The bound-state solution of the generalized inhomogeneous Hirota equation

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Abstract

We develop the Riemann-Hilbert (RH) method for the generalized inhomogeneous Hirota equation with zero boundary condition. The RH problem is related to two kinds of scattering data: N simple poles and one N-order pole. Here we consider that when the scattering data have one or more higher-order poles, the formulas of bound-state (BS) solitons and multiple BS solitons are obtained, and the interaction between solitons and BS solitons are shown.

Keywords: Generalized inhomogeneous Hirota equation, Riemann-Hilbert problem, bound-state soliton solution.

1. Introduction

Nonlinear evolution equations can describe the basic dynamics of real systems and have important applications in the fields of optics[1, 2], Bose–Einstein condensate[3], atmosphere[4] and plasma physics[5]. The soliton solutions of nonlinear partial differential equations have also attracted the attention of scholars [6–8]. A significant feature of the solitons is that after each interaction, the solitons maintain the same velocity and amplitude and travel in the same direction as before. Most of the literature has studied the behavior of solitons and their interactions in integrable systems, which are very valuable for integrable systems. When two or more basic solitons coexist at the same speed and position, bound-state (BS) solitons, also known as multipole soliton solutions, will be generated. The BS soliton is a special soliton with multiple same velocity components. In this paper, we plan to consider the generalized inhomogeneous Hirota equation[9],

$$\begin{aligned} & iq_t + i\mu_1 q + i(\upsilon_1 + \mu_1 x)q_x + (\upsilon_2 + \mu_2 x)(q_{xx} + 2|q|^2 q) \\ & +2\mu_2(q_x + q\int_{-\infty}^{x} |q|^2 dx') + i\upsilon\left(q_{xxx} + 6|q|^2 q_x\right) = 0. \end{aligned} \tag{1}$$

The Lax pair of Eq.(1) can be given by the AKNS method as follows

$$\begin{aligned} & \phi_x = P\phi, P = -i\lambda\sigma_3 + U, \\ & \phi_t = Q\phi, Q = -4i\upsilon\lambda^3\sigma_3 - 2i\left(\upsilon_2 + \mu_2 x\right)\lambda^2\sigma_3 + i\left(\upsilon_1 + \mu_1 x\right)\lambda\sigma_3 \\ & \qquad + 4\upsilon\lambda^2 U + 2\left(\upsilon_2 + \mu_2 x\right)\lambda U - 2i\upsilon\lambda\sigma_3 U^2 + 2i\upsilon\lambda\sigma_3 U_x + V, \end{aligned} \tag{2}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, U = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, V = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix}, \tag{3}$$

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and

$$\begin{aligned}A &= i\mu_2 \int_{-\infty}^{x} |q|^2 dx' + i(\upsilon_2 + \mu_2 x)|q|^2 + \upsilon\left(qq_x^* - q^* q_x\right), \\ B &= i(\upsilon_2 + \mu_2 x)q_x - \upsilon q_{xx} - (\upsilon_1 + \mu_1 x)\, q - 2\upsilon|q|^2 q + i\mu_2 q.\end{aligned} \tag{4}$$

where $q = q(x,t)$ represents the complex envelope, ϕ is the eigenfunction, λ is the spectral parameter, x and t represent the coordinates of time evolution and spatial distribution respectively, the subscript represents the partial derivative, * represents the complex conjugate, and $\upsilon, \upsilon_1, \mu_1, \mu_2$ are real numbers and $\upsilon_1 + \mu_1 x, \upsilon_2 + \mu_2 x$ are the linear inhomageneous coefficients. Eq.(1) can be obtained by verifying the compatibility condition $P_t - Q_x + [P, Q] = 0$.

Ref.[10] analyzed the Lax pair, B$\ddot{a}$cklund transformation and one-soliton solution of Eq.(1). In Ref.[9], the non-autonomous soliton solutions are obtained by Darboux transformation. In Ref.[11], the bilinear form, one-soliton solution, two-soliton solutions and N-soliton solutions of Eq.(1) are obtained by Hirota bilinear method and symbolic computation, respectively. In Ref.[12], the N-soliton matrix related to the simple zeros in the Riemann-Hilbert (RH) problem of the Hirota equation is constructed. Then, the N-soliton matrix of the inhomogeneous variable coefficient Hirota equation can be obtained by special relational transformation from the N-soliton matrix of the Hirota equation. Next, by using the generalized Darboux transformation, the higher-order soliton solutions corresponding to the elementary higher-order zeros in the RH problem of the Hirota equation are obtained. In Ref.[13], the quantitative equivalence between Eq.(1), Heisenberg ferromagnetic spin and Wadati-Konno-Ichikawa-Shimizu equation are proved by moving spiral space curve form and three-dimensional representation.

It is well known that the inverse scattering transformation (IST) method is an effective method for solving nonlinear integrable equations including NLS hierarchy equations. It can be used to solve the initial value problem of integrable systems proposed by Gardner, Greene, Kruskal and Miura [14, 15]. It is mainly based on the asymptotic properties of the Jost solution and the corresponding Gelfand-Levitan-Marchenko integral equation theorem. Later, Zakharov and Shabat developed the RH method to simplify the original IST method[16]. At present, the research of RH method has made many successful progresses in the field of nonlinear integrable systems, such as the matrix modified Korteweg-de Vries equation [17], the reverse space-time nonlocal Sasa–Satsuma equation[18] and the quartic NLS equation[19]. Recently, a robust IST is proposed for the focused NLS equation, which is an improvement of the standard IST to deal with severe spectral singularity [20, 21]. In this paper, we will construct the corresponding RH problem, and use the Laurent expansion to derive the BS soliton solution of Eq.(1)[22–25].

The rest of the paper is arranged as follows. In Sect.2, we perform spectral analysis on Eq.(1) and construct a suitable RH problem for Eq.(1). In Sect.3, we assume that the analytical scattering coefficient has high-order zeros, which leads to the high-order poles of the reflection coefficient in RH problem. By using the Laurent expansion, the BS soliton solutions and multi-BS soliton solutions of the Eq.(1) can be derived. In Sect.4, the interaction of high-order pole solitons can be shown. In Sect.5, we give the summary.

2. Riemann-Hilbert problem: zero boundary condition

2.1. Eigenfunction and asymptotic analysis

With the initial conditions of rapid decay[26]

$$q(x,0) = q_0(x), q_0(x) \to 0,\ |x| \to \infty, \tag{5}$$

the Lax pair have the following asymptotic Jost solution

$$\psi(x,t,\lambda) \to I,\ |x| \to \infty, \tag{6}$$

where $\theta(\lambda) = \lambda x + 4\upsilon\lambda^3 t + 2(\upsilon_2 + \mu_2 x)\lambda^2 t - (\upsilon_1 + \mu_1 x)\lambda t$. Therefore, it is easy to do the following transformation

$$\psi = \phi e^{i\theta(\lambda)\sigma_3}, \tag{7}$$

there is

$$\psi(x,t,\lambda) \to I,\ |x| \to \infty, \tag{8}$$

at the same time, the Lax pair equivalent to Eq.(2) is

$$\begin{aligned}
&\psi_x + i\lambda\left[\sigma_3, \psi\right] = U\psi, \\
&\psi_t + \left[4i\upsilon\lambda^3 + 2i(\upsilon_2 + \mu_2 x)\lambda^2 - i(\upsilon_1 + \mu_1 x)\lambda\right]\left[\sigma_3, \psi\right] \\
&= \left[4\upsilon\lambda^2 U + 2\left(\upsilon_2 + \mu_2 x\right)\lambda U - 2i\upsilon\lambda\sigma_3 U^2 + 2i\upsilon\lambda\sigma_3 U_x + V\right]\psi.
\end{aligned} \tag{9}$$

This Lax pair can be written in the form of total differential

$$\begin{aligned}
&d(e^{i\theta(\lambda)\hat{\sigma}_3}\psi) \\
&= e^{i\theta(\lambda)\hat{\sigma}_3}\{[Udx + (4\upsilon\lambda^2 U + 2\left(\upsilon_2 + \mu_2 x\right)\lambda U \\
&-2i\upsilon\lambda\sigma_3 U^2 + 2i\upsilon\lambda\sigma_3 U_x + V)dt]\psi\},
\end{aligned} \tag{10}$$

where $e^{i\theta(\lambda)\hat{\sigma}_3}\psi = e^{i\theta(\lambda)\sigma_3}\psi e^{i\theta(\lambda)\sigma_3}$.

