Linear growth of spherically symmetric solutions to Navier-Stokes equations in $\mathbb{R}^N$ with degenerate viscosity and free boundary

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Abstract: This paper is concerned with a class of spherically symmetric analytical solutions to the isentropic compressible Navier-Stokes equations with physical vacuum free boundary in $\mathbb{R}^N$, when the viscosity coefficients are proportional to the pressure function (see (1.2)-(1.4)). It was shown that the free boundary will grow linearly in time which is consistent with the linear growth properties of inviscid fluids (Euler equations). We derived a second order nonlinear ODE of the free boundary $r = a(t)$, and tracked the profile of $a(t)$ by studying directly the intrinsic structure of the ODE, instead of the usual energy methods used in the previous literature. In particular, these results can be applied to the Navier-Stokes equations with constant viscosity and the Euler equations.

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1 Introduction

The evolving boundary of a viscous fluid can be modeled by the following compressible Navier-Stokes free boundary problem:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } \tilde{\Omega}(t), \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(p(\rho)\right) - \operatorname{div}\Psi = 0 & \text{in } \tilde{\Omega}(t), \\ \rho > 0 & \text{in } \tilde{\Omega}(t), \\ \rho = 0 & \text{on } \Gamma\left(t\right) = \partial\tilde{\Omega}(t), \\ (\rho,\mathbf{u}) = (\rho_0,\mathbf{u}_0) & \text{on } \tilde{\Omega} := \Omega(0). \end{cases} \tag{1.1}$$

Here ρ, $\mathbf{u} = (u_1, u_2, ..., u_N) \in \mathbb{R}^N$ $(N \geq 2)$ and $p = p(\rho)$ denote, respectively, the density, the velocity field and pressure of the fluid, which are functions of the space and time variable $(\mathbf{x}, t) \in \mathbb{R}^N \times [0, \infty)$; $\tilde{\Omega}(t) \subset \mathbb{R}^N$ and $\Gamma\left(t\right)$ represent, respectively, the changing volume occupied by a fluid and the moving interface at time t, equation $(1.1)_4$ is called the vacuum free boundary condition (or continuous density condition). Equation (1.1) can be

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used to describe the boundary expansion of gaseous stars, liquid flow in pipes, atmospheric flow, etc. For the polytropic gases, the pressure satisfies the γ-law assumption

$$p(\rho) = K\rho^{\gamma}, \ \gamma > 1, \tag{1.2}$$

where $K > 0$ is a fixed constant and γ is the adiabatic exponent. The different values of γ imply different physical significance [2], for example, $\gamma = 5/3$, $\gamma = 7/5$ and $\gamma = 1$ correspond to a monatomic gas, a diatomic gas, and isothermal gas, respectively. In this paper, the viscosity tensor Ψ in $(1.1)_2$ is assumed to be of the following form:

$$\Psi = \lambda_1(\rho)\nabla\mathbf{u} + \lambda_2(\rho)\nabla\mathbf{u}^T + \lambda_3(\rho)\operatorname{div}\mathbf{u}\, I_N, \tag{1.3}$$

where I_N is the $N \times N$ identity matrix, and for simplicity, we set the viscosity coefficients

$$\lambda_i(\rho) = k_i\rho^{\gamma}, \ k_i > 0, \quad i = 1, 2, 3, \tag{1.4}$$

where k_i $(i = 1, 2, 3)$ are three constants. Indeed, as in [11] by Guo and Xin, the viscosity tensor can usually be given by the following form

$$\widetilde{\Psi} = \mu_1(\rho)\frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2} + \mu_2(\rho)\operatorname{div}\mathbf{u}\, I_2, \tag{1.5}$$

$$\mu_1(\rho) > 0, \quad \mu_1(\rho) + N\mu_2(\rho) \geq 0, \tag{1.6}$$

where μ_1 and μ_2 are the Lamé viscosity coefficients, and the inequality (1.6) is derived from physical constraints. Therefore, the assumptions of the viscosity coefficients in (1.3)-(1.4) falls within the scope of hypothesis (1.5)-(1.6), and they are physically meaningful.

