New generalized integral transform on Hilfer-Prabhakar fractional derivatives and its applications

MOHD KHALID, SUBHASH ALHA

Department of Mathematics, Maulana Azad National Urdu University, Gachibowli,Hyderabad-500032, India.

Email: khalid.jmi47@gmail.com, subhashalha@manuu.edu.in

Abstract: In this paper, we obtain the new generalized integral transform on Prabhakar integral, Hilfer-Prabhakar derivatives and regularized Hilfer-Prabhakar fractional derivatives. Next, we evaluate the solution of some Cauchy type fractional differential equation with Hilfer-Prabhakar fractional derivatives by applying the new integral transform and Fourier transforms which involves three parameter Mittage-Leffler function.

Keywords and phrases: Prabhakar integral, Hilfer-Prabhakar derivative, new generalized integral transform, Fourier transform, Mittage-Leffler functions.

2010 Mathematical Subject Classification : 44A35, 26A33, 42B10.

1 Introduction

The intention behind the fractional calculus is the study of fractional integrals and fractional derivatives of arbitrary order (real or complex). In the literature there are different kind of fractional integrals and derivatives are involved. Fractional calculus well established and proven to be very efficient to describe many phenomena with memory and inherent processes. These phenomena are very useful to solve different kind of problems like physics, chemistry, biology, control theory, economics, electronics and other areas of science and technology [13], [14], [15], [3], [16], [11], [23], .

The Hilfer-Prabhakar derivative defines the Prabhakar and its its Caputo type regularized Prabhakar derivatives which contains the Hilfer fractional derivative. First replacing the Reimann-Liouville integral with Prabhakar integral in the definition of Hilfer derivative, then modify of Reimann-Liouville integral by extended its kernal with the three parameter Mittage-Leffler function. The Hilfer-Prabhakar derivative and its regularized virsion was first intriduced in [6].

Hilfer-Prabhakar fractional derivatives have gained popularity among researchers due to their unique properties and ability to incorporate various integral transforms, such as Laplace, Fourier, Sumudu, Shehu, Elzaki, and Sawi, into their calculations [18], [19], [20], [21], [22], [24], [25], [26], [27], [28], [29] . Some authors applied Laplace, Sumudu, Elzaki and Shehu transforms to the Prabhakar and Hilfer-Prabhakar fractional derivatives and employed to find the solutions of some fractional differential equations in terms of Mittage-Leffler function [6], [8], [9], [10]. **Mahgoub et al. [17]** found new integral transform called as Sawi transform, this transform has a deeper connection with Laplace, Elzaki, Sumudu,Shehu, Kamal and Monand transform. In our earlier paper available on arXiv.org [30], we applied the Sawi transform on Hilfer-Prabhakar derivative and their regularization versions to applied these results on some Cauchy type fractional differential equations.

In 2021, a new integral transform was introduced by Jafari.H in [4], recently which has been used by Meddahi and Jafari in their paper [5] through Atangana-Baleanu fractional derivative and they obtained the existing sub cases such as Laplace, Elzaki, Sumudu and Shehu transforms on Atangana-Baleanu derivatives. This new transform, known as the new generalized integral transform, encompasses a wide range of integral transforms including those in the Laplace transform family, and it is particularly useful in solving differential equations and integral equations. In this paper, the authors aim to use the new generalized integral transform on Prabhakar integral, Prabhakar derivatives, Hilfer-Prabhakar derivative and their regularized versions.The authors also plan to use the results obtained from the new generalized integral transform to solve some Cauchy type fractional differential equations that involve Hilfer-Prabhakar fractional derivatives of fractional order, which are presented in the form of Mittag-Leffler type functions.

2 Definition and Preliminaries

Definition 2.1. *[2] The Reimann Liouville integral operator of order $\mu > 0$ of a function $z(t)$ is*

$$ {}_0\mathcal{I}_t^{\mu} z(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} z(\tau) d\tau, \mu \in \mathbb{C} \; and \; t > 0. \tag{2.1}$$

Definition 2.2. *[2] The Reimann Liouville Fraction derivative of order $\mu > 0$ of a function $z(t)$ is*

$$ {}_0\mathcal{D}_t^{\mu} z(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\mu-1} z(\tau) d\tau, n-1 < \mu < n, \;\; n \in \mathbb{N}. \tag{2.2}$$

Definition 2.3. *[2] Caputo frectional derivative of order $\mu > 0$ of a function $z(t)$ is*

$$ {}_0^C\mathcal{D}_t^{\mu} z(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} z^{(n)}(\tau) d\tau, \;\; n-1 < \mu < n, \;\; n \in \mathbb{N} \tag{2.3}$$

Definition 2.4. *[1] The Hilfer fractional derivative of order μ and β of a function $z(t)$ is*

$$ {}_0\mathcal{D}_t^{\mu,\beta} z(t) = \left({}_0\mathcal{I}_t^{\beta(1-\mu)} \frac{d}{dt} ({}_0\mathcal{I}_t^{(1-\mu)(1-\beta)} z(t)) \right) \tag{2.4}$$

where $0 < \mu \leq 1$, and $0 \leq \beta \leq 1$

Definition 2.5. *[3] A 1-parameter Mittag-Leffler function is obtained*

$$E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\mu k + 1)}, \quad z, \mu \in \mathbb{C}, \quad Re(\mu) > 0. \tag{2.5}$$