Letting ψ be Taylor expansion at ∞

$$\psi = \psi^{(0)} + \frac{\psi^{(1)}}{\lambda} + \cdots. \tag{11}$$

Then, we take the above formula into Eq.(9) and compare the power coefficient of λ to get

$x-$ part

$$O\left(\lambda\right) : i\left[\sigma_3, \psi^{(0)}\right] = 0, \tag{12}$$

$$O\left(\lambda^0\right) : \psi_x^{(0)} + i\left[\sigma_3, \psi^{(1)}\right] = U\psi^{(0)}; \tag{13}$$

$t-$ part

$$O\left(\lambda^3\right) : 4i\upsilon\left[\sigma_3, \psi^{(0)}\right] = 0, \tag{14}$$

$$O\left(\lambda^2\right) : 4i\upsilon\left[\sigma_3, \psi^{(0)}\right] + 2i\left(\upsilon_2 + \mu_2 x\right)\left[\sigma_3, \psi^{(0)}\right] = 4\upsilon U\psi^{(0)}. \tag{15}$$

It is inferred from Eq.(12) and Eq.(14) that $\psi^{(0)}$ is a diagonal matrix, from Eq.(13) and Eq.(15) that

$$\psi_x^{(0)} = 0. \tag{16}$$

Eq.(11) takes limits on both sides of x and λ at the same time. Since $\psi^{(0)}$ has nothing to do with x and λ, after exchanging order, it has

$$\lim_{\lambda\to\infty}\lim_{|x|\to\infty}\psi=\lim_{|x|\to\infty}\lim_{\lambda\to\infty}\left(\psi^{(0)}+\frac{\psi^{(1)}}{\lambda}+\cdots\right). \tag{17}$$

So we can get $\psi^{(0)}=I$, then get

$$\psi=I,\lambda\to\infty. \tag{18}$$

Then take $\psi^{(0)}=I$ into Eq.(15), and compare the corresponding elements of the matrix to get

$$q(x,t)=2i\left(\psi^{(1)}\right)_{12}=2i\lim_{\lambda\to\infty}(\lambda\psi)_{12.} \tag{19}$$

Since the integral of Eq.(10) is path independent, we can choose two special ways $(-\infty,t)\to(x,t)$ and $(+\infty,t)\to(x,t)$. Thus we can obtain two characteristic functions

$$\psi_{\pm}(x,t,\lambda)=I\pm\int_{\pm\infty}^{x}e^{-i\lambda(x-y)\,\hat{\sigma_3}}\,(-i\sigma_3U(y,t))\psi_{\pm}(y,t,\lambda)dy. \tag{20}$$

Theorem 1. *The jost solution $\psi_{\pm}\,(x,t,\lambda)$ have the following analytic properties:*

•$[\psi_-]_1,[\psi_+]_2$ *are analytic in* $\{\lambda|\mathrm{Im}\lambda>0\}$,

•$[\psi_-]_2,[\psi_+]_1$ *are analytic in* $\{\lambda|\mathrm{Im}\lambda<0\}$.

Proof. For the above integral equation, we directly calculate that

$$e^{-i\lambda(x-y)\,\hat{\sigma_3}}\,(-i\sigma_3U(y,t,\lambda))=\begin{pmatrix}0 & -iqe^{-2i\lambda(x-y)}\\ iq^*e^{2i\lambda(x-y)} & 0\end{pmatrix}, \tag{21}$$

and

$$e^{2i\lambda(x-y)}=e^{2i(x-y)\mathrm{Re}\lambda}e^{-2(x-y)\mathrm{Im}\lambda},e^{-2i\lambda(x-y)}=e^{-2i(x-y)\mathrm{Re}\lambda}e^{-2(x-y)\mathrm{Im}\lambda}, \tag{22}$$

because $y<x$, the first column of ψ_- is analytic in C_+ and the second column of ψ_- is analytic in C_-, denoted by

$$\psi_-=\left[[\psi_-]_1^+,[\psi_-]_2^-\right], \tag{23}$$

the same can prove the first column of ψ_+ is analytic in C_- and the second column of ψ_+ is analytic in C_+ , denoted by

$$\psi_+=\left[[\psi_+]_1^-,[\psi_+]_2^+\right]. \tag{24}$$

□

From Abel's theorem, we can get $\det(\phi)_x=\det(\phi)_t=0$. Based on Eq.(7), we have

$$\det(\psi)=\det(\phi)\det(e^{i\theta(\lambda)\sigma_3})=\det(\phi), \tag{25}$$

then

$$\det(\psi)_x=\det(\psi)_t=0, \tag{26}$$

it shows that $\det(\psi)$ has nothing to do with x, t, and then from progressive $\psi \to I\,(|x| \to \infty)$, we can know

$$\det(\psi) = \lim_{|x|\to\infty} \det(\psi) = \det\left(\lim_{|x|\to\infty} \psi\right) = 1, \tag{27}$$

specially

$$\det(\psi_\pm) = \det(\psi) = 1. \tag{28}$$

Since $\psi_\pm = \phi_\pm e^{i\theta(\lambda)\sigma_3}$ is the matrix solution of Eq.(9), the two solutions are linearly related, then

$$\psi_+(x,t,\lambda) = \psi_-(x,t,\lambda) e^{-i\theta(\lambda)\,\hat{\sigma}_3} S\,(\lambda)\,, \tag{29}$$

with

$$S\,(\lambda) = \begin{pmatrix} s_{11}\,(\lambda) & s_{12}\,(\lambda) \\ s_{21}\,(\lambda) & s_{22}\,(\lambda) \end{pmatrix}, \tag{30}$$

where $S\,(\lambda)$ is independent of x, t and is called spectral matrix function. Taking the determinant on both sides of Eq.(29) to get

$$\det(S\,(\lambda)) = 1. \tag{31}$$

Based on the symmetry condition

$$\sigma_2 \psi_\pm^*(x,t,\lambda^*)\sigma_2 = \psi_\pm(x,t,\lambda), \tag{32}$$

$$\sigma_2 S^*(\lambda^*)\sigma_2 = S\,(\lambda), \tag{33}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, we can get

$$s_{11}\,(\lambda) = s_{22}^*\,(\lambda^*)\,, s_{21}\,(\lambda) = -s_{12}^*\,(\lambda^*)\,. \tag{34}$$

Then $S\,(\lambda)$ can be written as

$$S\,(\lambda) = \begin{pmatrix} s_{22}^*\,(\lambda^*) & s_{12}\,(\lambda) \\ -s_{12}^*\,(\lambda^*) & s_{22}\,(\lambda) \end{pmatrix}, \tag{35}$$

with $s_{22}\,(\lambda)\,, s_{12}\,(\lambda)$ are called scattering data. Moreover, Eq.(7) and Eq.(29) lead to

$$S\,(\lambda) = e^{i\theta(\lambda)\,\hat{\sigma}_3}\psi_-^{-1}\psi_+ \tag{36}$$

and

$$s_{22}\,(\lambda) = \det\left([\psi_-]_1, [\psi_+]_2\right), s_{12}\,(\lambda) = e^{2i\theta(\lambda)} \det\left([\psi_+]_1, [\psi_-]_2\right). \tag{37}$$

Then, we will find that $s_{22}\,(\lambda)$ and $s_{12}\,(\lambda)$ can be represented by jost solution $\psi_\pm(x,t,\lambda)$. According to the analyticity of jost solution $\psi_\pm(x,t,\lambda)$, we can get

Theorem 2. $s_{22}\,(\lambda)\,, s_{12}\,(\lambda)$ *satisfy*

- $|s_{22}\,(\lambda)|^2 + |s_{12}\,(\lambda)|^2 = 1, \lambda \in R,$

• $s_{22}(\lambda) \to 1, s_{12}(\lambda) \to 0\,(\lambda \to \infty)$,

• $s_{22}(\lambda)$ *is analytic in* $\{\lambda | \mathrm{Im}\lambda > 0\}$.