Due to its importance and challenge in physics and mathematics, the vacuum free boundary problems are widely studied and many important developments on the well-posedness theory of weak or strong solutions for both inviscid and viscous flows have been established in recent years. The difficulty of the problem lies in the degeneration of the boundary, and the usual method of hyperbolic equations cannot be applied directly. The local well-posedness was only obtained recently for compressible inviscid flows (cf. [3, 4, 15]), and for compressible viscous flows (cf. [8, 14]). For the global existence results to the vacuum free boundary problem in multidimensional space, most of which are concerned with spherically symmetric solutions (cf. [11, 12, 22, 23, 37]) or affine ones (cf. [27, 29]). In general, it is still a challenging problem to obtain the global existence to vacuum free boundary problem of the system (1.1), (1.5) and (1.6) in $\mathbb{R}^N$ $(N \geq 2)$.

On the other hand, the construction of exact solutions is a very important part in mathematical physics, since that it can further understand the nonlinear behaviors of the system (see [35]). First, for non-rotational flows with $k_1 = k_2 = k_3 = 0$, the radially symmetric solutions or exact solutions for the Euler equations were established in [19, 28] and references therein, while the blowup of radial solutions to the compressible Euler equations with/without damping on some fixed bounded domains are given in [6]. For rotational analytical solutions to the Euler equations, one can refer to [35, 38] and references therein. Taking the self-gravity force into consideration, Makino [24] proved the blowup solutions to the 3D Euler-Poisson equations for $\gamma = 4/3$, the result was extended by Deng etal. [5] and Yuen [32] to the case $\gamma = (2N-2)/N$ $(N \geq 3)$ and the case allowing viscosity or frictional damping, respectively. For more results on the Euler equations and related equations, one may refer to [1, 10, 13, 20, 21, 31, 36] and references therein.

When $k_1 = \gamma > 1$, $k_2 = 0$ and $k_3 = \gamma - 1 > 0$, Guo and Xin [11] constructed spherically symmetric analytical solutions to the Navier-Stokes equations (1.1) in $\mathbb{R}^N$ $(N \geq 2)$ with both vacuum free boundary and stress free conditions, and showed that the large time expanding behaviors at an algebraic rate of the free boundary. Two related spherically symmetric results for the free boundary problem of damped Euler equations in $\mathbb{R}^3$ and Navier-Stokes equations with density-dependent viscosity coefficients that $k_1 = k_2 > 0$ and $k_3 = \rho^\beta$ $(1 < \beta < \gamma)$ in $\mathbb{R}^2$ were established in [7, 16]. Motivated by the results in [7, 11, 16], we choose $r = a(t)$ as the free boundary and construct a class of spherically symmetric analytical solutions for the Navier-Stokes equations (1.1) in $\mathbb{R}^N$ $(N \geq 2)$ space with a general viscosity coefficients satisfying (1.4). These spherically symmetric and linear-growth solutions will provide reference examples for numerical computation.

In the following sections, we will first reformulate the original free boundary problem in spherical coordinates and state the main results in Section 2, then prove the main theorems in Subsections 3.1-3.2. For the reader's convenience, we give an explicit expression for the viscous terms divΨ in spherical coordinates in appendix.

