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RESEARCH ARTICLE

Sharp estimates of solution of an elliptic problem on a family of open non-convex planar sectors

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Abstract

Based on partial Fourier series analysis, we adapt on a model case a new approach to classical results obtained in the literature describing the singularities of a family a solutions of a second order elliptic problems on open non-convex planar sectors. The method allows the exhibition of singular and regular frequencies, explicit decomposition and description of coefficients of singularities of the solution. As a main result, explicit and sharp estimates with respect to the opening angle parameter are obtained via this method. They are not uniform near π where corners have opening angle generating a jump of singularity in Sobolev exponent, contrarily to the results obtained in A. Tami (2016),(2019),(2021) for harmonic and/or biharmonic problems on a family of convex planar sectors.

KEYWORDS

Fourier series, Elliptic problem, Family of non-convex sectors, Regularity, Singularity, Sharp estimates.

MSC CLASSIFICATION

35J25, 35J40; 35J75; 35B45; 35Q99; 35B40

1 | INTRODUCTION

The behavior of solutions of elliptic problems on polygons near a corner has been investigated in the 60's. In engineering mathematics, analysis of singular solutions of partial differential equations in non-smooth domains is of great interest. It is well known that the presence of such singularities may severely reduce convergence in error estimates of standard numerical schemes of approximation. Thus, it is always interesting to study and provide explicit formulas, in particular, when it comes to a family of solutions dependent on a parameter ω (the opening angle at a corner) where these singularities as well as only a part of the regular solution may blow-up near some critical values of this parameter. Recently, Tami (2019)[1,2] has put into evidence the analogy of such decomposition locally with standard Taylor expansions in the vicinity of a convex corner. The author provided uniform estimates with respect to $\omega \in (2\pi/3, \pi)$ for both laplacian and the bi-Laplacian operators, i.e, when the corner is convex and the critical value of ω equals π. This property of behavior of solutions w.r.t ω allows the treatment of families of elliptic problems on families of open sets. To the knowledge of the authors of this paper, no such study was performed on a family of non-convex open sets, i.e. when $\omega \in (\pi, 2\pi)$, whether in the harmonic or biharmonic case. When a non-convex corner is considered, an additional difficulty, namely the existence of L^2 solutions, see for example Nazarov (2007)[3]Li, Hengguang et. al.[4], in the kernel of the Laplace operator and satisfying the boundary conditions of the problem. Nevertheless and Fortunately, the orthogonal space of the range of this operator is still finite-dimensional and it is possible to identify its basis. Therefore, it is still worthwhile and possible to study and provide explicit extraction formulas with explicit sharp estimates. In this paper, we aim at studying by means of partial Fourier analysis in polar coordinates, cf. Nkemzi (2023)[5] and references therein, w.r.t to the polar angle θ, the asymptotic behavior of solutions by deriving explicit computable formulas for the coefficients of the singularities with explicit estimates that show the behavior (in H^2 norm) of the family of solutions u_ω near the critical angle $\omega = \pi$. As a main result of our approach is the lack of uniformity in the estimates with respect to the angle parameter ω, contrarily to those obtained in the case on convex corners, cf. [1].

We consider a model case of Elliptic equation with homogeneous Dirichlet condition and source term f_ω square integrable on a planar polygonal domains withnon convex corner. Such problems, and more generally elliptic boundary value problem of partial differential equation, are known to exhibit singular behaviors near the corners or boundaries of non-smooth domains. Well posedeness of such problems and description of their singularities near corners with different boundary conditions were addressed, whether in the harmonic or biharmonic case, by many authors in the literature, cf., Grisvard ([6,7,8]), Kondrat'ev [9] , Blum [4], Maz'ya ([10,11]), Nicaise ([12,13]), Dauge ([14,15]), Stylianou [16], Gerasimov [17], Tami ([1,2,18]) and the references cited therein.

Throughout this paper, a generic constant $C > 0$ in all estimates that follow may be independent of ω and different at different occurrences. In the second section, we present the problem setting and the main result with a partial proof of H^1 uniform estimates w.r.t the opening angle ω of the family of solutions u_ω our problem. The proof of H^2 estimates will be justified gradually in the sections that follow. The third one contains some preliminaries such as Sobolev spaces in polar coordinates, Sobolev norms expressed via Fourier coefficients, and some fundamental tools useful for estimates of Fourier coefficients. Next, in the fourth section, formal determination of corner singularity via Fourier series is presented. In the fifth section, Fourier coefficients of the regular part of the solution and the coefficient of singularity obtained formerly are given explicitly. In addition, explicit and sharp or uniform estimates are given w.r.t to the opening angle parameter $\omega \in (\pi, 2\pi)$. These estimates are not uniform in the vicinity of π, even for the regular part taken separately in the case of the first frequency $k = 1$ in the Fourier series. The sixth section is devoted to the completion of the proof of the main result, in particular the characterization of coefficient of singularity and the estimates on of the regular part in the norm H^2. Concluding remarks and comments are presented in the last section.

2 | PROBLEM SETTING AND THE MAIN RESULT

By a localization technique around the reentering corner in question, we are interested in a family of boundary value problems of the following type. Let us denote by $\{\Omega_\omega\}_{\omega\in(\pi,2\pi)}$ a family of open bounded sectors of radius 1 centered at the origin O (here O represents the reentering corner where the localization has been performed). In polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, one has , cf. Figure 1,

$$\Omega_\omega = \{(x, y),\ 0 < r < 1,\ 0 < \theta < \omega\},$$

with boundary $\partial\Omega_\omega = \overline{\Gamma_0} \cup \overline{C_\omega} \cup \overline{\Gamma_\omega}$ where

$$\begin{aligned} \Gamma_\alpha &= \{(x, y),\ 0 < r < 1,\ \theta = \alpha\},\ \alpha = 0, \omega \\ C_\omega &= \{(x, y),\ r = 1,\ 0 < \theta < \omega\}. \end{aligned}$$

For a right hand side (r.h.s) $f_\omega \in L^2(\Omega_\omega)$ depending on the parameter $\omega \in (\pi, 2\pi)$, we look for solutions u_ω of the following Elliptic equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} \Delta u_\omega = f_\omega \text{ in } \Omega_\omega, \\ \quad u_\omega = 0 \text{ on } \partial\Omega_\omega. \end{cases} \tag{1}$$

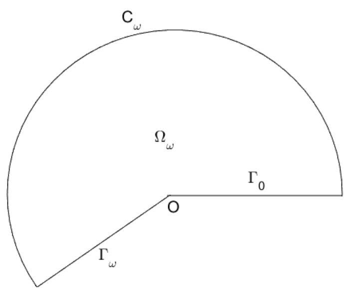

FIGURE 1 Planar sector Ω_ω with non-convex corner at the origin: Opening angle $\omega \in (\pi, 2\pi)$.

Following the results in the literature, cf. Grisvard (1986)[7], the solution u_ω of (1) admits near the origin the following decomposition (in regular / singular parts) that can be written in polar coordinates as follows:

$$u_\omega = u_{\omega,r} + \lambda_\omega r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta,$$

where $u_{\omega,r} \in H^2_{loc}(\Omega_\omega)$, and the singular part $u_{\omega,s}(r,\theta) = \lambda_\omega r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta \in H^{1+\sigma}(\Omega_\omega)$ for all $\sigma < \frac{\pi}{\omega}$. According to the general theory on H^2-regularity of elliptic boundary value problems, see the cited references, in two-dimensional domains with corners, the solution u_ω associated to a right-hand side f_ω square integrable on the family of planar sectors Ω_ω with non-convex corner exhibits a singularity at the origin whose effect is to limit its regularity, expressed in the scale of fractional order Sobolev spaces. In contrast, when $\omega = \pi$, there no singularity and the solution u_π is $H^2(\Omega_\omega)$. Hence, there is a jump in Sobolev exponents describing the regularity of solution when $\omega \to \pi$ on the side $\omega > \pi$.