2.2. The Riemann-Hilbert problem

Expanding Eq.(29), we have

$$[\psi_+]_1 - [\psi_-]_1 s_{22}^*(\lambda^*) = -[\psi_-]_2 e^{2i\theta(\lambda)} s_{12}^*(\lambda^*), \tag{38}$$

$$[\psi_+]_2 = [\psi_-]_1 e^{-2i\theta(\lambda)} s_{12}(\lambda) + [\psi_-]_2 s_{22}(\lambda), \tag{39}$$

furthermore, we have

$$[\psi_-]_1 = \frac{1}{s_{22}^*(\lambda^*)}[\psi_+]_1 + \frac{s_{12}^*(\lambda^*)}{s_{22}^*(\lambda^*)}[\psi_-]_2 e^{2i\theta(\lambda)}, \tag{40}$$

$$\frac{[\psi_+]_2}{s_{22}(\lambda)} = \frac{s_{12}(\lambda)}{s_{22}^*(\lambda^*)\, s_{22}(\lambda)}[\psi_+]_1 e^{-2i\theta(\lambda)} + \frac{1}{s_{22}^*(\lambda^*)\, s_{22}(\lambda)}[\psi_-]_2. \tag{41}$$

The above formula is changed into matrix form

$$\left([\psi_-]_1, \frac{[\psi_+]_2}{s_{22}(\lambda)}\right) = \left(\frac{[\psi_+]_1}{s_{22}^*(\lambda^*)}, [\psi_-]_2\right) e^{-2i\theta(\lambda)\hat{\sigma}_3} \begin{pmatrix} 1 & \frac{s_{12}(\lambda)}{s_{22}(\lambda)} \\ \frac{s_{12}^*(\lambda^*)}{s_{22}^*(\lambda^*)} & \frac{1}{s_{22}^*(\lambda^*) s_{22}(\lambda)} \end{pmatrix}. \tag{42}$$

It is known from Eq.(31) and Eq.(35) that

$$\det(S(\lambda)) = s_{22}^*(\lambda^*)\, s_{22}(\lambda) + s_{12}^*(\lambda^*)\, s_{12}(\lambda) = 1. \tag{43}$$

Next, letting $\rho(\lambda) = \frac{s_{12}(\lambda)}{s_{22}(\lambda)}$, we know that for $\lambda \in R$, there is

$$\frac{s_{12}^*(\lambda^*)}{s_{22}^*(\lambda^*)} = \rho^*(\lambda), \tag{44}$$

$$\begin{aligned} \frac{1}{s_{22}^*(\lambda^*)\, s_{22}(\lambda)} &= \frac{s_{22}^*(\lambda^*)\, s_{22}(\lambda) + s_{12}^*(\lambda^*)\, s_{12}(\lambda)}{s_{22}^*(\lambda^*)\, s_{22}(\lambda)} \\ &= 1 + \frac{s_{12}^*(\lambda^*)\, s_{12}(\lambda)}{s_{22}^*(\lambda^*)\, s_{22}(\lambda)} \\ &= 1 + |\rho(\lambda)|^2. \end{aligned} \tag{45}$$

So, Eq.(42) can be converted into

$$\left([\psi_-]_1, \frac{[\psi_+]_2}{s_{22}(\lambda)}\right) = \left(\frac{[\psi_+]_1}{s_{22}^*(\lambda^*)}, [\psi_-]_2\right) e^{-2i\theta(\lambda)\hat{\sigma}_3} \begin{pmatrix} 1 & \rho(\lambda) \\ \rho^*(\lambda) & 1 + |\rho(\lambda)|^2 \end{pmatrix}. \tag{46}$$

Define piecewise analytic functions

$$m(x,t,\lambda) = \begin{cases} \left[[\psi_-]_1, \frac{[\psi_+]_2}{s_{22}(\lambda)}\right], \mathrm{Im}\lambda > 0, \\ \left[\frac{[\psi_+]_1}{s_{22}^*(\lambda^*)}, [\psi_-]_2\right], \mathrm{Im}\lambda < 0. \end{cases} \tag{47}$$

$$m_{\pm}(x,t,\lambda) = \lim_{\varepsilon\to 0^{+}} m(x,t,\lambda \pm i\varepsilon), (\varepsilon,\lambda \in R). \tag{48}$$

From Eq.(46), we have

$$m_{+}(x,t,\lambda) = m_{-}(x,t,\lambda) J(\lambda), \tag{49}$$

and jump matrix

$$J(\lambda) = e^{-i\theta(\lambda)\hat{\sigma}_3} \begin{pmatrix} 1 & r(\lambda) \\ r^*(\lambda) & 1+|r(\lambda)|^2 \end{pmatrix} = \begin{pmatrix} 1 & r(\lambda)e^{-2i\theta(\lambda)} \\ r^*(\lambda)e^{2i\theta(\lambda)} & 1+|r(\lambda)|^2 \end{pmatrix}. \tag{50}$$

Theorem 3. *Based on the above situations, the following RH problem can be obtained*

- $m(x,t,\lambda)$ *is analytic in* $C\backslash R$*,*
- $m_{+}(x,t,\lambda) = m_{-}(x,t,\lambda) J(\lambda), \lambda \in R$*,*
- *Normalization:* $m(x,t,\lambda) \to I, \lambda \to \infty$*.*

From Eq.(19), the solution $q(x,t)$ of the generalized inhomogeneous Hirota equation can be given by $m(x,t,\lambda)$ as

$$q(x,t) = \lim_{\lambda\to\infty} (2i\lambda) m_{12}(x,t,\lambda) = 2i \lim_{\lambda\to\infty} (\lambda m(x,t,\lambda))_{12}. \tag{51}$$

It is worth noting that if the reflection coefficient $s_{22}(\lambda) \neq 0 (\forall\lambda)$, the above RH problem can be transformed into the regular RH problem by the method of the falling line, and then solved by the Plemelj formula [27]. If the reflection coefficient $s_{22}(\lambda) = 0$ and its zero point is simple, we can apply the residue condition to solve the problem. If it has multiple high-order poles, the residue condition is not sufficient to obtain the solution of RH problem, which will involve the coefficients of the singular part of the Laurent expansion at the corresponding high-order poles [23, 25, 28]. Therefore, in the next section, we will solve the RH problem with multiple higher-order poles.

3. The RH problem with multiple higher-order poles

Since $s_{22}(\lambda)$ is analytic in the upper half plane, $s_{22}(\lambda)$ has finite zeros in the upper half plane, but there are no singular points. Here, we consider the case of no reflection, that is, $s_{12}(\lambda) = 0$. Suppose $s_{22}(\lambda)$ has N higher order poles and $s_{22}^*(\lambda^*)$ has N poles $\left\{\lambda_j^*, \mathrm{Im}\lambda_j^* < 0\right\}_{j=1}^{N}$, so $s_{22}(\lambda)$ and $s_{22}^*(\lambda^*)$ can be expanded as follows

$$s_{22}(\lambda) = (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2} \times \cdots (\lambda-\lambda_N)^{n_N} s_0(\lambda), \tag{52}$$

$$s_{22}^*(\lambda^*) = \left(\lambda-\lambda_1^*\right)^{n_1}\left(\lambda-\lambda_2^*\right)^{n_2} \times \cdots \left(\lambda-\lambda_N^*\right)^{n_N} s_0^*(\lambda^*), \tag{53}$$

where $s_0(\lambda) \neq 0$, $s_0^*(\lambda^*) \neq 0 (\forall\lambda \in \{\lambda|\mathrm{Im}\lambda > 0\})$.