2 Formulation in spherical coordinates and main results

In spherical coordinates, the region surrounded by vacuum can be described as

$$\Omega(t) := \left\{(r,t) \in \mathbb{R}^+ \times [0,\infty) | 0 \leq r \leq a(t), t \geq 0\right\}, \tag{2.1}$$

where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, the center of the region (0,0) is fixed and the free boundary $r = a(t)$ satisfies

$$\frac{d}{dt}a(t)) = u\left(a(t), t\right) \text{ with } a(0) = a_0 > 0, \tag{2.2}$$

where the positive and bounded constant a_0 represents the initial location of the free boundary. The density and velocity field of the fluid have the following form

$$\rho(\mathbf{x}, t) = \rho(r, t), \ \mathbf{u}(\mathbf{x}, t) = u(r,t)\mathbf{e}_r, \tag{2.3}$$

where $\mathbf{e}_r = \frac{\mathbf{x}}{r} = \frac{(x_1, x_2, \ldots, x_N)}{r}$ is the outward unit vector along the radial direction, and the dissipative term divΨ in the equation $(1.1)_2$ in the spherical coordinates reads

$$\begin{aligned}
&\operatorname{div}\Psi = \operatorname{div}\left(\lambda_1(\rho)\nabla\mathbf{u}\right) + \operatorname{div}\left(\lambda_2(\rho)\nabla\mathbf{u}^T\right) + \nabla\left(\lambda_3(\rho)\operatorname{div}\mathbf{u}\right) \\
&= \left[\left(\lambda_1(\rho) + \lambda_2(\rho) + \lambda_3(\rho)\right)\left(u_r + \frac{(N-1)u}{r}\right)_r \right. \\
&\quad \left. + \left(\lambda_1(\rho) + \lambda_2(\rho) + \lambda_3(\rho)\right)_r u_r + (\lambda_3)_r \frac{(N-1)u}{r}\right]\mathbf{e}_r.
\end{aligned} \tag{2.4}$$

(The derivation of Eq (2.4) is shown in the appendix.) Thus, the equations (1.1)-(1.4) can be rewritten in spherical coordinates as follows:

$$\begin{cases}
\rho_t + (\rho u)_r + (N-1)\frac{\rho u}{r} = 0, \\
\rho\left[u_t + uu_r\right] + p_r - \left[(\lambda_1 + \lambda_2 + \lambda_3)\left(u_r + \frac{(N-1)u}{r}\right)_r \right. \\
\quad \left. + (\lambda_1 + \lambda_2 + \lambda_3)_r u_r + (\lambda_3)_r \frac{(N-1)u}{r}\right] = 0,
\end{cases} \tag{2.5}$$

or equivalently as

$$\begin{cases} \rho_t + (\rho u)_r + (N-1)\frac{\rho u}{r} = 0, \\ \rho\left[u_t + uu_r\right] + K\gamma\rho^{\gamma-1}\rho_r - \Big[(k_1+k_2+k_3)\rho^{\gamma}\left(u_r + \frac{(N-1)u}{r}\right)_r \\ \quad + (k_1+k_2+k_3)\gamma\rho^{\gamma-1}\rho_r u_r + k_3\gamma\rho^{\gamma-1}\rho_r\frac{(N-1)u}{r}\Big] = 0, \end{cases} \tag{2.6}$$

with the initial conditions

$$(\rho,u)(r,t)|_{t=0} = (\rho_0,u_0)(r), \quad \text{on} \quad (0,a_0), \tag{2.7}$$

and the Dirichlet boundary condition on the center of the region and the vacuum boundary condition on the free boundary

$$u(r,t)|_{r=0} = 0, \quad \rho(a(t),t) = 0. \tag{2.8}$$

In the following, we will use $C > 0$ to denote the generic constant which only depend on γ, k_i $(i = 1,2,3)$ and the initial data such as a_0, a_1 and H_0 appearing in Theorem 2.1, but are independent of t, and they may change from one line to another. The labels "$x \lesssim y$" and "$x \sim y$" represent "$x \le Cy$" and $C_1 y \le x \le C_2 y$, respectively. The main results read:

Theorem 2.1 *The problem (2.6)-(2.8) has a global solution of the form*

$$\rho(r,t) = \frac{\left[\frac{\widetilde{k}(\gamma-1)}{2}\left(1 - \frac{r^2}{a^2(t)}\right)\right]^{\frac{1}{\gamma-1}}}{a^N(t)}, \quad u(r,t) = \frac{a'(t)}{a(t)}r, \tag{2.9}$$

where constants $\gamma > 1$, $\widetilde{k} > 0$ is an arbitrary constant, and the free boundary $a(t) \in C^2([0,+\infty))$ satisfies the Emden equation

$$a''(t) - K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)} + (k_1+k_2+Nk_3)\gamma\widetilde{k}\frac{a'(t)}{a^{N(\gamma-1)+2}(t)} = 0, \tag{2.10}$$

with initial values

$$a_0 = a(0) > 0, \ a_1 = a'(0) \in \mathbb{R}. \tag{2.11}$$

Remark 2.2 *In the form of the equation, (2.10) generalized the one studied in [34, Theorem 1] for the case that $k_1 = k_2 = 0$:*

$$a''(t) - K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)} + Nk_3\gamma\widetilde{k}\frac{a'(t)}{a^{N(\gamma-1)+2}(t)} = 0. \tag{2.12}$$

In addition, setting $N = 2$ in (2.10) to deduce that

$$a''(t) - K\gamma\widetilde{k}\frac{1}{a^{2\gamma-1}(t)} + (k_1+k_2+2k_3)\gamma\widetilde{k}\frac{a'(t)}{a^{2\gamma}(t)} = 0. \tag{2.13}$$

which is a special case studied in [17]. Therefore, the properties of the solution of equation (2.13) (see (2.14)-(2.16)), can be studied by using a similar analytical method in [17].

Theorem 2.3 *For the Emden equation (2.10), it has a unique and positive solution $a(t)$ such that*

$$0<\underline{a}\le a(t)\le \bar{C}\,(1+t)\,,\ \ for\ t>0, \tag{2.14}$$

where $\underline{a}=\left(\frac{K\gamma\widetilde{k}}{2(\gamma-1)H_0}\right)^{1/[N(\gamma-1)]}$, $H_0=\frac{1}{2}\left[a_1^2+\frac{2K\gamma\widetilde{k}}{N(\gamma-1)}{a_0}^{-N(\gamma-1)}\right]$ *and* $\bar{C}=\max\left\{a_0,(2H_0)^{1/2}\right\}$. *Furthermore, the large time behaviors of $a(t)$ and $a'(t)$ can be described as follows*

$$\lim_{t\to+\infty} a(t)/t=\lim_{t\to+\infty} a'(t)=C_0>0, \tag{2.15}$$

$$a(t)\sim C_0t+a_0\ for\ a\ suitably\ large\ t>0, \tag{2.16}$$

with constant

$$C_0=a_1-\frac{(k_1+k_2+Nk_3)\,\gamma\widetilde{k}}{N(\gamma-1)+1}{a_0}^{-[N(\gamma-1)+1]}+\int_0^{+\infty}\left(\frac{K\gamma\widetilde{k}}{a^{N(\gamma-1)+1}(t)}\right)dt.$$

Remark 2.4 *The constant C_0 appears in (2.15) is well-defined through (3.34) and (3.35). Similar to the derivation of equation (2.10), the following two special cases hold:*

Case (1): $\lambda_1(\rho)=k_1$, $\lambda_2(\rho)=k_2$ *and* $\lambda_3(\rho)=k_3\rho^\gamma$, *then $a(t)$ satisfies that*

$$a''(t)-K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)}+Nk_3\gamma\widetilde{k}\frac{a'(t)}{a^{N(\gamma-1)+2}(t)}=0. \tag{2.17}$$

Case (2): $\lambda_i(\rho)=k_i$ $(i=1,2,3)$, *then $a(t)$ satisfies that*

$$a''(t)-K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)}=0. \tag{2.18}$$

By comparing equations (2.10), (2.17) and (2.18), it can be seen that the different effects of viscosity. Moreover, Theorems 2.1 and 2.3 also apply to equations (2.17)-(2.18), except that C_0 needs to be made with small adjustments. Indeed, (2.9) also gives a special solution to the Euler equations, due to the fact that the viscosity term vanished ($(u+\frac{u}{r})_r=0$, see $(2.6)_2$). Note that the spherically symmetric solution in (2.9) is exactly the simplest affine solution, and we guess that the results in Theorems 2.1-2.3 can be extended to the general affine solutions without symmetry, by using the matrix and curve integration theories, as has been done in [9, 27, 29, 30].