A 2-parameter Mittag-Leffler function is obtained

$$E_{\mu,\beta}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\mu k + \beta)}, \quad z, \mu, \beta \in \mathbb{C}, \quad Re(\mu) > 0. \tag{2.6}$$

Definition 2.6. *[3][* ***Mittag-Leffler function]:*** *A 3-parameter Mittag-Leffler function, which is also known as the Prabhakar function is represented as follows*

$$E^{\gamma}_{\mu,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\mu k + \beta)} \frac{(z)^k}{k!}, \quad z, \mu, \beta, \gamma \in \mathbb{C}, \mu > 0, \tag{2.7}$$

for applications purpose we will use further generalization of (2.7) which is obtained

$$e^{\gamma}_{\mu,\beta,\varpi} = t^{\beta-1} E^{\gamma}_{\mu,\beta}(\varpi t^{\mu}) \tag{2.8}$$

the parameter ϖ is a complex number and the variable t is a positive real number.

Definition 2.7. *[3] Let f be a function that belongs to the space $L^1[0,1]$, the Prabhakar fractional integral can be represented in the form where $0 < t < b < \infty$*

$$\begin{aligned} \mathcal{I}^{\gamma}_{\mu,\beta,\varpi,0^+} z(t) &= \int_0^t (t-\tau)^{\beta-1} E^{\gamma}_{\mu,\beta}(\varpi (t-\tau)^{\mu}) z(\tau) d\tau \\ &= (f * e^{\gamma}_{\mu,\beta,\varpi})(t) \end{aligned} \tag{2.9}$$

where the parameters $\mu, \beta, \gamma, \varpi$ with $Re(\mu), Re(\beta) > 0$, are complex numbers.

Definition 2.8. *[3] If f is a function that belongs to the space $L^1[0,1]$, the Prabhakar fractional derivative can be written in the form where $0 < t < b < \infty$, and $f * e^{\gamma}_{\mu,\beta,\varpi}$ belongs to $W^{n,1}[0,b]$, and $n = \lceil \beta \rceil$*

$$\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+} z(t) = \frac{d^n}{dt^n} \mathcal{I}^{-\gamma}_{\mu,n-\beta,\varpi,0^+} z(t), \tag{2.10}$$

where the parameters $\mu, \beta, \gamma, \varpi$ are complex numbers, $Re(\mu), Re(\beta) > 0$ and $W^{n,1}[a,b] = \{z \in L^1[a,b] \ s.t \ \frac{d^n}{dt^n} z \in L^1[a,b]\}$.

The Reimann Liouville Fractional Derivative in (2.2) can be represented in the following form

$$\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+} z(t) = \mathcal{D}^{\beta+\theta}_{0^+} \mathcal{I}^{-\gamma}_{\mu,\theta,\varpi,0^+} z(t), \tag{2.11}$$

Definition 2.9. *[6] When f is a function that belongs to the space of absolutely continuous functions defined on $[0,b]$, such that $0 < b < \infty$, and $n = \lceil \beta \rceil$, then the regularized Prabhakar fractional derivative is represented in the following form*

$${}^{C}\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+} z(t) = \mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+} \frac{d^n}{dt^n} z(t), \ t > 0 \tag{2.12}$$

where the parameters $\mu, \beta, \gamma, \varpi$ with $Re(\mu), Re(\beta) > 0$, are complex numbers.

Definition 2.10. *[6, 7] Let $f \in L^1[a,b], \beta$ belongs to $(0,1), \alpha$ belongs to $[0,1], 0 < b \leq \infty,\ f * e^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi}(.)$ belongs to $AC^1[a,b]$. The Hilfer-Prabhakar fractional derivative is obtained*

$$\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+} z(t) = \left(\mathcal{I}^{-\gamma\alpha}_{\mu,\alpha(1-\beta),\varpi,0^+} \frac{d}{dt} (\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+} z) \right)(t),\ t > 0 \tag{2.13}$$

Definition 2.11. *[7] Let f belongs to $L^1[a,b], \beta$ belongs to $(0,1), \alpha$ belongs to $[0,1], 0 < b \leq \infty$. The regularized Hilfer-Prabhakar fractional derivative of z(t) is obtained*

$$\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+} z(t) = \left(\mathcal{I}^{-\gamma\alpha}_{\mu,\alpha(1-\beta),\varpi,0^+} \mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+} \frac{d}{dt} z \right)(t) = \mathcal{I}^{-\gamma}_{\mu,1-\beta,\varpi,0^+} \frac{d}{dt} z(t), t > 0 \tag{2.14}$$

Definition 2.12. *[4] The new generalized integral transform denoted by $\mathcal{Z}(s)$ for the function $z(t)$ which is given as:*

$$T[z(t),s] = \mathcal{Z}(s) = \phi(s) \int_0^\infty z(t) exp(-\psi(s)t) dt, \tag{2.15}$$

over the set of functions

$$\mathcal{K} = \{z(t)\ s.t.\ there\ exist\ A > 0, k > 0, |z(t)| \leq A\ exp(kt), if\ t \geq 0\},$$

New general integral transform (2.15) is well defined for all values of $\psi(s) > k$. It can be easily established that this general integral transform is a linear operator like Laplace, Sumudu, Elzaki and other integral transforms.