3.1. The RH problem with one higher-order pole

We first consider the case of one higher-order pole $\lambda = \lambda_0$, so we have

$$s_{22}(\lambda) = (\lambda-\lambda_0)^N s_0(\lambda), \tag{54}$$

where $s_0(\lambda)$ is analytic in C_+ and $s_0(\lambda) \neq 0\,(\forall k \in \{k|\mathrm{Im}k > 0\})$. Based on Laurent series expansion at the higher-order pole, $\rho(\lambda)$ and $\rho^*(\lambda^*)$ can be expressed as

$$\rho(\lambda) = \rho_0(\lambda) + \sum_{l=1}^{N} \frac{\rho_l}{(\lambda - \lambda_0)^l}, \rho^*(\lambda^*) = \rho_0^*(\lambda^*) + \sum_{l=1}^{N} \frac{\rho_l^*}{(\lambda - \lambda_0^*)^l}, \tag{55}$$

where

$$\rho_l = \lim_{\lambda \to \lambda_0} \frac{1}{(N-l)!} \frac{\partial^{N-l}}{\partial k^{N-l}} \left[(\lambda - \lambda_0)^N \rho(\lambda)\right], l = 1, 2 \cdots N. \tag{56}$$

Therefore m_{11} and m_{12} are meromorphic functions with one higher-pole λ_0 and λ_0^*, respectively. According to the normalization condition, the decomposition of m_{11} and m_{12} can be defined as

$$m_{11}(x,t,\lambda) = \sum_{n=1}^{N} \frac{H_n(x,t)}{(\lambda - \lambda_0^*)^n} + 1, m_{12}(x,t,\lambda) = \sum_{n=1}^{N} \frac{L_n(x,t)}{(\lambda - \lambda_0)^n}, \tag{57}$$

where $H_n(x,t), L_n(x,t)\,(n = 1, 2 \cdots N)$ are unknown functions. If $H_n(x,t), L_n(x,t)$ are addressed, we can derive one high-order pole solution $q(x,t)$ of the generalized inhomogeneous Hirota equation. In order to find m_{11} and m_{12}, according to Taylor expansion at the analytic point, we have

$$e^{-2i\theta(\lambda)} = \sum_{m=0}^{+\infty} g_m(x,t)(\lambda - \lambda_0)^m, \tag{58}$$

$$e^{-2i\theta(\lambda)} = \sum_{m=0}^{+\infty} g_m^*(x,t)\left(\lambda - \lambda_0^*\right)^m, \tag{59}$$

$$m_{11}(x,t,\lambda) = \sum_{m=0}^{+\infty} \varsigma_m(x,t)(\lambda - \lambda_0)^m, \tag{60}$$

$$m_{12}(x,t,\lambda) = \sum_{m=0}^{+\infty} \tau_m(x,t)\left(\lambda - \lambda_0^*\right)^m, \tag{61}$$

where

$$g_m(x,t) = \lim_{\lambda \to \lambda_0} \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} e^{-2i\theta(\lambda)}, \tag{62}$$

$$\varsigma_m(x,t) = \lim_{\lambda \to \lambda_0} \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} m_{11}(x,t,\lambda), \tag{63}$$

$$\tau_m(x,t) = \lim_{\lambda \to \lambda_0^*} \frac{1}{m!} \frac{\partial^m}{\partial \lambda^m} m_{12}(x,t,\lambda), \tag{64}$$

with $m = 0, 1, 2 \cdots$. Based on Eq.(46) and Eq.(47), we have scattering relation in the form

$$m_{11}(x,t,\lambda) = [\psi_-]_1 = \frac{[\psi_+]_1}{s_{22}^*(\lambda^*)} + [\psi_-]_2 e^{-2i\theta(\lambda)} \rho^*(\lambda), \tag{65}$$

$$m_{12}(x,t,\lambda) = \frac{[\psi_+]_2}{s_{22}(\lambda)} = \frac{[\psi_+]_1}{s_{22}^*(\lambda^*)} e^{-2i\theta(\lambda)} \rho(\lambda) + [\psi_-]_2 \left(1 + |\rho(\lambda)|^2\right). \tag{66}$$

Then we bring the above formulas into Taylor expansion. Letting $|\lambda| \to \infty$, by comparing the coefficients of $\left(\lambda - \lambda_0^*\right)^{-n}$, $H_n(x,t)$ $(n = 1, 2, \cdots N)$ can be expressed as

$$H_n(x,t) = -\sum_{i=n}^{N}\sum_{m=0}^{i-n} \rho_i^* g_{i-n-m}^* \tau_m. \tag{67}$$

Similarly, by comparing the coefficients of $(\lambda - \lambda_0)^{-n}$, $L_n(x,t)$ $(n = 1, 2, \cdots N)$ can be expressed as

$$L_n(x,t) = \sum_{i=n}^{N}\sum_{m=0}^{i-n} \rho_i g_{i-n-m} \varsigma_m. \tag{68}$$

Consequently, $H_n(x,t)$ and $L_n(x,t)$ can be expressed by $\tau_m(x,t)$ and $\varsigma_m(x,t)$ respectively. But we still can not get $H_n(x,t)$ and $L_n(x,t)$. Now, we bring $m_{12}(x,t,\lambda)$ in Eq.(57) into $\tau_m(x,t)$ to get

$$\tau_m(x,t) = \sum_{n=1}^{N} \begin{pmatrix} m+n-1 \\ m \end{pmatrix} \frac{(-1)^m L_n(x,t)}{\left(\lambda_0^* - \lambda_0\right)^{m+n}}, \tag{69}$$

where $m = 0, 1, 2. \cdots$, likewise, we bring $m_{11}(x,t,\lambda)$ in Eq.(57) into $\varsigma_m(x,t)$ to get

$$\varsigma_m(x,t) = \begin{cases} 1 + \sum\limits_{n=1}^{N} \frac{L_n(x,t)}{\left(\lambda_0 - \lambda_0^*\right)^{m+n}}, m = 0, \\ \sum\limits_{n=1}^{N} \begin{pmatrix} m+n-1 \\ m \end{pmatrix} \frac{(-1)^m H_n(x,t)}{\left(\lambda_0 - \lambda_0^*\right)^{m+n}}, m = 1, 2, 3, \cdots. \end{cases} \tag{70}$$

Taking the above equations into Eq.(67) and Eq.(68), we can get the representation of $H_n(x,t)$ and $L_n(x,t)$ as follows

$$H_n(x,t) = -\sum_{i=n}^{N}\sum_{m=0}^{i-n}\sum_{j=1}^{N} \begin{pmatrix} m+j-1 \\ m \end{pmatrix} \frac{(-1)^m L_j(x,t)}{\left(\lambda_0^* - \lambda_0\right)^{m+j}} \rho_i^* g_{i-n-m}^*, \tag{71}$$

$$L_n(x,t) = \sum_{i=n}^{N} \rho_i g_{i-n} + \sum_{i=n}^{N}\sum_{m=0}^{i-n}\sum_{j=1}^{N} \begin{pmatrix} m+j-1 \\ m \end{pmatrix} \frac{(-1)^m H_j(x,t)}{\left(\lambda_0 - \lambda_0^*\right)^{m+j}} \rho_i g_{i-n-m}. \tag{72}$$

