frequency  $\Omega_p$  about  
 223 the Laplace pole.

224 The specific orientation of each of the vectors  $\hat{e}_3^p, \Omega, \Omega_f, \Omega_s$  and  $\hat{e}_3^s$  in the Cassini state  
 225 equilibrium depends on the mean solar torque (time-averaged over one orbit) applied on Mer-  
 226 cury’s instantaneous figure and on internal torques that arise from the misalignment between  
 227 its interior regions. The rotational model in D21 solves for these orientations. It consists of a  
 228 linear system of five equations written in terms of five rotational variables,  $\tilde{\epsilon}_m, \tilde{m}, \tilde{m}_f, \tilde{m}_s$  and  
 229  $\tilde{n}_s$ , which are projections of the five orientation vectors in the equatorial plane of Mercury’s ro-  
 230 tating frame.

231 In the absence of dissipation, the vectors  $\hat{e}_3^p, \Omega, \Omega_f, \Omega_s$  and  $\hat{e}_3^s$  all lie in the Cassini plane.  
 232 Viewed in the inertial frame, the Cassini plane is rotating in a retrograde direction at frequency  
 233  $\Omega_p$ . The equations of the rotational model of D21 are developed in a frame attached to the man-  
 234 tle and crust rotating at sidereal frequency  $\Omega_o$ . Viewed in this frame, the Cassini plane is ro-  
 235 tating in a retrograde direction at frequency  $\omega\Omega_o$  (see Figure 2b of D21), where  $\omega$ , expressed  
 236 in cycles per Mercury day, is equal to (Equation 21 of D21)

$$\omega = -1 - \delta\omega \cos I, \quad (1)$$

237 where  $I = 8.5330^\circ$  is the inclination of the orbital plane. The factor  $\delta\omega = \Omega_p/\Omega_o = 4.933 \times$   
 238  $10^{-7}$  is the Poincaré number, the ratio of the forced precession to sidereal rotation frequencies.  
 239 The mean solar torque is pointing in the same direction as the vector connecting the Sun to  
 240 the descending node of Mercury’s orbit (see Figure 1), so from the mantle-fixed frame the ori-  
 241 entation of this mean torque is periodic, rotating at frequency  $\omega\Omega_o$ . Setting the equatorial di-  
 242 rections  $\hat{e}_1^p$  and  $\hat{e}_2^p$  to correspond with the real and imaginary axes of the complex plane, re-  
 243 spectively, the equatorial components of the mean solar torque is written in a compact form  
 244 as

$$\Gamma_1(t) + i\Gamma_2(t) = -i\tilde{\Gamma}(\omega) \exp[i\omega\Omega_o t], \quad (2)$$

245 where  $i = \sqrt{-1}$  and  $\tilde{\Gamma}(\omega)$  represents the amplitude of the torque at frequency  $\omega\Omega_o$ . The ro-  
 246 tational variables  $\tilde{\epsilon}_m, \tilde{m}, \tilde{m}_f, \tilde{m}_s$  and  $\tilde{n}_s$  are complex amplitudes, also proportional to  $\exp[i\omega\Omega_o t]$ ,  
 247 in response to this applied external torque. Their real parts correspond to the angles of the five  
 248 rotational vectors in the Cassini plane (i.e. in-plane components), the response that is in-phase  
 249 with the applied solar torque. Their imaginary parts reflect the component of these angles out  
 250 of the Cassini plane (out-of-plane components), the out-of-phase response to the applied torque  
 251 as a result of dissipation. A positive imaginary part corresponds to a phase lag, a negative imag-  
 252 inary part to phase lead.

253 The rotational model of D21 includes a parameterization for the viscous and EM torques  
 254 at the CMB and ICB expressed as

$$\tilde{\Gamma}_{cmb} = i\Omega_o^2 \bar{A}_f K_{cmb} \tilde{m}_f, \quad (3a)$$

$$\tilde{\Gamma}_{icb} = i\Omega_o^2 \bar{A}_s K_{icb} (\tilde{m}_f - \tilde{m}_s), \quad (3b)$$

where  $K_{cmb}$  and  $K_{icb}$  are dimensionless complex coupling constants. Specific expressions for the viscous and EM coupling models are given further ahead in the results sections. These torques generate both an in-phase and out-of-phase response.

The model of D21 assumes a rigid outer shell (mantle and crust) and a rigid inner core. Here, we take into account viscoelastic deformations within each interior region in response to gravitational and centrifugal forces. Such deformations induce a perturbation in the moment of inertia tensors of each region and therefore a modification of both the solar torque and Mercury’s angular momentum response. The details of how the rotational model is adapted to include these are presented in Appendix B. Deformations are characterized by a set of compliances  $\mathcal{S}_{ij}$  which quantify the changes in the moment of inertia tensors of each region.

Elastic tidal deformations of a planetary body are typically expressed by the Love number  $k_2$ . The latter represents the fractional change in the gravitational potential of degree 2 at the surface induced by global deformations. Viscous or anelastic deformations are captured by a quality factor  $Q$ , with  $Q^{-1}$  representing the fraction of the total energy that is dissipated over one cycle. A low (high)  $Q$  value indicates a high (low) dissipation.  $k_2$  and  $Q^{-1}$  characterize, respectively, deformations that are in-phase and out-of-phase with the tidal potential. In our rotational model, these are connected to the compliance  $\mathcal{S}_{11}$  through

$$Re[\mathcal{S}_{11}] = k_2 \frac{R^5 \Omega_0^2}{3G\bar{A}}, \quad Im[\mathcal{S}_{11}] = \frac{k_2}{Q} \frac{R^5 \Omega_0^2}{3G\bar{A}}, \quad (4)$$

where  $G$  is the gravitational constant. Recent estimates of  $k_2$  are  $0.569 \pm 0.025$  [*Genova et al.*, 2019] and  $0.53 \pm 0.03$  [*Konopliv et al.*, 2020]. We do not have direct observational constraints on  $Q$ .

The method to compute the compliances  $\mathcal{S}_{ij}$  is presented in Appendix C. Their numerical values depend on the rheology assumed in the solid regions (crust, mantle and inner core). We assume a Maxwell solid rheology, and constrain this rheology such that  $k_2$  in all our interior models matches  $k_2 = 0.55$ , a value at the mid-point of the recent estimates given above. The quality factor  $Q$  depends on the uniform viscosity assumed within the mantle and inner core; we present results for a range of possible values. To give a sense of the amplitude of  $\mathcal{S}_{11}$ , we can approximate  $\bar{A}$  to be equal to the mean (spherical) moment of inertia and take the latter to be  $0.346 \cdot MR^2$  [*Margot et al.*, 2012]. Using the parameters from Table 1 of D21, a tidal Love number  $k_2 = 0.55$  (the value that we use for all our results), corresponds to  $Re[\mathcal{S}_{11}] = 5.37 \times 10^{-7}$ . For  $Q = 100$ , this gives  $Im[\mathcal{S}_{11}] = 5.37 \times 10^{-9}$ .

### 2.3 Approximate solutions

The set of equations that enter the rotational model is presented in Appendix B. Substituting  $\omega = -1 - \delta\omega \cos I$  (Eq. 1) in Equations (B.4e) and (B.4d) provides the following two kinematic relationships, relating  $\tilde{m}$  to  $\tilde{\epsilon}_m$  and  $\tilde{m}_s$  to  $\tilde{n}_s$ :

$$\tilde{m} = \delta\omega(\sin I + \tilde{\epsilon}_m \cos I), \quad (5a)$$

$$\tilde{m}_s = (1 + \delta\omega \cos I)\tilde{n}_s. \quad (5b)$$

With  $I = 8.5330^\circ$ ,  $\delta\omega = 4.9327 \times 10^{-7}$  and taking  $\tilde{\epsilon}_m = 2.04$  arcmin, this gives  $\tilde{m} = 0.0151$  arcsec: the offset of the spin axis of the mantle with respect to its symmetry axis is very small.

291 Similarly, the misalignment between the spin axis of the inner core ( $\tilde{m}_s$ ) and its symmetry axis  
 292 ( $\tilde{n}_s$ ) is also very small: as an indication, for an inner core tilt with respect to the mantle of  $\tilde{n}_s =$   
 293 1 arcmin,  $\tilde{m}_s$  is offset from  $\tilde{n}_s$  by approximately 0.03 milliarcsec.

294 For the purpose of building an approximate analytical solution, we can simply assume  $\tilde{m}_s =$   
 295  $\tilde{n}_s$ . However, we cannot set  $\tilde{m} = 0$ . This is because our system of equations is developed in  
 296 the frame of the rotating mantle. In this frame,  $\tilde{m}$  captures the change in mantle angular mo-  
 297 mentum induced by the solar torque. To express this change in terms the orientation of Mer-  
 298 cury's figure in the inertial (Laplace) frame, we substitute  $\tilde{m}$  with Equation 5a.

299 Approximate solutions for the obliquity and phase lag of the mantle can be constructed  
 300 from the angular momentum equation for the whole of Mercury (Equation B.4a). All compli-  
 301 ances  $\mathcal{S}_{ij}$  are of the order of  $10^{-7}$  or smaller; the term  $\tilde{c}/\bar{A}$  can be neglected when compared  
 302 to other terms on the left-hand side. By substituting Eq. 5a and setting  $\tilde{m}_s = \tilde{n}_s$ , we can sim-  
 303 plify Eq. (B.4a) to

$$-\frac{C}{\bar{A}} \delta\omega \left( \sin I + \tilde{\varepsilon}_m \cos I \right) - \delta\omega \cos I \left[ \frac{\bar{A}_f}{\bar{A}} \tilde{m}_f + \frac{\bar{A}_s}{\bar{A}} \tilde{n}_s \right] = \frac{1}{i\Omega_o^2 \bar{A}} \left( \tilde{\Gamma}_{sun} + \tilde{\Gamma}_t \right), \quad (6)$$

304 where we have used  $C = \bar{A}(1 + e)$ , and where the torques  $\tilde{\Gamma}_{sun}$  and  $\tilde{\Gamma}_t$  are given by Equa-  
 305 tions (B.12) and (B.18). Keeping only the largest terms in the former, these are given by

$$\frac{\tilde{\Gamma}_{sun}}{i\Omega_o^2 \bar{A}} = - \left[ \phi_m^{el} \tilde{\varepsilon}_m + \frac{\bar{A}_s}{\bar{A}} \alpha_3 \phi_s^{el} \tilde{n}_s + \frac{\phi_m}{e} \left( \mathcal{S}_{12} \tilde{m}_f + \mathcal{S}_{14} \tilde{n}_s \right) \right], \quad (7a)$$

$$\frac{\tilde{\Gamma}_t}{i\Omega_o^2 \bar{A}} = iIm[\mathcal{S}_{11}] \left[ \phi_m^{t3} \tilde{\varepsilon}_m + \phi_m^{t2} \cos I \sin I \right], \quad (7b)$$

306 where  $\alpha_3 = 1 - \rho_f/\rho_s$  is the density contrast at the ICB. The definitions of the torque fac-  
 307 tors  $\phi_m$ ,  $\phi_m^{el}$ ,  $\phi_s^{el}$ ,  $\phi_m^{t2}$  and  $\phi_m^{t3}$  are given in Appendix B. In addition to the compliance  $\mathcal{S}_{11}$ , the  
 308 two additional compliances that have the largest influence on the solutions are  $\mathcal{S}_{12}$  and  $\mathcal{S}_{14}$ . These  
 309 capture the global viscoelastic deformations of Mercury in response to internal forcing. For  $\mathcal{S}_{12}$ ,  
 310 it is the centrifugal force on the CMB by the misaligned spin axis of the fluid core. For  $\mathcal{S}_{14}$ , it  
 311 is the gravitational force from the tilted inner core. The compliances are complex: their real  
 312 and imaginary parts capture, respectively, elastic and anelastic deformations.

313 Using  $\delta\omega = \Omega_p/\Omega_o$ , with Equations (7a-7b), Equation (6) can be written as

$$\begin{aligned} C\Omega_p \left( \sin I + \tilde{\varepsilon}_m \cos I \right) + \Omega_p \cos I \left( \bar{A}_f \tilde{m}_f + \bar{A}_s \tilde{n}_s \right) = \\ \bar{A}\Omega_o \phi_m^{el} \tilde{\varepsilon}_m + \bar{A}_s \Omega_o \alpha_3 \phi_s^{el} \tilde{n}_s + \bar{A}\Omega_o \frac{\phi_m}{e} \left( \mathcal{S}_{12} \tilde{m}_f + \mathcal{S}_{14} \tilde{n}_s \right) \\ - i\bar{A}\Omega_o Im[\mathcal{S}_{11}] \left( \phi_m^{t3} \tilde{\varepsilon}_m + \phi_m^{t2} \sin I \cos I \right). \end{aligned} \quad (8)$$

314 From this latter equation, we can derive approximate solutions for both the obliquity (in-plane  
 315 component)  $\varepsilon_m = Re[\tilde{\varepsilon}_m]$  and the phase lag (out-of-plane component)  $\zeta_m = Im[\tilde{\varepsilon}_m]$ .

## 2.4 Obliquity

Although our study focuses on the phase lag, the introduction of viscoelastic deformations in the rotational model alters the obliquity solutions presented in D21. For completeness, let us first consider predictions of the obliquity, which can be computed from the real part of Equation (8), and can be written as

$$\varepsilon_m = \varepsilon_m^t + \varepsilon_m^{L,c} + \varepsilon_m^{t,s} + \varepsilon_m^{t,e} + \varepsilon_m^{t,a}, \quad (9)$$

where

$$\varepsilon_m^t = \frac{C\Omega_p \sin I}{\mathcal{L}_m}, \quad (10a)$$

$$\varepsilon_m^{L,c} = \frac{\bar{A}\Omega_p \cos I}{\mathcal{L}_m} \left[ \frac{\bar{A}_f}{\bar{A}} \text{Re}[\tilde{m}_f] + \frac{\bar{A}_s}{\bar{A}} \text{Re}[\tilde{n}_s] \right], \quad (10b)$$

$$\varepsilon_m^{t,s} = \frac{\bar{A}_s\Omega_o}{\mathcal{L}_m} \left[ -\alpha_3\phi_s^{el} \text{Re}[\tilde{n}_s] \right], \quad (10c)$$

$$\varepsilon_m^{t,e} = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \frac{\phi_m}{e} \left[ -\text{Re}[S_{12}] \text{Re}[\tilde{m}_f] - \text{Re}[S_{14}] \text{Re}[\tilde{n}_s] \right], \quad (10d)$$

$$\varepsilon_m^{t,a} = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \frac{\phi_m}{e} \left[ \text{Im}[S_{12}] \text{Im}[\tilde{m}_f] + \text{Im}[S_{14}] \text{Im}[\tilde{n}_s] \right], \quad (10e)$$

and

$$\mathcal{L}_m = \bar{A}\Omega_o\phi_m^{el} - C\Omega_p \cos I. \quad (10f)$$

Each of the terms on the right-hand side of Equation (9) captures a contribution to  $\varepsilon_m$  from a different origin.  $\varepsilon_m^t$  captures the obliquity resulting from the solar torque acting on the ellipsoidal shape of Mercury.  $\varepsilon_m^{L,c}$  captures the contribution to the obliquity connected with the angular momentum carried by the fluid and solid cores. These result from internal torques between the mantle, fluid core and solid core; this term captures then the mantle obliquity generated by internal torques. The remaining three contributions result from the solar torque acting on additional aspherical features of Mercury's shape. In  $\varepsilon_m^{t,s}$ , it is on the tilt of the ellipsoidal figure of the inner core with respect to the mantle. In  $\varepsilon_m^{t,e}$ , it is on the global elastic deformation caused by the in-plane components of the misaligned fluid core spin axis ( $\tilde{m}_f$ ) and inner core tilt ( $\tilde{n}_s$ ). In  $\varepsilon_m^{t,a}$ , it is on the delayed, anelastic deformation in response to the out-of-plane components of  $\tilde{m}_f$  and  $\tilde{n}_s$ .

