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Supporting Information for

Changes in Crack Shape and Saturation in Laboratory-Induced Seismicity by Water Infiltration in the Transversely Isotropic Case with Vertical Cracks

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Elastic constants of rock material for the case of transversely isotropic symmetry along the x-3 axis (z-axis) with randomly distributed vertical cracks

Here we describe the method of calculation of elastic constants for the case in which plane normals of cracks are randomly distributed in directions perpendicular to the x-3 axis (z-axis). We also show that the ratio of the elastic constant of rock material that includes cracks to that of the matrix or the square of velocity ratio is expressed as $(V/V_0)^2 = 1 - p_i\,\varepsilon$, where V and V_0 are the elastic-wave velocities with and without cracks, respectively, and ε is the crack density parameter defined by

$$\varepsilon = \frac{3\,\emptyset}{4\,\pi\,\alpha},$$

where ∅ is the porosity and α = c/a is the aspect ratio of the crack ($a = b \gg c$). In addition, we derive the coefficients p_i.

The focus of this study is on a transversely isotropic medium with the x-3 axis (z-axis) as the axis of symmetry and with a vertical crack distribution in which the plane normals of the cracks are randomly distributed in horizontal directions (directions parallel to the x-1,2 plane, x-y plane). The right-handed rectangular coordinate system is used in this study (Figure S1a).

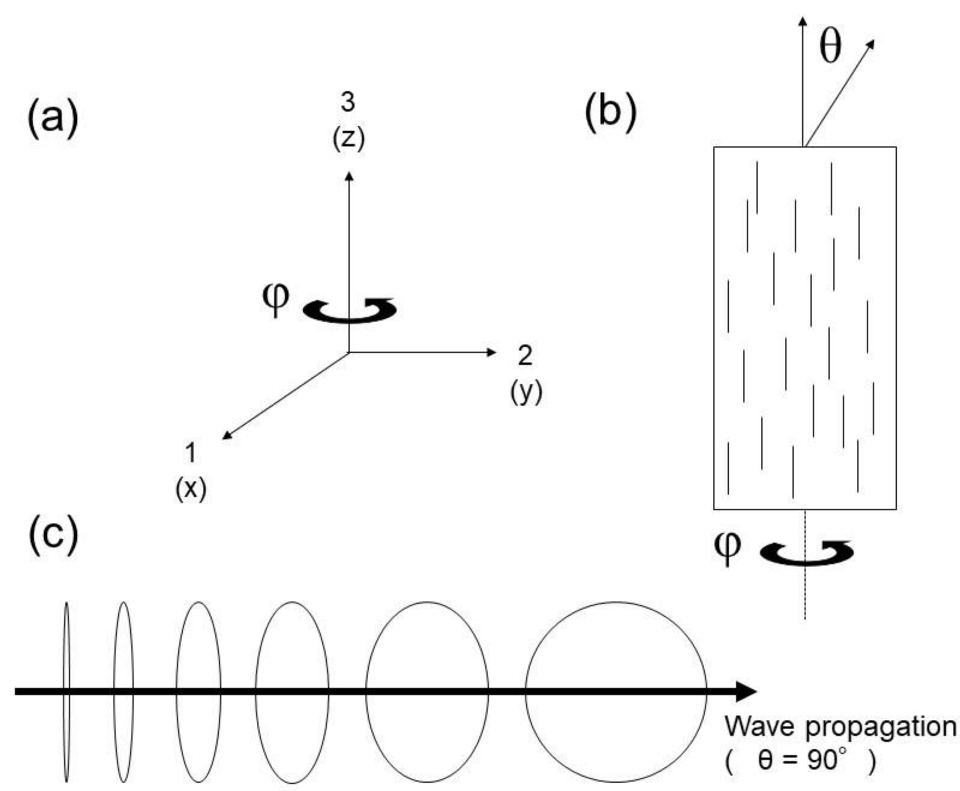

Figure S1

Figure S1. The basic assumptions in this study: (a) the coordinate system, (b) vertical cross section of transversely isotropic rock with vertical cracks, and (c) side view of the direction of wave propagation and crack distribution.

First, based on the method of Hudson (1981), we calculated C_{ij} for a material that includes vertical cracks that are plane normal along the x-1 axis (x-axis) (Figure S1b). Next, we took the rotational average of C_{ij} around the x-3 axis (z-axis), resulting in $\hat{C}_{ij}$, which were shown to be transversely isotropic with the x-3 axis (z-axis) by using a method similar to that of Nishizawa and Masuda (1991).