Proposition 2.1. *[5] If $\mathcal{F}(s)$ and $\mathcal{G}(s)$ be the new generalized integral transforms of the functions $z(t)$ and $g(t)$ respectively then the new generalized integral of their convolution*

$$T[z(t) * g(t)), s] = \frac{1}{\phi(s)} \mathcal{F}(s)\mathcal{G}(s), \tag{2.16}$$

or equivalently,

$$T^{-1}\left[\frac{1}{\phi(s)}\mathcal{F}(s)\mathcal{G}(s), t\right] = (z(t) * g(t)) \tag{2.17}$$

where

$$z(t) * g(t) = \int_0^\infty z(\tau) g(t-\tau) d\tau$$

Theorem 2.1. *[4] Suppose $\mathcal{Z}(s)$ is the new generalized integral transform of $z(t)$, then the new generalized integral of m^{th} derivative $z^{(m)}(t)$ is denoted by $\mathcal{Z}_m(s)$ and*

$$\mathcal{Z}_m(s) = T[z^{(m)}(t), s] = \psi(s)^m T[z(t), s] - \phi(s) \sum_{k=0}^{m-1} \psi(s)^{m-1-k} z^{(k)}(0), m \geq 0 \tag{2.18}$$

Lemma 2.1. *[5] If $0 < \mu < 1$ and $\varpi \in R$ such that $\psi(s) < |\varpi|^{\frac{1}{\mu}}$, the new generalized integral transform of $t^{\beta-1} E^{\gamma}_{\mu,\beta}(\varpi t^\mu)$ is given as*

$$T\left[t^{\beta-1} E^{\gamma}_{\mu,\beta}(\varpi t^\mu), s\right] = \frac{\phi(s)}{\psi(s)^\beta} \left(1 - \frac{\varpi}{\psi(s)^\mu}\right)^{-\gamma}, \tag{2.19}$$

Theorem 2.2. ***[New generalized integral transform of Prabhakar integral]:*** *If a new generalized transform to a function $z(t)$ is represented as $\mathcal{Z}(s)$, using (2.16) and (2.19), the new generalized transform to the Prabhakar fractional integral (2.9) can be written in the form*

$$T[\mathcal{I}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\gamma}\mathcal{Z}(s), \tag{2.20}$$

Proof.

$$\begin{aligned}
T[\mathcal{I}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s] &= T\left[\int_0^t (t-\tau)^{\beta-1}E^{\gamma}_{\mu,\beta}[\varpi(t-\tau)^{\mu}]z(\tau)d\tau, s\right]\\
&= T\left[(f * e^{\gamma}_{\mu,\beta,\varpi})(t), s\right]\\
&= \frac{1}{\phi(s)}T\left[t^{\beta-1}E^{\gamma}_{\mu,\beta}(\varpi t^{\mu}), s\right]T[z(t),s]\\
&= \frac{1}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\gamma}T[z(t),s],
\end{aligned}$$

□

3 Main results

Theorem 3.1. ***[New generalized integral transform of Prabhakar derivative]:*** *Let $\mathcal{Z}(s)$ be the new generalized integral transform of $z(t)$, then the new generalized integral transform to the fractional derivative of Prabhakar is represented as follows.*

$$\begin{aligned}
T[\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\mathcal{Z}(s)&\\
-\phi(s)\sum_{k=0}^{m-1}\psi(s)^{m-1-k}\mathcal{D}^{\gamma}_{\mu,k-m+\beta,\varpi,0^+}z(t)|_{t=0}&
\end{aligned} \tag{3.1}$$

Proof. By using equations (2.18), (2.20), and the convolution (2.16), the new generalized integral transform can be applied to the Prabhakar fractional derivative (2.10) w.r.t. variable t, resulting in the following expression

$$\begin{aligned}
&T[\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s]\\
&\quad= T\left[\frac{d^m}{dt^m}\mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}z(t), s\right]\\
&\quad= T\left[\frac{d^m}{dt^m}g(t), s\right], \quad where \quad g(t) = \mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}z(t)\\
&\quad= \psi(s)^m T[g(t),s] - \phi(s)\sum_{k=0}^{m-1}\psi(s)^{m-1-k}g^{(k)}(0), \;\; g^{(k)}(0) = \frac{d^k}{dt^k}\mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}z(0)\\
&\quad= \psi(s)^m\frac{1}{\psi(s)^{m-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s] - \phi(s)\sum_{k=0}^{m-1}\psi(s)^{m-1-k}\frac{d^k}{dt^k}\mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}z(t)\big|_{t=0}\\
&\quad= \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s] - \phi(s)\sum_{k=0}^{m-1}\psi(s)^{\beta-1-k}\left[\mathcal{D}^{\gamma}_{\mu,k-m+\beta,\varpi,0^+}z(t)\right]_{t=0}
\end{aligned}$$

$\square$

Theorem 3.2. ***[New generalized integral transform of regularised Prabhakar derivative]:*** *Let $\mathcal{Z}(s)$ be the new generalized integral transform of $z(t)$, then the new generalized integral transform to the regularised Prabhakar fractional derivative is represented as*

$$\begin{aligned} T[{}^{C}\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\mathcal{Z}(s) \\ -\phi(s)\sum_{k=0}^{m-1}\psi(s)^{\beta-1-k}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}z^{(k)}(0^+) \end{aligned} \tag{3.2}$$