In the absence of a fluid core and inner core,

$$\varepsilon_m = \varepsilon_m^t = \frac{C\Omega_p \sin I}{\mathcal{L}_m} = \frac{C\Omega_p \sin I}{\bar{A}\Omega_o\phi_m^{el} - C\Omega_p \cos I}. \quad (11)$$

This is identical to Equation (26) of D21, except that  $\phi_m$  has been replaced by  $\phi_m^{el}$ ; the latter is a modification of the former by elastic deformation (see Equation B.11a). We also retrieve, in our notation, the solution given in Equation (64) of *Baland et al.* [2017], where their definition of  $\dot{\Omega}$  is equal to  $-\Omega_p$ . [Note also that their definition of  $\tilde{C}$  is equal, in our notation, to  $C - \bar{A}(\phi_m/e)\text{Re}[S_{11}]$ , which differs from  $C$  only by a few parts in  $10^7$  and can be neglected.]

340 The real and imaginary parts of  $\tilde{m}_f$  and  $\tilde{n}_s$  can be similar in magnitude for sufficiently  
 341 strong viscous or EM coupling at the ICB and CMB. However, the imaginary parts of the com-  
 342 pliances are smaller than their real parts by a factor approximately equal to the quality fac-  
 343 tor  $Q$ . Hence, provided that  $Q > 10$ , this implies that  $\varepsilon_m^{t,e} \gg \varepsilon_m^{t,a}$ . For a small or no inner  
 344 core,  $\bar{A}_s \ll \bar{A}_f$ ,  $\mathcal{S}_{14} \ll \mathcal{S}_{12}$  and the prediction of the obliquity is

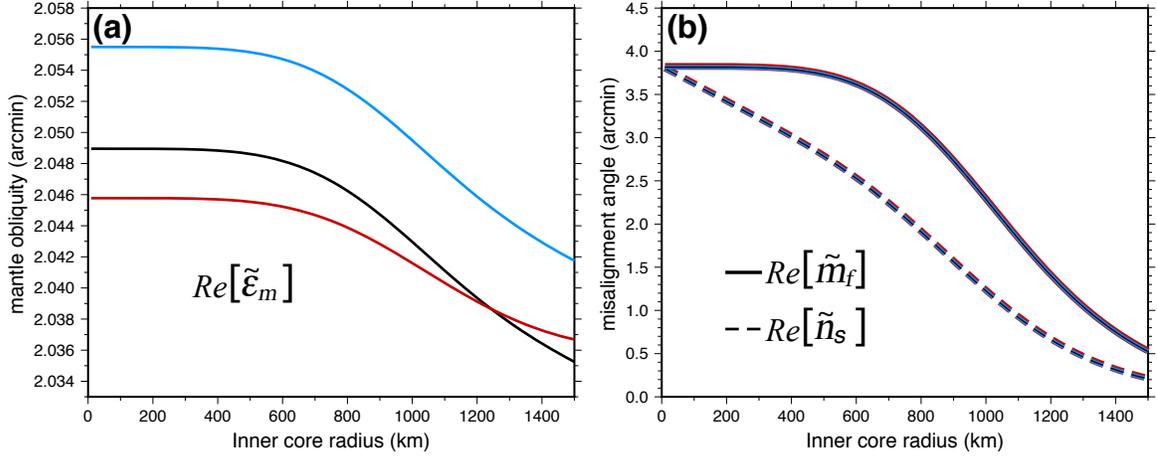
$$\varepsilon_m = \frac{C\Omega_p \sin I}{\mathcal{L}_m} + \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \left( \frac{\bar{A}_f}{\bar{A}} \frac{\Omega_p}{\Omega_o} \cos I - \frac{\phi_m}{e} \text{Re}[\mathcal{S}_{12}] \right) \text{Re}[\tilde{m}_f]. \quad (12)$$

345 The second term on the right-hand side, connected to the misaligned spin axis of the core, is  
 346 comprised of two parts with opposite signs; an angular momentum part, and a global deforma-  
 347 tion part. Both  $\bar{A}_f/\bar{A}$  and  $\phi_m/e$  are fractions smaller than 1 (and of order 1), and the Poincaré  
 348 number ( $\delta\omega = \Omega_p/\Omega_o = 4.93 \times 10^{-7}$ ) is of the same order as  $\text{Re}[\mathcal{S}_{12}]$  which is approximately  
 349 equal to  $3.5 \times 10^{-7}$ . Since  $\bar{A}_f/\bar{A} < \phi_m/e$ , not only is the term related to  $\text{Re}[\mathcal{S}_{12}]$  non-negligible,  
 350 it is larger in magnitude than the angular momentum part, and changes the sign of the cor-  
 351 rection to  $\varepsilon_m$  associated with the misaligned spin axis of the fluid core.

352 This is also true for the correction to  $\varepsilon_m$  associated with the misaligned inner core: the  
 353 part related to  $\text{Re}[\mathcal{S}_{14}]$  is larger than the part related to its angular momentum. In the con-  
 354 tributions to  $\varepsilon_m$ , we thus have that  $\varepsilon_m^{t,e} > \varepsilon_m^{L,c}$ . Elastic deformations induced by the misaligned  
 355 fluid core and solid core have to be taken into account in order to properly predict Mercury's  
 356 obliquity.

357 We can illustrate for a specific example how the solutions presented in D21 are affected  
 358 by the inclusion of the compliances  $\mathcal{S}_{11}$ ,  $\mathcal{S}_{12}$  and  $\mathcal{S}_{14}$ . Figure 3 shows how the real parts of  $\tilde{\varepsilon}_m$ ,  
 359  $\tilde{m}_f$  and  $\tilde{n}_s$  vary with inner core size. These solutions are computed with a viscosity in all solid  
 360 regions equal to  $10^{20}$  Pa s (i.e. in the elastic limit),  $k_2 = 0.55$ , a turbulent kinematic viscos-  
 361 ity of  $\nu = 10^{-4}$  m<sup>2</sup> s<sup>-1</sup> at both the ICB and CMB, an electrically insulating lowermost man-  
 362 tle (so that EM coupling at the CMB vanishes), an electrical conductivity of  $10^6$  S m<sup>-1</sup> in both  
 363 the solid and fluid cores, and a magnetic field strength at the ICB of  $\langle B_r \rangle = 0.1$  mT. Three  
 364 solutions are shown in Figure 3. First, a solution where all compliances  $\mathcal{S}_{ij}$  are set to zero (black  
 365 lines); the rotational model in this case is equivalent to that used in D21 and corresponds to  
 366 a case where the crust, mantle and inner core are rigid. Second, a solution where only the com-  
 367 pliance  $\mathcal{S}_{11}$  is retained (light blue lines). Third, a solution that includes all compliances (red  
 368 lines).

369 Compared with the rigid case, the mantle obliquity  $\text{Re}[\tilde{\varepsilon}_m]$  is increased by 0.0065 arcmin  
 370 = 0.39 arcsec when the compliance  $\mathcal{S}_{11}$  is introduced, reflecting the change in  $\text{Re}[\tilde{\varepsilon}_m]$  caused  
 371 by tidal deformations. This is consistent with the results presented in *Baland et al.* [2017] (see  
 372 their Figure 7), who also considered how tidal deformations (through the Love number  $k_2$ ) af-  
 373 fect the obliquity. With the addition of all other compliances, compared to the solution when  
 374 only  $\mathcal{S}_{11}$  is retained, the mantle obliquity is reduced by 0.01 arcmin (for a small inner core) to  
 375 0.005 arcmin (for a large inner core). It is dominantly the compliances  $\mathcal{S}_{11}$ ,  $\mathcal{S}_{12}$  and  $\mathcal{S}_{14}$  that  
 376 have an effect on the resulting mantle obliquity (the difference in the solution is virtually un-  
 377 changed if only these three compliances are kept). This third solution shows that elastic de-  
 378 formations induced by the misaligned spin axis of the fluid core (through  $\mathcal{S}_{12}$ ) and the misaligned  
 379 figure axis of the inner core (through  $\mathcal{S}_{14}$ ) are as important as those from tidal forces on the  
 380 resulting mantle obliquity. Present-day observations are not sufficiently precise to differenti-  
 381 ate between the different solutions shown in Figure 3a. In other words, the observed mantle obliq-



388 **Figure 3.** (a) Mantle obliquity ( $Re[\tilde{\epsilon}_m]$ ) and (b) misalignment angles of the fluid core spin axis  
 389 ( $Re[\tilde{m}_f]$ , solid lines) and inner core figure axis ( $Re[\tilde{n}_s]$ , dashed lines) in the Cassini plane as a function  
 390 of inner core radius. Different colored lines correspond to solutions when all compliances  $\mathcal{S}_{ij}$  are set to  
 391 zero (black), when only  $\mathcal{S}_{11}$  is retained (light blue), and when all compliances are included (red). The  
 392 thicknesses of the red and light blues curves have been increased in panel (b) to show that the different  
 393 solutions of  $Re[\tilde{m}_f]$  and  $Re[\tilde{n}_s]$  are indistinguishable from one another on the scale of the figure.

382 uity cannot be used to further constrain Mercury’s rheology. But if precision improves, our re-  
 383 sults illustrate that to do so properly, incorporating deformations caused by the misaligned fluid  
 384 core and inner core in rotational models of the Mercury is necessary. Finally, we note that the  
 385 solutions of  $\tilde{m}_f$  and  $\tilde{n}_s$  (Figure 3b) for these three different cases are virtually indistinguish-  
 386 able from one another; solutions of  $\tilde{m}_f$  and  $\tilde{n}_s$  for a rigid planet are not substantially differ-  
 387 ent from those for a deformable planet.

## 394 2.5 Phase lag

395 The imaginary part of Eq. (8) gives an approximate solution for the phase lag  $\zeta_m = Im[\tilde{\epsilon}_m]$ ,  
 396 which can be written in a similar form as for the obliquity prediction,

$$\zeta_m = \zeta_m^t + \zeta_m^{L,c} + \zeta_m^{t,s} + \zeta_m^{t,a} + \zeta_m^{t,e}, \quad (13)$$

397 where

$$\zeta_m^t = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \text{Im}[\mathcal{S}_{11}] \left[ \phi_m^{t3} \text{Re}[\tilde{\varepsilon}_m] + \phi_m^{t2} \sin I \cos I \right], \quad (14a)$$

$$\zeta_m^{L,c} = \frac{\bar{A}\Omega_p \cos I}{\mathcal{L}_m} \left[ \frac{\bar{A}_f}{A} \text{Im}[\tilde{m}_f] + \frac{\bar{A}_s}{A} \text{Im}[\tilde{n}_s] \right], \quad (14b)$$

$$\zeta_m^{t,s} = \frac{\bar{A}_s \Omega_o}{\mathcal{L}_m} \left[ -\alpha_3 \phi_s^{el} \text{Im}[\tilde{n}_s] \right], \quad (14c)$$

$$\zeta_m^{t,a} = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \frac{\phi_m}{e} \left[ -\text{Im}[S_{12}] \text{Re}[\tilde{m}_f] - \text{Im}[S_{14}] \text{Re}[\tilde{n}_s] \right], \quad (14d)$$

$$\zeta_m^{t,e} = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \frac{\phi_m}{e} \left[ -\text{Re}[S_{12}] \text{Im}[\tilde{m}_f] - \text{Re}[S_{14}] \text{Im}[\tilde{n}_s] \right]. \quad (14e)$$

398 The different contributions to  $\zeta_m$  have similar physical interpretations to their counterparts  
 399 for  $\varepsilon_m$ .  $\zeta_m^t$ ,  $\zeta_m^{t,s}$ ,  $\zeta_m^{t,e}$  and  $\zeta_m^{t,a}$  capture the contributions to the phase lag from the solar torque  
 400 acting on different out-of-plane aspherical features of Mercury. In  $\zeta_m^t$ , it is on the delayed, anelas-  
 401 tic tidal bulge of Mercury in response to the external gravitational force from the Sun. In  $\zeta_m^{t,s}$ ,  
 402 it is on the out-of-plane tilt of the inner core. In  $\zeta_m^{t,a}$ , it is on the delayed, anelastic deforma-  
 403 tion in response to the in-plane components of  $\tilde{m}_f$  and  $\tilde{n}_s$ . In  $\zeta_m^{t,e}$ , it is on the elastic deforma-  
 404 tion in response to out-of-plane components of  $\tilde{m}_f$  and  $\tilde{n}_s$ .  $\zeta_m^{L,c}$  captures the contribution to  
 405 the phase lag connected with the out-of-plane angular momentum carried by the fluid and solid  
 406 cores.

407 If we set  $\tilde{m}_f = \tilde{n}_s = 0$ , which amounts to neglecting all contributions associated with  
 408 the misaligned fluid core and solid inner core, the only contribution to the phase lag is from  
 409  $\zeta_m^t$ , and so

$$\zeta_m = \zeta_m^t = \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \text{Im}[\mathcal{S}_{11}] \left[ \phi_m^{t3} \text{Re}[\tilde{\varepsilon}_m] + \phi_m^{t2} \sin I \cos I \right]. \quad (15)$$

410 To a good approximation, this is equal to

$$\zeta_m \approx \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \text{Im}[\mathcal{S}_{11}] \phi_m^{t2} \sin I, \quad (16)$$

411 and is equivalent, in our notation, to the expression given in Equation (70) of *Baland et al.* [2017],  
 412 where they have made the further approximation  $\mathcal{L}_m \approx \bar{A}\Omega_o \phi_m$ .

413 Provided  $Q > 10$ ,  $\zeta_m^{t,e} \gg \zeta_m^{t,a}$ . For a small or no inner core,  $\bar{A}_s \ll \bar{A}_f$ ,  $\mathcal{S}_{14} \ll \mathcal{S}_{12}$  and  
 414 the phase lag can be approximated by

$$\zeta_m \approx \frac{\bar{A}\Omega_o}{\mathcal{L}_m} \left[ \text{Im}[\mathcal{S}_{11}] \phi_m^{t2} \sin I + \text{Im}[\tilde{m}_f] \left( \frac{\bar{A}_f}{A} \frac{\Omega_p}{\Omega_o} \cos I - \frac{\phi_m}{e} \text{Re}[\mathcal{S}_{12}] \right) \right]. \quad (17)$$

415 The term proportional to  $\text{Im}[\tilde{m}_f]$  captures the contribution to the phase lag from the out-of-  
 416 plane component of the spin vector of the fluid core. It involves the same factor as in the pre-  
 417 diction for the obliquity in Equation (12). If the global elastic deformations caused by the mis-  
 418 aligned fluid core are neglected,  $\text{Im}[\tilde{m}_f]$  contributes to a positive phase lag. But since  $\frac{\phi_m}{e} \text{Re}[\mathcal{S}_{12}] >$   
 419  $\frac{\bar{A}_f}{A} \frac{\Omega_p}{\Omega_o} \cos I$ ,  $\text{Im}[\tilde{m}_f]$  actually contributes to a negative phase lag (i.e. a phase lead). For a large  
 420 inner core, terms that involve  $\text{Im}[\tilde{n}_s]$  are also important, and so are the global deformations  
 421 captured by the compliance  $\mathcal{S}_{14}$ . Just like for the prediction of the obliquity, a proper predic-  
 422 tion of the phase lag must include global deformations induced by  $\tilde{m}_f$  and  $\tilde{n}_s$ .