The above two equations form a linear system. In order to find the exact forms of $m_{11}(x,t,\lambda)$ and $m_{12}(x,t,\lambda)$ more conveniently, we introduce some definitions of symbols as follows

$$|H\rangle = (H_1, H_2, \cdots H_N)^T, \; |L\rangle = (L_1, L_2, \cdots L_N)^T, \tag{73}$$

$$|\gamma\rangle = (\gamma_1, \gamma_2, \cdots \gamma_N)^T, \gamma_n(x,t) = \sum_{i=n}^{N} \rho_i g_{-n}, \tag{74}$$

$$\Omega = [\Omega_{n,j}]_{N\times N} = \left[-\sum_{i=n}^{N}\sum_{m=0}^{i-n} \begin{pmatrix} m+j-1 \\ m \end{pmatrix} \frac{(-1)^m \rho_i^* g_{i-n-m}^*}{\left(\lambda_0^* - \lambda_0\right)^{m+j}} \right]_{N\times N}, \tag{75}$$

where $n, j = 1, 2, \cdots N$ and T denotes transposition of matrix. So $H_n(x,t)$ and $L_n(x,t)$ can also be expressed as

$$|H\rangle = \Omega\,|\gamma\rangle\,,\,|L\rangle = |\gamma\rangle - \Omega^*\,|H\rangle\,. \tag{76}$$

Combining the above two formulas, we can get

$$|H\rangle = \Omega(I+\Omega^*\Omega)^{-1}\,|\gamma\rangle\,,\,|L\rangle = (I+\Omega^*\Omega)^{-1}\,|\gamma\rangle\,. \tag{77}$$

Substituting the above formulas into the expression of $m_{11}(x,t,\lambda)$ and $m_{12}(x,t,\lambda)$ in Eq.(57). Letting

$$\langle\Lambda(\lambda)| = \left(\frac{1}{(\lambda-\lambda_0^*)}, \frac{1}{(\lambda-\lambda_0^*)^2}\cdots\frac{1}{(\lambda-\lambda_0^*)^N}\right), \tag{78}$$

then we can get

$$m_{11}(x,t,\lambda) = 1 + \langle\Lambda(\lambda)|\,\Omega(I+\Omega^*\Omega)^{-1}\,|\gamma\rangle = \frac{\det\left(I+\Omega^*\Omega+|\gamma\rangle\langle\Lambda(\lambda)|\,\Omega\right)}{\det\left(I+\Omega^*\Omega\right)}, \tag{79}$$

$$m_{12}(x,t,\lambda) = \langle\Lambda^*(\lambda^*)|\,(I+\Omega^*\Omega)^{-1}\,|\gamma\rangle = \frac{\det\left(I+\Omega^*\Omega+|\gamma\rangle\langle\Lambda^*(\lambda^*)|\right)}{\det\left(I+\Omega^*\Omega\right)} - 1. \tag{80}$$

Based on Eq.(51), N-order BS solition of the generation inhomogeneous Hirota equation can be obtained, which can be expressed as follows

$$\begin{aligned} q(x,t) &= 2i\lim_{\lambda\to\infty}\left(\lambda m(x,t,\lambda)\right)_{12} \\ &= \lim_{\lambda\to\infty} 2i\lambda\left(\langle\Lambda^*(\lambda^*)|\,(I+\Omega^*\Omega)^{-1}\,|\gamma\rangle\right) \\ &= 2i\left(\frac{\det\left(I+\Omega^*\Omega+|\gamma\rangle\langle\Lambda^0|\right)}{\det\left(I+\Omega^*\Omega\right)} - 1\right), \end{aligned} \tag{81}$$

where $\langle\Lambda^0| = (1, 0, 0\cdots 0)$.

3.2. RH problem with N higher-order poles

In this section, we study the scattering coefficient with N different higher-order poles $\lambda_1, \lambda_{2,}\cdots\lambda_N$, the order of $\lambda_1, \lambda_{2,}\cdots\lambda_N$ are $n_1, n_2\cdots n_N$, respectively.The method is similar to that of a higher order pole, then the expansion of $s_{22}(\lambda)$ is Eq.(52). The Laurent series of $\rho(\lambda)$ is expanded to

$$\rho(\lambda) = \rho_0(\lambda) + \sum_{i=1}^{N}\sum_{m_i=1}^{n_i}\frac{\rho_{i,m_i}}{(\lambda-\lambda_i)^{m_i}}, \rho^*(\lambda^*) = \rho_0^*(\lambda^*) + \sum_{i=1}^{N}\sum_{m_i=1}^{n_i}\frac{\rho^*_{i,m_i}}{(\lambda-\lambda_i^*)^{m_i}}, \tag{82}$$

where

$$\rho_{i,m_i} = \lim_{\lambda\to\lambda_i}\frac{1}{(n_i-m_i)!}\frac{\partial^{n_i-m_i}}{(\lambda-\lambda_i)^{n_i-m_i}}\left[(\lambda-\lambda_i)^{n_i}\rho(\lambda)\right], (i = 1, 2, \cdots N), \tag{83}$$

and $\rho_0(\lambda)$ $(\forall\lambda \in \{\lambda|\mathrm{Im}\lambda > 0\})$ is analytic in C_+. According to the normalization condition, $\lambda_1, \lambda_{2,} \cdots \lambda_N$ are also the poles of m_{11} and m_{12}. Then m_{11} and m_{12} can be defined as

$$m_{11}(x,t,\lambda) = \sum_{r=1}^{n_i}\sum_{i=1}^{N} \frac{H_{i,r}(x,t)}{\left(\lambda - \lambda_i^*\right)^r} + 1, m_{12}(x,t,\lambda) = \sum_{r=1}^{n_i}\sum_{i=1}^{N} \frac{L_{i,r}(x,t)}{\left(\lambda - \lambda_i\right)^r}, \tag{84}$$

where $i = 1, 2, \cdots N$ and $r = 1, 2, \cdots n_i$. If we solve $H_{i,r}(x,t)$ and $L_{i,r}(x,t)$, m_{12} can be obtained. Then m_{11} and m_{12}, then we can get the solution $q(x,t)$ of the generalized inhomogeneous Hirota equation. By using the same method as the previous section, we have

$$H_{i,r}(x,t) = -\sum_{m_i=r}^{n_i}\sum_{n=0}^{m_i-r}\sum_{q=1}^{n_l}\sum_{l=1}^{N} \begin{pmatrix} n+q-1 \\ n \end{pmatrix} \frac{(-1)^n L_{l,q}(x,t)}{\left(\lambda_i^* - \lambda_l\right)^{n+q}} \rho^*_{i,m_i} g^*_{i,m_i-n-r}, \tag{85}$$

$$L_{i,r}(x,t) = \sum_{m_i=r}^{n_i} \rho_{i,m_i} g_{i,m_i-r} + \sum_{m_i=r}^{n_i}\sum_{n=0}^{m_i-r}\sum_{q=1}^{n_l}\sum_{l=1}^{N} \begin{pmatrix} n+q-1 \\ n \end{pmatrix} \frac{(-1)^n H_{l,q}(x,t)}{\left(\lambda_i - \lambda_l^*\right)^{n+q}} \rho_{i,m_i} g_{i,m_i-n-r}. \tag{86}$$