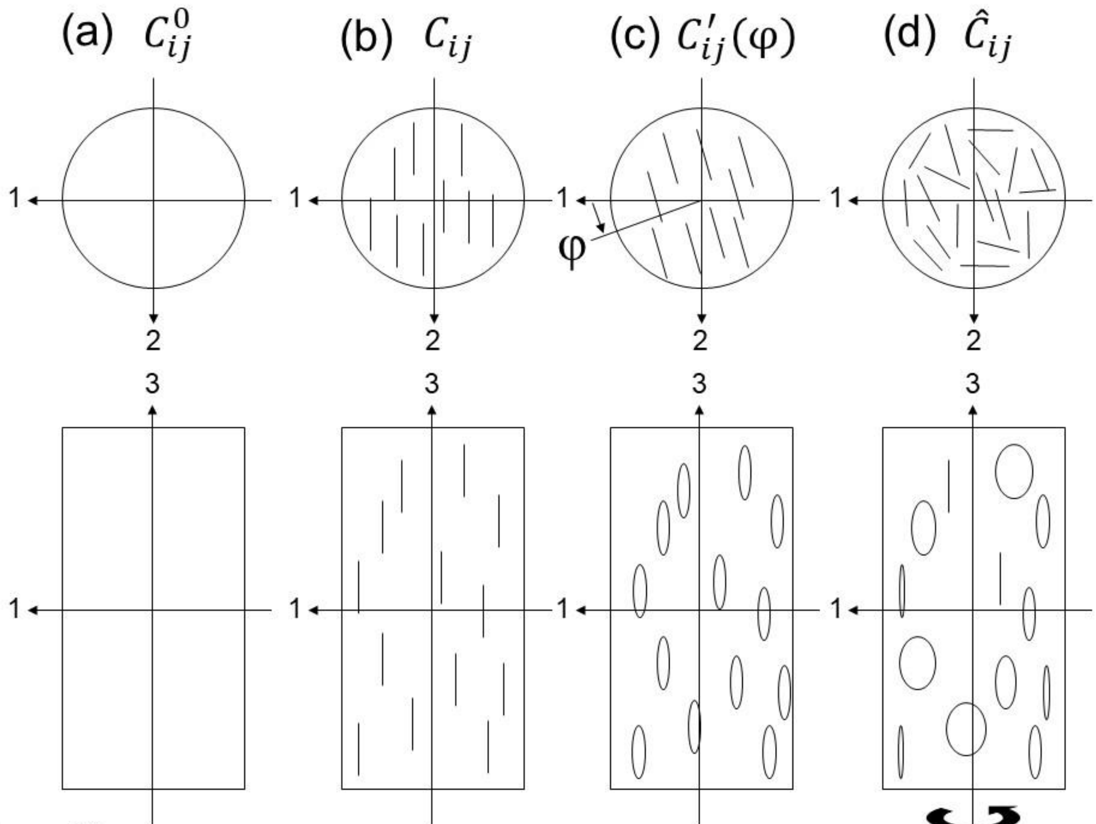

Figure S2

Figure S2. Procedure for calculating the elastic constants in the transversely isotropic rock with vertical cracks: (a) $\boldsymbol{C_{ij}^{0}}$ elastic constants of the rock matrix; (b) $\boldsymbol{C_{ij}}$ elastic constants for the rock material with vertical cracks for which the plane-normal direction is along the x-1 axis (x-axis); (c) $\boldsymbol{C'_{ij}(\varphi)}$ elastic constants for rock material with vertical cracks for which the angle between the plane-normal direction and the x-1 axis (x-axis) is $\boldsymbol{\varphi}$; and (d) $\boldsymbol{\hat{C}_{ij}}$ elastic constants for rock material with transversely isotropic symmetry along the x-3 axis (z-axis) with vertical cracks with random values of $\boldsymbol{\varphi}$.