In this work, we will retrieve explicit decomposition by the partial Fourier series method which allows the extraction of the singularity systematically. Moreover, explicit estimates w.r.t the opening angle ω are obtained on $u_{\omega,r}$ and the coefficient of singularity λ_ω. The main result of this paper is given bellow and its proof will be given later. Notice that the proof of the first result (i) in the main theorem, i.e. existence in H^1_0 is evident and standard, unless the uniformity of the constant w.r.t ω that comes directly from Poincare's inequality, cf. [18], for all $u \in H^1_0(\Omega_\omega)$,

$$\|u\|_{H^1(\Omega_\omega)} \leq \sqrt{1+\omega}\|\nabla u\|_{L^2(\Omega_\omega)},$$

which yields equivalence, with uniformly bounded constant, between the norm and semi-norm H^1. Hence, only proofs of (ii) and (iii) are needed and will completed later at the end of the paper.

Theorem 1 (Main Theorem). *Let $\omega \in (\pi, 2\pi)$ and $f_\omega \in L^2(\Omega_\omega)$ with Fourier coefficients*

$$c_{k,\omega}(r) = \frac{2}{\omega}\int_0^\omega f_\omega(r,\theta)\sin\frac{k\pi}{\omega}\theta d\theta,\ k \geq 1.$$

i) Problem (1) admits a unique solution in $H^1_0(\Omega_\omega)$ that depends continuously on the r.h.s $f_\omega \in L^2(\Omega_\omega)$ and uniformly on the parameter $\omega \in (\pi, 2\pi)$, i.e. there exists a constant independent of ω, $C > 0$, such that:

$$\|u_\omega\|_{H^1(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{2}$$

ii) $u_\omega \in H^{1+\sigma}(\Omega_\omega) \cap H^1_0(\Omega_\omega)$, for all $\sigma < \frac{\pi}{\omega}$, and u_ω admits in Ω_ω the following decomposition

$$u_\omega = \lambda_\omega r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta + u^I_{\omega,r} + u^{II}_{\omega,r}, \tag{3}$$

where $u^I_{\omega,r}, u^{II}_{\omega,r} \in H^2(\Omega_\omega)$ and the coefficient of singularity λ_ω are given explicitly as follows:

$$\lambda_\omega = -\frac{\omega}{2\pi}\left(\int_0^1 c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds - \int_0^1 c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds\right), \tag{4}$$

$$u^I_{\omega,r}(r,\theta) = \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds\right)\sin\frac{\pi}{\omega}\theta, \tag{5}$$

$$u^{II}_{\omega,r}(r,\theta) = \sum_{k\geq 2}\frac{\omega}{2k\pi}\left(-r^{\frac{k\pi}{\omega}}\int_r^1 c_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}ds - r^{-\frac{k\pi}{\omega}}\int_0^r c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}ds + r^{\frac{k\pi}{\omega}}\int_0^1 c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}ds\right)\sin\frac{k\pi}{\omega}\theta. \tag{6}$$

iii) There exists $C > 0$ independent of $\omega \in (\pi, 2\pi)$ and $f_\omega \in L^2(\Omega_\omega)$ such that the following estimate holds and is sharp:

$$\|u^{II}_{\omega,r}\|_{H^2(\Omega_\omega)} + \sqrt{\omega-\pi}\left(\|u^I_{\omega,r}\|_{H^2(\Omega_\omega)} + |\lambda_\omega|\right) \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{7}$$

Let us denote by $(.,.)_\omega$ the natural scalar product of $L^2(\Omega_\omega)$, defined by $(f,g)_\omega := \int_{\Omega_\omega} fg d\Omega_\omega$ and denote by

$$\xi_\omega(r,\theta) = \frac{r^{\frac{\pi}{\omega}} - r^{-\frac{\pi}{\omega}}}{\pi}\sin\frac{\pi}{\omega}\theta, \tag{8}$$

a $L^2(\Omega_\omega)$ solution, since $\omega \in (\pi, 2\pi)$, of the homogeneous Dirichlet problem

$$\begin{cases} \Delta\xi_\omega = 0 \text{ in } \Omega_\omega, \\ \xi_\omega = 0 \text{ on } \partial\Omega_\omega. \end{cases}$$

Remark that ξ_ω is not unique as a $L^2(\Omega_\omega)$ solution since the trivial case $\xi_\omega = 0$ is also solution. Note also that the boundary condition $\xi_\omega = 0$ on $\partial\Omega_\omega$ is defined in the trace sense, namely $\widetilde{H}^{-\frac{1}{2}}$ on each side of $\partial\Omega_\omega$, , cf.[7,8,4]. Let us denote by

$$L^2_{\xi_\omega^\perp}(\Omega_\omega) := \left\{\phi \in L^2(\Omega_\omega),\ (\phi, \xi_\omega)_\omega = 0\right\}$$

We know that the mapping $-\Delta_D : H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega) \to L^2(\Omega_\omega)$ is injective and has a closed range (Grisvard, 1992)[8]. The following corollary shows that its image equals $L^2_{\xi_\omega^\perp}(\Omega_\omega)$ and that

$$-\Delta_D : H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega) \to L^2_{\xi_\omega^\perp}(\Omega_\omega)$$

is a uniformly bounded family of isomorphisms w.r.t to the opening angle parameter $\omega \in (\pi, 2\pi)$.

Corollary 1. *Let $\omega \in (\pi, 2\pi)$ and $f_\omega \in L^2(\Omega_\omega)$. Then, $u_\omega \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ if and only if the $(f_\omega, \xi_\omega)_\omega = 0$. In this case, there exists $C > 0$ independent of $\omega \in (\pi, 2\pi)$ such that:*

$$\|u_\omega\|_{H^2(\Omega_\omega)} \leq C\|f_\omega\|_{L^2(\Omega_\omega)}. \tag{9}$$

3 | PRELIMINARY RESULTS

3.1 | Sobolev norms in polar coordinates via Fourier series

According to previous studies on partial Fourier series in Sobolev spaces in polar coordinates, see for example[4] and references therin, we will use the same notation $G(r, \theta) := G(r\cos\theta, r\sin\theta)$. As far as we work with Dirichlet boundary condition on the boundary $\partial\Omega_\omega$, for any $G \in L^2(\Omega_\omega)$, let us denote by $G(r, \theta) = \sum_{k\geq 1} G_k(r) \sin \dfrac{k\pi}{\omega}\theta$ a.e in Ω_ω the partial Fourier series of G in θ, where $G_k(r) = \frac{2}{\omega}\int_0^\omega G(r, \theta) \sin \frac{k\pi}{\omega}\theta d\theta$ is the k^{th} Fourier coefficient seen as a function of $r \in (0, 1)$. We have:

$$\|G\|^2_{L^2(\Omega_\omega)} = \int_{\Omega_\omega} \left|G\right|^2 r dr d\theta, \tag{10}$$

$$\|\nabla G\|^2_{L^2(\Omega_\omega)} = \int_{\Omega_\omega} \left(\left|\frac{\partial G}{\partial r}\right|^2 + \left|\frac{1}{r}\frac{\partial G}{\partial \theta}\right|^2 \right) r dr d\theta, \tag{11}$$

$$\||\nabla^2 G|\|^2_{L^2(\Omega_\omega)} = \int_{\Omega_\omega} \left(\left|\frac{\partial^2 G}{\partial r^2}\right|^2 + 2\left|\frac{1}{r}\frac{\partial^2 G}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial G}{\partial\theta}\right|^2 + \left|\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2} + \frac{1}{r}\frac{\partial G}{\partial r}\right|^2 \right) r dr d\theta. \tag{12}$$