Remark 2.5 *If one sets $k_1=\gamma>1$, $k_2=0$ and $k_3=\gamma-1>0$, then (2.17) reduces to*

$$a''(t)-K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)}+(\gamma+N(\gamma-1))\,\gamma\widetilde{k}\frac{a'(t)}{a^{N(\gamma-1)+2}(t)}=0, \tag{2.19}$$

which is a generalization of equation (40) studied by Guo and Xin [11] for $N=3$. In fact, they [11] assumed that the Bresch-Desjardins equality

$$\mu_2(\rho)=\rho\mu_1'(\rho)-\mu_1(\rho), \tag{2.20}$$

by setting that $\mu_1(\rho)=\rho^\gamma$ and $\mu_2(\rho)=(\gamma-1)\rho^\gamma$. Here, we remove the Bresch-Desjardins equality condition and obtain a global solution to equation (2.10) by studying the equation directly instead of the normal energy method.

Remark 2.6 *We remark that the solution in (2.9) satisfies the physical vacuum boundary conditions (see [18, 26, 36]). Indeed, (3.2), (3.3), (3.7) and (2.10) give that*

$$\begin{aligned} K\gamma\rho^{\gamma-1}\rho_r &= -\frac{\rho r}{a(t)}\left[a''(t) - (k_1+k_2+Nk_3)\gamma\frac{\rho^{\gamma-2}\rho_r}{r}a'(t)\right] \\ &= -\frac{\rho r}{a(t)}K\gamma\widetilde{k}\frac{1}{a^{N(\gamma-1)+1}(t)} = -\frac{K\gamma\widetilde{k}\rho}{a^{N(\gamma-1)+2}(t)}r, \end{aligned}$$

which implies that

$$\frac{K\gamma}{\gamma-1}\left(\rho^{\gamma-1}\right)_r = K\gamma\rho^{\gamma-2}\rho_r = -\frac{K\gamma\widetilde{k}}{a^{N(\gamma-1)+2}(t)}r,$$

and

$$p'(\rho) = K\gamma\rho^{\gamma-1} = \frac{K\gamma\widetilde{k}(\gamma-1)(a(t)+r)}{2a^{N(\gamma-1)+2}(t)}(a(t)-r). \tag{2.21}$$

This, together with (2.14), implies that the sound speed $c = \sqrt{p'(\rho)}$ is $C^{1/2}$-Hölder continuous across the vacuum boundary (the physical vacuum boundary).

Remark 2.7 *In this paper, we have selected special viscosity coefficients in (1.4):*

$$\lambda_i(\rho) = k_i\rho^{\gamma},\ k_i > 0, \quad i = 1,2,3. \tag{2.22}$$

The remain problem is how to extend the range of parameters in equation (2.22), or to investigate a more general form of the viscosity coefficient as follows:

$$\lambda_1(\rho) = k_i\rho^{\theta_i},\ i = 1,2,3, \tag{2.23}$$

with some constants $\theta_i > 0$ and $k_i > 0$ $(i = 1,2,3)$, and it will motivate our future work.