We introduce

$$\begin{aligned} &|H\rangle = (H_1, H_2, \cdots, H_N)^T, \ |H_i\rangle = (H_{i,1}, H_{i,2}, \cdots, H_{i,n_i})^T, \\ &|L\rangle = (L_1, L_2, \cdots .L_N)^T, \ |L_i\rangle = (L_{i,1}, L_{i,2}, \cdots, L_{i,n_i})^T, \\ &|\gamma\rangle = (\gamma_1, \gamma_2, \cdots, \gamma_N)^T, \ |\gamma_i\rangle = (\gamma_{i,1}, \gamma_{i,2}, \cdots, \gamma_{i,n_i})^T, \end{aligned} \tag{87}$$

$$\Omega = [\Omega_{i,l}]_{n_i \times n_l} = \left[-\sum_{m_i=r}^{n_i}\sum_{n=0}^{m_i-r} \begin{pmatrix} n+q+1 \\ n \end{pmatrix} \frac{(-1)^n \rho^*_{i,m_i} g^*_{i,m_i-n-r}}{\left(\lambda_i^* - \lambda_l\right)^{n+q}} \right]_{n_i \times n_l}, \tag{88}$$

$$\langle\Lambda_0| = \left(\langle\Lambda_0^1|, \langle\Lambda_0^2|, \cdots, \langle\Lambda_0^n|\right), \langle\Lambda_0^i| = (1, 0, \cdots, 0)_{1\times n_i}. \tag{89}$$

Then, the multiple BS solitons of the generalized inhomogeneous Hirota equation are derived as

$$q(x,t) = 2i\left[\frac{\det\left(I + \Omega^*\Omega + |\Lambda_0\rangle\right)}{\det\left(I + \Omega^*\Omega\right)} - 1\right]. \tag{90}$$

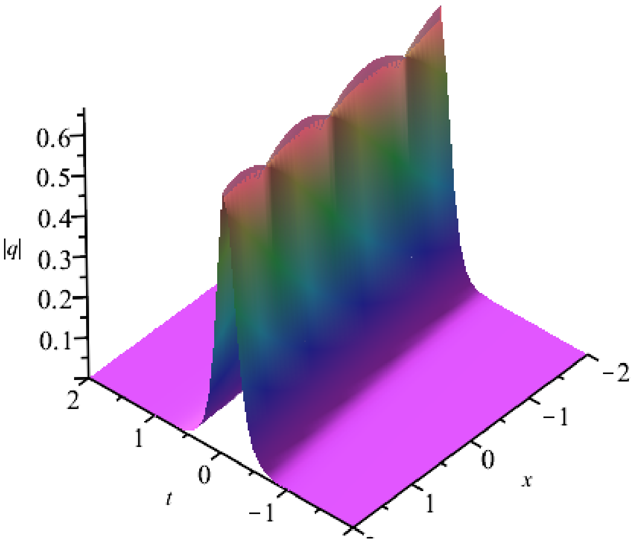
4. Some solutions of the generalized inhomogeneous Hirota equation

4.1. Solution related to one simple pole

For $N = 1$, letting $\lambda_0 = \xi + i\eta$ be a first order pole of $s_{22}(\lambda)$, then, based on Eq.(81) we can easily get the first-order BS soliton solution of the generalized inhomogeneous Hirota equation is

$$q = 2i\frac{\rho_1 g_0}{1 - \frac{\rho_1^* g_0^* \rho_1 g_0}{\left(\lambda_0^* - \lambda_0\right)^2}} = 2i\eta e^{-2\alpha i}\mathrm{sech}\left(2\beta + \ln\left|\frac{\rho_1}{2\eta}\right|\right), \tag{91}$$

where $\alpha = \xi x + 4\upsilon\left(\xi^3 - 3\xi\eta^2\right)t + 2\left(\upsilon_2 + \mu_2 x\right)\left(\xi^2 - \eta^2\right)t - \xi\left(\upsilon_1 + \mu_1 x\right)t$ and $\beta = \eta x + 4\upsilon\left(3\xi^2\eta - \eta^3\right)t + 4\xi\eta\left(\upsilon_2 + \mu_2 x\right)t - \eta\left(\upsilon_1 + \mu_1 x\right)t$. Here we set $\rho_1 = 1, \upsilon = \upsilon_1 = \upsilon_2 = \mu_1 = 1, \mu_2 = 0$ and $\xi = 1, \eta = -\frac{1}{3}$. In fig.1 and fig.2 ,the patterns of first-order BS solution are shown.



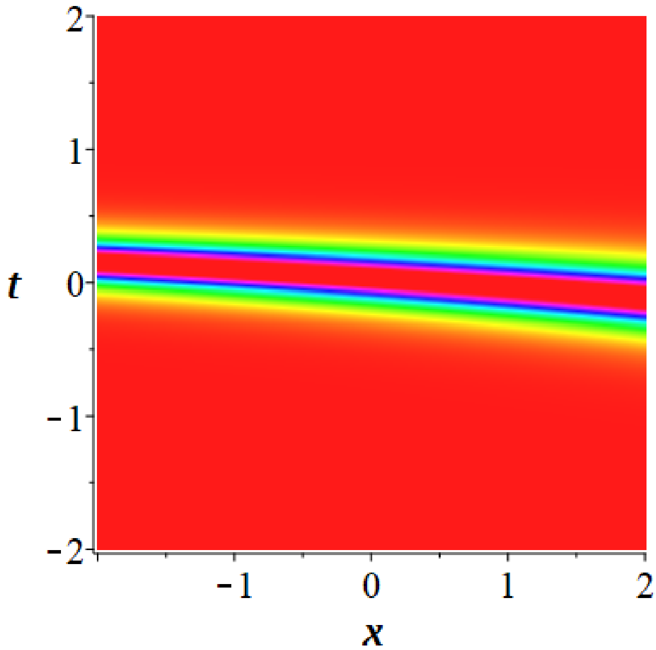


Figure 1: The soliton solution with one simple pole of the generalized inhomogeneous Hirota equation with $\lambda_1 = 1 - \frac{1}{3}i$.

Figure 2: The density plot with one simple pole of the generalized inhomogeneous Hirota equation with $\lambda_1 = 1 - \frac{1}{3}i$.

4.2. Solution related to one second-order pole

For $N = 2$, letting $\lambda = \lambda_1$ be a second-order pole of $s_{22}(\lambda)$, then, we can easily get

$$\rho(\lambda) = \rho_0(\lambda) + \frac{\rho_1}{\lambda - \lambda_1} + \frac{\rho_2}{(\lambda - \lambda_1)^2}, \tag{92}$$

$$\Omega = \begin{pmatrix} -\frac{\rho_1^* g_0^*}{\lambda_1^* - \lambda_1} - -\frac{\rho_2^* g_1^*}{\lambda_1^* - \lambda_1} + \frac{\rho_2^* g_0^*}{(\lambda_1^* - \lambda_1)^2} & -\frac{\rho_1^* g_0^*}{(\lambda_1^* - \lambda_1)^2} - \frac{\rho_2^* g_1^*}{(\lambda_1^* - \lambda_1)^2} + \frac{2\rho_2^* g_0^*}{(\lambda_1^* - \lambda_1)^3} \\ -\frac{\rho_2^* g_0^*}{\lambda_1^* - \lambda_1} & -\frac{\rho_2^* g_0^*}{(\lambda_1^* - \lambda_1)^2} \end{pmatrix}, \tag{93}$$

and

$$|\gamma\rangle = \begin{pmatrix} \rho_1 g_0 + \rho_2 g_1 & \rho_2 g_0 \end{pmatrix}^T, \; |\Lambda_0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{94}$$