By orthogonality, we have also the following Parseval identities :

$$\|G\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left|G_k\right|^2 r dr, \tag{13}$$

$$\|\nabla G\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left(\left|G'_k\right|^2 + \left(\frac{k\pi}{\omega}\right)^2 \left|\frac{G_k}{r}\right|^2 \right) r dr, \tag{14}$$

$$\|\nabla^2 G\|^2_{L^2(\Omega_\omega)} = \frac{\omega}{2}\sum_{k\geq 1}\int_0^1 \left(\left|G''_k\right|^2 + 2\left(\frac{k\pi}{\omega}\right)^2 \left|\frac{G'_k}{r} - \frac{G_k}{r^2}\right|^2 + \left|\frac{G'_k}{r} - \left(\frac{k\pi}{\omega}\right)^2\frac{G_k}{r^2}\right|^2 \right) r dr. \tag{15}$$

According to the Dirichlet boundary condition taken partially w.r.r θ, for $m = 0, 1, 2$, the space of distributions $G(r, \theta)$ such that $\|\nabla^m G\|^2_{L^2(\Omega_\omega)} < +\infty$ and that satisfy the boundary condition $G(r, 0) = G(r, \omega) = 0$,a.e in $(0, 1)$ will be denoted by $H^m_D(\Omega_\omega)$ and equipped with the standard Sobolev norm, such that

$$\|G\|^2_{H^m(\Omega_\omega)} := \sum_{l=0}^{m} \|\nabla^l G\|^2_{L^2(\Omega_\omega)}. \tag{16}$$

3.2 | Two fundamental Lemmas for the Fourier coefficients analysis

In what follows, we use the standard notation $\int_0^1 \left|\phi(r)\right|^2 rdr := \|\phi\|^2_{L^2(rdr)}$. The two following lemma are essential for the estimation of the norms in $H^2(\Omega_\omega)$ of the Fourier coefficients of the regular and singular parts parts of the solution u_ω of problem (1). Moreover, they make the Fourier series method efficient to obtain the decomposition of u_ω into its regular/singular part $u_\omega = u_{\omega,r} + u_{\omega,s}$ just by handling some critical powers of r and balancing integral limits between those from 0 to r and others from r to 1 in order to isolate what is called the roots of a transcendent equation as obtained in the literature. We will see in the following section how these roots are extracted directly by imposing the condition on the singular part $u_{\omega,s}$ to belong to the space $H^\sigma(\Omega_\omega)$ with $1 \le \sigma < 2$. The two following lemmas which result from a variant of Hardy's type weighted inequalities, (G. Hardy 1927)[19], are fundamental in the uniform estimates for the Fourier coefficients that will be given later.

Lemma 1. *For any $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(rdr)$, let $F(r) := r^\alpha \int_0^r f(s)s^\beta ds$ defined for $r \in (0, 1)$. If $\beta > 0$ and $\alpha + \beta \ge -1$ then $F \in L^2(rdr)$ and we have:*

$$\|F\|_{L^2(rdr)} \le \frac{1}{2\sqrt{\beta(\alpha+\beta+1)}}\|f\|_{L^2(rdr)}, \text{ if } \alpha + \beta > -1, \tag{17}$$

$$\|F\|_{L^2(rdr)} \le \frac{1}{\beta}\|f\|_{L^2(rdr)}, \text{ if } \alpha + \beta = -1. \tag{18}$$

Proof. The case $\alpha + \beta > -1$ is trivial as a direct consequence of Cauchy-Schwartz inequality. Now, if $\alpha + \beta = -1$, then we have from Cauchy-Schwartz (C.S) inequality and Fubini's theorem. In fact,

$$\begin{aligned}\int_0^1 \left(\int_0^r f(s)s^\beta ds\right)^2 r^{2\alpha} rdr &= \int_0^1 \left(\int_0^r f(s)s^{\frac{1+\beta}{2}} s^{-\frac{1-\beta}{2}} ds\right)^2 r^{2\alpha+1}dr \\ &\overset{(C.S)}{\le} \int_0^1 \left(\int_0^r f^2(s)s^{1+\beta}ds\right)\left(\int_0^r s^{-1+\beta}ds\right) r^{2\alpha+1}dr \\ &\le \frac{1}{\beta}\int_0^1 \left(\int_0^r f^2(s)s^{1+\beta}ds\right) r^\beta r^{2\alpha+1}dr \\ &\le \frac{1}{\beta}\int_0^1 \left(\int_0^r f^2(s)s^{1+\beta}ds\right) r^{-1-\beta}dr\end{aligned}$$

since $\alpha = -1 - \beta$. Next, with the help of Fubini's theorem, $0 < s < r < 1$, we obtain

$$\begin{aligned}\frac{1}{\beta}\int_0^1 \left(\int_0^r f^2(s)s^{1+\beta}ds\right) r^{-1-\beta}dr &= \frac{1}{\beta}\int_0^1 \left(f^2(s)s^{1+\beta}\int_s^1 r^{-1-\beta}dr\right) ds \\ &\le \frac{1}{\beta}\int_0^1 \left(f^2(s)s^{1+\beta}\int_s^{+\infty} r^{-1-\beta}dr\right) ds \\ &\le \frac{1}{\beta^2}\int_0^1 f^2(s)sds\end{aligned}$$

hence, (18) holds. □

Lemma 2. *For any $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(rdr)$, let $G(r) := r^\alpha \int_r^1 f(s)s^\beta ds$ defined for $r \in (0, 1)$. If $\beta < 0$ and $\alpha + \beta \ge -1$ then $G \in L^2(rdr)$ and we have*

$$\|G\|_{L^2(rdr)} \le \frac{1}{2\sqrt{|\beta|(\alpha+\beta+1)}}\|f\|_{L^2(rdr)}, \text{ if } \alpha + \beta > -1, \tag{19}$$

$$\|G\|_{L^2(rdr)} \le \frac{1}{|\beta|}\|f\|_{L^2(rdr)}, \text{ if } \alpha + \beta = -1. \tag{20}$$