Proof. By using equations (2.20), (2.18), and the convolution (2.16), the new generalized integral transform can be applied to the fractional derivative of regularized Prabhakar derivative (2.12) w.r.t. variable t, resulting in the following expression

$$\begin{aligned} &T[{}^{C}\mathcal{D}^{\gamma}_{\mu,\beta,\varpi,0^+}z(t),s] \\ &= T\left[\mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}\frac{d^m}{dt^m}z(t),s\right] \\ &= T\left[\mathcal{I}^{-\gamma}_{\mu,m-\beta,\varpi,0^+}h(t),s\right], \quad where \quad h(t)=\frac{d^m}{dt^m}z(t) \\ &= \frac{1}{\psi(s)^{m-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[h(t),s] \\ &= \frac{1}{\psi(s)^{m-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\left[\psi(s)^{m}T[z(t),s]-\phi(s)\sum_{k=0}^{m-1}\psi(s)^{m-1-k}z^{(k)}(0)\right] \\ &= \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s]-\phi(s)\sum_{k=0}^{m-1}\psi(s)^{m-1-k}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}z^{(k)}(0) \end{aligned}$$

$\square$

Theorem 3.3. ***[New generalized integral transform to the Hilfer-Prabhakar derivative]:*** *Let $\mathcal{Z}(s)$ be the new generalized integral transform of $z(t)$, then the new generalized integral transform to the fractional derivative of Hilfer Prabhakar derivative is represented as*

$$\begin{aligned} T[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\mathcal{Z}(s) \\ -\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0} \end{aligned} \tag{3.3}$$

Proof. By using equations (2.20), (2.18) and convolution (2.16), the new generalized integral transform of the Hilfer-Prabhakar fractional derivative (2.13) w.r.t. variable t can be obtained as follows

$$
\begin{aligned}
&T[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s]\\
&\quad= T\left[\left(\mathcal{I}^{-\gamma\alpha}_{\mu,\alpha(1-\beta),\varpi,0^+}\frac{d}{dt}(\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z)\right)(t),s\right]\\
&\quad= T\left[\mathcal{I}^{-\gamma\alpha}_{\mu,\alpha(1-\beta),\varpi,0^+}k(t),s\right],\ \ where\ \ k(t)=\frac{d}{dt}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)\\
&\quad= \frac{1}{\psi(s)^{\alpha(-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}T[k(t),s]\\
&\quad= \frac{1}{\psi(s)^{\alpha(-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\\
&\quad\times\left[\psi(s)T[\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t),s]-\phi(s)\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(0)\right]\\
&\quad= \frac{1}{\psi(s)^{\alpha(-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\\
&\quad\times\left[\frac{\psi(s)}{\psi(s)^{(1-\alpha)(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma(1-\alpha)}T[z(t),s]-\phi(s)\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(0)\right]\\
&\quad= \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s]-\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}
\end{aligned}
$$

□

Theorem 3.4. ***[New generalized integral transform to the regularized Hilfer-Prabhakar derivative]:*** *Let $\mathcal{Z}(s)$ be the new generalized integral transform of $z(t)$, then the new generalized integral transform of regularized Hilfer-Prabhakar fractional derivative is represented as*

$$
T[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s]=\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\mathcal{Z}(s)-\frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}z(0^+)
\tag{3.4}
$$

Proof. By using equations (2.20), (2.18), the new generalized integral transform can be applied to the regularized Hilfer-Prabhakar fractional derivative(2.14) w.r.t. variable t, which results in the following expression

$$
\begin{aligned}
&T[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s]\\
&\qquad= T\left[\mathcal{I}^{-\gamma}_{\mu,1-\beta,\varpi,0^+}\frac{d}{dt}z(t),s\right]\\
&\qquad= T\left[\mathcal{I}^{-\gamma}_{\mu,1-\beta,\varpi,0^+}\xi(t),s\right],\ \ \xi(t)=\frac{d}{dt}z(t)\\
&\qquad= \frac{1}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}[\psi(s)T[z(t),s]-\phi(s)z(0^+)]\\
&\qquad= \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s]-\frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}z(0^+)
\end{aligned}
$$

□

3.1 Special cases:

Corollary 1

The new generalized integral transform of the Hilfer-Prabhakar fractional derivative is represented as.

$$T[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s] - \frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}$$

$*_1$ When $\psi(s)=s$ and $\phi(s)=1$, the new generalized integral transform will result in the Laplace transform to the Hilfer-Prabhakar fractional derivative, which was studied in [6]. If the Laplace transform of $z(t)$ is denoted by $L[z(t),s]$ then

$$L[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{\beta}(1-\varpi s^{-\mu})^{\gamma}L[z(t),s] - s^{-\alpha(1-\beta)}(1-\varpi s^{-\mu})^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}$$

$*_2$ When $\psi(s)=\frac{1}{u}$ and $\phi(s)=\frac{1}{u}$, the new generalized integral transform will result in the Sumudu transform to the Hilfer-Prabhakar fractional derivative, which was studied in [8]. If the Sumudu transform of $z(t)$ is denoted by $G(u)$ then

$$S[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),u] = u^{-\beta}(1-\varpi u^{\mu})^{\gamma}G(u) - u^{\alpha(1-\beta)-1}(1-\varpi u^{\mu})^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}z(0^+)$$