1. C_{ij}^0 elastic constants of the isotropic rock matrix (Figure S2a)

In this study, we use abbreviated 2-index Voigt notation to express elastic constants such as C_{ij} instead of 4-index notation for the fourth-rank tensor c_{ijkl}. We assume that the matrix of rock without cracks or inclusions is isotropic with two independent constants:

$$C_{ij}^0 = \begin{pmatrix} C_{11}^0 & C_{12}^0 & C_{12}^0 & 0 & 0 & 0 \\ C_{12}^0 & C_{11}^0 & C_{12}^0 & 0 & 0 & 0 \\ C_{12}^0 & C_{12}^0 & C_{11}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44}^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44}^0 \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

$$C_{12}^0 = C_{11}^0 - 2C_{44}^0$$

The relationships between the elements C_{ij}^0 and Lame's parameters λ and μ of isotropic linear elasticity are

$$C_{11}^0 = \lambda + 2\mu, \quad C_{12}^0 = \lambda, \quad C_{44}^0 = \mu.$$

2. C_{ij} elastic constants for rock material with cracks that are plane normal along the x-1 axis (x-axis) (Figure S2b)

Hudson (1981) modeled fractured rock as an elastic solid with thin, penny-shaped ellipsoidal cracks or inclusions. The effective moduli C_{ij} are given as

$$C_{ij} = C_{ij}^0 + C_{ij}^1,$$

where C^0_{ij} are the isotropic background moduli and C^1_{ij} are the first-order corrections. For the case in which the vertical cracks have crack normals along the x-1 axis (x-axis), the axis of symmetry of the material lies along the x-1 axis (x-axis), which has hexagonal symmetry with five independent constants as

$$C_{ij}^1 = \begin{pmatrix} C_{11}^1 & C_{12}^1 & C_{12}^1 & 0 & 0 & 0 \\ C_{12}^1 & C_{22}^1 & C_{23}^1 & 0 & 0 & 0 \\ C_{12}^1 & C_{23}^1 & C_{22}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55}^1 \end{pmatrix}, \quad C_{44}^1 = \frac{1}{2}\left(C_{22}^1 - C_{23}^1\right).$$

The following correction terms are given by Schön (2011, Table 6.15) for the case in which the crack normals are aligned along the x-1 axis (x-axis), including the vertical cracks:

$$C_{11}^{1} = -\frac{(\lambda + 2\mu)^2}{\mu}\,\varepsilon\,U_3$$

$$C_{13}^{1} = -\frac{\lambda\,(\lambda + 2\mu)^2}{\mu}\,\varepsilon\,U_3$$

$$C_{33}^{1} = -\frac{\lambda^2}{\mu}\,\varepsilon\,U_3$$

$$C_{44}^{1} = 0$$

$$C_{66}^{1} = -\mu\,\varepsilon\,U_1$$

in which the correction terms C_{ij}^{1} are negative; thus, the elastic properties decrease with fracturing. U_1 and U_3 depend on the crack conditions (Mavko et al., 2009; Schon 2011). For dry cracks,

$$U_1 = \frac{16(\lambda + 2\mu)}{3(3\lambda + 4\mu)}; \qquad U_3 = \frac{4(\lambda + 2\mu)}{3(\lambda + \mu)}.$$

For wet cracks, Hudson's expressions for infinitely thin fluid-filled cracks are

$$U_1 = \frac{16(\lambda + 2\mu)}{3(3\lambda + 4\mu)}; \qquad U_3 = 0\,.$$

Therefore, for the dry case, C_{ij} are

$$C_{11} = C_{11}^{0} + C_{11}^{1} = (\lambda + 2\mu)(1 - 6\,\varepsilon)$$

$$C_{13} = C_{13}^{0} + C_{13}^{1} = \lambda(1 - 6\,\varepsilon)$$

$$C_{33} = C_{33}^{0} + C_{33}^{1} = (\lambda + 2\mu)(1 - \frac{2}{3}\,\varepsilon)$$

$$C_{44} = C_{44}^{0} + C_{44}^{1} = \mu$$

$$C_{66} = C_{66}^{0} + C_{66}^{1} = \mu\left(1 - \tfrac{16}{7}\,\varepsilon\right).$$

For the wet case,

$$C_{11} = C_{11}^{0} + C_{11}^{1} = \lambda + 2\mu$$

$$C_{13} = C_{13}^{0} + C_{13}^{1} = \lambda$$

$$C_{33} = C_{33}^0 + C_{33}^1 = \lambda + 2\mu$$

$$C_{44} = C_{44}^0 + C_{44}^1 = \mu$$

$$C_{66} = C_{66}^0 + C_{66}^1 = \mu(1 - \frac{16}{7}\varepsilon) .$$

C_{ij} has hexagonal symmetry with the x-1 axis (x-axis) expressed with five independent moduli.