Proof. The case $\alpha + \beta > -1$ is trivial as a direct consequence of Cauchy-Schwartz inequality. Now, if $\alpha + \beta = -1$, then we have:

$$\begin{aligned}
\int_0^1 \left(\int_r^1 f(s) s^{\beta} ds \right)^2 r^{2\alpha} r dr &= \int_0^1 \left(\int_r^1 f(s) s^{\frac{1+\beta}{2}} s^{\frac{-1+\beta}{2}} ds \right)^2 r^{2\alpha+1} dr \\
&\overset{(C.S)}{\leq} \int_0^1 \left(\int_r^1 f^2(s) s^{1+\beta} ds \right) \left(\int_r^1 s^{-1+\beta} ds \right) r^{2\alpha+1} dr \\
&\leq \int_0^1 \left(\int_r^1 f^2(s) s^{1+\beta} ds \right) \left(\int_r^{+\infty} s^{-1+\beta} ds \right) r^{2\alpha+1} dr \\
&\leq \frac{1}{-\beta} \int_0^1 \left(\int_r^1 f^2(s) s^{1+\beta} ds \right) r^{-1-\beta} dr
\end{aligned}$$

where we have used the inequality $\int_r^1 s^{-1+\beta} ds \leq \int_r^{+\infty} s^{-1+\beta} ds$ since $\beta < 0$. Next, by Fubini's theorem, $0 < r < s < 1$, we obtain,

$$\begin{aligned}
\frac{1}{-\beta} \int_0^1 \left(\int_r^1 f^2(s) s^{1+\beta} ds \right) r^{-1-\beta} dr &= \frac{1}{-\beta} \int_0^1 \left(f^2(s) s^{\beta+1} \int_0^s r^{-1-\beta} dr \right) ds \\
&= \frac{1}{\beta^2} \int_0^1 f^2(s) s ds
\end{aligned}$$

hence, (20) holds. □

Remark 1. The estimates given by Lemmas 1 and 2 are optimal in the sens that one can not, for example, expect better that $1/|\beta|$ in the inequalities (18) and (20), in particular in a critical case such as $\beta \to 0$. This is a consequence of optimality results of Hardy's inequalities.

4 CORNER SINGULARITIES VIA FOURIER SERIES DECOMPOSITION

Following the results in the literature, cf Grisvard (1986)[7], we can summarize that a solution u_ω of (1) admits near the origin the following decomposition (in regular / singular parts and written in polar coordinates) as follows:

$$u_\omega = u_{\omega,r} + u_{\omega,s},$$

such that

$$u_{\omega,r} \in H^2_{loc}(\Omega_\omega), \text{ and } u_{\omega,s}(r,\theta) = \sum_{-1<\Im m \zeta_k<0} r^{i\zeta_k} \psi_k(\theta).$$

where the ζ_k are roots of the transcendent equation with imaginary part in $]-1,0[$ and the ψ_k are C^∞ functions of θ. In this section, we will retrieve such a decomposition by the Fourier series method which allows us to extract the singularity systematically. Moreover, both regular and singular parts are given explicitly and some explicit estimates w.r.t the opening angle ω are given. Since singularities are caused by the geometry of the domain, it follows that they are found in the kernel of the harmonic operator, i.e., they are solutions to the homogeneous equation

$$\Delta u_{\omega,s} = 0 \text{ in } L^2(\Omega_\omega). \tag{21}$$

Let

$$u_{\omega,s}(r,\theta) = \sum_{k\geq 1} a_{k,\omega}(r) \sin \frac{k\pi}{\omega}\theta,$$

be the Fourier series of $u_{\omega,s}(r,\theta)$ in polar coordinates. Thus, one has, at least formally,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) u_{\omega,s}(r,\theta) = 0,$$

which implies, by putting the Fourier coefficients of $\Delta u_{\omega,s}$ all equal zero, that $a_{k,\omega}(r)$ is solution to the following ordinary differential equation, for all $k \geq 1$,

$$a''_{k,\omega}(r) + \frac{1}{r}a'_{k,\omega}(r) - \frac{\pi^2}{\omega^2}a_{k,\omega}(r) = 0, \tag{22}$$

whose general solution is given by

$$a_{k,\omega}(r) = \lambda_{k,\omega}r^{\frac{k\pi}{\omega}} + \mu_{k,\omega}r^{-\frac{k\pi}{\omega}}, \tag{23}$$

where $\lambda_{k,\omega}, \mu_{k,\omega}$ are constants that can be determined by the boundary condition and by imposing the regularity condition on the singular part $u_{\omega,s}$ to belong to the space $H^{\sigma}(\Omega_\omega)$ with $1 \leq \sigma < 2$. As far as we know, the power function in r, $(r,\theta) \mapsto r^{\alpha_k} \sin \frac{k\pi}{\omega}\theta$, α_k not integer, belongs to $H^{\sigma}(\Omega_\omega)$ as long as $\sigma < \alpha_k + 1$, henceforth we look for the Fourier coefficient of $u_{\omega,s}$ that satisfy

$$a_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta \in H^{\sigma}(\Omega_\omega),\ 1 \leq \sigma < 2.$$

In other words, the problem turns out to find non integer powers α_k in the r.h.s of (23) such that $0 < \alpha_k < 1$, which yields the unique possible power $\alpha_1 = \frac{\pi}{\omega}$ for all $\omega \in (\pi, 2\pi)$. It follows immediately $\mu_{k,\omega} = 0$ for all k and that $a_{k,\omega}(r)$ will be balanced into the regular part for all $k \geq 2$. More precisely, we conclude that the singular part of u_ω takes the following expression:

$$u_{\omega,s}(r,\theta) = a_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta = \lambda_{1,\omega}r^{\frac{\pi}{\omega}}\sin\frac{k\pi}{\omega}\theta$$

$\lambda_{1,\omega}$ is called the coefficient of singularity and it can be determined by the boundary condition on the solution $u_\omega = u_{\omega,r} + u_{\omega,s}$ after having given the expression of the regular part $u_{\omega,r}$.

5 | REGULARITY AND EXPLICIT ESTIMATES VIA FOURIER SERIES DECOMPOSITION

We will look for the regular part $u_{\omega,r}$ of u_ω as the particular solution of the problem (1) with the regularity property of being in $H^2(\Omega_\omega) \cap H^1_D(\Omega_\omega)$ for all $\omega \in (\pi, 2\pi)$. Thus, $u_{\omega,r}$ is solution to the non homogeneous equation $\Delta u_{\omega,r} = f_\omega$ in $L^2(\Omega_\omega)$ such that the global solution $u_\omega \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$. Observe that one does not need homogeneous Dirichlet boundary condition $u_{\omega,r} = 0$ on the curved boundary C_ω. We start by Fourier series decomposition as follows:

$$f_\omega(r,\theta) = \sum_{k\geq 1} c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta,\ c_{k,\omega}(r) = \frac{2}{\omega}\int_0^\omega f_\omega(r,\theta)\sin\frac{k\pi}{\omega}\theta d\theta, \tag{24}$$

$$u_{\omega,r}(r,\theta) = \sum_{k\geq 1} b_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta,\ b_{k,\omega}(r) = \frac{2}{\omega}\int_0^\omega u_\omega(r,\theta)\sin\frac{k\pi}{\omega}\theta\ d\theta. \tag{25}$$

Plugging the Fourier series in the non homogeneous equation $\Delta u_{\omega,r} = f_\omega$ in $L^2(\Omega_\omega)$, we look for $b_{k,\omega}$ by identifying all the Fourier coefficients in this equation written in polar coordinates. We obtain immediately that $b_{k,\omega}(r)$ is solution to the following ordinary differential equation, for all $k \geq 1$,

$$b''_{k,\omega}(r) + \frac{1}{r}b'_{k,\omega}(r) - \frac{\pi^2}{\omega^2}b_{k,\omega}(r) = c_{k,\omega}(r). \tag{26}$$

The general form of solution of this equation is:

$$b_{k,\omega}(r) = \frac{\omega}{2k\pi}\left(r^{\frac{k\pi}{\omega}}\int_a^r c_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}ds - r^{-\frac{k\pi}{\omega}}\int_b^r c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}ds\right) + \alpha_{k,\omega}r^{\frac{k\pi}{\omega}} + \beta_{k,\omega}r^{-\frac{k\pi}{\omega}} \tag{27}$$

where a, b are some constants in $(0,1)$ that may be determined together with the coefficients $\alpha_{k,\omega}$,$\beta_{k,\omega}$, either with the help of boundary conditions on $\partial\Omega_\omega$ and/or the expected regularity of $u_{\omega,r}$ in H^2. Since $\frac{1}{2} < \frac{\pi}{\omega} < 1$ then $r^{-\frac{k\pi}{\omega}}\sin\frac{k\pi}{\omega}\theta$ is never in $H^1_D(\Omega_\omega)$ for any $k \geq 1$, which implies that all the coefficients $\beta_{k,\omega}$ must be zero for all $k \geq 1$. In addition, $\alpha_{1,k} = 0$ in the expression of $b_{k,\omega}$ since $r^{\frac{\pi}{\omega}}$ is reserved in the Fourier coefficient of the singular part.