$*_3$ When $\psi(s)=\frac{s}{u}$ and $\phi(s)=1$, the new generalized integral transform will result in the Shehu transform to the Hilfer-Prabhakar fractional derivative, which was studied in [10]. If the Shehu transform of $z(t)$ is denoted by $V(s,u)$ then

$$SH[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t)](s,u) = \left(\frac{u}{s}\right)^{-\beta}\left(1-\varpi\left(\frac{u}{s}\right)^{\mu}\right)^{\gamma}V(s,u) - \left(\frac{u}{s}\right)^{\alpha(1-\beta)}\left(1-\varpi\left(\frac{u}{s}\right)^{\mu}\right)^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}z(0^+)$$

$*_4$ When $\psi(s)=\frac{1}{s}$ and $\phi(s)=s$, the new generalized integral transform will result in the Elzaki transform to the Hilfer-Prabhakar fractional derivative, which was studied in [9]. If the Elzaki transform of $z(t)$ is denoted by $\mathcal{T}(s)$ then

$$E[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{-\beta}(1-\varpi s^{\mu})^{\gamma}\mathcal{T}(s) - s^{\alpha(1-\beta)+1}(1-\varpi s^{\mu})^{\gamma\alpha}\,\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}$$

$*_5$ When $\psi(s)=\frac{1}{s}$ and $\phi(s)=\frac{1}{s^2}$, the new generalized integral transform will result in the Sawi transform to the Hilfer-Prabhakar fractional derivative, which is studied in a recent paper available on arXiv.org [30]. If the Sawi transform of $z(t)$ is denoted by $\mathcal{R}(s)$ then

$$Sa[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{-\beta}(1-\varpi s^{\mu})^{\gamma}\mathcal{R}(s) - s^{\alpha(1-\beta)-2}(1-\varpi s^{\mu})^{\gamma\alpha}\,\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}z(t)|_{t=0}$$

Corollary 2

The new generalized integral transform to the regularized Hilfer-Prabhakar fractional derivative is represented as.

$$T[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = \frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[z(t),s] - \frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}z(0^+)$$

$*_1$ When $\psi(s)=s$ and $\phi(s)=1$, the new generalized integral transform will result in the Laplace transform to the regularized Hilfer-Prabhakar fractional derivative, which was studied in [6]. If the Laplace transform of $z(t)$ is denoted by $L[z(t),s]$

$$L[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{\beta}(1-\varpi s^{-\mu})^{\gamma}L[z(t),s] - s^{\beta-1}(1-\varpi s^{-\mu})^{\gamma}z(0^+)$$

$*_2$ When $\psi(s)=\frac{1}{u}$ and $\phi(s)=\frac{1}{u}$, the new generalized integral transform will result in the Sumudu transform to the regularized Hilfer-Prabhakar fractional derivative, which was studied in [8]. If the Sumudu transform of $z(t)$ is denoted by $G(u)$ then

$$S[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),u] = u^{-\beta}(1-\varpi u^{\mu})^{\gamma}G(u) - u^{-\beta}(1-\varpi u^{\mu})^{\gamma}z(0^+)$$

$*_3$ When $\psi(s)=\frac{s}{u}$ and $\phi(s)=1$, the new generalized integral transform will result in the Shehu transform to the regularized Hilfer-Prabhakar fractional derivative, which was studied in [10]. If the Shehu transform of $z(t)$ is denoted by $V(s,u)$

$$\begin{aligned} SH[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t)](s,u) = {} & (\frac{u}{s})^{-\beta}\left(1-\varpi(\frac{u}{s})^{\mu}\right)^{\gamma}V(s,u) \\ & - (\frac{u}{s})^{1-\beta}\left(1-\varpi(\frac{u}{s})^{\mu}\right)^{\gamma}z(0^+) \end{aligned}$$

$*_4$ When $\psi(s)=\frac{1}{s}$, and $\phi(s)=1$, the new generalized integral transform will result in the Elzaki transform to the regularized Hilfer-Prabhakar fractional derivative, which was studied in [9]. If the Elzaki transform of $z(t)$ is denoted by $\mathcal{T}(s)$

$$E[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{-\beta}(1-\varpi s^{\mu})^{\gamma}\mathcal{T}(s) - s^{2-\beta}(1-\varpi s^{\mu})^{\gamma}z(0^+)$$

$*_5$ When $\psi(s)=\frac{1}{s}$ and $\phi(s)=\frac{1}{s^2}$, the new generalized integral transform will result in the Sawi transform to the regularized Hilfer-Prabhakar fractional derivative, which was studied in a recent paper available on arXiv.org [30]. If the Sawi transform of $z(t)$ is denoted by $\mathcal{R}(s)$

$$Sa[{}^{C}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}z(t),s] = s^{-\beta}(1-\varpi s^{\mu})^{\gamma}\mathcal{R}(s) - s^{-\beta-1}(1-\varpi s^{\mu})^{\gamma}z(0^+)$$

4 Application

In this section, the authors will demonstrate the use of Hilfer-Prabhakar and regularized Hilfer-Prabhakar fractional derivatives, which have been obtained using the new generalized integral transform, to solve some Cauchy type fractional differential equations.