### 3 Results

#### 3.1 Viscous dissipation

We first investigate the dissipation due to viscous coupling at the CMB and ICB in isolation. EM coupling is turned off and the imaginary parts of all compliances are set to zero. The real parts of compliances are retained so elastic deformations are part of the solutions, but there are no anelastic deformations and so no tidal dissipation. The parameterization of the viscous coupling constants  $K_{cmb}$  and  $K_{icb}$  is the same as that used in D21 (based on *Mathews and Guo* [2005]),

$$K_{cmb} = \frac{\pi \rho_f r_f^4}{A_f} \sqrt{\frac{\nu}{2\Omega_o}} (0.195 - 1.976i), \quad (18a)$$

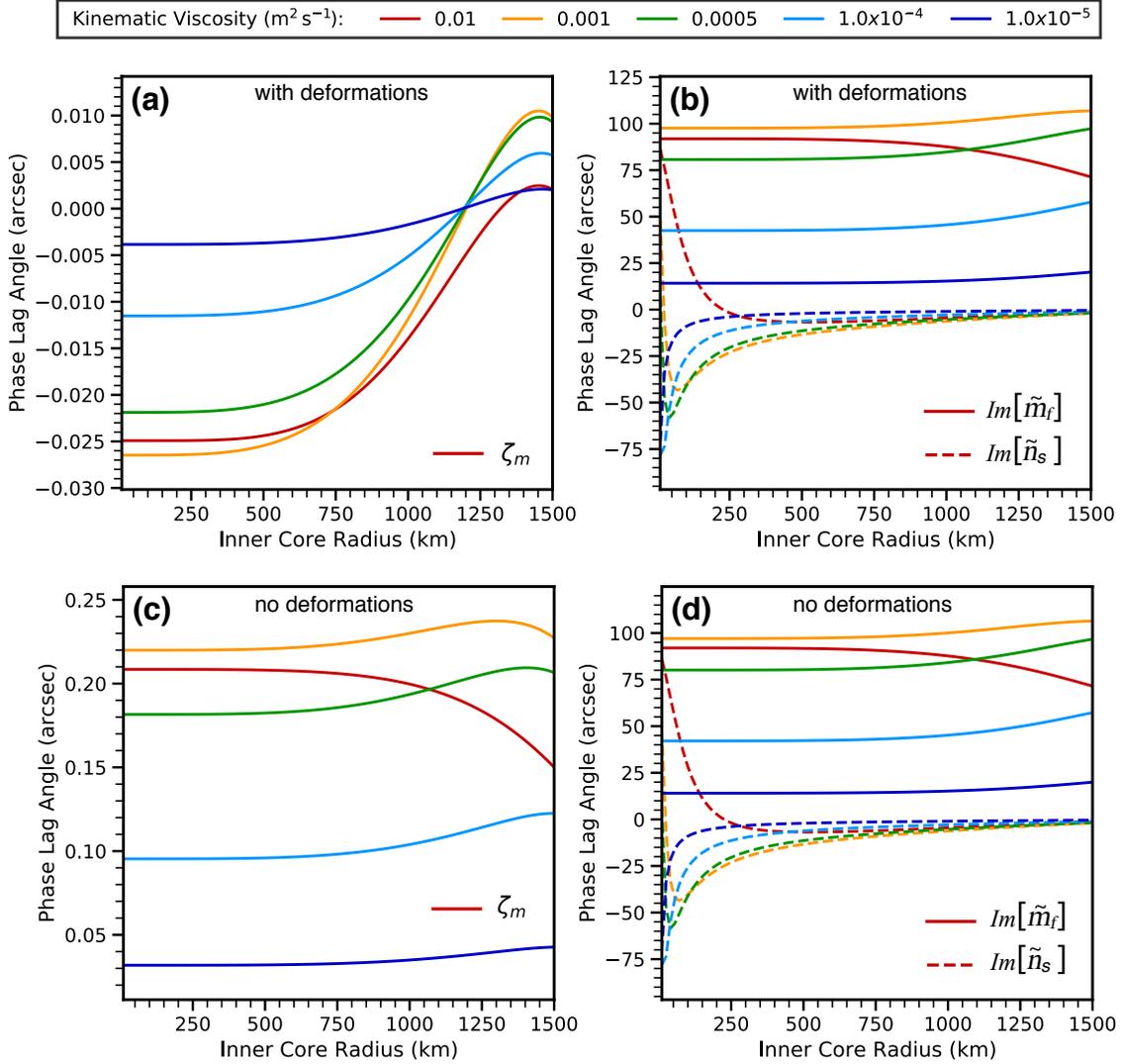
$$K_{icb} = \frac{\pi \rho_f r_s^4}{A_s} \sqrt{\frac{\nu}{2\Omega_o}} (0.195 - 1.976i), \quad (18b)$$

where  $\nu$  is the kinematic viscosity. These expressions are valid provided the flow in the boundary layer remains laminar. As detailed in D21, the boundary layer flow is expected to be in a turbulent regime. We take the same simple approach as that taken in D21; we use the above laminar model with the understanding that  $\nu$  represents an effective turbulent viscosity.

Figure 4ab shows how the mantle phase-lag  $\zeta_m$  and the imaginary parts (out-of-plane components) of  $\tilde{m}_f$  and  $\tilde{n}_s$  vary as a function of inner core radius for different choices of the kinematic viscosity,  $\nu$ . Let us first concentrate on results for a small inner core (radius  $< 500$  km).  $\zeta_m$  is negative for all choices of  $\nu$ : the spin axis of the mantle is ahead of the Cassini plane (a phase lead). The spin axis of the fluid core lags behind the Cassini plane ( $Im[\tilde{m}_f] > 0$ ). Starting from  $\nu = 10^{-5} \text{ m}^2 \text{ s}^{-1}$ , viscous dissipation increases with increasing  $\nu$ , which leads to an increase in the magnitudes of  $\zeta_m$  and  $Im[\tilde{m}_f]$ . The dissipation peaks to a maximum value when  $\nu$  is approximately equal to  $10^{-3} \text{ m}^2 \text{ s}^{-1}$ . With a further increase in  $\nu$  beyond this value, viscous dissipation decreases, and so do the magnitudes of  $\zeta_m$  and  $Im[\tilde{m}_f]$ .

The peak in dissipation is connected to the viscous torque at the CMB, proportional to  $\sqrt{\nu} \tilde{m}_f$ . In the Cassini state equilibrium, with weak or no viscous coupling, the obliquity of the spin axis of the fluid outer core,  $Re[\tilde{m}_f]$ , is offset from the mantle by approximately 4 arcmin (see Figures 4 and 5 of D21). For a very small  $\nu$ , the viscous torque is weak, and so is the resulting viscous dissipation. As  $\nu$  is increased,  $Re[\tilde{m}_f]$  is reduced; the spin axis of the fluid core is brought into an alignment with the mantle's rotation (see Figure 5 of D21). When  $\nu$  is very large, the differential velocity at the CMB is very small and, consequently, viscous dissipation is also weak. The dissipation is then maximized when  $\nu$  is sufficiently large to generate a large viscous torque, yet not so large as to prevent a misalignment between the spin axes of the fluid core and mantle. For  $\nu \approx 10^{-3} \text{ m}^2 \text{ s}^{-1}$ , which optimizes viscous dissipation, the mantle phase lead is  $\sim 0.027$  arcsec and the fluid core phase lag is  $\sim 100$  arcsec ( $\sim 1.7$  arcmin).

Our results for a small or no inner core differ from those obtained by *Peale et al.* [2014]. First, we use a different parameterization of the viscous torque, so for the same choice of  $\nu$  the numerical values of the out-of-plane components of the mantle and fluid core spin orientations that we obtain are different. But our results are also qualitatively different: in contrast to *Peale et al.* [2014], we find that the net effect of viscous coupling at the CMB is to generate a mantle phase lead instead of a phase lag. The reason for this difference can be understood from the



455 **Figure 4.** a) Phase lag of the mantle spin axis ( $\zeta_m$ ) b) fluid core spin axis ( $Im[\tilde{m}_f]$ , solid lines)  
 456 and inner core symmetry axis ( $Im[\tilde{n}_s]$ , dashed lines) as a function of inner core radius and for different  
 457 choices of kinematic viscosity (colour in legend). c) and d) idem, but with no deformations (all compli-  
 458 ances  $\mathcal{S}_{ij}$  set to zero).

465 prediction of the mantle phase lag given by the approximate solution of Equation (17) in section  
 466 2.5. As we explained in that section, the solar torque acting on the elastic deformations  
 467 induced by the out-of-plane component of the fluid core spin axis (through the compliance  $\mathcal{S}_{12}$ )  
 468 acts akin to a tidal torque. This contribution to  $\zeta_m$  is opposite to that caused by the viscous  
 469 torque at the CMB and larger in magnitude. As a result, the net effect of viscous coupling at  
 470 the CMB is to generate a mantle phase lead. Figure 4cd shows how the results are altered when  
 471 all compliances are set to zero (no deformations). The mantle phase lag is now positive, con-  
 472 sistent with the results of *Peale et al.* [2014], and is increased in magnitude by approximately  
 473 a factor 10.

474 As observed in Figure 4ab, when the inner core radius exceeds 500 km, its presence al-  
 475 ters the resulting mantle phase lead, reducing its magnitude. For a very large inner core,  $\zeta_m$   
 476 can be positive (a mantle phase lag), with a magnitude peaking at 0.01 arcsec. The influence  
 477 of the inner core on  $\zeta_m$  occurs through several mechanisms, as discussed in section 2.5. First,  
 478 as shown in Figure 4b, the viscous torque at the ICB entrains a phase lead of the inner core  
 479 spin axis (recall that the spin and symmetry axes of the inner core are virtually in alignment,  
 480  $\tilde{n}_s \approx \tilde{m}_s$ ). This induces a gravitational torque on the mantle which contributes to a mantle  
 481 phase lead (the contribution from the term  $\zeta_m^{L,c}$  in the prediction given by Equation 13). The  
 482 solar torque acting on the tilted inner core (the term  $\zeta_m^{t,s}$ ) and the elastic deformation result-  
 483 ing from the latter (the term  $\zeta_m^{t,e}$ ) both contribute to a phase lag. These latter two contribu-  
 484 tions are more important than that from the gravitational torque, so the net effect of viscous  
 485 coupling at the ICB is to generate a mantle phase lag. When the inner core radius is  $> 500$   
 486 km, the magnitude of the net mantle phase lead (from viscous coupling at the CMB) is reduced.  
 487 For a very large inner core, the net effect from viscous coupling at both the CMB and ICB is  
 488 a mantle phase lag.

489 Just as elastic deformations induced by the out-of-plane component of the fluid core spin  
 490 cannot be neglected, those induced by the out-of-plane component of the inner core tilt can-  
 491 not either. A convenient way to demonstrate this is to write the total perturbation in the mo-  
 492 ment of inertia produced by an inner core tilt in the form  $\bar{A}_s \alpha_3 e_s (1 + k_s) \tilde{n}_s$ , where  $k_s$  is the  
 493 equivalent of a Love number, capturing the added contribution to the change in moment of in-  
 494 ertia induced by deformations (see Appendix C).  $k_s$  depends on inner core size and the rheol-  
 495 ogy of the solid regions. The sum of the contributions  $\zeta_m^{t,s}$  and  $\zeta_m^{t,e}$  from the inner core can then  
 496 be written as  $\zeta_m^{t,s} (1 + k_s)$ . For a rheology that is constrained to match  $k_2 = 0.55$ ,  $k_s$  falls be-  
 497 tween 0.6 and 0.9 (see Figure C.1). Hence, elastic deformations cannot be neglected in the pre-  
 498 diction of  $\zeta_m$ . The contrast in the results of Figures 4ab and 4cd indeed illustrates the impor-  
 499 tance of including elastic deformations induced by the misaligned fluid core and inner core in  
 500 the prediction of  $\zeta_m$ . (Note though that the solutions for  $Im[\tilde{m}_f]$  and  $Im[\tilde{n}_s]$  are virtually un-  
 501 changed; these solutions are not altered significantly by elastic deformations.)

502 In summary, viscous coupling at the CMB and ICB generate a mantle phase lead for a  
 503 small inner core, and a mantle phase lag for a large inner core. As argued in D21, a conserva-  
 504 tive upper bound for the effective turbulent viscosity is  $\nu \approx 5 \times 10^{-4} \text{ m}^2 \text{ s}^{-1}$ . This places an  
 505 upper limit of 0.02 arcsec on the mantle phase lead. The out-of-plane components of the spin  
 506 axes of the fluid and solid cores are substantially larger. The spin axis of the fluid core lags be-  
 507 hind the Cassini plane, with a maximum phase lag that can approach 100 arcsec. The inner  
 508 core leads ahead of the Cassini plane, with a phase lead of a few 10s of arcsec for a small in-

509 ner core, and limited to a few arcsec for a large inner core. Note that these amplitudes are of  
510 the same order as their in-plane components (see Figure 5 of D21).

### 511 3.2 Electromagnetic dissipation

512 We now investigate dissipation caused by EM coupling. We set viscous coupling to zero  
513 and again set the imaginary parts of all compliances to zero. The differential velocity at the  
514 CMB and ICB shears the local radial magnetic field  $B_r$ . This induces a secondary magnetic  
515 field which leads to a tangential force resisting the differential motion. This magnetic “friction”  
516 depends on the radial magnetic field strength  $B_r$  and the electrical conductivity  $\sigma$  on either side  
517 of the boundary [Rochester, 1960, 1962, 1968].

518 As argued in section 3.4 of D21, at the CMB of Mercury, EM coupling is expected to be  
519 much weaker than viscous coupling. For simplicity, we simply assume no EM coupling at the  
520 CMB ( $K_{cmb} = 0$ ) and concentrate our efforts on the dissipation induced by EM coupling at  
521 the ICB. We follow D21 and assume a parameterization for  $K_{icb}$  given by

$$K_{icb} = \frac{5}{4}(1 - i)\mathcal{F}_{icb} \langle B_r \rangle^2, \quad (19)$$

522 where  $\langle B_r \rangle$  is the r.m.s. strength of the radial component of the field at the ICB and

$$\mathcal{F}_{icb} = \frac{\sigma\delta}{\Omega_o\rho_s r_s}, \quad (20)$$

523 where  $\sigma$  is the electrical conductivity (assumed equal in the fluid and solid core) and  $\delta = \sqrt{2/(\sigma\mu\Omega_o)}$   
524 is the magnetic skin depth, with  $\mu = 4\pi \times 10^{-7}$  N A<sup>-1</sup> the magnetic permeability of free space.  
525 We use  $\sigma = 10^6$  S m<sup>-1</sup>, a reasonable value for Mercury’s core [e.g. Berrada and Secco, 2021].  
526 This parameterization is valid provided EM coupling remains in a weak-field regime which, as  
527 detailed in D21, is a reasonable assumption for Mercury.

528 Figure 5ab shows how  $\zeta_m$  and the imaginary parts of  $\tilde{m}_f$  and  $\tilde{n}_s$  vary as a function of in-  
529 ner core radius for different choices of  $\langle B_r \rangle$ . The net effect of EM coupling at the ICB is to gen-  
530 erate a mantle phase lag ( $\zeta_m > 0$ ). The EM torque (and dissipation) increases with the size  
531 of the inner core; the resulting mantle phase lag remains small ( $< 0.01$  arcsec) for an inner core  
532 radius  $< 500$  km. For a large inner core, the magnitude of  $\zeta_m$  can be considerably larger than  
533 that from viscous coupling, as high as  $\sim 0.08$  arcsec for  $\langle B_r \rangle = 0.03$  mT.

534 The EM torque is proportional to  $\langle B_r \rangle^2 (\tilde{m}_s - \tilde{m}_f)$ . EM dissipation is weak when  $\langle B_r \rangle$   
535 is small, and also weak when  $\langle B_r \rangle$  is large, as then a strong EM coupling prevents a large dif-  
536 ferential rotation at the ICB (i.e.  $\tilde{m}_s \approx \tilde{m}_f$ ). Hence, just as for viscous coupling, EM dissi-  
537 pation is characterized by a saturation effect; it is maximized when  $\langle B_r \rangle$  is sufficiently large to  
538 generate a large EM torque but not too large as to prevent differential rotation. This maximum  
539 dissipation is produced when  $\langle B_r \rangle$  is of the order 0.03–0.1 mT and also depends on inner core  
540 size.