3 Proof of Theorems

3.1 Proof of Theorem 2.1

Now, we are ready to show the proof by some direct calculations. Firstly, we quote a result of the self-similar solution for the continuity equation of mass obtained by Yuen:

Lemma 3.1 (Lemma 3 of Ref. [33]) *For the equation of conservation of mass $(2.6)_1$ in radial symmetry,*

$$\rho_t + (\rho u)_r + (N-1)\frac{\rho u}{r} = 0,$$

there exist solutions,

$$\rho(r,t) = \frac{f(r/a(t))}{a^N(t)},\ u(r,t) = \frac{a'(t)}{a(t)}r, \tag{3.1}$$

with the form $f \geq 0 \in C^1$ and $a(t) > 0 \in C^1$.

Next, we will seek two suitable functions $f(s)$ of self-similar variable $s = \frac{r}{a(t)}$, and $a(t)$ satisfies the Emden equation (2.10). Substituting (3.1) into the equation $(2.6)_2$ to deduce that

$$\begin{aligned}
&\rho\left[u_t + uu_r\right] + K\gamma\rho^{\gamma-1}\rho_r - \left[(k_1+k_2+k_3)\rho^{\gamma}\left(u_r + \frac{(N-1)u}{r}\right)_r\right. \\
&\left. + (k_1+k_2+k_3)\gamma\rho^{\gamma-1}\rho_r u_r + k_3\gamma\rho^{\gamma-1}\rho_r\frac{(N-1)u}{r}\right] \\
=&\rho\left[\left(\frac{a'(t)}{a(t)}r\right)_t + \frac{a'(t)}{a(t)}r\frac{a'(t)}{a(t)}\right] + K\gamma\rho^{\gamma-1}\rho_r \\
&-\left[(k_1+k_2+Nk_3)\gamma\rho^{\gamma-1}\rho_r\frac{a'(t)}{a(t)}\right] = 0,
\end{aligned} \tag{3.2}$$

which is equivalent to the following

$$\frac{a''(t)}{a(t)} + K\gamma\frac{\rho^{\gamma-2}\rho_r}{r} - (k_1+k_2+Nk_3)\gamma\frac{\rho^{\gamma-2}\rho_r}{r}\frac{a'(t)}{a(t)} = 0. \tag{3.3}$$

In view of (3.1), the third term on the righthand side of the equation above can be rewritten as

$$\frac{\rho^{\gamma-2}\rho_r}{r} = \frac{1}{r}\left(\frac{f(s)}{a^N(t)}\right)^{\gamma-2}\frac{f'(s)}{a^N(t)}\frac{1}{a(t)} = \frac{1}{r}\frac{f^{\gamma-2}(s)f'(s)}{a^{N(\gamma-1)+1}(t)}. \tag{3.4}$$

To seek a solution to equations (3.3)-(3.4), similar to that in [7, 11], we set

$$f^{\gamma-2}(s)f'(s) = -\widetilde{k}s, \quad \widetilde{k} > 0, \tag{3.5}$$

integrating it over $(s,1)$ and using the boundary condition that $f(1) = 0$ (due to (2.8)) to get

$$f(s) = \left[\frac{\widetilde{k}(\gamma-1)}{2}\left(1-s^2\right)\right]^{\frac{1}{\gamma-1}} = \left[\frac{\widetilde{k}(\gamma-1)}{2}\left(1-\frac{r^2}{a^2(t)}\right)\right]^{\frac{1}{\gamma-1}}. \tag{3.6}$$

Hence, (3.4) can be rewritten as follows

$$\frac{\rho^{\gamma-2}\rho_r}{r} = \frac{1}{r}\frac{-\widetilde{k}\frac{r}{a(t)}}{a^{N(\gamma-1)+1}(t)} = -\frac{\widetilde{k}}{a^{N(\gamma-1)+2}(t)}. \tag{3.7}$$

Thus, inserting (3.7) into (3.3), one gets

$$\frac{a''(t)}{a(t)} - \frac{K\gamma\widetilde{k}}{a^{N(\gamma-1)+2}(t)} + (k_1+k_2+Nk_3)\gamma\frac{\widetilde{k}}{a^{N(\gamma-1)+2}(t)}\frac{a'(t)}{a(t)} = 0.$$

which implies that the Emden equation (2.10) holds. So, (3.1) and (3.6) are solutions to system (2.6)-(2.8), the proof of Theorem 2.1 is complete.