On the basis of Eq.(81), letting $\rho_1 = \rho_2 = 1$, $\upsilon = \upsilon_1 = \upsilon_2 = \mu_2 = 1$, $\mu_1 = 0$ and $\lambda_1 = \frac{1}{3} + \frac{\sqrt{3}}{3}i$, the second-order BS soliton solutions of the generalized inhomageneous Hirota equation is obtained as follows

$$q(x,t) = \frac{A_1(x,t)\, e^{B_1(x,t)} + A_2(x,t)\, e^{B_2(x,t)}}{A_3(x,t)\, e^{B_3(x,t)} + 243 e^{B_4(x,t)} + 768}, \tag{95}$$

where

$$
\begin{aligned}
A_1(x,t) &= 4096\sqrt{3}it + 37888t + 1536i + 3072x,\\
B_1(x,t) &= \tfrac{\sqrt{3}(8t+6x)}{9} - \tfrac{2i(23t+3x)}{9},\\
A_2(x,t) &= 2304\left[\sqrt{3}\left(\tfrac{3}{4} + ti\right) + \tfrac{3i}{8} - \tfrac{37t}{4} - \tfrac{3}{4}x\right],\\
B_2(x,t) &= \tfrac{\sqrt{3}(24t+18x)}{9} - \tfrac{46it}{9} - \tfrac{2ix}{3},\\
A_3(x,t) &= (-25344t - 2304x)\sqrt{3} + 362752t^2 + 56832tx + 2304x^2 + 3168,\\
B_3(x,t) &= \tfrac{4\sqrt{3}(4t+3x)}{9},\\
B_4(x,t) &= \tfrac{8\sqrt{3}(4t+3x)}{9}.
\end{aligned}
\tag{96}
$$

The fig.3 and fig.4 illustrate the interaction of two solitons. Without the influence of inhomogeneous parameters, two solitons pass through each other, and their shapes remain unchanged except for phase shift. Before and after the interaction, the two solitons have the same amplitude, and the amplitude will change during the interaction.

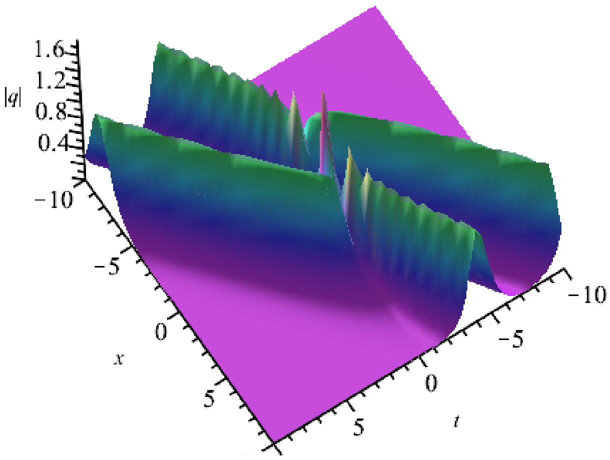


Figure 3: The second-order BS soliton solution of the generalized inhomogeneous Hirota equation with $\lambda_1 = \frac{1}{3} + \frac{\sqrt{3}}{3}i$.

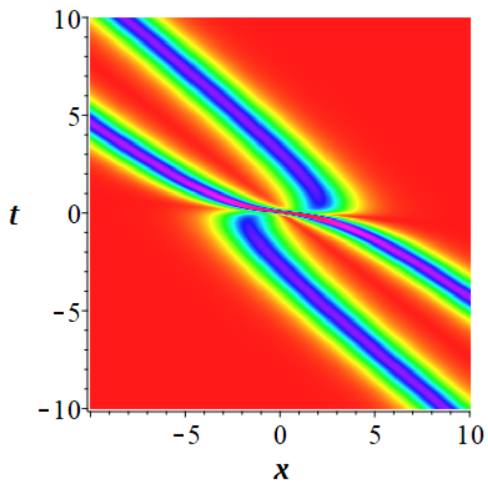


Figure 4: The density plot of the generalized inhomogeneous Hirota equation with $\lambda_1 = \frac{1}{3} + \frac{\sqrt{3}}{3}i$.

4.3. Solution related to one simple pole and one second-order pole

Letting $\lambda = \lambda_1$ be a simple pole of $s_{22}(\lambda)$ and $\lambda = \lambda_2$ be a second-order pole of $s_{22}(\lambda)$, then we can easily get

$$\rho(\lambda) = \rho_0(\lambda) + \frac{\rho_{1,1}}{\lambda - \lambda_1} + \frac{\rho_{1,2}}{(\lambda - \lambda_1)^2} + \frac{\rho_{2,1}}{\lambda - \lambda_2}, \tag{97}$$

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \tag{98}$$

$$|\gamma\rangle = \begin{pmatrix} \rho_{1,1}g_{1,0} + \rho_{1,2}g_{1,1} & \rho_{1,2}g_{1,0} & \rho_{2,1}g_{2,0} \end{pmatrix}^T, \tag{99}$$

$$|\Lambda_0\rangle = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \tag{100}$$

with

$$
\begin{aligned}
\Omega_{11} &= \begin{pmatrix} -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_1}-\frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\lambda_1^{*}-\lambda_1}+\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_1\right)^2} & -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_1\right)^2}-\frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\left(\lambda_1^{*}-\lambda_1\right)^2}+\frac{2\rho_{1,2}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_1\right)^3} \\ -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_1} & -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_1\right)^2} \end{pmatrix}, \\
\Omega_{12} &= \begin{pmatrix} -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_2}-\frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\lambda_1^{*}-\lambda_2}+\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_2\right)^2} \\ -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\left(\lambda_1^{*}-\lambda_2\right)^2} \end{pmatrix}, \\
\Omega_{21} &= \begin{pmatrix} -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_1} & -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{\left(\lambda_2^{*}-\lambda_1\right)^2} \end{pmatrix}, \Omega_{22} = -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_2}.
\end{aligned}
\tag{101}
$$

Here, we let $\rho_{1,1}=\rho_{1,2}=\rho_{2,1}=1, \upsilon=\upsilon_1=\upsilon_2=1, \mu_1=\mu_2=0$. Based on Eq.(90), we can derive the explicit solution for the generalized inhomogeneous Hirota equation. The interaction between a soliton and a BS soliton are shown in fig.5 and fig.6.

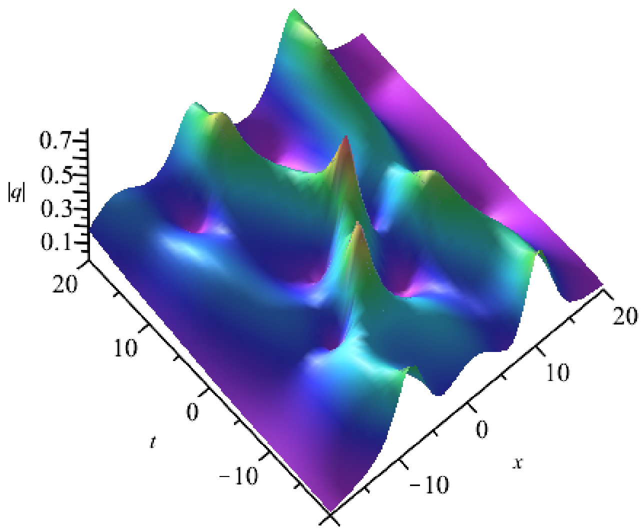


Figure 5: One soliton solution and one second-order BS soliton solution of the generalized inhomogeneous Hirota equation with $\lambda_1=\frac{1}{5}-\frac{1}{4}i, \lambda_2=\frac{1}{5}-\frac{1}{8}i$.

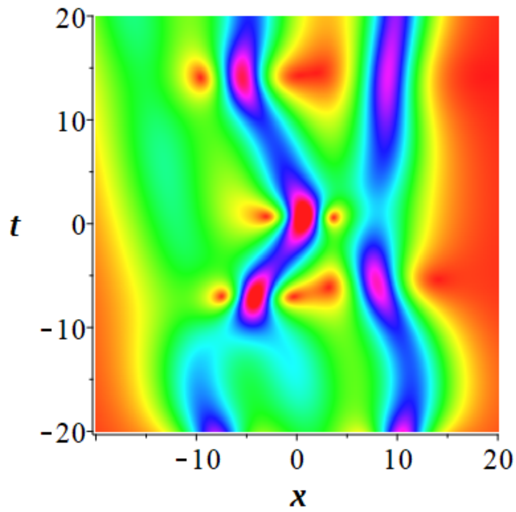


Figure 6: The density plot of the generalized inhomogeneous Hirota equation with $\lambda_1=\frac{1}{5}-\frac{1}{4}i, \lambda_2=\frac{1}{5}-\frac{1}{8}i$.