541 The spin axis of the fluid inner core lags behind the Cassini plane, while the spin axis of  
542 the inner core is displaced ahead of it. The amplitude of their offsets is of the order of a few  
543 10s of arcsec. The inner core phase lead results in a mantle phase lag for the same reasons as  
544 explained in the previous section; the gravitational torque by the inner core generates a man-

545 tle phase lead, but the solar torque acting on the tilted inner core and the global deformations  
546 that it entrains produce a phase lag, and the latter contribution is larger in magnitude.

547 As in the case of viscous coupling, elastic deformations induced by both the misaligned  
548 fluid core (through the compliance  $\mathcal{S}_{12}$ ) and inner core (through  $\mathcal{S}_{14}$ ) have a first order influ-  
549 ence on the prediction of  $\zeta_m$ . To illustrate this, Figure 5cd shows how the results are altered  
550 when all compliances are set to zero. The solutions for  $\zeta_m$  are qualitatively similar, but their  
551 amplitudes are different. Note again that, as observed in the case of viscous coupling, the so-  
552 lutions for  $Im[\tilde{m}_f]$  and  $Im[\tilde{n}_s]$  are not altered significantly by elastic deformations.

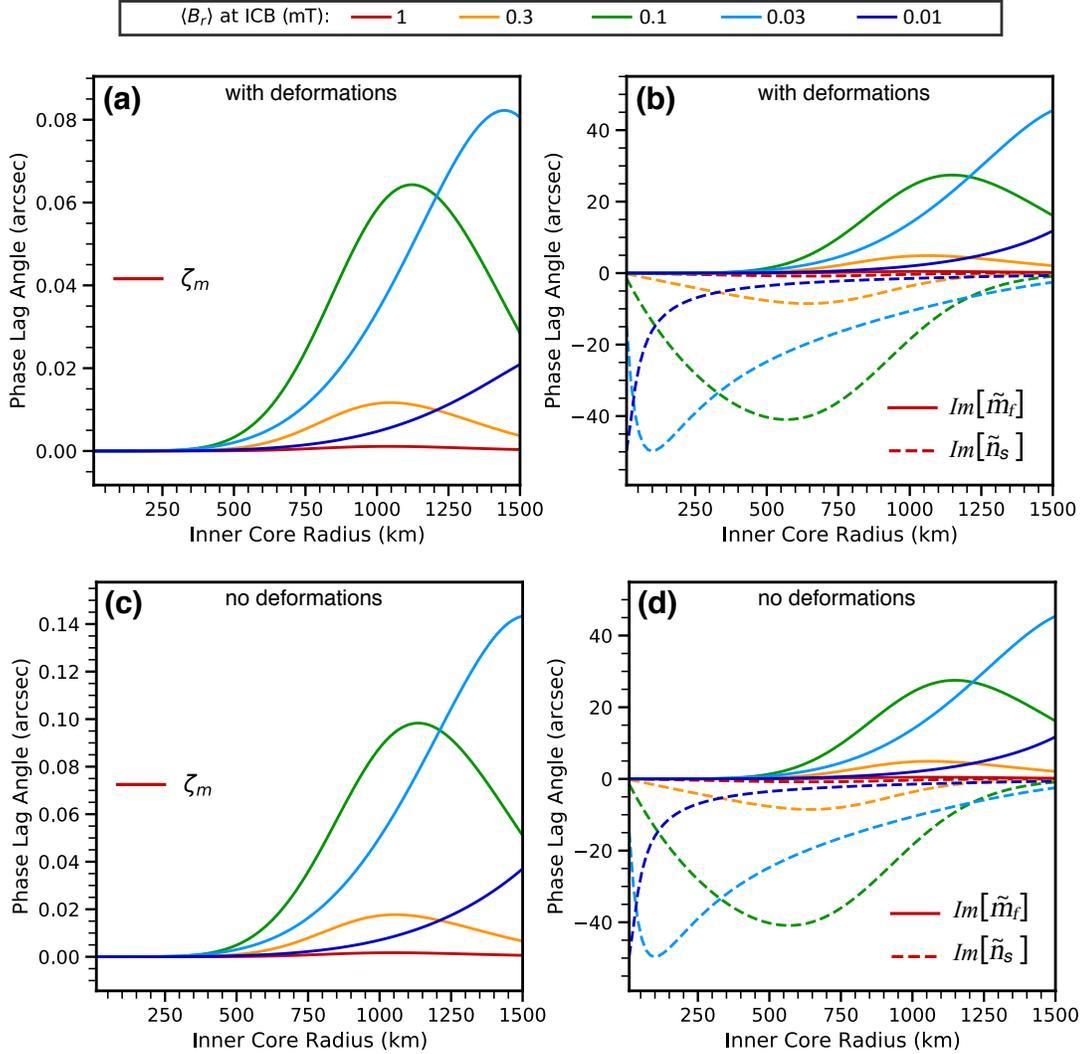
557 In summary, EM coupling at the ICB generates a mantle phase lag which, for an inner  
558 core radius of 1000 km or larger, can be as high as 0.08 arcsec for a  $B_r$  field close to 0.03 mT  
559 that optimizes dissipation. Such a field strength is a factor 100 larger than the field measured  
560 at Mercury’s surface [e.g. *Anderson et al.*, 2012], but it is not an unreasonable estimate if the  
561 field geometry deep within the core is dominated by small length scales [e.g. *Christensen*, 2006].  
562 Hence, it may well be that dissipation at the ICB from EM coupling is close to its optimal value  
563 at present-day. If the inner core radius is 1000 km or larger, the mantle phase lag resulting from  
564 EM coupling at the ICB is substantially larger than the maximum phase lag or lead generated  
565 by viscous coupling.

### 566 3.3 Tidal dissipation

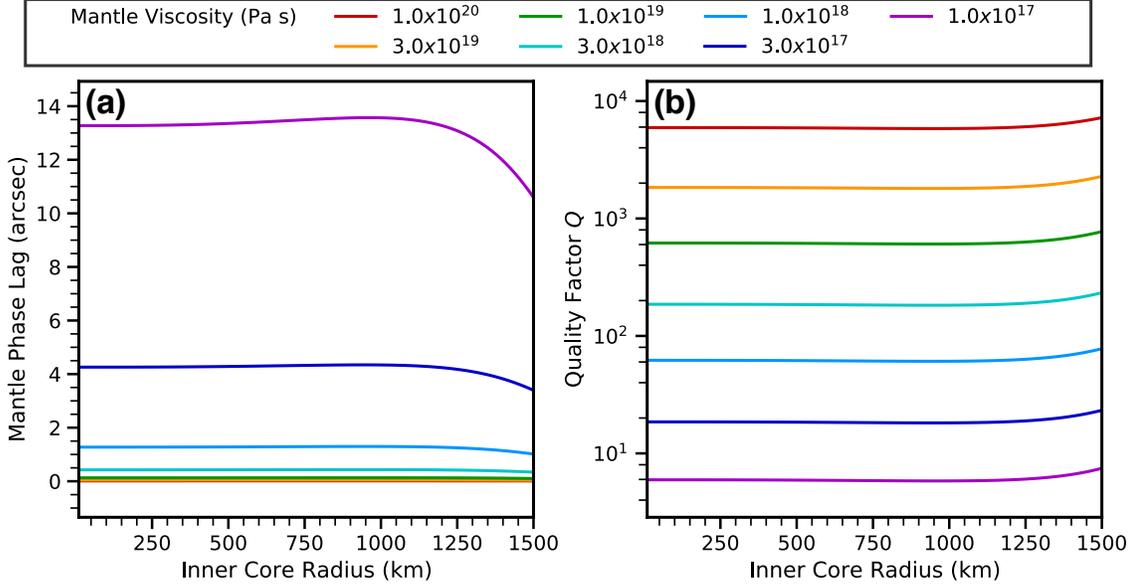
567 We now turn to the dissipation resulting from anelastic deformations. To isolate their ef-  
568 fect on the mantle phase lag, we set both viscous and EM coupling to zero. The delayed, anelas-  
569 tic response of Mercury to tidal forces depends on the ratio  $k_2/Q$  which, in our formulation,  
570 is captured by the imaginary component of the compliance  $\mathcal{S}_{11}$  (see Equation 4). We do not  
571 prescribe values of  $Q$ ; instead, we specify the viscosity of each solid regions, and calculate the  
572 resulting  $Q$  on the basis of  $Im[\mathcal{S}_{11}]$ . We recall that we assume a Maxwell rheology in solid re-  
573 gions, see Appendix C for the computation of the compliances. Global anelastic deformations  
574 also occur in response to the pressure force at the CMB from the misaligned fluid core spin axis  
575 and from the gravitational force induced by a tilted inner core. These are captured by the imag-  
576 inary parts of the compliances  $\mathcal{S}_{12}$  and  $\mathcal{S}_{14}$ , respectively.

577 Figure 6 shows how  $\zeta_m$  and  $Q$  vary as a function of inner core radius for different choices  
578 of mantle viscosity; these values refer to the bulk viscosity of the whole of the mantle. In all  
579 cases, the inner core viscosity is fixed at  $10^{20}$  Pa s. Tidal deformations result in a positive  $\zeta_m$ ,  
580 in other words a mantle phase lag. For our largest choice of mantle viscosity,  $10^{20}$  Pa s,  $Q$  is  
581 approximately 6000 and the phase lag is very small, approximately 0.01 arcsec. As the viscos-  
582 ity of the mantle is decreased,  $Q$  is reduced and the phase lag increases in amplitude. An ap-  
583 proximate empirical relationship between  $\zeta_m$  and  $Q$  based on our results is  $\zeta_m \sim (80/Q)$  arc-  
584 sec. When  $Q$  is of the order of 100, the mantle phase-lag is of the order of 1 arcsec, consistent  
585 with the results obtained by *Baland et al.* [2017]. Unless  $Q$  is larger than a few hundred, the  
586 deviation of the mantle spin axis from the Cassini plane caused by anelastic deformations is sig-  
587 nificantly larger in magnitude than that from EM and viscous coupling at the fluid core bound-  
588 aries.

591 For all cases in Figure 6a, the dominant contribution to  $\zeta_m$  is from tidal dissipation (the  
592 term  $\zeta_m^t$  in Equation 13). There is a small secondary contribution (of the order of 1%) from



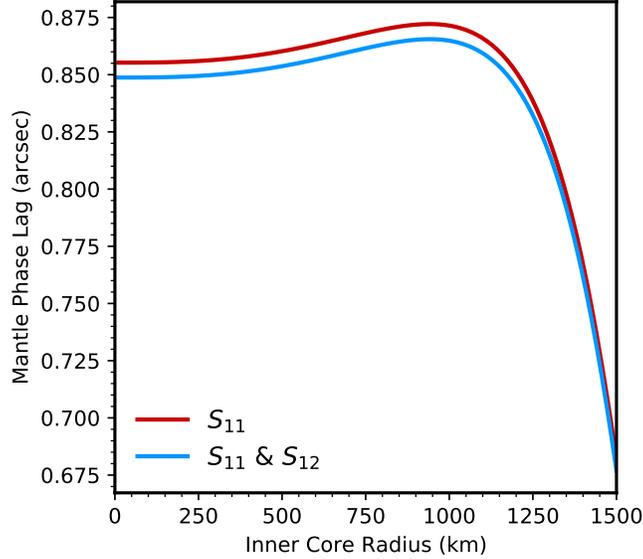
553 **Figure 5.** a) Phase lag of the mantle spin axis ( $\zeta_m$ ) b) fluid core spin axis ( $Im[\tilde{m}_f]$ , solid lines)  
 554 and inner core symmetry axis ( $Im[\tilde{m}_s]$ , dashed lines) as a function of inner core radius and for different  
 555 choices of  $\langle B_r \rangle$  at the ICB (colour in legend). c) and d) idem, but with no deformations (all compliances  
 556  $S_{ij}$  set to zero).



589 **Figure 6.** (a) Phase lag of the mantle  $\zeta_m$  and (b) tidal quality factor  $Q$  as a function of inner core  
 590 radius and for different choices of mantle viscosity.

593 the term  $\zeta_m^{t,a}$ , the delayed anelastic response of the mantle to the misaligned obliquity of the  
 594 fluid core and inner core. In the absence of viscous and EM coupling,  $Re[\tilde{m}_f]$  is of the order  
 595 of 4 arcmin, while  $Re[\tilde{n}_s]$  is very small, approximately 1.5 arcsec (see Figure 4 of D21), so it  
 596 is predominantly the part from  $Im[\mathcal{S}_{12}]Re[\tilde{m}_f]$  that contributes to  $\zeta_m^{t,a}$ . To illustrate this, Fig-  
 597 ure 7 shows an example of how the solution for  $\zeta_m$  as a function inner core radius differs when  
 598 only  $\mathcal{S}_{11}$  is retained, versus when both  $\mathcal{S}_{11}$  and  $\mathcal{S}_{12}$  are retained, with all other compliances set  
 599 to zero. For these solutions, the bulk viscosity of the mantle is set to  $1.5 \times 10^{18}$  Pa s and gives  
 600 a  $Q$  of approximately 100. The delayed, anelastic response of the mantle to the pressure force  
 601 at the CMB does affect the resulting mantle phase lag, but it only reduces it by a small amount  
 602 (not more than 0.0125 arcsec on Figure 7). When all other compliances are included, the so-  
 603 lutions is virtually identical to that shown in Figure 7 when only  $\mathcal{S}_{11}$  and  $\mathcal{S}_{12}$  are retained.

604 Both  $Q$  and  $\zeta_m$  are affected by the size of the inner core. This can be observed in Fig-  
 605 ure 6 but is better highlighted by Figure 7. A large, stiff inner core implies smaller global anelas-  
 606 tic deformations, leading to an increase in  $Q$  with inner core size, and a decrease in  $\zeta_m$ . In turn,  
 607 the inner core viscosity can also influence  $Q$  and the resulting  $\zeta_m$ . Keeping the mantle viscos-  
 608 ity fixed at  $1.5 \times 10^{18}$  Pa s, Figure 8 shows how  $\zeta_m$  and  $Q$  vary as a function of inner core ra-  
 609 dius for different choices of inner core viscosity. Provided the inner core radius is smaller than  
 610 approximately 1000 km, the inner core viscosity has a negligible influence on  $Q$  and  $\zeta_m$ . How-  
 611 ever, for a large inner core (radius > 1000 km) and a low viscosity ( $< 10^{17}$  Pa s),  $Q$  can be  
 612 substantially reduced and  $\zeta_m$  substantially increased. Note that the empirical relation  $\zeta_m \sim$   
 613  $(80/Q)$  arcsec remains applicable in all cases shown in Figures 6-8.



614 **Figure 7.** Phase lag of the mantle  $\zeta_m$  as a function of inner core radius when only the imaginary  
 615 part(s) of  $S_{11}$  (solid red line) and,  $S_{11}$  and  $S_{12}$  (solid blue line) are retained.