Remark 2. Integrals in the r.h.s of (27) and their derivatives w.r.t r until order 2 have all the same forms as those in fundamental lemmas 1 and 2 where the left powers α and right ones β all satisfy the condition $\alpha+\beta \geq -1$. Hence, the set of parameters a, b can already be refined according to the signs of β, i.e., missing integral's limit will be 0 if $\beta > 0$ and 1 if $\beta < 0$.

5.1 | First frequency term, $k = 1$ and determination of $\lambda_{1,\omega}$

According to Remark 2, and the fact that $\frac{1}{2} < \frac{\pi}{\omega} < 1$, all thes powers β in the two integrals in expression of $b_{1,\omega}(r)$ given by (27) have positive sign, hence one takes $a = b = 0$ and writes:

$$b_{1,\omega}(r) = \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds\right). \tag{28}$$

The Fourier series terms in the decomposition

$$u_\omega = u_{\omega,r} + u_{\omega,s} = \sum_{k\geq 1} u_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta = \sum_{k\geq 1} U_{k,\omega}(r,\theta),$$

satisfy, for each $k \geq 1$, the boundary value problem,

$$\begin{cases} \Delta U_{k,\omega}(r,\theta) = f_{k,\omega}(r) \text{ in } \Omega_\omega, \\ U_{k,\omega} = 0 \text{ on } \partial\Omega_\omega, \end{cases} \tag{29}$$

where $f_{k,\omega}(r,\theta) = c_{k,\omega}(r)\sin\dfrac{k\pi}{\omega}\theta$ is the k^{th} term of the Fourier series of f_ω, such that $f_\omega(r,\theta) = \sum_{k\geq 1} f_{k,\omega}(r,\theta)$. It follows that the first term in the Fourier series of the global solution has the following expression, (recalling that $a_{1,\omega}(r)$ is the first and unique non zero Fourier coefficient of the singular part $u_{\omega,s}$):

$$\begin{aligned} U_{1,\omega}(r,\theta) &= b_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta + a_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta \\ &= \frac{\omega}{2\pi}\left(r^{\frac{\pi}{\omega}}\int_0^r c_{k,\omega}(s)s^{1-\frac{\pi}{\omega}}ds - r^{-\frac{\pi}{\omega}}\int_0^r c_{k,\omega}(s)s^{1+\frac{\pi}{\omega}}ds\right)\sin\frac{\pi}{\omega}\theta + \lambda_{1,\omega}r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta. \end{aligned} \tag{30}$$

Applying the boundary conditions $U_{1,\omega} = 0$ at $r = 1$, we obtain the expression of $\lambda_{1,\omega}$:

$$\lambda_{1,\omega} = -\frac{\omega}{2\pi}\left(\int_0^1 c_{1,\omega}(s)s^{1-\frac{\pi}{\omega}}ds - \int_0^1 c_{1,\omega}(s)s^{1+\frac{\pi}{\omega}}ds\right). \tag{31}$$

Theorem 2. *Let $U_{1,\omega}(r,\theta)$ given by (30) the solution of (29) ($k = 1$) with r.h.s $f_{1,\omega}(r,\theta) = c_{1,\omega}(r)\sin\frac{\pi}{\omega}\theta$. There exists $C > 0$ uniform in ω, such that the following explicit estimate holds :*

$$\left|\lambda_{1,\omega}\right| + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^2(\Omega_\omega)} \leq \frac{C}{\sqrt{1-\frac{\pi}{\omega}}}\left\|f_{1,\omega}\right\|_{L^2(\Omega_\omega)}. \tag{32}$$

Moreover, this estimate is sharp, i.e, there exists $f_{1,\omega}$ such that $\|f_{1,\omega}\|_{L^2(\Omega_\omega)} = 1$ and

$$\left|\lambda_{1,\omega}\right| + \left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^2(\Omega_\omega)} = O\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}}\right), \text{ as } \omega \to \pi. \tag{33}$$

Proof. **Proof of estimate (32)**: we use the definition of $H_D^2(\Omega_\omega)$ norm defined in Subsection 3.1 and according to Lemma 1 the notation

$$I_k^{\alpha,\beta}(r) = r^\alpha\int_0^r c_{k,\omega}(s)s^\beta ds,\ k \geq 1,$$

then

$$\left(I_k^{\alpha,\beta}\right)'(r) = \alpha r^{\alpha-1}\int_0^r c_{k,\omega}(s)s^\beta ds + r^{\alpha+\beta}c_{k,\omega}(r) = \alpha I_k^{\alpha-1,\beta}(r) + r^{\alpha+\beta}c_{k,\omega}(r).$$

Proof of (32): Using the previous notation with $k = 1$and expression (28) of $b_{1,\omega}$, one has after some computation and simplification:

$$b_{1,\omega}(r) = \frac{\omega}{2\pi}\left(I_1^{\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) - I_1^{-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{34}$$

$$\frac{b_{1,\omega}(r)}{r} = \frac{\omega}{2\pi}\left(I_1^{-1+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) - I_1^{-1-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{35}$$

$$\frac{b_{1,\omega}(r)}{r^2} = \frac{\omega}{2\pi}\left(I_1^{-2+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) - I_1^{-2-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{36}$$

$$b'_{1,\omega}(r) = \frac{1}{2}\left(I_1^{-1+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) + I_1^{-1-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{37}$$

$$\frac{b'_{1,\omega}(r)}{r} = \frac{1}{2}\left(I_1^{-2+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) + I_1^{-2-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{38}$$

$$b''_{1,\omega}(r) = c_{1,\omega}(r) - \frac{1}{2}\left(\left(1-\frac{\pi}{\omega}\right) I_1^{-2+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) + \left(1+\frac{\pi}{\omega}\right) I_1^{-2-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right) \tag{39}$$

Now, observe that expressions in the r.h.s of (34),...,(39), contain all linear combinations, with uniform bounded coefficients w.r.t ω, of expressions $I_1^{\alpha,\beta}$, exactly like those in fundamental Lemma 1, all with $\beta > 0$ and $\alpha + \beta \geq -1$. Henceforth, one has by Lemma 1:

$$\|b_{1,\omega}\|_{L^2(rdr)} \leq \frac{\omega}{4\pi\sqrt{2}}\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}} + \frac{1}{\sqrt{1+\frac{\pi}{\omega}}}\right)\|c_{1,\omega}\|_{L^2(rdr)} \leq \frac{C_1}{\sqrt{1-\frac{\pi}{\omega}}}\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{40}$$

$$\left\|b'_{1,\omega}\right\|_{L^2(rdr)} \leq \frac{1}{4}\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}} + \frac{1}{\sqrt{1+\frac{\pi}{\omega}}}\right)\|c_{1,\omega}\|_{L^2(rdr)} \leq \frac{C_2}{\sqrt{1-\frac{\pi}{\omega}}}\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{41}$$

$$\left\|\frac{b_{1,\omega}}{r}\right\|_{L^2(rdr)} \leq \frac{\omega}{4\pi}\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}} + \frac{1}{\sqrt{1+\frac{\pi}{\omega}}}\right)\|c_{1,\omega}\|_{L^2(rdr)} \leq \frac{C_3}{\sqrt{1-\frac{\pi}{\omega}}}\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{42}$$