3.2 Proof of Theorem 2.3

Indeed, the global existence and the large time asymptotic behavior of solution to equation (3.8), as described in theorem 2.3, can be obtained by a similar way as that done in [17].

For the convenience of the reader, we give the detailed proof and which consists of three steps. We first rewrite equation (2.10) as follows

$$a''(t) - C_1 \frac{1}{a^{N(\gamma-1)+1}(t)} + C_2 \frac{a'(t)}{a^{N(\gamma-1)+2}(t)} = 0, \tag{3.8}$$

where two positive constants

$$C_1 = K\gamma\widetilde{k} > 0 \text{ and } C_2 = (k_1 + k_2 + Nk_3)\gamma\widetilde{k} > 0. \tag{3.9}$$

Step 1. In this step, we show the local and global existence of solution to (3.8). To this end, one can rewrite (3.8) as follows

$$\left(a'(t) - \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t)\right)_t = \frac{C_1}{a^{N(\gamma-1)+1}(t)}, \tag{3.10}$$

which gives that

$$\begin{aligned} a'(t) = {} & a_1 - \frac{C_2}{N(\gamma-1)+1} a_0^{-[N(\gamma-1)+1]} \\ & + \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t) + \int_0^t \left(\frac{C_1}{a^{N(\gamma-1)+1}(t)}\right) dt, \end{aligned} \tag{3.11}$$

Notice the equivalence of (3.8) and (3.11), the local existence of solution to equation (3.8) that there exists a small T such that (3.8) has a positive solution $a(t)$, which is unique in $C^2([0,T])$ and satisfies $0 < a_0/2 \le a(t) \le 2a_0$, can be obtained by using the contraction mapping principle as in [7, 11], we omit the details here. Then, the global existence of solutions to Eq (3.8) can be proved by establishing the a priori estimates using the standard continuity argument:

Lemma 3.2 (Global existence) *The Emden equation (3.8) has a positive solution $a(t)$, which is unique in $C^2([0,+\infty))$ and satisfies (2.14):*

$$0 < \underline{a} \le a(t) \le \bar{C}(1+t), \ \textit{for } t > 0,$$

where $\bar{C} = \max\left\{a_0, (2H_0)^{1/2}\right\}$, $\underline{a} = \left(\frac{C_1}{N(\gamma-1)H_0}\right)^{1/[N(\gamma-1)]}$, C_1 and C_2 are given by (3.9), and H_0 is defined by (3.15).

Proof: Assume $a(t) \in C^1([0,T])$ is a solution to (3.8), we first prove the a priori estimate

$$0 < \underline{a} \le a(t) \le \bar{C}(1+t), \text{ for all } t \in [0,T]. \tag{3.12}$$

Multiplying (3.8) by $a'(t)$ yields

$$a''(t)a'(t) - C_1 a^{-[N(\gamma-1)+1]}(t)a'(t) + C_2 \frac{(a'(t))^2}{a^{N(\gamma-1)+2}(t)} = 0,$$

then it follows that

$$\frac{1}{2}\left[(a'(t))^2 + \frac{2C_1}{N(\gamma-1)} a^{-N(\gamma-1)}(t)\right]' + C_2 \frac{(a'(t))^2}{a^{N(\gamma-1)+2}(t)} = 0. \tag{3.13}$$