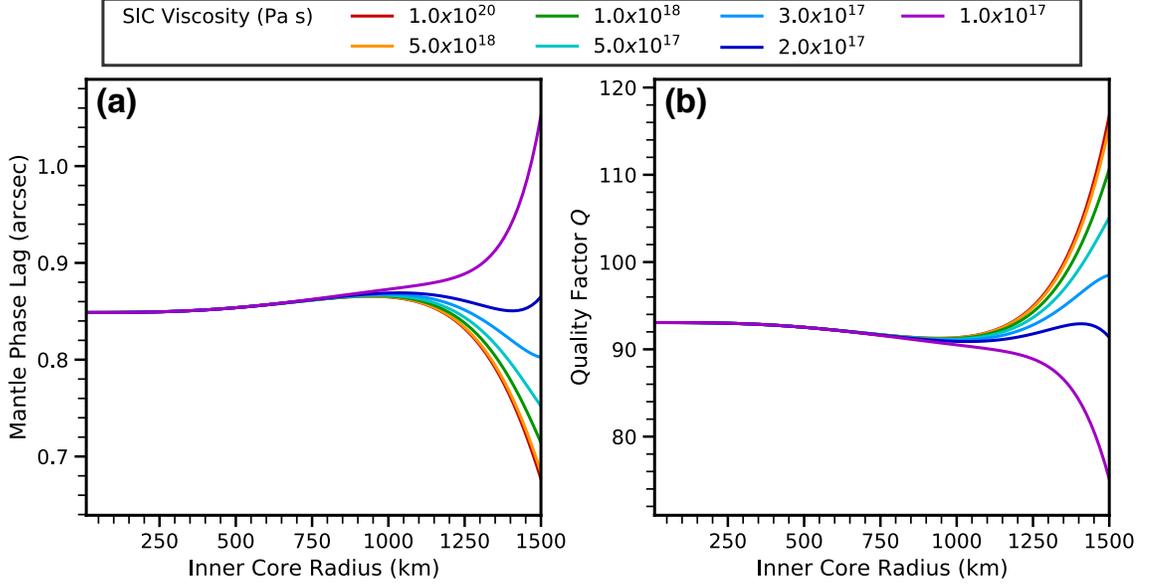
#### 618 4 Discussion

619 We have shown that viscous coupling at the CMB results in a mantle phase lead ahead  
 620 of the Cassini plane, while viscous and/or EM coupling at the ICB results in a mantle phase  
 621 lag. Elastic deformations induced by the misaligned spin axes of the fluid core and inner core  
 622 play a first order role in the resulting mantle phase  $\zeta_m$ . The influence on  $\zeta_m$  from EM and vis-  
 623 cous coupling at the ICB gets proportionally more important the larger the inner core is. The  
 624 net phase that results from dissipation at both the CMB and ICB depends then on the inner  
 625 core size and on the parameters on which the viscous and EM torques depend, notably, on the  
 626 kinematic viscosity and the amplitude of the radial magnetic field at the ICB. Importantly, a  
 627 saturation effect limits the dissipation and thus the maximum phase lead or lag that can be gen-  
 628 erated by either viscous and EM drag.

629 Overall, viscous and EM coupling at the fluid core boundaries generate only a small de-  
 630 viation of no more than 0.1 arcsec of the mantle spin away from the Cassini plane. This is a  
 631 factor 10 smaller than the smallest measurement error on the different estimates of the man-  
 632 tle spin position, which is  $\sim 1$  arcsec (see Table 1). Hence, unless measurement errors can be  
 633 reduced by more than a factor 10, it is unlikely that observations of the mantle phase lag can  
 634 yield useful constraints on inner core size and/or viscous and EM coupling at the fluid core bound-  
 635 aries.

636 From our results shown in Figures 4 and 5, we can compute the dissipation at the CMB  
 637 and ICB, respectively, from

$$Q_{cmb} = \Omega_o^3 \bar{A}_f \text{Im}[K_{cmb}] |\tilde{m}_f|^2, \quad (21a)$$



616 **Figure 8.** (a) Phase lag of the mantle  $\zeta_m$  and (b) tidal quality factor  $Q$  as a function of inner core  
 617 radius and for different choices of solid inner core viscosity.

$$Q_{icb} = \Omega_o^3 \bar{A}_s \text{Im}[K_{icb}] |\tilde{m}_f - \tilde{m}_s|^2. \quad (21b)$$

638 At its peak value, the dissipation from viscous coupling at the CMB is approximately  $2.5 \times 10^7$   
 639 W, and that at the ICB is  $5 \times 10^6$  W. Expressed in terms of heat fluxes, these correspond to  
 640  $q_{cmb} = Q_{cmb}/4\pi r_f^2 \approx 5 \times 10^{-7}$  W m $^{-2}$  and  $q_{icb} = Q_{icb}/4\pi r_s^2 \approx 2 \times 10^{-7}$  W m $^{-2}$  (the latter  
 641 based on an inner core radius of  $r_s = 1500$  km). The peak dissipation at the ICB from EM  
 642 coupling is approximately  $6 \times 10^6$  W ( $q_{icb} \approx 2 \times 10^{-7}$  W m $^{-2}$ ). These are small compared to  
 643 estimates of the heat flow out of the core, which are of the order of  $10^{11}$  W, corresponding to  
 644 a heat flux of  $2 \times 10^{-3}$  W m $^{-2}$  [e.g. *Knibbe and van Westrenen, 2018; Tosi et al., 2013; Grott*  
 645 *et al., 2011*]. Dissipation at the CMB and ICB from viscous and/or EM coupling contributes  
 646 to only a very small fraction of the internal heat budget of Mercury. Furthermore, the heat re-  
 647 leased at the ICB from viscous and EM dissipation is very small compared to the latent heat  
 648 associated with inner core growth [of the order of  $10^{11}$  W, e.g. *Knibbe and van Westrenen, 2018*]  
 649 and adds a negligible contribution to the convective power in Mercury’s fluid core and to the  
 650 power required to generate its dynamo.

651 Tidal dissipation generates a mantle phase lag with a magnitude inversely proportional  
 652 to the quality factor  $Q$ . An approximate empirical relationship derived from our results is  $\zeta_m \sim$   
 653  $(80/Q)$  arcsec. For  $Q$  of the order of 100, the phase lag is approximately 1 arcsec. Unless  $Q >$   
 654 1000, the phase lag produced by tidal dissipation dominates that due to viscous and EM cou-  
 655 pling at the fluid core boundaries.  $Q$  is proportional to the bulk mantle viscosity; a  $Q$  value of  
 656 100 corresponds to a bulk mantle viscosity of approximately  $10^{18}$  Pa s, based on a Maxwell rhe-  
 657 ology.

658 Thermal evolution and mantle convection models tuned to match Mercury’s history of mag-  
 659 matism and radial contraction tend to favor a stiff mantle with high viscosities in the range of  
 660  $10^{19}$ – $10^{22}$  Pa s [Grott *et al.*, 2011; Tosi *et al.*, 2013; Michel *et al.*, 2013; Ogawa, 2016; Knibbe  
 661 and van Westrenen, 2018]. A high mantle viscosity is also required to maintain deep seated mass  
 662 anomalies so as to explain Mercury’s long wavelength topography [James *et al.*, 2015] and non-  
 663 hydrostatic shape [Matsuyama and Nimmo, 2009]. Based on these, we expect then a small phase  
 664 lag angle of the order of 0.1 arcsec or smaller from tidal dissipation. However, the viscosity of  
 665 the lower mantle that is compatible with observations of  $k_2$  falls in the range of  $10^{13}$  –  $10^{18}$   
 666 Pa s [e.g. Steinbrügge *et al.*, 2021]. The viscosity in the top part of the mantle is expected to  
 667 be higher, as temperature decreases with radius, so a bulk viscosity of  $10^{18}$  Pa s in order to fit  
 668  $k_2$  may not be unreasonable. If so, the phase lag from tidal dissipation can be expected to be  
 669 of the order of 1 arcsec.

670 The precession of the pericentre causes a deviation of the spin pole from the Cassini plane  
 671 equivalent to a phase lag of 0.85 arcsec [Baland *et al.*, 2017]. With a  $Q$  of approximately 80,  
 672 we expect the net phase lag of the spin pole to be  $\sim 1.85$  arcsec. All measurements of the spin  
 673 pole position listed in Table 1 are consistent with this. Even the measurement by Mazarico *et al.*  
 674 [2014], which suggests a phase lead of approximately 7.8 arcsec, remains within its error bar  
 675 consistent with a small phase lag. The largest possible phase lag allowed by the different spin  
 676 pole measurements is approximately 12 arcsec. This provides a lower bound for  $Q$  in the vicin-  
 677 ity of 10. If we take the most recent measurement of Bertone *et al.* [2021] as a benchmark, the  
 678 largest phase lag allowed by the measurement error is approximately 1.8 arcsec. Removing the  
 679 contribution from the precession of the pericentre, this leaves a maximum of 1 arcsec caused  
 680 by tidal dissipation, elevating the lower bound for  $Q$  to  $\sim 80$ .

681 As these simple calculations show, an improved measurement of the mantle spin position  
 682 can yield a constraint on  $Q$ , and in turn, on the mantle viscosity. Lower bounds on  $Q$  of 10 and  
 683 100 corresponds to lower bounds on the bulk mantle viscosity of  $10^{17}$  and  $10^{18}$  Pa s, respec-  
 684 tively. It is worth emphasizing that these viscosity values are based on a Maxwell rheology in  
 685 the mantle. Using an Andrade-pseudoperiod model, believed to capture better the rheology of  
 686 planetary mantles [e.g. Padovan *et al.*, 2014; Steinbrügge *et al.*, 2021], the viscosity would be  
 687 higher for the same  $Q$ , so the values quoted above remain lower bounds. As we have shown,  
 688 a large inner core (radius  $> 1000$  km) with a bulk viscosity lower than  $10^{17}$  Pa s can reduce  
 689 the global  $Q$  and increase the phase lag. A large inner core with a very low viscosity would then  
 690 permit to achieve the same  $Q$  with a higher bulk mantle viscosity, though the values quoted  
 691 above remain lower bounds.

692 We have shown that the delayed, anelastic deformations caused by the pressure force at  
 693 the CMB from the misaligned rotation vector of the fluid core contribute to the total mantle  
 694 phase lag. However, this is a small contribution, of the order of 1% compared to the anelastic  
 695 response of the mantle to tidal forcing. We note though that our results are based on a uni-  
 696 form mantle viscosity; the amplitude of this contribution may be increased if the viscosity is  
 697 weakest at the bottom of the mantle – which is indeed what we expect. An improvement on  
 698 our model would be to consider radial variations in the material properties in the mantle, in  
 699 particular its viscosity.

## 5 Conclusion

In this study, we computed predictions of the deviation of Mercury’s spin axis from the Cassini plane (out-of-plane component) from different dissipation mechanisms. Viscous coupling at the CMB results in a phase lead, viscous and EM coupling at the ICB produce a phase lag, and tidal dissipation produces a phase lag.

The magnitude of the mantle phase lead or lag from viscous and EM coupling depends on the inner core size, the kinematic viscosity, and magnetic field strength, though it cannot exceed a maximum value. For a small inner core, viscous drag at the CMB dominates and produces a maximum phase lead of 0.027 arcsec. For a large inner core (radius > 1000 km), EM drag at the ICB can exceed viscous coupling at both the ICB and CMB, and produces a phase lag that does not exceed 0.1 arcsec. For both viscous and EM coupling, the solar torque acting on the global elastic deformations induced by the out-of-plane components of the spin axes of the fluid core and inner core play a first order role in the resulting mantle phase. Tidal dissipation in the mantle produces a phase lag with a magnitude inversely proportional to the quality factor  $Q$ . For a  $Q$  of the order of 100, the phase lag is approximately 1 arcsec.

Our results suggest that dissipation should not displace Mercury’s mantle spin axis away from the Cassini plane by more than a few arcsec. This is indeed in agreement with observations. In turn, the limited phase lag suggested by observations ( $\sim 1$  to 10 arcsec) implies lower limits on  $Q$  and the bulk mantle viscosity which cannot be much smaller than 10 and  $10^{17}$  Pa s, respectively. A more precise measurement of the position of the spin axis can in principle provide a constraint on  $Q$  and thus on the bulk mantle viscosity.

### A: Calculation of the phase lag angle

The classical Cassini State of Mercury is characterized by the co-planar precession of the orbit and spin poles of the planet about the Laplace pole. The Cassini plane is defined as the plane spanned by the axes of the orbit and Laplace poles (the normals to the orbital and Laplace planes, respectively). If Mercury’s spin pole were to obey a classical Cassini state exactly, it should lie in the Cassini plane. Dissipation induces a misalignment of the spin pole away from the Cassini plane characterized by an angle of offset  $\zeta_m$ , defined positive and corresponding to a phase lag if it trails behind the Cassini plane. Conversely, a negative  $\zeta_m$  corresponds to a spin pole that is ahead of the Cassini plane and to a phase lead. In this Appendix, we explain how we calculate the phase lag angles  $\zeta_m$  and their errors that are listed in Table 1 based on measurements of the orientation of the spin pole.

The orientation of the spin pole is given in terms of its right ascension ( $\alpha$ ) and declination ( $\delta$ ) angles with respect to the International Celestial Reference Frame (ICRF). The Cartesian components of a unit vector  $\mathbf{u} = (u_x, u_y, u_z)$  pointing to a coordinate  $(\alpha, \delta)$  on this imaginary celestial sphere are

$$u_x = \cos(\delta) \cos(\alpha), \quad u_y = \cos(\delta) \sin(\alpha), \quad u_z = \sin(\delta), \quad (\text{A.1})$$

where the  $z$ -axis is aligned with the celestial pole ( $\delta = \frac{\pi}{2}$ ) and the  $x$ -axis is aligned with zero right ascension ( $\alpha = 0$ ). The orientations of the Laplace pole ( $\alpha_L, \delta_L$ ) and orbit pole ( $\alpha_O, \delta_O$ ) at epoch J2000 are calculated in *Baland et al.* [2017], and are

$$\alpha_L = (273.811048 \pm 0.324494)^\circ, \quad \delta_L = (69.457475 \pm 0.259017)^\circ, \quad (\text{A.2a})$$

$$\alpha_O = (280.987906 \pm 0.000009)^\circ, \quad \delta_O = (61.447794 \pm 0.000006)^\circ. \quad (\text{A.2b})$$

739 The unit vectors derived from the central values of the right-ascension and declination mea-  
 740 surements of the Laplace and orbit poles are denoted with  $\mathbf{u}_L$  and  $\mathbf{u}_O$  respectively (these are  
 741 denoted by  $\hat{\mathbf{e}}_3^L$  and  $\hat{\mathbf{e}}_3^I$ , respectively, in the main text). The Cassini plane corresponds to the  
 742 plane that passes through the origin of the ICRF and whose great circle on the celestial sphere  
 743 joins both the Laplace and orbit poles. To define this great circle as a function of  $\delta$  and  $\alpha$ , one  
 744 must first determine the unit normal to the Cassini plane, defined by

$$\mathbf{u}_C = \frac{\mathbf{u}_L \times \mathbf{u}_O}{\sqrt{1 - (\mathbf{u}_L \cdot \mathbf{u}_O)^2}}. \quad (\text{A.3})$$

745 The function of  $\delta$  and  $\alpha$  that defines the great circle can be found from the criteria that  $\mathbf{u}_C \cdot$   
 746  $\mathbf{u} = 0$  with  $\mathbf{u}$  defined as in Equation (A.1). In this manner, one can construct the great cir-  
 747 cle of the Cassini plane on the celestial sphere. Figure A.1a shows how this great circle maps  
 748 on a two dimensional projection of the celestial sphere. Figure A.1b shows a close up view in  
 749 the vicinity of the Laplace and orbit poles.

750 For a measurement of the spin pole orientation given as a pair  $(\alpha, \delta)$ , its corresponding  
 751 unit vector is denoted by  $\mathbf{u}_S$ . The phase lag angle,  $\zeta_m$ , between the great circle of the Cassini  
 752 plane and the orientation of the spin pole is obtained from [e.g. Eq. 41 of *Baland et al.*, 2017]

$$\sin(\zeta_m) = \frac{\mathbf{u}_S \cdot (\mathbf{u}_L \times \mathbf{u}_O)}{\sqrt{1 - (\mathbf{u}_L \cdot \mathbf{u}_O)^2}}. \quad (\text{A.4})$$

753 The numerical values for  $\zeta_m$  given in Table 1 in the main text are calculated from Equation (A.4),  
 754 using the central values of the the Laplace and orbit poles given in Equation (A.2b) and the  
 755 central values of the spin pole measurements.