On the other hand,

$$\frac{b'_{1,\omega}(r)}{r} - \frac{b_{1,\omega}(r)}{r^2} = \frac{1}{2}\left(\left(1-\frac{\omega}{\pi}\right) I_1^{-2+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) + \left(1+\frac{\omega}{\pi}\right) I_1^{-2-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right),$$

$$\frac{b'_{1,\omega}(r)}{r} - \left(\frac{\pi}{\omega}\right)^2\frac{b_{1,\omega}(r)}{r^2} = \frac{1}{2}\left(\left(1-\frac{\pi}{\omega}\right) I_1^{-2+\frac{\pi}{\omega},1-\frac{\pi}{\omega}}(r) + \left(1+\frac{\pi}{\omega}\right) I_1^{-2-\frac{\pi}{\omega},1+\frac{\pi}{\omega}}(r)\right),$$

implies

$$\left\|\frac{b'_{1,\omega}}{r} - \frac{b_{1,\omega}}{r^2}\right\|_{L^2(rdr)} \leq \frac{\omega}{\pi}\|c_{1,\omega}\|_{L^2(rdr)} \leq C_4\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{43}$$

$$\left\|\frac{b'_{1,\omega}}{r} - \left(\frac{\pi}{\omega}\right)^2\frac{b_{1,\omega}}{r^2}\right\|_{L^2(rdr)} \leq \frac{1}{2}\left(\frac{\omega}{\pi} + \frac{\pi}{\omega}\right)\|c_{1,\omega}\|_{L^2(rdr)} \leq C_5\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{44}$$

$$\left\|b''_{1,\omega}\right\|_{L^2(rdr)} \leq 2\|c_{1,\omega}\|_{L^2(rdr)} \leq C_6\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{45}$$

where $C_j > 0, j = 1, 2, ..., 6$, are constants all independent of ω. Thus, since $b_{1,\omega}\sin\frac{\pi}{\omega}\theta$ is a single term of its Fourier series, by definition (16) of the norm $H^2_D(\Omega_\omega)$, see (13),(14) and (15), we obtain

$$\left\|b_{1,\omega}\sin\frac{\pi}{\omega}\theta\right\|_{H^2(\Omega_\omega)} \leq \frac{C_6}{\sqrt{1-\frac{\pi}{\omega}}}\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{46}$$

where $C > 0$ is uniformly bounded in $\omega \in (\pi, 2\pi)$.

On an other hand, it is easy to check that $\left|\lambda_{1,\omega}\right|$can be estimated using the Cauchy-Schwartz inequality to obtain

$$\left|\lambda_{1,\omega}\right| \leq \frac{C_7}{\sqrt{1-\frac{\pi}{\omega}}}\|f_{1,\omega}\|_{L^2(\Omega_\omega)}, \tag{47}$$

where $C_7 > 0$ is uniformly bounded in $\omega \in (\pi, 2\pi)$. Finally, Inequality (32) holds with a constant $C > 0$ independent of ω.

Proof of sharpness of estimate (32): In fact, this is a consequence of optimality results of Hardy's inequalities as mentioned in Remark 1. More precisely, since $-1 < -\frac{\pi}{\omega}$, one can find, for example, a r.h.s such as

$$f_{1,\omega}(r,\theta) = \frac{2}{\omega}\sqrt{\omega-\pi}\, r^{-\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right),$$

$$\|f_{1,\omega}\|_{L^2(\Omega_\omega)} = 1,\ \forall \omega \in (\pi, 2\pi),$$

$$c_{1,\omega}(r) = \frac{2}{\omega}\sqrt{\omega-\pi}\, r^{-\frac{\pi}{\omega}},$$

which gives after computation of $b_{1,\omega}(r,\theta)$ from its expression given by (28),

$$b_{1,\omega}(r) = \frac{r^{2-\frac{\pi}{\omega}}}{2\sqrt{\omega-\pi}},$$

$$\left\| b_{1,\omega}(r)\sin\left(\frac{\pi}{\omega}\theta\right)\right\|_{L^2(\Omega_\omega)} = \frac{\omega}{4\sqrt{(\pi-3\omega)(\pi-\omega)}} = O\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}}\right) \to +\infty \text{ as } \omega \to \pi,$$

$$\left|\lambda_{1,\omega}\right| = \frac{1}{2\sqrt{\omega-\pi}} = O\left(\frac{1}{\sqrt{1-\frac{\pi}{\omega}}}\right) \to +\infty \text{ as } \omega \to \pi.$$

The proof of the theorem is ended. □

5.2 | Regular frequency terms, $k \geq 2$

We first need the following Lemma which gives, in the case of a sector, the uniformity w.r.t ω of the elliptic estimate "second fundamental inequality" , cf. (Stylianou (2010), Corollary 2.3.6 p.31)[16], and which can be reformulated in the case of a planar sector Ω_ω as follows:

Lemma 3. *Let Ω_ω planar sector defined as in Section 2, $\omega \in (0, 2\pi)$. Then, there exists $C > 0$ constant independent of ω, such that for all $u \in H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$,*

$$\|u\|_{H^2(\Omega_\omega)} \leq C\|\Delta u\|_{L^2(\Omega_\omega)}. \tag{48}$$

Proof. In fact, it comes from Poincare's inequality on the one hand, cf. Tami & Tlemcani (2021): for all $u \in H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$, one has

$$\|u\|_{H^2(\Omega_\omega)} \leq \sqrt{1+(1+\omega)^2}\,|||\nabla^2 u|||_{L^2(\Omega_\omega)},$$

which yields equivalence, with uniformly bounded constant w.r.t ω, between the norm and semi-norm H^2 in the space $H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$, and, on the other hand, the "second fundamental inequality" , cf. (Stylianou (2010), Corollary 2.3.6 p.31), which can be reformulated in the case of a planar sector Ω_ω as follows: Using the Green's formula in $H^3(\Omega_\omega) \cap H_0^1(\Omega_\omega)$, as in (Stylianou (2010), p.29),

$$\int_{\Omega_\omega} (\Delta u)^2\, d\Omega_\omega = \int_{\partial\Omega_\omega} \kappa(m)\, (\partial_n u)^2\, dm + |||\nabla^2 u|||^2_{L^2(\Omega_\omega)} \geq |||\nabla^2 u|||^2_{L^2(\Omega_\omega)} \tag{49}$$

where ∂_n represents the normal derivative operator outward to $\partial\Omega_\omega$, where in our case,

$$\kappa(m) = \begin{cases} 0 & \text{if } m \in \Gamma_0 \cup \Gamma_\omega \\ 1 & \text{if } m \in C_\omega \end{cases}$$

designates the curvature of $\partial\Omega_\omega$ and which is essentially positive in the case of the planar sector Ω_ω, no matter if it is non-convex or convex. Hence, and by same arguments as in Stylianou (2010), based on the density of $H^3(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ in $H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$, Inequality (48) follows for all $u \in H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$. □

Following Remark 2, and the fact that $1 \leq \frac{k}{2} < \frac{k\pi}{\omega} < k$ for $k \geq 2$, the powers β in the two integrals in expression of $b_{k,\omega}(r)$ given by (27), to know $1 - \frac{k\pi}{\omega} < 0$ and $1 + \frac{k\pi}{\omega} > 0$, have respectively negative and positive sign for all $k \geq 2$, hence one takes the integrals limits $a = 1, b = 0$ in (27) and writes:

$$b_{k,\omega}(r) = \frac{\omega}{2k\pi}\left(r^{\frac{k\pi}{\omega}}\int_1^r c_{k,\omega}(s)s^{1-\frac{k\pi}{\omega}}ds - r^{-\frac{k\pi}{\omega}}\int_0^r c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}ds \; + r^{\frac{k\pi}{\omega}}\int_0^1 c_{k,\omega}(s)s^{1+\frac{k\pi}{\omega}}ds,\right) \tag{50}$$

where the last term in the r.h.s of (50) is added in order to satisfy the homogeneous Dirichlet boundary condition at $r = 1$.