3. $C'_{ij}(\varphi)$ elastic constants for rock material with vertical cracks that have an angle φ between the plane-normal direction and the x-1 axis (x-axis) (Figure S2c)

When we rotate C_{ij} around the x-3 axis (z-axis) by an angle of φ from the x-1 axis (x axis), C_{ij} is a function of φ, as expressed by $C'_{ij}(\varphi)$.

Regarding coordinate transformations, the elastic compliances c_{ijkl} are, in general, fourth-rank tensors and hence transform according to

$$c'_{ijkl} = \beta_{ip}\beta_{jq}\beta_{kr}\beta_{ls}c_{pqrs},$$

where c'_{ijkl} and c_{pqrs} are the elastic compliances after and before the coordinate transformation, respectively. For rotation around the x-3 axis (z-axis), β_{ij} is the following matrix element

$$\begin{pmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this study, we use the abbreviated 2-index Voigt notation C_{ij} instead of c'_{ijkl} and c_{ijkl}. Although an elastic constant looks like a second-rank tensor (C_{ij}) with this notation, it is indeed a fourth-rank tensor; when one performs a coordinate transformation, one must go back to the full notation and follow the transformation rules for a fourth-rank tensor. The usual tensor transformation law is no longer valid. However, the change of coordinates for C_{ij} is more efficiently performed with the 6 × 6 Bond Transformation Matrices, **M** (Mavko et al., 2009). The advantage of the Bond method for transforming compliances is that it can be applied directly to the elastic constants given in 2-index notation, as expressed as follows:

$$[C'] = [M][C][M]^T$$

M

$$
= \begin{bmatrix}
\beta_{11}^2 & \beta_{12}^2 & \beta_{13}^2 & 2\beta_{12}\beta_{13} & 2\beta_{13}\beta_{11} & 2\beta_{11}\beta_{12} \\
\beta_{21}^2 & \beta_{22}^2 & \beta_{23}^2 & 2\beta_{22}\beta_{23} & 2\beta_{23}\beta_{21} & 2\beta_{21}\beta_{22} \\
\beta_{31}^2 & \beta_{32}^2 & \beta_{33}^2 & 2\beta_{32}\beta_{33} & 2\beta_{33}\beta_{31} & 2\beta_{31}\beta_{32} \\
\beta_{21}\beta_{31} & \beta_{22}\beta_{32} & \beta_{23}\beta_{33} & \beta_{22}\beta_{33}+\beta_{23}\beta_{32} & \beta_{21}\beta_{33}+\beta_{23}\beta_{31} & \beta_{22}\beta_{31}+\beta_{21}\beta_{32} \\
\beta_{31}\beta_{11} & \beta_{32}\beta_{12} & \beta_{33}\beta_{13} & \beta_{12}\beta_{33}+\beta_{13}\beta_{32} & \beta_{11}\beta_{33}+\beta_{13}\beta_{31} & \beta_{11}\beta_{32}+\beta_{12}\beta_{31} \\
\beta_{11}\beta_{21} & \beta_{12}\beta_{22} & \beta_{13}\beta_{23} & \beta_{22}\beta_{13}+\beta_{12}\beta_{23} & \beta_{11}\beta_{23}+\beta_{13}\beta_{21} & \beta_{22}\beta_{11}+\beta_{12}\beta_{21}
\end{bmatrix}.
$$

Then, we obtain $C'_{ij}(φ)$ as

$$C'_{11}(φ) = cos^4φ\, C_{11} + 2sin^2φ\, cos^2φ\, C_{12} + sin^4φ\, C_{22} + 4sin^2φ cos^2φ\, C_{55}$$

$$C'_{22}(φ) = sin^4φ\, C_{11} + 2sin^2φ\, cos^2φ\, C_{12} + cos^4φ\, C_{22} + 4sin^2φ cos^2φ\, C_{55}$$

$$\begin{aligned} C'_{12}(φ) = C'_{21}(φ) \\ &= sin^2φ\, cos^2φ\, C_{11} + (sin^4φ + cos^4φ)\, C_{12} + sin^2φ\, sin^2φ\, C_{22} \\ &- 4sin^2φ\, cos^2φ\, C_{55} \end{aligned}$$

$$C'_{13}(φ) = C'_{31}(φ) = cos^2φ\, C_{12} + sin^2φ\, C_{23}$$

$$C'_{23}(φ) = C'_{32}(φ) = sin^2φ\, C_{12} + cos^2φ\, C_{23}$$

$$C'_{33}(φ) = C_{22}$$

$$C'_{44}(φ) = cos^2φ\, C_{44} + sin^2φ\, C_{55}$$

$$C'_{55}(φ) = sin^2φ\, C_{44} + cos^2φ\, C_{55}$$

$$C'_{66}(φ) = sin^2φ\, cos^2φ\, C_{11} - 2sin^2φ\, cos^2φ\, C_{12} + sin^2φ\, cos^2φ\, C_{22} + (cos^2φ - sin^2φ)^2\, C_{55}.$$