756 The error in the phase lag is constructed from the errors in right ascension ( $\Delta\alpha$ ) and dec-  
 757 lination ( $\Delta\delta$ ). For each spin pole measurement, an ellipse of error can be drawn around the cen-  
 758 tral value. The phase lag error corresponds to the distance  $\Delta\zeta_m$  between the central value and  
 759 a point on this ellipse, in the direction perpendicular to the great circle of the Cassini plane.  
 760 We express this direction by an angle  $\theta_o$  between  $\mathbf{u}_C$  and the local unit vector in the direction  
 761 of the increasing right ascension ( $\hat{\alpha} = \hat{x} \sin \alpha_o + \hat{y} \cos \alpha_o$ ), at the location of the spin pole. Graph-  
 762 ically, on Figure A.1b,  $\theta_o$  corresponds to the angle between the x-axis and the direction per-  
 763 pendicular to the great circle of the Cassini plane at the location of the orbit pole. We take  $\alpha_o =$   
 764  $281.0075^\circ$  as our reference spin pole position, which gives  $\theta_o = 17.17^\circ$ . The distance  $\Delta\zeta_m$  is  
 765 then found by

$$\Delta\zeta_m = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (\text{A.5a})$$

766 where

$$\Delta x = 3600 \cdot \Delta\alpha \cdot \cos\theta_o \cdot \cos\delta_o, \quad (\text{A.5b})$$

$$\Delta y = 3600 \cdot \Delta\delta \cdot \sin\theta_o. \quad (\text{A.5c})$$

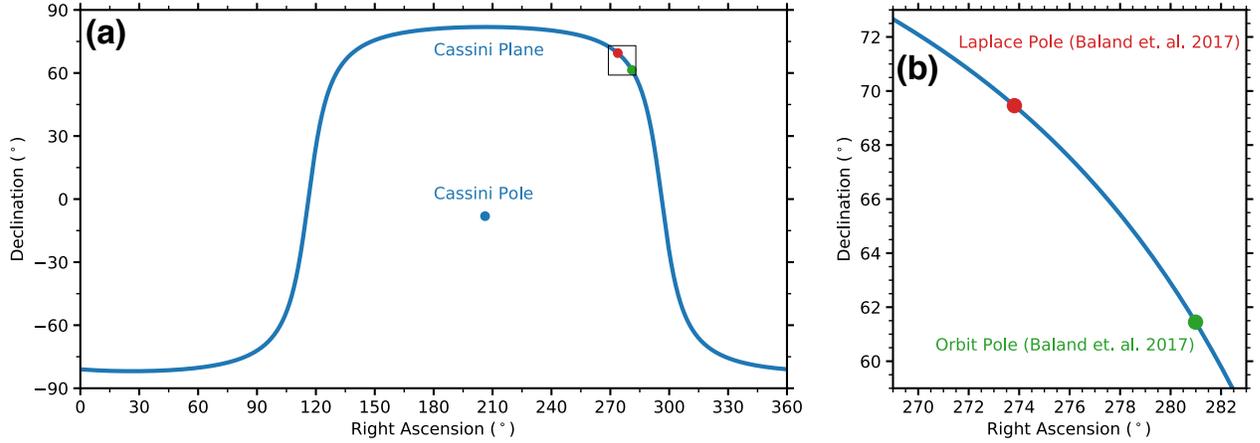
767 The factor 3600 converts degrees to arcseconds and the factor  $\cos\delta_o$  in the expression for  $\Delta x$   
 768 scales the angular error  $\Delta\alpha$  at declination  $\delta_o$  to its proper angular arc distance in right ascen-  
 769 sion. We take  $\delta_o = 61.415^\circ$ .

770 The phase lag errors calculated by this method are based solely on the uncertainty in the  
 771 position of the spin pole at epoch J2000 reported in different studies. Uncertainties in the de-  
 772 termination of the Laplace pole, orbit pole and precession rate translate to an error in the pre-  
 773 cise location of the great circle of the Cassini plane on the Celestial sphere, both today and back  
 774 at epoch J2000, and consequently to an additional error on the phase lag angle. Depending on  
 775 the method used to retrieve these orbital elements, at the location of the spin pole, this cor-  
 776 responds to a phase lag error of the order of 0.02 arcsec [*Baland et al.*, 2017] to 0.2 arcsec [*Stark*  
 777 *et al.*, 2015b]. Spin pole measurements reported in different studies are made at different epochs  
 778 (or more precisely over a time span with respect to a mean epoch) and not all studies give the  
 779 details of how the projection back to epoch J2000 is carried out. The phase lag error connected  
 780 to the uncertainties in orbital elements may then be larger than 0.2 arcsec in individual stud-  
 781 ies. Nevertheless, this error is typically an order of magnitude smaller than that connected to  
 782 the spin pole positions reported in Table 1 and we simply neglect it here.

783 For the same spin pole positions, the phase lags that we calculate in Table 1 are slightly  
 784 different than those given in Table C.2 of *Baland et al.* [2017]. This is because of the choice made  
 785 in the specific values of the Laplace pole. Note also that our phase lag errors are smaller than  
 786 those given in *Baland et al.* [2017]. The method to calculate  $\Delta\zeta_m$  is not detailed in *Baland et al.*  
 787 [2017], so the reason for this difference is unknown. We note however that if the factor  $\cos\delta_o$   
 788 is omitted in Equation (A.5b) the  $\Delta\zeta_m$  that we obtain are closer to those given in Table C.2  
 789 of *Baland et al.* [2017], so a part of the discrepancy may be due to this. We also note that es-  
 790 timates of  $\alpha$  and  $\delta$  are correlated in some studies, which causes the ellipse of error to be tilted  
 791 in 2D plots like the one we show in Figure 2 of the main text. We do not take this tilt into ac-  
 792 count in our calculations of  $\Delta\zeta_m$ . Instead, we simply assume an ellipse with semi-major (semi-  
 793 minor) axis equal to the largest (smallest) value between  $\Delta\alpha \cos\delta_o$  and  $\Delta\delta$ .

## 796 B: Modification of the rotational model

797 Tidal deformations of Mercury’s figure occur in response to the imposed solar gravitational  
 798 potential. The deformations are of spherical harmonic degree 2 and hence induce a perturba-  
 799 tion in the moment of inertia tensor. The reshaping of Mercury’s figure alters the amplitude  
 800 of the solar torque acting on it and it also alters Mercury’s angular momentum response. For  
 801 a purely elastic deformation, the tidal bulge is aligned with the line connecting the centre of  
 802 Mercury to the Sun. Anelastic deformations from internal dissipation results in delayed response  
 803 and to a misalignment of the tidal bulge. The solar torque acting on the delayed part of the  
 804 deformation is referred to as the tidal torque. We show in this Appendix how the rotational  
 805 model of D21 is modified to take into account viscoelastic deformations. For brevity, we do not  
 806 repeat the whole presentation of the model but only point out its modifications. All variable  
 807 names and symbol that are not explicitly defined here are identical to those used in D21.



794 **Figure A.1.** (a) The great circle of the Cassini plane on the celestial sphere as a function of right  
 795 ascension and declination angles. (b) Close-up view in the vicinity of the Laplace and orbit poles.

### 808 B.1 Perturbation in the moment of inertia tensor

809 As seen in the mantle frame, the inner core figure axis and the rotation vectors of the man-  
 810 tle, fluid core and inner core all precess in the retrograde direction. The periodic changes in the  
 811 gravitational and centrifugal potential associated with these lead to global deformations, and  
 812 thus to a perturbation in the moment of inertia tensor of Mercury  $\Delta\mathcal{I}$ . These involve the off-  
 813 diagonal terms  $(\Delta\mathcal{I})_{13}$  and  $(\Delta\mathcal{I})_{23}$ . In the complex notation used in D21, we write

$$(\Delta\mathcal{I})_{13}(t) + i(\Delta\mathcal{I})_{23}(t) = \tilde{c} \exp[i\omega\Omega_o t], \quad (\text{B.1})$$

814 where  $\tilde{c} \equiv \tilde{c}(\omega\Omega_o)$  is the amplitude of the perturbation at frequency  $\omega\Omega_o$ , where  $\omega$  is given by  
 815 Equation (1). Equivalent definitions are used for the perturbation in the moment of inertia ten-  
 816 sors of the fluid core and inner core, with  $\tilde{c}_f$  and  $\tilde{c}_s$  denoting their amplitudes, respectively. The  
 817 amplitudes  $\tilde{c}$ ,  $\tilde{c}_f$  and  $\tilde{c}_s$  are expressed as a linear combination of the rotation variables and a  
 818 set of compliances. Following the notation introduced by *Buffett et al. [1993]*, we denote these  
 819 compliances by  $\mathcal{S}_{ij}$ . The perturbation in the moment of inertia tensors from internal contri-  
 820 butions are defined as

$$\tilde{c}^i = \bar{A}(\mathcal{S}_{11}\tilde{m} + \mathcal{S}_{12}\tilde{m}_f + \mathcal{S}_{13}\tilde{m}_s + \mathcal{S}_{14}\tilde{n}_s), \quad (\text{B.2a})$$

$$\tilde{c}_f^i = \bar{A}_f(\mathcal{S}_{21}\tilde{m} + \mathcal{S}_{22}\tilde{m}_f + \mathcal{S}_{23}\tilde{m}_s + \mathcal{S}_{24}\tilde{n}_s), \quad (\text{B.2b})$$

$$\tilde{c}_s^i = \bar{A}_s(\mathcal{S}_{31}\tilde{m} + \mathcal{S}_{32}\tilde{m}_f + \mathcal{S}_{33}\tilde{m}_s + \mathcal{S}_{34}\tilde{n}_s), \quad (\text{B.2c})$$

821 where  $\bar{A}$ ,  $\bar{A}_f$  and  $\bar{A}_s$  are the mean equatorial moments of inertia of the whole planet, the fluid  
 822 core and inner core, respectively. The perturbation in the moment of inertia tensors from ex-  
 823 ternal contributions (i.e. due to tidal forces) are written as

$$\tilde{c}^e = -\bar{A} \frac{\phi_m}{e} \mathcal{S}_{11} \tilde{\epsilon}_m, \quad \tilde{c}_f^e = -\bar{A}_f \frac{\phi_m}{e} \mathcal{S}_{21} \tilde{\epsilon}_m, \quad \tilde{c}_s^e = -\bar{A}_s \frac{\phi_m}{e} \mathcal{S}_{31} \tilde{\epsilon}_m, \quad (\text{B.3})$$

824 where  $\phi_m$  is given by Equation (B.6) below and  $e = (C - \bar{A}) / \bar{A}$  is the dynamic ellipticity (Equa-  
825 tion 3a of D21).

## 826 B.2 The linear system of equations

827 Equations (12a-12c) of D21 describe, respectively, the time rate of change of the angu-  
828 lar momenta of the whole of Mercury, the fluid core, and the inner core in the reference frame  
829 of the rotating mantle. Viscoelastic deformations modify these three equations to

$$(\omega - e)\tilde{m} + (1 + \omega) \left[ \frac{\bar{A}_f}{\bar{A}} \tilde{m}_f + \frac{\bar{A}_s}{\bar{A}} \tilde{m}_s + \alpha_3 e_s \frac{\bar{A}_s}{\bar{A}} \tilde{n}_s + \frac{\tilde{c}}{\bar{A}} \right] = \frac{1}{i\Omega_o^2 \bar{A}} (\tilde{\Gamma}_{sun} + \tilde{\Gamma}_t), \quad (\text{B.4a})$$

$$\omega \tilde{m} + (1 + \omega + e_f) \tilde{m}_f - \omega \alpha_1 e_s \frac{\bar{A}_s}{\bar{A}_f} \tilde{n}_s + \omega \frac{\tilde{c}_f}{\bar{A}_f} = \frac{1}{i\Omega_o^2 \bar{A}_f} (-\tilde{\Gamma}_{cmb} - \tilde{\Gamma}_{icb}), \quad (\text{B.4b})$$

$$(\omega - \alpha_3 e_s) \tilde{m} + \alpha_1 e_s \tilde{m}_f + (1 + \omega) \tilde{m}_s + (1 + \omega - \alpha_2) \left[ e_s \tilde{n}_s + \frac{\tilde{c}_s}{\bar{A}_s} \right] = \frac{1}{i\Omega_o^2 \bar{A}_s} (\tilde{\Gamma}_{sun}^s + \tilde{\Gamma}_{ts} + \tilde{\Gamma}_{icb}), \quad (\text{B.4c})$$

830 where  $\tilde{\Gamma}_{sun}, \tilde{\Gamma}_{sun}^s$  are the gravitational torques by the Sun on the whole of Mercury and on the  
831 inner core alone, respectively, and  $\tilde{\Gamma}_{cmb}, \tilde{\Gamma}_{icb}$  are the torques from tangential stresses by the fluid  
832 core on the mantle at the CMB and on the inner core at the ICB, respectively. We have also  
833 introduced the torques associated with tidal dissipation (the tidal torque) acting on the whole  
834 of Mercury,  $\tilde{\Gamma}_t$ , and on its inner core,  $\tilde{\Gamma}_{ts}$ ; these are developed in section B.4.

835 The two additional equations of the system are kinematic relations, one that expresses  
836 the change in the orientation of the inner core figure as a result of its own rotation, and a sec-  
837 ond that expresses the invariance of the Laplace pole as seen in the mantle frame. These are  
838 unaffected by deformations and are

$$\tilde{m}_s + \omega \tilde{n}_s = 0, \quad (\text{B.4d})$$

$$\tilde{m} + (1 + \omega) \tilde{\epsilon}_m = -(1 + \omega) \tan I. \quad (\text{B.4e})$$

## 839 B.3 Modification of the solar torque

840 For a small mantle obliquity  $\tilde{\epsilon}_m$ , the (rigid) gravitational torque by the Sun on the whole  
841 of Mercury is given by Equation (14) of D21,

$$\tilde{\Gamma}_{sun} = -i\Omega_o^2 [\bar{A} \phi_m \tilde{\epsilon}_m + \bar{A}_s \alpha_3 \phi_s \tilde{n}_s], \quad (\text{B.5})$$

842 where

$$\phi_m = \frac{3}{2} \frac{n^2}{\Omega_o^2} \left[ G_{210} e + \frac{1}{2} G_{201} \gamma \right], \quad \phi_s = \frac{3}{2} \frac{n^2}{\Omega_o^2} \left[ G_{210} e_s + \frac{1}{2} G_{201} \gamma_s \right], \quad (\text{B.6})$$

843 and where  $e$ ,  $\gamma$  and  $e_s$ ,  $\gamma_s$  are dynamical ellipticities (defined by Equations 3a and 3b of D21),  
 844  $G_{210}$  and  $G_{201}$  are functions of the orbital eccentricity  $e_c$  (defined by Equations 16a and 16b  
 845 of D21),  $n$  is the mean motion and  $\Omega_o$  is the rotation frequency.