Theorem 3. *Let $U_{k,\omega}(r,\theta) = b_{k,\omega}(r)\sin\left(\frac{k\pi}{\omega}\theta\right)$, $k \geq 2$, the solution of (1) with r.h.s $f_{k,\omega}(r,\theta) = c_{k,\omega}(r)\sin\frac{k\pi}{\omega}\theta$. Then, $U_{k,\omega} \in H^2(\Omega_\omega)\cap H^1_0(\Omega_\omega)$, and there exists $C > 0$ independent of $\omega \in (\pi, 2\pi)$ and $k \geq 2$, , such that:*

$$\sum_{k\geq 2}\|U_{k,\omega}\|_{H^2(\Omega_\omega)} \leq C\sum_{k\geq 2}\|f_{k,\omega}\|_{L^2(\Omega_\omega)}. \tag{51}$$

Proof. The proof is similar to the case $k = 1$. In addition to the notation $I_k^{\alpha,\beta}$ in the previous proof, and according to Lemma 20 we add the notation

$$J_k^{\alpha,\beta}(r) = r^\alpha\int_1^r c_{k,\omega}(s)s^\beta ds,\ k \geq 2,$$

then

$$\left(J_k^{\alpha,\beta}\right)'(r) = \alpha r^{\alpha-1}\int_1^r c_{k,\omega}(s)s^\beta ds + r^{\alpha+\beta}c_{k,\omega}(r) = \alpha J_k^{\alpha-1,\beta}(r) + r^{\alpha+\beta}c_{k,\omega}(r).$$

Moreover, let

$$H_k^{\alpha,\beta}(r) := r^\alpha\int_0^1 c_{k,\omega}(s)s^\beta ds.$$

By computation and simplification, $\forall k \geq 2$:

$$b_{k,\omega}(r) = \frac{\omega}{2k\pi}\left(J_k^{\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) - I_k^{-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) + H_k^{\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r)\right), \tag{52}$$

$$\frac{b_{k,\omega}(r)}{r} = \frac{\omega}{2k\pi}\left(J_k^{-1+\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) - I_k^{-1-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) + H_k^{\frac{k\pi}{\omega}-1,1+\frac{k\pi}{\omega}}(r)\right), \tag{53}$$

$$\frac{b_{k,\omega}(r)}{r^2} = \frac{\omega}{2k\pi}\left(J_k^{-2+\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) - I_k^{-2-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) + H_k^{\frac{k\pi}{\omega}-2,1+\frac{k\pi}{\omega}}(r)\right), \tag{54}$$

$$b'_{k,\omega}(r) = \frac{1}{2}\left(J_k^{-1+\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) + I_k^{-1-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) + H_k^{\frac{k\pi}{\omega}-1,1+\frac{k\pi}{\omega}}(r)\right), \tag{55}$$

$$\frac{b'_{k,\omega}(r)}{r} = \frac{1}{2}\left(J_k^{-2+\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) + I_k^{-2-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) + H_k^{\frac{k\pi}{\omega}-2,1+\frac{k\pi}{\omega}}(r)\right), \tag{56}$$

$$b''_{k,\omega}(r) = \frac{1}{2}\left(\left(\frac{k\pi}{\omega}-1\right)J_k^{-2+\frac{k\pi}{\omega},1-\frac{k\pi}{\omega}}(r) - \left(1+\frac{k\pi}{\omega}\right)I_k^{-2-\frac{k\pi}{\omega},1+\frac{k\pi}{\omega}}(r) \; + \left(\frac{k\pi}{\omega}-1\right)H_k^{\frac{k\pi}{\omega}-2,1+\frac{k\pi}{\omega}}(r)\right) + c_{k,\omega}(r). \tag{57}$$

Henceforth, fundamental Lemmas, 1 applied to $I_k^{\alpha,\beta}$ and 2 applied to $J_k^{\alpha,\beta}$, and the Cauchy-Schwartz inequality applied to the terms $H_k^{\alpha,\beta}$ which gives a constant $C_0(\omega,k) > 0$,

$$\left\|H_k^{\alpha,\beta}\right\|_{L^2(rdr)} \leq C_0(\omega,k)\|f_{k,\omega}\|_{L^2(rdr)}, \tag{58}$$

yield,

$$\|b_{k,\omega}\|_{L^2(rdr)} \leq C_1(\omega,k)\|f_{k,\omega}\|_{L^2(\Omega_\omega)}, \tag{59}$$

$$\left\|b'_{1,\omega}\right\|_{L^2(rdr)} \leq C_2(\omega,k)\|f_{k,\omega}\|_{L^2(\Omega_\omega)}, \tag{60}$$

$$\left\|\frac{b_{1,\omega}}{r}\right\|_{L^2(rdr)} \leq C_3(\omega,k)\|f_{k,\omega}\|_{L^2(\Omega_\omega)}, \tag{61}$$

$$\left\| \frac{b'_{k,\omega}}{r} - \frac{b_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \leq C_4(\omega, k) \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)}, \tag{62}$$

$$\left\| \frac{b'_{k,\omega}}{r} - \left(\frac{k\pi}{\omega}\right)^2 \frac{b_{k,\omega}}{r^2} \right\|_{L^2(rdr)} \leq C_5(\omega, k) \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)}, \tag{63}$$

$$\left\| b''_{k,\omega} \right\|_{L^2(rdr)} \leq C_6(\omega, k) \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)}, \tag{64}$$

where $C_j(\omega, k) > 0, j = 0, 1, 2, ..., 6$, are constants depending on ω and k. Thus, since $b_{k,\omega} \sin \dfrac{k\pi}{\omega}\theta$ is a single term of its Fourier series, by definition (16) of the norm $H^2_D(\Omega_\omega)$, see (13),(14) and (15), we obtain

$$\left\| U_{k,\omega} \right\|_{H^2(\Omega_\omega)} = \left\| b_{k,\omega} \sin \frac{k\pi}{\omega}\theta \right\|_{H^2(\Omega_\omega)} \leq C(\omega, k) \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)}. \tag{65}$$

As a result, $U_{k,\omega} \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$, then Lemma 3 implies that there exists $C > 0$ independent of $\omega \in (\pi, 2\pi)$ and $k \geq 2$, , such that:

$$\left\| U_{k,\omega} \right\|_{H^2(\Omega_\omega)} \leq C \left\| \Delta U_{k,\omega} \right\|_{L^2(\Omega_\omega)} = C \left\| f_{k,\omega} \right\|_{L^2(\Omega_\omega)},$$

since $\Delta U_{k,\omega} = f_{k,\omega}$. Finally, by taking the sum over $k \geq 2$, one obtains the uniform estimate (51), which ends the proof. □

6 | PROOF OF THE MAIN RESULTS

In this section, we end the proof of the main Theorem 1 and its Corollary 1 stated in Section 2.