Theorem 4.1. *The solution of Hilfer-Prabhakar fractional differential equation*

$$\left.\begin{array}{r}\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}y(t)=\lambda\mathcal{I}^{\delta}_{\mu,\beta,\varpi,0^+}y(t)+z(t)\\ \left(\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}y(t)\right)|_{t=0}=K\end{array}\right\};\tag{4.1}$$

where $z(t)\in L_1[0,\infty)$, β *belongs to* $(0,1)$, α *belongs to* $[0,1]$; ϖ,λ *are complex numbers,* $t,\mu>0, K,\gamma,\delta\geq 0$, *is obtained*

$$\begin{aligned}y(t)=\lambda^n\sum_{n=0}^{\infty}\mathcal{I}^{\gamma+n(\delta+\gamma)}_{\mu,\beta(2n+1),\varpi,0^+}z(t)+K\lambda^n\sum_{n=0}^{\infty}&t^{\beta(2n+1)+\alpha(1-\beta)-1}\\ &\times\mathcal{I}^{\delta n+\gamma n+\gamma-\gamma\alpha}_{\mu,\alpha(1-\beta)+\beta(2n+1)}(\varpi t^{\mu})\end{aligned}\tag{4.2}$$

Proof Applying new generalized integral transform on both side of equation (4.1) and using (2.13), (3.3),(2.20) then

$$\begin{aligned}T[\lambda\mathcal{I}^{\delta}_{\mu,\beta,\varpi,0^+}y(t)+z(t)]&=T\left[\lambda\mathcal{I}^{\delta}_{\mu,\beta,\varpi,0^+}y(t),s\right]+T[z(t),s]\\ &=\lambda T\left[\lambda\mathcal{I}^{\delta}_{\mu,\beta,\varpi,0^+}y(t),s\right]+T[z(t),s]\\ &=\frac{\lambda}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta}T[y(t),s]+T[z(t),s]\end{aligned}$$

$$\begin{aligned}T\left[\mathcal{D}^{\gamma,\beta,\alpha}_{\mu,\varpi,0^+}y(t),s\right]&=\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[y(t),s]\\ &\quad-\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\mathcal{I}^{-\gamma(1-\alpha)}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}y(0)\\ &=\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}T[y(t),s]-\frac{K\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\end{aligned}$$

therefore

$$\begin{aligned}T[y(t),s]&=\frac{T[z(t),s]+\frac{K\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\left[1-\frac{\frac{\lambda}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right]}\\ &=\frac{T[z(t),s]+\frac{K\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\sum_{n=0}^{\infty}\left[\frac{\frac{\lambda}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right]^{n}\\ &=\left[T[z(t),s]+\frac{K\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\right]\sum_{n=0}^{\infty}\frac{\lambda^n}{\psi(s)^{\beta(2n+1)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta n-\gamma n-\gamma}\\ &=T[z(t),s]\sum_{n=0}^{\infty}\frac{\lambda^n}{\psi(s)^{\beta(2n+1)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta n-\gamma n-\gamma}\\ &\quad+\frac{\lambda^n\phi(s)}{\psi(s)^{\beta(2n+1)+\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha-(\delta n+\gamma n+\gamma)}\end{aligned}$$

for $\left[\frac{\frac{\lambda}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\delta}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right] < 1$, and by taking the inverse of new generalized integral transform on both sides of the above equation, we will get the desired result

$$y(t) = \lambda^{n}\sum_{n=0}^{\infty}\mathcal{I}_{\mu,\beta(2n+1),\varpi,0^{+}}^{\gamma+n(\delta+\gamma)}z(t) + K\lambda^{n}\sum_{n=0}^{\infty}t^{\beta(2n+1)+\alpha(1-\beta)-1} \times \mathcal{I}_{\mu,\alpha(1-\beta)+\beta(2n+1)}^{\delta n+\gamma n+\gamma-\gamma\alpha}(\varpi t^{\mu})$$

Theorem 4.2. *The solution of regularized Hilfer-Prabhakar fractional differential equation*

$$^{C}\mathcal{D}_{\mu,\varpi,0^{+}}^{\gamma,\beta,\alpha}u(x,t) = K\frac{\partial^{2}}{\partial x^{2}}u(x,t) \tag{4.3}$$

$$\left.\begin{array}{r} u(x,0) = g(x) \\ \lim_{x\to\infty}u(x,t) = 0 \end{array}\right\}; \tag{4.4}$$

with β *belongs to* $(0,1), \alpha$ *belongs to* $[0,1]$; ϖ, x *are real numbers* $K, \mu > 0, \gamma \geq 0$, *is obtained*

$$u(x,t) = \frac{1}{2\pi}\int_{\infty}^{\infty}\sum_{n=0}^{\infty}E_{\mu,\beta n+1}^{\gamma n}(\varpi t^{\mu})e^{-inx}(-Kk^{2}t^{\beta})^{n}g(k)dk \tag{4.5}$$