4.4. Solution related to two second-order poles

Letting $\lambda=\lambda_1$ and $\lambda=\lambda_2$ be second-order pole of $s_{22}(\lambda)$, then we can easily get

$$
\rho(\lambda)=\rho_0(\lambda)+\frac{\rho_{1,1}}{\lambda-\lambda_1}+\frac{\rho_{1,2}}{(\lambda-\lambda_1)^2}+\frac{\rho_{2,1}}{\lambda-\lambda_2}+\frac{\rho_{2,1}}{(\lambda-\lambda_2)^2}, \tag{102}
$$

$$
\Omega=\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \tag{103}
$$

$$
|\gamma\rangle=\begin{pmatrix} \rho_{1,1}g_{1,0}+\rho_{1,2}g_{1,1} & \rho_{1,2}g_{1,0} & \rho_{2,1}g_{2,0}+\rho_{2,2}g_{2,1} & \rho_{2,2}g_{2,0} \end{pmatrix}^T, \tag{104}
$$

$$
|\Lambda_0\rangle=\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}, \tag{105}
$$

with

$$
\begin{aligned}
\Omega_{11} &= \begin{pmatrix} -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_1} - \frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\lambda_1^{*}-\lambda_1} + \frac{\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_1)^2} & -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_1)^2} - \frac{\rho_{1,2}^{*}g_{1,1}^{*}}{(\lambda_1^{*}-\lambda_1)^2} + \frac{2\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_1)^3} \\ -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_1} & -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_1)^2} \end{pmatrix}, \\
\Omega_{12} &= \begin{pmatrix} -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_2} - \frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\lambda_1^{*}-\lambda_2} + \frac{\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_2)^2} & -\frac{\rho_{1,1}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_2} - \frac{\rho_{1,2}^{*}g_{1,1}^{*}}{\lambda_1^{*}-\lambda_2} + \frac{2\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_2)^2} \\ -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{\lambda_1^{*}-\lambda_2} & -\frac{\rho_{1,2}^{*}g_{1,0}^{*}}{(\lambda_1^{*}-\lambda_2)^2} \end{pmatrix}, \\
\Omega_{21} &= \begin{pmatrix} -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_1} - \frac{\rho_{2,2}^{*}g_{2,1}^{*}}{\lambda_2^{*}-\lambda_1} + \frac{\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_1)^2} & -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_1)^2} - \frac{\rho_{2,2}^{*}g_{2,1}^{*}}{(\lambda_2^{*}-\lambda_1)^2} + \frac{2\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_1)^3} \\ -\frac{\rho_{2,2}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_1} & -\frac{\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_1)^2} \end{pmatrix}, \\
\Omega_{22} &= \begin{pmatrix} -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_2} - \frac{\rho_{2,2}^{*}g_{2,1}^{*}}{\lambda_2^{*}-\lambda_2} + \frac{\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_2)^2} & -\frac{\rho_{2,1}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_2)^2} - \frac{\rho_{2,2}^{*}g_{2,1}^{*}}{(\lambda_2^{*}-\lambda_2)^2} + \frac{2\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_2)^3} \\ -\frac{\rho_{2,2}^{*}g_{2,0}^{*}}{\lambda_2^{*}-\lambda_2} & -\frac{\rho_{2,2}^{*}g_{2,0}^{*}}{(\lambda_2^{*}-\lambda_2)^2} \end{pmatrix}.
\end{aligned}
\tag{106}
$$

Here we let $\rho_{1,1} = \rho_{1,2} = \rho_{2,1} = 1, \upsilon = \upsilon_1 = \upsilon_2 = 1, \mu_1 = \mu_2 = 0$. Based on Eq.(90), we can get the explicit solution of the generalized inhomogeneous Hirota equation. The interaction between the two BS solitons are shown in fig.7 and fig.8.

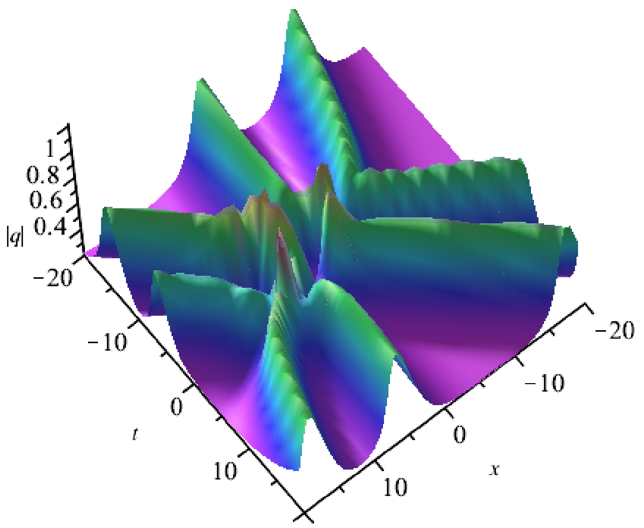


Figure 7: Two second-order BS solitons solution of the generalized inhomogeneous Hirota equation with $\lambda_1 = \frac{1}{3} + \frac{1}{3}i, \lambda_2 = \frac{1}{5} + \frac{1}{3}i$.

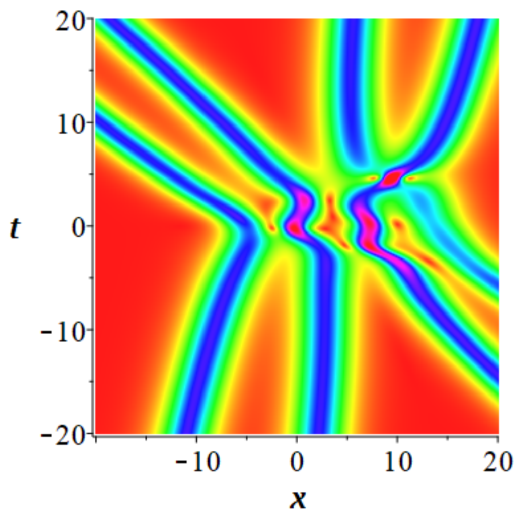


Figure 8: The density plot of the generalized inhomogeneous Hirota equation with $\lambda_1 = \frac{1}{3} + \frac{1}{3}i, \lambda_2 = \frac{1}{5} + \frac{1}{3}i$.

5. Summary

In this paper, based on the spectral analysis of Lax pair, the Riemann-Hilbert (RH) method is used to study the generalized inhomogeneous Hirota equation with one or more high-order poles. By the analyticity and symmetry of the characteristic function, we transform the zero boundary initial value problem of Eq.(1) into RH problem. When the scattering data have one high-order pole and multiple high-order poles respectively, the RH problem is solved by its Laurent expansion, and the formula of bound-state (BS) soliton corresponding to one high-order pole is obtained, as shown in Eq.(81), and the formula of BS soliton corresponding to multiple high-order poles is obtained, as shown in Eq.(90), and the interaction between solitons are shown in

Fig.(1-8).

The one-soliton solution and two-soliton solution of the generalized inhomogeneous Hirota equation can be given by the most basic Darboux transformation [9], but this method can not find the high-order soliton solution, so it needs to be generalized. The generalized Darboux transformation can derive the high-order soliton solution corresponding to the basic high-order zeros in the RH problem of the Hirota equation, and then the high-order soliton solution of the generalized inhomogeneous variable coefficient Hirota equation can be obtained by using the corresponding transformation relationship [12], but this method is more complicated. In this paper, we directly solve the RH problem with high-order poles, and the formulas of BS solitons and multiple solitons are expressed. The obtained solution has more extensive significance and richer content. In addition, the method proposed in this paper can be further extended to solve some other nonlinear systems, and can be optimized to improve future results.

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