846 We adapt Equation (B.5) to include the perturbation in the moment of inertia caused by  
 847 elastic tidal deformations. To do so, we follow *Baland et al.* [2017]. Their model does not take  
 848 into account the misalignment of the inner core tilt (i.e. they assume  $\tilde{n}_s = 0$ ). They write the  
 849 rigid torque  $\tilde{\Gamma}_{sun}$  as

$$\tilde{\Gamma}_{sun} = -i \frac{3}{2} n (\kappa_{20} + \kappa_{22}) \tilde{\epsilon}_m, \quad (\text{B.7})$$

850 where the parameters  $\kappa_{20}$  and  $\kappa_{22}$  are defined in their Equations (24-25). The connection be-  
 851 tween Equations (B.5) and (B.7) implies that  $\frac{3}{2} n (\kappa_{20} + \kappa_{22}) = \Omega_o^2 \bar{A} \phi_m$  in our notation. *Ba-*  
 852 *land et al.* [2017] then show how elastic deformations induced by solar tides modify  $\kappa_{20}$  and  $\kappa_{22}$   
 853 (their Equations 53-54), and alter the solar torque to

$$\begin{aligned} \tilde{\Gamma}_{sun} &= -i \frac{3}{2} n \left( \kappa_{20} + \kappa_{22} + k_2 M R^2 q_t n \left( \frac{1}{6} + \frac{1}{2} e_c^2 + \frac{49}{24} e_c^2 \right) \right) \tilde{\epsilon}_m, \\ &= -i \frac{3}{2} n \left( \kappa_{20} + \kappa_{22} + k_2 M R^2 q_t n \left( \frac{1}{6} + \frac{61}{24} e_c^2 \right) \right) \tilde{\epsilon}_m, \end{aligned} \quad (\text{B.8})$$

854 where  $q_t = -3R^3 n^2 / (GM)$  is a tidal parameter. Substituting  $q_t$  and  $k_2$  (from Equation 4) into  
 855 Equation (B.8), we get

$$\tilde{\Gamma}_{sun} = -i \frac{3}{2} n \left( \kappa_{20} + \kappa_{22} - 9 \bar{A} n \frac{n^2}{\Omega_o^2} \text{Re}[\mathcal{S}_{11}] \left( \frac{1}{6} + \frac{61}{24} e_c^2 \right) \right) \tilde{\epsilon}_m. \quad (\text{B.9})$$

856 The difference between Equations (B.9) and (B.7) captures the modification of the torque by  
 857 elastic deformations. Re-introducing the part of the torque associated with a tilted inner core,  
 858 and modifying the latter to take into account elastic deformations in the same manner (though  
 859 it involves the compliance  $\mathcal{S}_{31}$  instead of  $\mathcal{S}_{11}$ ), we write the modified torque in our notation as

$$\tilde{\Gamma}_{sun} = -i \Omega_o^2 \left[ \bar{A} \phi_m^{el} \tilde{\epsilon}_m + \bar{A}_s \alpha_3 \phi_s^{el} \tilde{n}_s \right], \quad (\text{B.10})$$

860 with

$$\phi_m^{el} = \phi_m - \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{11}], \quad \phi_s^{el} = \phi_s - \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{31}], \quad (\text{B.11a})$$

861 and where

$$\mathcal{F}(e_c) = 9 \frac{n^4}{\Omega_o^4} \left( \frac{1}{4} + \frac{61}{16} e_c^2 \right). \quad (\text{B.11b})$$

862 The expression for the torque in Equation (B.10) includes the effect of elastic deforma-  
 863 tions associated with the external gravitational potential from the Sun (captured by Equation

864 B.3). We further modify the torque to also take into account elastic deformations from inter-  
 865 nal contributions (captured by Equation B.2). For this, we follow section 2.4 of *Organowski and*  
 866 *Dumberry* [2020] and our final expression of the solar torque is

$$\tilde{\Gamma}_{sun} = -i\Omega_o^2 \left[ \bar{A}\phi_m^{el} \tilde{\varepsilon}_m + \bar{A}_s \alpha_3 \phi_s^{el} \tilde{n}_s + \phi_m \frac{\tilde{c}^i}{e} + \alpha_s \phi_s \frac{\tilde{c}_s^i}{e_s} \right]. \quad (\text{B.12})$$

867 The solar torque on a rigid inner core is given by Equation (17) of D21. Following the same  
 868 procedure as above, elastic deformations modify this torque to

$$\tilde{\Gamma}_{sun}^s = -i\Omega_o^2 \left[ \bar{A}_s \alpha_3 \phi_s^{el} (\tilde{\varepsilon}_m + \tilde{n}_s) + \alpha_s \phi_s \frac{\tilde{c}_s^i}{e_s} \right]. \quad (\text{B.13})$$

#### 869 B.4 Tidal torque

870 We adopt a weak friction tidal model in which the deformed surface of Mercury due to  
 871 the solar tide matches that based on a purely elastic planet, but delayed by a time lag [*Dar-*  
 872 *win*, 1879; *Alexander*, 1973]. The torque associated with tidal dissipation is [e.g. *Levrard et al.*,  
 873 2007, Equation 1],

$$\Gamma_{\mathbf{t}} = 3 \frac{k_2}{Q} \frac{GM_s^2 R^5}{a^6} \left[ \left( f_1 - \frac{f_2 \Omega_o}{2n} \hat{\Omega} \cdot \hat{\mathbf{e}}_3^I \right) \hat{\mathbf{e}}_3^I + \left( f_1 - \frac{f_2 \Omega_o}{2n} \left( 1 + (\hat{\Omega} \cdot \hat{\mathbf{e}}_3^I)^2 \right) \right) \hat{\Omega} \right], \quad (\text{B.14})$$

874 where  $M_s$  is mass of the Sun,  $a$  is the semi-major axis of Mercury's orbit,  $\hat{\Omega} = \mathbf{\Omega}/\Omega_o$  is the  
 875 planetary rotation unit vector, and the functions of the eccentricities  $f_1$  and  $f_2$  are given by

$$f_1 = \frac{1 + \frac{15}{2}e_c^2 + \frac{45}{8}e_c^4}{(1 - e_c^2)^6}, \quad f_2 = \frac{1 + 3e_c^2 + \frac{3}{8}e_c^4}{(1 - e_c^2)^{9/2}}. \quad (\text{B.15})$$

876 Writing  $k_2/Q$  in terms of  $Im[\mathcal{S}_{11}]$  using Equation (4), and using the definition of the mean mo-  
 877 tion  $n^2 = GM_s/a^3$ ,  $\Omega_o = \frac{3}{2}n$  and  $\hat{\Omega} \cdot \hat{\mathbf{e}}_3^I = \cos(Re[\tilde{\varepsilon}_m]) \approx 1$ , we can write the tidal torque as

$$\Gamma_{\mathbf{t}} = 9\bar{A} \frac{n^4}{\Omega_o^2} Im[\mathcal{S}_{11}] \left[ \left( f_1 - \frac{3}{4}f_2 \right) \hat{\mathbf{e}}_3^I + \left( f_1 - \frac{3}{2}f_2 \right) \hat{\Omega} \right]. \quad (\text{B.16})$$

878 We now project this torque onto the equatorial components of the frame attached to Mer-  
 879 cury. If we chose  $t = 0$  to correspond to when the Cassini plane coincides with the real axis,  
 880 then with respect to  $\hat{\mathbf{e}}_3^P$ , the projection of the  $\hat{\mathbf{e}}_3^I$  component of the tidal torque onto the com-  
 881 plex plane involves a factor  $-\sin \tilde{\varepsilon}_m \approx -\tilde{\varepsilon}_m$  (see Figure 1b). The part of the torque directed  
 882 along the rotation vector  $\hat{\Omega}$  can be divided into a part pointing in the direction of the Laplace  
 883 pole  $\hat{\mathbf{e}}_3^L$  and a part directed in the Laplace plane. The former is responsible for a secular change  
 884 in the orbit of Mercury; as we assume no change in any orbital quantity, we set this part equal  
 885 to zero. The remaining part, directed along the Laplace plane, participates in the precession  
 886 torque. With the same choice of  $t = 0$  as above, its projection onto the complex plane of the  
 887 equator of Mercury involves a factor

$$\cos(I + \tilde{\varepsilon}_m) \sin(I + \tilde{\varepsilon}_m) \approx \cos I \sin I + (\cos^2 I - \sin^2 I) \tilde{\varepsilon}_m. \quad (\text{B.17})$$

888 Using these projections, the tidal torque is expressed as

$$\tilde{\Gamma}_t = -\Omega_o^2 \bar{A} \text{Im}[\mathcal{S}_{11}] \left[ \phi_m^{t3} \tilde{\varepsilon}_m + \phi_m^{t2} \cos I \sin I \right], \quad (\text{B.18})$$

889 where

$$\phi_m^{t3} = \left( \phi_m^{t1} + \phi_m^{t2} (\cos^2 I - \sin^2 I) \right), \quad \phi_m^{t1} = 9 \frac{n^4}{\Omega_o^4} \left( f_1 - \frac{3}{4} f_2 \right), \quad \phi_m^{t2} = 9 \frac{n^4}{\Omega_o^4} \left( -f_1 + \frac{3}{2} f_2 \right). \quad (\text{B.19})$$

890 Truncated to  $e_c^2$ , we can write

$$\left( f_1 - \frac{3}{4} f_2 \right) = \frac{1}{8} (2 + 63e_c^2), \quad \left( -f_1 + \frac{3}{2} f_2 \right) = \frac{1}{4} (2 - 9e_c^2), \quad (\text{B.20})$$

891 and the expression for  $\phi_m^{t1}$  and  $\phi_m^{t2}$  directly in terms of  $e_c$  are

$$\phi_m^{t1} = \frac{9}{4} \frac{n^4}{\Omega_o^4} \left( 1 + \frac{63}{2} e_c^2 \right), \quad \phi_m^{t2} = \frac{9}{4} \frac{n^4}{\Omega_o^4} (2 - 9e_c^2). \quad (\text{B.21})$$

892 In principle, for a planet with an inner core whose rotation vector is misaligned with that  
893 of the mantle, then the deviation from  $\hat{\Omega}$  within the inner core introduces a correction term to  
894 the expression of the torque given by Equation (B.18). However, the misalignment of the in-  
895 ner core rotation vector is small and we neglect this correction term.

896 The torque on the inner core alone can be constructed in exactly the same manner as that  
897 for the whole of Mercury. The torque has a similar form as that of Equation (B.18), except it  
898 involves the density contrast at the ICB  $\alpha_3$  and we must replace  $\bar{A}$  with  $\bar{A}_s$  and  $\mathcal{S}_{11}$  with  $\mathcal{S}_{31}$ :

$$\tilde{\Gamma}_{ts} = -\Omega_o^2 \bar{A}_s \alpha_3 \text{Im}[\mathcal{S}_{31}] \left[ \phi_m^{t3} \tilde{\varepsilon}_m + \phi_m^{t2} \cos I \sin I \right]. \quad (\text{B.22})$$

## 899 B.5 Modified matrix elements

900 The linear system given by Equations (B.4a-B.4e) can be written in matrix form as  $\mathbf{M} \cdot$   
901  $\mathbf{x} = \mathbf{y}$  (Equation 22a of D21). The elements of the vector  $\mathbf{x}$  (Equation 22b of D21) are the  
902 5 unknown rotational variables; solutions for  $\mathbf{x}$  are found by solving this linear system. With  
903 the addition of elastic deformations, the matrix  $\mathbf{M}$  and right-hand side vector  $\mathbf{y}$  given by Equations  
904 (22d) and (22c) of D21, respectively, are modified to  $\mathbf{M} + \delta\mathbf{M}$  and  $\mathbf{y} + \delta\mathbf{y}$ . The non-zero  
905 elements of  $\delta\mathbf{M}$  and  $\delta\mathbf{y}$  are:

$$\delta\mathbf{M}_{1,1-3} = \left(1 + \omega + \frac{\phi_m}{e}\right) \mathcal{S}_{1,1-3} + \alpha_3 \frac{\bar{A}_s}{A} \frac{\phi_s}{e_s} \mathcal{S}_{3,1-3}, \quad (\text{B.23a})$$

$$\delta\mathbf{M}_{1,4} = \left(1 + \omega + \frac{\phi_m}{e}\right) \mathcal{S}_{14} + \alpha_3 \frac{\bar{A}_s}{A} \left(\frac{\phi_s}{e_s} \mathcal{S}_{34} - \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{31}]\right), \quad (\text{B.23b})$$

$$\delta\mathbf{M}_{1,5} = -(1 + \omega) \frac{\phi_m}{e} \mathcal{S}_{11} - \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{11}] - i\phi_m^{t3} \text{Im}[\mathcal{S}_{11}], \quad (\text{B.23c})$$

$$\delta\mathbf{M}_{2,1-4} = \omega \mathcal{S}_{2,1-4}, \quad (\text{B.23d})$$

$$\delta\mathbf{M}_{2,5} = -\omega \frac{\phi_m}{e} \mathcal{S}_{21}, \quad (\text{B.23e})$$

$$\delta\mathbf{M}_{3,1-3} = \left(1 + \omega - \alpha_2 + \alpha_3 \frac{\phi_s}{e_s}\right) \mathcal{S}_{3,1-3}, \quad (\text{B.23f})$$

$$\delta\mathbf{M}_{3,4} = \left(1 + \omega - \alpha_2 + \alpha_3 \frac{\phi_s}{e_s}\right) \mathcal{S}_{34} - \alpha_3 \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{31}], \quad (\text{B.23g})$$

$$\delta\mathbf{M}_{3,5} = -(1 + \omega - \alpha_2) \frac{\phi_m}{e} \mathcal{S}_{31} - \alpha_3 \mathcal{F}(e_c) \text{Re}[\mathcal{S}_{31}] - i\alpha_3 \phi_m^{t3} \text{Im}[\mathcal{S}_{31}], \quad (\text{B.23h})$$

$$\delta\mathbf{y}_1 = i\phi_m^{t2} \text{Im}[\mathcal{S}_{11}] \cos I \sin I, \quad (\text{B.23i})$$

$$\delta\mathbf{y}_3 = i\alpha_3 \phi_m^{t2} \text{Im}[\mathcal{S}_{31}] \cos I \sin I. \quad (\text{B.23j})$$

906

## C: Computation of the Compliances

907

908 The compliances connected to the misaligned rotation vectors of the whole planet ( $\mathcal{S}_{i1}$ ),  
 909 of the fluid core ( $\mathcal{S}_{i2}$ ) and of the inner core ( $\mathcal{S}_{i3}$ ) are computed with the standard method pre-  
 910 sented in many studies [e.g. *Buffett et al.*, 1993; *Dehant and Mathews*, 2015].

911 To compute the compliances associated with the inner core tilt ( $\mathcal{S}_{i4}$ ), we follow the method  
 912 presented in Appendix A of *Dumberry* [2008]. This method applies for Earth, and it is mod-  
 913 ified here for Mercury. A tilt by an angle  $\theta_n$  of an elliptical inner core (with geometrical ellip-  
 914 ticity  $\epsilon_s$ ) produces a radial displacement of degree 2 at the ICB (radius  $r_s$ ) of amplitude  $\Delta =$   
 915  $r_s \epsilon_s \sin \theta_n$ . Because we use a simplified Mercury model with uniform density in each region, the  
 916 only perturbation in mass produced by a tilted inner core is at the ICB, a mass load equal to  
 917  $(\rho_s - \rho_f) \Delta$ . The forcing vector inside the inner core [Equation A16 of *Dumberry*, 2008] is set  
 918 to zero. We then model the viscoelastic response of a reference spherical planet to this degree  
 919 2 mass load at the ICB. Written in terms of the standard set of 6 linear variables  $y_{1-6}$  [see their  
 920 definitions in *Dumberry*, 2008], the mass load boundary conditions at the ICB are

$$y_1^s = -\frac{y_5^f}{g} + A_1, \quad (\text{C.1a})$$

$$y_2^s = A_1 \rho_f g - (\rho_s - \rho_f) g \Delta, \quad (\text{C.1b})$$

$$y_3^s = A_2, \quad (\text{C.1c})$$

$$y_4^s = 0, \quad (\text{C.1d})$$

$$y_5^s = y_5^f, \quad (\text{C.1e})$$

$$y_6^s = y_6^f + 4\pi G \rho_f A_1 - 4\pi G (\rho_s - \rho_f) \Delta, \quad (\text{C.1f})$$

921 where  $g$  is the gravitational acceleration, superscripts  $s$  and  $f$  denote quantities on the solid  
 922 and fluid side of the ICB, respectively, and  $A_1$  and  $A_2$  are constants of integration. Five more

923 constants of integrations are introduced by the boundary conditions at the centre and at the  
 924 CMB [unchanged from those used in *Dumberry, 2008*]. Solutions for the viscoelastic deforma-  
 925 tions of the whole planet are found for an assumed (non-dimensional) radial displacement equal  
 926 to 1, and the compliances  $\mathcal{S}_{14}$ ,  $\mathcal{S}_{24}$  and  $\mathcal{S}_{34}$  are then computed from the perturbation in the mo-  
 927 ments of inertia of the whole planet, the fluid core and the inner core, respectively.