Proof Let $\overline{u}(x,s)$ be the new generalized integral transform of $u(x,t)$ with respect to t and $u^{*}(k,t)$ be the Fourier transform of $u(x,t)$ w.r.t. variable x. Now applying the Fourier and new generalized integral transform on equation (4.3) by using the equations (2.14), (3.4) and (2.20), then we get

$$\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\overline{u}^{*}(k,s) - \frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}g^{*}(k) = K(-k^{2})\overline{u}^{*}(k,s)$$

$$\overline{u}^{*}(k,s)\left(\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma} + Kk^{2}\right) = \frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}g^{*}(k)$$

$$\overline{u}^{*}(k,s) = \frac{\frac{\phi(s)}{\psi(s)^{1-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}g^{*}(k)}{\left(\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma} + Kk^{2}\right)}$$

$$\overline{u}^{*}(k,s) = \frac{\phi(s)g^{*}(k)}{\psi(s)}\left(1 + \frac{Kk^{2}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right)^{-1}$$

$$\overline{u}^{*}(k,s) = \sum_{n=0}^{\infty}\left(\frac{-Kk^{2}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right)^{n}\frac{g^{*}(k)\phi(s)}{\psi(s)}$$

$$\overline{u}^{*}(k,s) = \sum_{n=0}^{\infty}(-Kk^{2})^{n}\frac{\phi(s)}{\psi(s)^{\beta n+1}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-\gamma n}g^{*}(k)$$

for $\left(\frac{Kk^{2}}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right) < 1$, taking the inverse of Fourier and new integral transform, we will get the desired result

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}E_{\mu,\beta n+1}^{\gamma n}(\varpi t^{\mu})e^{-inx}(-Kk^2t^{\beta})^n g(k)dk$$

Theorem 4.3. *The solution of Hilfer-Prabhakar fractional differential equation*

$$\mathcal{D}_{\mu,\varpi,0^+}^{\gamma,\beta,\alpha}u(x,t)=K\frac{\partial^2}{\partial x^2}u(x,t) \tag{4.6}$$

$$\left.\begin{array}{r}\mathcal{I}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}^{-\gamma(1-\alpha)}u(x,t)|_{t=0}=g(x)\\ \lim_{x\to\infty}u(x,t)=0\end{array}\right\}; \tag{4.7}$$

with β *belongs to* $(0,1),\alpha$ *belongs to* $[0,1]$; ϖ, x *are real numbers,* $K,\mu>0,\gamma\geq 0$, *is obtained*

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}e^{-ikx}t^{\beta(n+1)-\alpha(\beta-1)-1}(-k^2K)^nE_{\mu,\beta(n+1)+\alpha(1-\beta)}^{\gamma(n+1-\alpha)}(\varpi t^{\mu})g(k)dk \tag{4.8}$$

Proof. Let $\overline{u}(x,s)$ be the new generalized transform of $u(x,t)$ with respect to t and $u^*(k,t)$ be the Fourier transform of $u(x,t)$ w.r.t. variable x. Now applying the Fourier and new generalized integral transform on equation (4.6) by using the equation (3.3), then we get

$$\begin{aligned}
&\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\overline{u}^*(k,s)-\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}\mathcal{I}_{\mu,(1-\alpha)(1-\beta),\varpi,0^+}^{-\gamma(1-\alpha)}u^*(x,0)\\
&\qquad=-k^2K\overline{u}^*(k,s)\\
&\qquad\overline{u}^*(k,s)\left[\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}+k^2K\right]=\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}g^*(k)\\
&\qquad\overline{u}^*(k,s)=\frac{\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha}g^*(k)}{\frac{1}{\psi(s)^{\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}\left(1+\frac{k^2K}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right)}\\
&\qquad\overline{u}^*(k,s)=\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)+\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha-\gamma}\left(1+\frac{k^2K}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right)^{-1}g^*(k)\\
&\qquad\overline{u}^*(k,s)=\frac{\phi(s)}{\psi(s)^{\alpha(1-\beta)+\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha-\gamma}\sum_{n=0}^{\infty}(-k^2K)^n\frac{1}{\psi(s)^{\beta n}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{-n\gamma}g^*(k)\\
&\qquad\overline{u}^*(k,s)=\sum_{n=0}^{\infty}(-k^2K)^n\frac{\phi(s)}{\psi(s)^{\beta n+\alpha(1-\beta)+\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma\alpha-n\gamma-\gamma}g^*(k)
\end{aligned}$$

for $\left(\frac{-k^2K}{\frac{1}{\psi(s)^{-\beta}}\left(1-\frac{\varpi}{\psi(s)^{\mu}}\right)^{\gamma}}\right)<1$, taking the inverse of Fourier and new generalized integral transforms, we will get the desired result

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}e^{-ikx}t^{\beta(n+1)-\alpha(\beta-1)-1}(-k^2K)^nE_{\mu,\beta(n+1)+\alpha(1-\beta)}^{\gamma(n+1-\alpha)}(\varpi t^{\mu})g(k)dk$$

5 Conclusion

In this work, the authors first obtain the results of the new generalized integral transform on Hilfer-Prabhakar fractional derivatives and their regularization. Next, they demonstrate its applications in solving some Cauchy type fractional differential equations with Hilfer-Prabhakar fractional derivatives. The authors apply the new generalized and Fourier transforms, which involve the three parameter Mittag-Leffler function, to find solutions.The outcome of this research indicates that the new generalized integral transform is highly effective in solving fractional differential equations.

References

[1] Hilfer, R., Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore-New jersey-Hong Kong (2000), 87-130.