The following non-zero elements are zero in the next step 4, taking the rotational average:

$$C'_{16}(φ), C'_{61}(φ), C'_{26}(φ), C'_{62}(φ), C'_{36}(φ), C'_{63}(φ), C'_{45}(φ), C'_{54}(φ).$$

4. $\hat{C}_{ij}$ elastic constants for rock material with transversely isotropic symmetry along the x-3 axis (z-axis) and a vertical crack distribution (Figure S2d)

We took the rotational average of C_{ij} around the x-3 axis (z-axis) to obtain $\hat{C}_{ij}$ that showed transversely isotropic symmetry along the x-3 axis (z-axis) in the case of a random vertical crack distribution as follows:

$$\hat{C}_{ij} = \frac{1}{2\pi} \int_0^{2\pi} C'_{ij}(φ)\; dφ,$$

which uses

$$\frac{1}{2\pi}\int_0^{2\pi} sin^4\varphi\ \mathrm{d}\varphi = \frac{3}{8},\ \ \frac{1}{2\pi}\int_0^{2\pi} cos^4\varphi\ \mathrm{d}\varphi = \frac{3}{8},\ \ \frac{1}{2\pi}\int_0^{2\pi} sin^2\varphi\ cos^2\varphi\ \mathrm{d}\varphi = \frac{1}{8},$$

$$\frac{1}{2\pi}\int_0^{2\pi} sin^2\varphi\ \mathrm{d}\varphi = \frac{1}{2},\ \ \frac{1}{2\pi}\int_0^{2\pi} cos^2\varphi\ \mathrm{d}\varphi = \frac{1}{2}\ .$$

$\hat{C}_{ij}$ shows hexagonal symmetry or transversely isotropic symmetry with the x-3 axis (z-axis) in which there are five independent constants:

$$\hat{C}_{11} = \frac{1}{2\pi}\int_0^{2\pi} C'_{11}(\varphi)\ d\varphi = \frac{3}{8}\,C_{11} + \frac{1}{4}\,C_{12} + \frac{3}{8}\,C_{22} + \frac{1}{2}\,C_{55}$$

$$\hat{C}_{12} = \frac{1}{2\pi}\int_0^{2\pi} C'_{12}(\varphi)\ d\varphi = \frac{1}{8}\,C_{11} + \frac{3}{4}\,C_{12} + \frac{1}{8}\,C_{22} - \frac{1}{2}\,C_{55}$$

$$\hat{C}_{13} = \frac{1}{2\pi}\int_0^{2\pi} C'_{13}(\varphi)\ d\varphi = \frac{1}{2}\,C_{12} + \frac{1}{2}\,C_{23}$$

$$\hat{C}_{33} = C_{22}$$

$$\hat{C}_{44} = \frac{1}{2\pi}\int_0^{2\pi} C'_{44}(\varphi)\ d\varphi = \frac{1}{2}\,C_{44} + \frac{1}{2}\,C_{55}$$

$$\hat{C}_{66} = \frac{1}{2\pi}\int_0^{2\pi} C'_{66}(\varphi)\ d\varphi = \frac{1}{8}\,C_{11} - \frac{1}{4}\,C_{12} + \frac{1}{8}\,C_{22} + \frac{1}{2}\,C_{55} = \frac{1}{2}\left(\hat{C}_{11} - \hat{C}_{12}\right).$$

5. Wave velocities which propagate in the horizontal directions

In the material with transversely isotropic symmetry, there are three modes of wave propagation, and their velocities are dependent on the angle θ between the axis of symmetry (in this case, x-3 axis or z-axis) and the direction of the wave vector:

$$V_P = \sqrt{\frac{\hat{C}_{11}\ sin^2\theta + \hat{C}_{33}\ cos^2\theta + \hat{C}_{44} + A}{2\rho}}$$

$$V_{SV} = \sqrt{\frac{\hat{C}_{11}\ sin^2\theta + \hat{C}_{33}\ cos^2\theta + \hat{C}_{44} - A}{2\rho}}$$

$$V_{SH} = \sqrt{\frac{\hat{C}_{66}\, sin^2\theta + \hat{C}_{44}\, cos^2\theta}{2\rho}}\ ,$$

where $A = \sqrt{\left[\left(\hat{C}_{11} - \hat{C}_{44}\right)sin^2\theta + \left(\hat{C}_{33} - \hat{C}_{44}\right)cos^2\theta\right]^2 + \left(\hat{C}_{13} + \hat{C}_{44}\right)^2 sin^2 2\theta}$.