Proof of Theorem 1. The proof now is a direct consequence of theorems 2 and 3 and the former Fourier series analysis. In fact, we write the Fourier series expansion of f_ω separating the singular frequency $k = 1$ from the regular ones $k \geq 2$, as follows: $f_\omega = f^I_\omega + f^{II}_\omega$ where

$$f^I_\omega(r, \theta) = c_{1,\omega}(r) \sin \frac{\pi}{\omega}\theta, \text{ and } f^{II}_\omega(r, \theta) = \sum_{k\geq 2} c_{k,\omega}(r) \sin \frac{k\pi}{\omega}\theta. \tag{66}$$

By Theorem 2, $\lambda_\omega r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta + u^I_{\omega,r} \in H^1_0(\Omega_\omega)$ represents the solution of Problem (1) with r.h.s f^I_ω corresponding to the singular frequency $k = 1$. Thus, we put $\lambda_\omega = \lambda_{1,\omega}$ as given by (31) and $u^I_\omega(r, \theta) = b_{1,\omega}(r) \sin \dfrac{\pi}{\omega}\theta$ as given by (28), and we have $u^I_{\omega,r} \in H^2(\Omega_\omega)$.

Next, by Theorem 3, $u^{II}_{\omega,r}$ represents the solution of Problem 1 with r.h.s f^{II}_ω corresponding to a superposition of all regular frequency $k \geq 2$. We obtain the expression of $u^{II}_{\omega,r}$ as a Fourier series

$$\sum_{k\geq 2} b_{k,\omega}(r) \sin \frac{k\pi}{\omega}\theta = \sum_{k\geq 2} U_{k,\omega}(r, \theta),$$

where $U_{k,\omega}(r, \theta) = b_{k,\omega}(r) \sin \dfrac{k\pi}{\omega}\theta$ and $b_{k,\omega}(r)$ is given by (50). Thus, by Theorem 3-Estimate (51), $u^{II}_{\omega,r} \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$.

On the other hand, as a power function of r, the singular part $(r, \theta) \mapsto r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$, $\frac{\pi}{\omega}$ not integer, belongs to the Sobolev space $H^{1+\sigma}(\Omega_\omega)$ for all $\sigma < \frac{\pi}{\omega}$.

Henceforth, $u_\omega \in H^{1+\sigma}(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ and the decomposition (3) follows with explicit expressions as given by (5) and (6).

Finally, the sharp Estimate (7) follows directly from theorems 2 - Inequality (32) and Theorem 3 - Inequality (51). The proof of the theorem is achieved. □

Proof of Corollary 1. This is a consequence of the main theorem 1 and Lemma 3. In fact, $u_\omega \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ if and only if the singular part $\lambda_\omega r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$ vanishes on Ω_ω, i.e $\lambda_\omega = 0$. We have, with the help of the definition of the Fourier coefficient

$c_{1,\omega}(r) = \frac{2}{\omega}\int_0^\omega f_\omega(r,\theta)\sin\frac{\pi}{\omega}\theta d\theta$ and the definition of ξ_ω given by (8),

$$\begin{aligned}\lambda_\omega &= -\frac{\omega}{2}\left(\int_0^1 c_{1,\omega}(r)\left(\frac{r^{-\frac{\pi}{\omega}} - r^{\frac{\pi}{\omega}}}{\pi}\right) rdr\right)\\ &= \int_0^1\int_0^\omega f_\omega(r,\theta)\left(\frac{r^{\frac{\pi}{\omega}} - r^{-\frac{\pi}{\omega}}}{\pi}\right)\sin\frac{\pi}{\omega}\theta\ rdrd\theta\\ &= (f_\omega, \xi_\omega)_\omega .\end{aligned}$$

Hence, $u_\omega \in H^2(\Omega_\omega)\cap H_0^1(\Omega_\omega)$ if and only if $(f_\omega, \xi_\omega)_\omega = 0$. Assume now that this condition is satisfied, then $\lambda_\omega = 0$ which implies that the decomposition becomes $u_\omega = u^I_{\omega,r} + u^{II}_{\omega,r}$, where $u^I_{\omega,r}$ is the unique $H^2(\Omega_\omega)\cap H_0^1(\Omega_\omega)$ solution to problem

$$\begin{cases}\Delta u^I_{\omega,r} = f^I_\omega & \text{in } \Omega_\omega,\\ u^I_{\omega,r} = 0 & \text{on } \partial\Omega_\omega,\end{cases}$$

with r.h.s f^I_ω defined by (66). Therefore, by Lemma 3, there exists $C_1 > 0$ constant independent of ω, such that

$$\|u^I_\omega\|_{H^2(\Omega_\omega)} \le C_1\|\Delta u^I_\omega\|_{L^2(\Omega_\omega)} = C_1\|f^I_\omega\|_{L^2(\Omega_\omega)}. \tag{67}$$

On the other hand, by Inequality 7, one has $\|u^{II}_{\omega,r}\|_{H^2(\Omega_\omega)} \le C_2\|f^{II}_\omega\|_{L^2(\Omega_\omega)}$,where $C_2 > 0$ constant independent of ω, henceforth:

$$\begin{aligned}\|u_\omega\|_{H^2(\Omega_\omega)} &= \|u^I_{\omega,r} + u^{II}_{\omega,r}\|_{H^2(\Omega_\omega)},\\ &\le \|u^I_{\omega,r}\|_{H^2(\Omega_\omega)} + \|u^{II}_{\omega,r}\|_{H^2(\Omega_\omega)},\\ &\le C_1\|f^I_\omega\|_{L^2(\Omega_\omega)} + C_2\|f^{II}_\omega\|_{L^2(\Omega_\omega)} \le C\|f_\omega\|_{L^2(\Omega_\omega)},\end{aligned}$$

with $C = C_1 + C_2 > 0$, constant independent of ω. The proof of the Corollary is ended. □

7 | CONCLUSION

Throughout this paper, we have given explicit extraction formulas via Fourier analysis of the coefficients of singularity and regular part of the solutions of a family of Poisson equations with Dirichlet boundary conditions on a family of open non-convex planar sectors. We have shown that explicit and sharp estimates can be obtained by highlighting the decomposition of the solution into three parts whose behavior in the vicinity of the critical angle ω is as follows: a stable regular part in the norm H^2, an unstable regular part in the norm H^2 and an unbounded coefficient of singularity in the vicinity of π. However and fortunately, the global solution remains stable in the H^1 norm from standard uniform estimates of the weak variational solution. This problem is actually quite similar to that of Babuška, cf.[11], when additional regularity on the source term f_ω is assumed at the origin. In fact, in the case of convex corners, and L^2r.h.s f_ω uniformly bounded in ω, a full answer to this problem was given in the case of both harmonic and biharmonic problem, cf.[1,2,18]. Thus, the question of existence of stable H^2 decomposition near a non-convex corner still an open problem. On the other hand, it was observed via an orthogonality criteria, see the corollary of the main result, that in the absence of the first singular frequency $k = 1$ in the Fourier series of f_ω, one retrieves uniform H^2 estimates and the problem turns out to be similar to the convex case. Finally, possible extension of the results herein are envisaged for boundary value problems with general (mixed) boundary conditions. This will be of great interest in the case of hyperbolic (wave equation), cf.[4], and/or parabolic (heat equation) problems where the time variable adds a new drawback in the analysis.

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