[2] Kilbas, A.A., Srivastava, H.M. and Trujillo, J. Theory and Applications of Fractional Differential Equations, Elsevier Science, USA (2006).

[3] Prabhakar, T.R., *et al.*, A singular integral equation with a generalized Mittag Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7-15.

[4] Jafari, H., A new general integral transform for solving integral equations., J. Adv. Res. (2020). https://doi.org/10.1016/j.jare.2020.08.016

[5] Meddahi, M., Jafari, H. and Ncube, M.N., New general integral transform via Atangana–Baleanu derivatives., Advances in Difference Equations, (2021), 1-14.

[6] Garra, R., et al., Hilfer–Prabhakar derivatives and some applications., Applied mathematics and computation, 242 (2014), 576-589.

[7] Panchal, S.K., Dole, P.V., and Khandagale, A.D., k-Hilfer-Prabhakar Fractional Derivatives and Applications, Indian journal of mathematics, 59, (2017), 367-383

[8] Panchal, S.K., Khandagale, A.D. and Dole, P.V., Sumudu transform of Hilfer-Prabhakar fractional derivatives and applications. arXiv preprint arXiv:1608.08017, (2016).

[9] Singh, Y., Gill, V., Kundu, S. and Kumar, D., On the Elzaki transform and its applications in fractional free electron laser equation. Acta Universitatis Sapientiae., Mathematica, 11(2) (2019), 419-429.

[10] Belgacem, R., Bokhari, A. and Sadaoui, B., "Shehu transform of Hilfer-Prabhakar fractional derivatives and applications on some Cauchy type problems." Advances in the Theory of Nonlinear Analysis and its Application 5(2) (2021), 203-214.

[11] Agrawal, O.P., Fractional optimal control of a distributed system using eigenfunctions. Journal of Computational and Nonlinear Dynamics, 3(2) (2008).

[12] Liu, F., Zhuang, P. and Burrage, K., Numerical methods and analysis for a class of fractional advection–dispersion models. Computers & Mathematics with Applications, 64(10) (2012), 2990-3007.

[13] Mainardi, F., Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models., World Scientific, (2010).

[14] Miller, K.S. and Ross, B., An introduction to the fractional integrals and derivatives-theory and applications., John Willey and Sons, New York (1993).

[15] Oldham, K. and Spanier, J., The fractional calculus theory and applications of differentiation and integration to arbitrary order., Elsevier, (1974).

[16] Samko, S.G., Kilbas, A.A. and Marichev, O.I., Fractional integrals and derivatives. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon, 1 (1993).

[17] Mahgoub, M.A. and Mohand, M., The new integral transform"Sawi Transform". Advances in Theoretical and Applied Mathematics, 14(1) (2019), 81-87.

[18] De Oliveira, E.C., Jarosz, S. and Vaz Jr, J., Fractional calculus via Laplace transform and its application in relaxation processes. Communications in Nonlinear Science and Numerical Simulation, 69 (2019), 58-72.

[19] Magin, R., Fractional calculus in bioengineering, part 1., Critical Reviews in Biomedical Engineering, 32(1) (2004).

[20] Axtell, M. and Bise, M.E., Fractional calculus application in control systems. In IEEE Conference on Aerospace and Electronics. IEEE., (1990), 563-566.

[21] Katatbeh, Q. D., & Belgacem, F. B. M., Applications of the Sumudu transform to fractional differential equations., Nonlinear Studies, 18(1) (2011), 99-112.

[22] Singh, J., Kumar, D., and Kılıçman, A., Homotopy perturbation method for fractional gas dynamics equation using Sumudu transform., Abstract and Applied Analysis. 2013 Hindawi, (2013).

[23] Elbeleze, A.A., Kılıçman, A., and Taib, B.M., "Homotopy perturbation method for fractional Black-Scholes European option pricing equations using Sumudu transform." Mathematical problems in engineering 2013 (2013).

[24] Rashid, S., Kubra, K.T., and Khadijah M. Abualnaja. "Fractional view of heat-like equations via the Elzaki transform in the settings of the Mittag–Leffler function." Mathematical Methods in the Applied Sciences (2021).

[25] Khalid, M., et al., Application of Elzaki transform method on some fractional differential equations, Math. Theory Model, 5(1) (2015), 89-96.

[26] Qureshi, S. and Kumar, P., Using Shehu integral transform to solve fractional order Caputo type initial value problems, Journal of Applied Mathematics and Computational Mechanics, 18(2) (2019).

[27] Shah, R., Saad A.A., and Weera, W ”A semi-analytical method to investigate fractional-order gas dynamics equations by Shehu transform.” Symmetry 14(7) (2022), 1458.

[28] Bhuvaneswari, R., and Bhuvaneswari, K., ”Application of Sawi transform in Cryptography.” International Journal of Modern Agriculture 10(1) (2021), 185-189.

[29] Sahoo, M. and Chakraverty, S., Sawi Transform Based Homotopy Perturbation Method for Solving Shallow Water Wave Equations in Fuzzy Environment., Mathematics, 10(16) (2022), 2900.

[30] Khalid, M. and Alha, S., On Hilfer-Prabhakar fractional derivatives Sawi transform and its applications to fractional differential equations. arXiv preprint arXiv:2301.06797, (2023)