For θ = 90°, the relationship simplifies to $A = \hat{C}_{33} - \hat{C}_{44}$ and the wave velocity vectors that propagate perpendicular to the x-3 axis in horizontal directions (Figure S1c) are

$$V_P = \sqrt{\frac{\hat{C}_{11}}{\rho}}\ , \quad V_{SV} = \sqrt{\frac{\hat{C}_{44}}{\rho}}\ , \quad V_{SH} = \sqrt{\frac{\hat{C}_{66}}{\rho}}\ ,$$

where V_P, V_{SV}, and V_{SH} are the longitudinal-wave velocity, shear-wave velocity with vertical polarization, and shear-wave velocity with horizontal polarization, respectively.

We consider low-porosity aggregate and flat cracks, and have ignored the effect of porosity on the density of the composite (Anderson et al., 1974).
For the dry case, using $\lambda = \mu$,

$$V_P^2 = \frac{\hat{C}_{11}}{\rho} = \frac{\lambda + 2\mu}{\rho}\left(1 - \frac{71}{21}\varepsilon\right) = V_{P0}^2\left(1 - \frac{71}{21}\varepsilon\right)$$

$$V_{SV}^2 = \frac{\hat{C}_{44}}{\rho} = \frac{\mu}{\rho}\left(1 - \frac{8}{7}\varepsilon\right) = V_{SV0}^2\left(1 - \frac{8}{7}\varepsilon\right)$$

$$V_{SH}^2 = \frac{\hat{C}_{66}}{\rho} = \frac{\mu}{\rho}\left(1 - \frac{15}{7}\varepsilon\right) = V_{SH0}^2\left(1 - \frac{15}{7}\varepsilon\right),$$

where V with a subscript 0 are the velocities without cracks.

For the wet case,

$$V_P^2 = \frac{\hat{C}_{11}}{\rho} = \frac{\lambda + 2\mu}{\rho}\left(1 - \frac{8}{21}\varepsilon\right) = V_{P0}^2\left(1 - \frac{8}{21}\varepsilon\right)$$

$$V_{SV}^2 = \frac{\hat{C}_{44}}{\rho} = \frac{\mu}{\rho}\left(1 - \frac{8}{7}\varepsilon\right) = V_{SV0}^2\left(1 - \frac{8}{7}\varepsilon\right)$$

$$V_{SH}^2 = \frac{\hat{C}_{66}}{\rho} = \frac{\mu}{\rho}\left(1 - \frac{8}{7}\varepsilon\right) = V_{SH0}^2\left(1 - \frac{8}{7}\varepsilon\right).$$

The effect of cracks on velocity, in terms of the ratio of velocities with and without cracks, is proportional to the crack density parameter ε at small values of ε:

$$\left(\frac{V}{V_0}\right)^2 = 1 - p_i\, \varepsilon\,.$$

Figure S1. The basic assumptions in this study: (a) the coordinate system, (b) vertical cross section of transversely isotropic rock with vertical cracks, and (c) side view of the direction of wave propagation and crack distribution.

Figure S2. Procedure for calculating the elastic constants in the transversely isotropic rock with vertical cracks: (a) $\boldsymbol{C}_{\boldsymbol{ij}}^{\mathbf{0}}$ elastic constants of the rock matrix; (b) $\boldsymbol{C}_{\boldsymbol{ij}}$ elastic constants for the rock material with vertical cracks for which the plane-normal direction is along the x-1 axis (x-axis); (c) $\boldsymbol{C'}_{\boldsymbol{ij}}(\boldsymbol{\varphi})$ elastic constants for rock material with vertical cracks for which the angle between the plane-normal direction and the x-1 axis (x-axis) is $\boldsymbol{\varphi}$; and (d) $\widehat{\boldsymbol{C}}_{\boldsymbol{ij}}$ elastic constants for rock material with transversely isotropic symmetry along the x-3 axis (z-axis) with vertical cracks with random values of $\boldsymbol{\varphi}$.