928 The numerical values of all compliances depend on the choice of a reference model of den-  
 929 sity and viscoelastic parameters (the Lamé parameter  $\lambda$  and shear modulus  $\mu$ ) as a function  
 930 of radius. We assume uniform values in each of the inner core, the fluid core, the mantle and  
 931 the crust. The density of the crust is taken as  $2974 \text{ kg m}^{-3}$ , that of the inner core as  $8800 \text{ kg}$   
 932  $\text{m}^{-3}$ . The densities of the fluid core and mantle depend on inner core size and are specified by  
 933 the method detailed in section 3.1 of D21. In the crust, mantle and inner core, the moduli  $\lambda$   
 934 and  $\mu$  are frequency dependent. We assume a Maxwell rheology, in which  $\lambda$  and  $\mu$  depend on  
 935 the viscosity  $\eta$  and the frequency of the deformation  $\omega'$  through [e.g. *Wu and Peltier, 1982*]

$$\lambda = \frac{(i\omega'\lambda_o + \frac{\kappa}{\eta}\mu_o)}{(i\omega' + \frac{1}{\eta}\mu_o)}, \quad \mu = \frac{i\omega'\mu_o}{(i\omega' + \frac{1}{\eta}\mu_o)}, \quad (\text{C.2})$$

936 where  $\lambda_o$  and  $\mu_o$  denote the moduli in the elastic limit ( $\omega' \gg \mu_o/\eta$ ) and  $\kappa = \lambda_o + \frac{2}{3}\mu_o$  is the  
 937 bulk modulus. For deformations connected to the Cassini state, the forcing frequency is  $\omega' =$   
 938  $\omega\Omega_o$ , where  $\omega$  is given by Equation (1) and  $\Omega_o = 2\pi/58.64623 \text{ day}^{-1}$  is the sidereal frequency.  
 939  $\lambda_o$  and  $\mu_o$  are specified in terms of uniform compressional ( $V_p$ ) and shear ( $V_s$ ) seismic wave ve-  
 940 locities and density  $\rho$  within each region. They are computed from,

$$\mu_o = \rho V_s^2, \quad \lambda_o = \rho V_p^2 - 2\mu_o. \quad (\text{C.3})$$

941 In doing so, we make the implicit assumption that the timescale of propagation of seismic waves  
 942 within the solid regions of Mercury is sufficiently short that deformations are in the elastic limit.  
 943 The  $V_p$  and  $V_s$  values that we use are listed in Table C.1 and are based on those presented in  
 944 *Rivoldini et al. [2009, 2011]*, except for  $V_s$  in the mantle and crust. The common numerical value  
 945 of the latter two is computed by ensuring that, for each choice of inner core size, for chosen val-  
 946 ues of the viscosity in each of the solid regions, and with  $\omega' = \omega\Omega_o$ , the  $\mu$  and  $\lambda$  values that  
 947 are calculated via Equation (C.2) yield a second degree tidal Love number  $k_2$  which is equal  
 948 to 0.55. This ensures that all interior models that we consider in our study are consistent with  
 949 recent observations of tidal deformations [*Konopliv et al., 2020; Genova et al., 2019*]. Note that  
 950 the observed value of  $k_2 = 0.55$  is based on sectorial tides whose frequency is equal to the mean  
 951 motion  $n = 2\pi/87.96935 \text{ day}^{-1}$ . Our computation is carried instead at a frequency close to  
 952  $\Omega_o$ , so in effect we make the assumption that  $k_2 \approx 0.55$  also at a frequency of  $\Omega_o$ .

956 Figure C.1a shows an example of how the seismic shear wave velocity  $V_s$  in the mantle  
 957 and crust changes as a function of inner core size ( $r_s$ ) in order to match  $k_2 = 0.55$ . This is  
 958 for a calculation where the viscosity in the crust, mantle and inner core is set to  $\eta = 10^{20} \text{ Pa s}$ ;  
 959 with this choice, deformations in the solid regions are firmly in the elastic limit.  $V_s$  is mod-  
 960 ified from  $3.93 \text{ km s}^{-1}$  for a small or no inner core, to  $3.37 \text{ km s}^{-1}$  for  $r_s = 1500 \text{ km}$ . We also  
 961 show on Figure C.1 how  $V_s$  is modified for a range of  $k_2$  values between  $0.52 - 0.58$  (for the  
 962 same viscosity  $\eta = 10^{20} \text{ Pa s}$  in all solid regions). For other choices of viscosity, for instance  
 963 a lower value in the mantle, the profile of  $V_s$  as a function of inner core size would be modified,  
 964 as then a different value of  $\mu_o$  is required in order to match  $k_2 = 0.55$ .

Seismic parameter	Crust	Mantle	Fluid core	Inner core
$V_p$ (m s <sup>-1</sup> )	8000	8500	5000	7000
$V_s$ (m s <sup>-1</sup> )	calculated	calculated	0	3800
$\rho$ (kg m <sup>-3</sup> )	2974	calculated	calculated	8800

953 **Table C.1.** Seismological parameters used in our calculations.  $V_p$  and  $V_s$  are, respectively, the com-  
954 pressional and shear seismic velocities. The density ( $\rho$ ) for the mantle and fluid core and the shear  
955 seismic wave ( $V_s$ ) for the mantle and crust depend on inner core size.

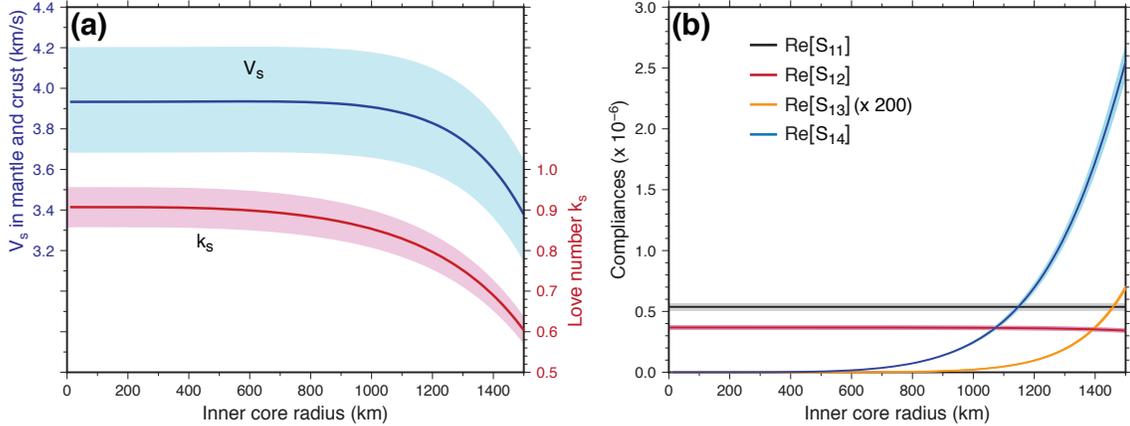
965 Since  $\lambda$  and  $\mu$  calculated from Equation (C.2) are complex, the compliances  $\mathcal{S}_{ij}$  are also  
966 complex. Their real parts capture the deformations that are in-phase with the applied forcing  
967 and their imaginary parts, those that are out-of-phase by a quarter of a cycle. Figure C.1b shows  
968 how the real parts of the compliances  $\mathcal{S}_{1,1-4}$  change as a function of  $r_s$ . These are the four com-  
969 pliances that have the largest influence on the Cassini state solution. As we enforce  $k_2$  to re-  
970 main fixed at 0.55, regardless of inner core size,  $Re[\mathcal{S}_{11}]$  (connected to  $k_2$  through Equation 4)  
971 also remains constant and is equal to  $5.38 \times 10^{-7}$ .  $Re[\mathcal{S}_{12}]$  is reduced slightly from  $3.68 \times 10^{-7}$   
972 for a small inner core to  $3.43 \times 10^{-7}$  for a large inner core. The two compliances connected to  
973 the inner core,  $\mathcal{S}_{13}$  and  $\mathcal{S}_{14}$ , are both very small for a small inner core and increase substan-  
974 tially with inner core size.  $Re[\mathcal{S}_{13}]$  remains small in amplitude; it is multiplied by a factor 200  
975 on Figure C.1b so as to be visible and its maximum value is  $3.47 \times 10^{-9}$  for  $r_s = 1500$  km.  
976  $Re[\mathcal{S}_{14}]$  becomes larger than both  $Re[\mathcal{S}_{11}]$  and  $Re[\mathcal{S}_{12}]$  once  $r_s > 1150$  km, reaching an am-  
977 plitude of  $2.55 \times 10^{-6}$  for  $r_s = 1500$  km. We also show on Figure C.1 how  $Re[\mathcal{S}_{1,1-4}]$  are mod-  
978 ified for a range of  $k_2$  values between 0.52–0.58. The important point to note is that choos-  
979 ing a different reference  $k_2$  value does not induce a large change in these compliances; the choice  
980 of inner core size has a much larger effect on  $Re[\mathcal{S}_{13}]$  and  $Re[\mathcal{S}_{14}]$ .

981 Because each of our interior model is constrained to match  $k_2 = 0.55$ , the real parts of  
982  $\mathcal{S}_{1,1-4}$  do not change when the viscosity of the mantle and/or inner core is reduced. The imag-  
983 inary parts of  $\mathcal{S}_{1,1-4}$ , however, increase in amplitude when the viscosity of the mantle is reduced.  
984 Figure C.2 shows how they change as a function of  $r_s$  for two different choices of mantle vis-  
985 cosity,  $10^{18}$  and  $10^{17}$  Pa s. The imaginary parts of  $\mathcal{S}_{1,1-4}$  vary with  $r_s$  in a way which is sim-  
986 ilar to their real parts. Their amplitudes increase in proportion with the decrease in mantle vis-  
987 cosity. The quality factor  $Q$  is connected to  $\mathcal{S}_{11}$  by  $Q = Re[\mathcal{S}_{11}]/Im[\mathcal{S}_{11}]$  (see Equation 4);  
988 a reduction in mantle viscosity leads to an increase in  $Im[\mathcal{S}_{11}]$  and to a lower  $Q$ .

989 The perturbation in the moment of inertia tensor of the whole planet caused by a tilt of  
990 an elliptical rigid inner core with dynamical ellipticity  $e_s$  is  $\bar{A}_s \alpha_3 e_s \tilde{n}_s$ . The additional pertur-  
991 bation caused by global deformations is  $\bar{A}(\mathcal{S}_{13} + \mathcal{S}_{14}) \tilde{n}_s$  (from Equation B.2a and using  $\tilde{m}_s =$   
992  $\tilde{n}_s$ ). Since  $\mathcal{S}_{13} \ll \mathcal{S}_{14}$ , we can approximate the total moment of inertia perturbation induced  
993 by an inner core tilt as,

$$(\bar{A}_s \alpha_3 e_s + \bar{A} \mathcal{S}_{14}) \tilde{n}_s = \bar{A}_s \alpha_3 e_s (1 + k_s) \tilde{n}_s, \quad (\text{C.4})$$

994 where the Love number  $k_s$  is given by

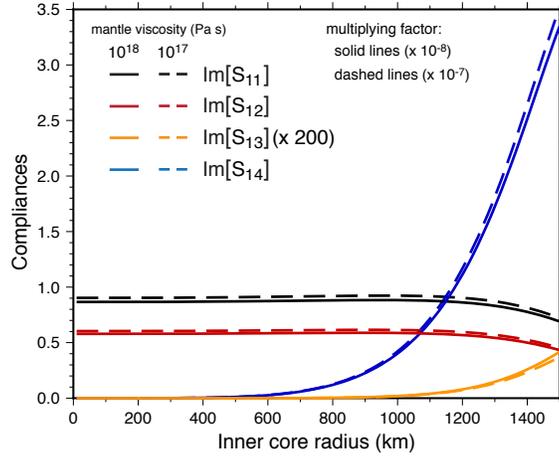


999 **Figure C.1.** (a) Shear wave seismic velocity  $V_s$  in the mantle and crust (blue), the real part of the  
 1000 Love number  $k_s$  (red) and (b) the real parts of the compliances  $\mathcal{S}_{11}$  (black),  $\mathcal{S}_{12}$  (red),  $\mathcal{S}_{13}$  (orange),  
 1001  $\mathcal{S}_{14}$  (blue) as a function of inner core radius. The viscosity is set to  $10^{20}$  Pa s in all solid regions. In  
 1002 both panels, the solid lines for each variables are for a Mercury model with  $k_2$  set equal to 0.55, and the  
 1003 coloured shaded areas bracket a range of  $k_2$  between 0.52 and 0.58.

$$k_s = \frac{\bar{A}}{A_s} \frac{\mathcal{S}_{14}}{\alpha_3 e_s}. \quad (\text{C.5})$$

995 We show in Figure C.1a how the real part of  $k_s$  varies with  $r_s$ . The Love number  $k_s$  is of order  
 996 1 and provides a convenient way to add the contribution from deformations to the change  
 997 in the moment of inertia of the whole planet caused by a tilted inner core, as it was done in the  
 998 case of Earth in *Dumberry* [2008].

1008 A Maxwell model is likely not a very accurate representation of the rheology of Mercury's  
 1009 mantle. A better choice would be to use an Andrade-pseudoperiod model [e.g. *Padovan et al.*,  
 1010 2014; *Steinbrügge et al.*, 2018]. Our choice is instead one of convenience. A Maxwell model pro-  
 1011 vides a simple way to characterize viscoelastic deformations directly in terms of viscosity val-  
 1012 ues, thus limiting the number of model parameters. Furthermore, a Maxwell model is also straight-  
 1013 forward to incorporate in the framework of our rotational model; the same strategy was used  
 1014 in previous studies using the same framework [e.g. *Greff-Lefftz et al.*, 2000; *Koot and Dumb-*  
 1015 *erry*, 2011; *Organowski and Dumberry*, 2020]. Our primary goal is to recover a first order con-  
 1016 nection between the phase lag angle and the bulk viscosities of the mantle and inner core. As  
 1017 we are focused on one single frequency, that associated with the Cassini state, assuming a Maxwell  
 1018 model is sufficient to accomplish this task. Moreover, because we assume uniform material prop-  
 1019 erties in the mantle, instead of taking into account their radial variations, the viscosity that we  
 1020 recover represents a bulk value averaged over the entire mantle, so it can be regarded at best  
 1021 as an order of magnitude estimate. In this spirit, using a Maxwell model rather than a more  
 1022 accurate rheology is sufficient, although we need to remain alert to the fact that the viscosity  
 1023 values that we recover do depend on this choice. As an example, a given rigidity is achieved  
 1024 with a higher viscosity in an Andrade rheology compared to a Maxwell rheology [e.g. *Padovan*



1004 **Figure C.2.** The imaginary parts of the compliances  $\mathcal{S}_{11}$  (black),  $\mathcal{S}_{12}$  (red),  $\mathcal{S}_{13}$  (orange),  $\mathcal{S}_{14}$  (blue)  
 1005 as a function of inner core radius for two choices of mantle viscosity:  $10^{18}$  Pa s (solid lines) and  $10^{17}$   
 1006 Pa s (dashed lines). Numerical values must be multiplied by a factor  $10^{-8}$  for solid lines, and  $10^{-7}$  for  
 1007 dashed lines. The viscosity of the inner core and crust is set to  $10^{20}$  Pa s.

1025 *et al.*, 2014]. In order to obtain the same tidal quality factor  $Q$  – the parameter ultimately tied  
 1026 to the mantle phase lag – the mantle viscosity would need to be larger in an Andrade model.

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