Now, we define $H(t)$ as follows

$$H(t)=\frac{1}{2}\left((a'(t))^2+\frac{2C_1}{N(\gamma-1)}\frac{1}{a^{N(\gamma-1)}(t)}\right), \tag{3.14}$$

which, together with (3.13), for all $t\in[0,T]$, gives that

$$H(t)+\int_0^t\left(C_2\frac{(a'(t))^2}{a^{N(\gamma-1)+2}(t)}\right)dt=H_0, \tag{3.15}$$

where $H_0=\frac{1}{2}\left[a_1^2+\frac{2C_1}{N(\gamma-1)}a_0^{-N(\gamma-1)}\right]$. Obviously, (3.14) and (3.15) imply that

$$(a'(t))^2\le 2H_0,\ \left(\frac{C_1}{N(\gamma-1)H_0}\right)^{1/(\gamma-1)}\le a^N(t). \tag{3.16}$$

Due to $a_0>0$ and the continuity property, one derives from (3.16) that

$$a(t)>0,\ \text{for all } t\in[0,T]. \tag{3.17}$$

Thus, (3.16) and (3.17) yield that

$$-(2H_0)^{1/2}\le a'(t)\le(2H_0)^{1/2},\ \left(\frac{C_1}{N(\gamma-1)H_0}\right)^{1/[N(\gamma-1)]}\le a(t). \tag{3.18}$$

It follows that

$$a(t)\le a_0+(2H_0)^{1/2}t\le\bar{C}(1+t),\ \text{for all } t\in[0,T], \tag{3.19}$$

where $\bar{C}=\max\left\{a_0,(2H_0)^{1/2}\right\}$. Thus, (3.12) follows from (3.18) and (3.19). Therefore, combining the local existence, the a priori estimates in (3.12), and the standard continuity argument, we know that the equation (3.8) has a globally defined positive solution $a(t)$ satisfying (2.14). Thus, the proof of Lemma 3.2 is complete. □

Step 2. In this step, we show the monotonically increasing property of $a(t)$ after time t_0 which is determined by (3.25). We define

$$h(t)=a'(t)-\frac{C_2}{N(\gamma-1)+1}a^{-[N(\gamma-1)+1]}(t),\ h(0)=a_1-\frac{C_2}{2\gamma-1}a_0^{-[N(\gamma-1)+1]}. \tag{3.20}$$

It follows from (3.10) and (2.14) that

$$(h(t))_t=\frac{C_1}{a^{N(\gamma-1)+1}(t)}>0, \tag{3.21}$$

and

$$h(t)=h(0)+\int_0^t\left(\frac{C_1}{a^{N(\gamma-1)+1}(t)}\right)dt\ge h(0). \tag{3.22}$$

According to the sign of initial value $h(0)$, there are roughly two kinds of profiles of $a(t)$.

If $h(0)<0$, due to the monotonicity and continuity property of $h(t)$, (3.21) implies that $h(t)$ will increase in a time interval until some finite time $t_0>0$ (If $t_0=+\infty$, (3.20) implies that

$$h(t)\le 0 \text{ for } t>0, \tag{3.23}$$

then it holds that

$$a'(t) \le \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t), \quad \left(a^{N(\gamma-1)+2}(t)\right)' \le \frac{C_2[N(\gamma-1)+2]}{N(\gamma-1)+1},$$

and hence

$$\begin{aligned} a(t) &\le \left(\frac{C_2[N(\gamma-1)+2]}{N(\gamma-1)+1} t + a_0^{2\gamma}\right)^{\frac{1}{N(\gamma-1)+2}} \\ &\le \left(\frac{C_2[N(\gamma-1)+2]}{N(\gamma-1)+1} + a_0^{2\gamma}\right)^{\frac{1}{N(\gamma-1)+2}} (1+t)^{\frac{1}{N(\gamma-1)+2}} \quad \text{for } t>0. \end{aligned} \tag{3.24}$$

Inserting (3.24) into (3.22) to get, for a suitably large $t^* > 0$, that

$$\begin{aligned} h(t) &\ge h(0) + \int_0^t \left(\frac{C_1}{a^{N(\gamma-1)+1}(t)}\right) dt \\ &\ge h(0) + \left(\frac{C_2[N(\gamma-1)+2]}{N(\gamma-1)+1} + a_0^{2\gamma}\right)^{-\frac{N(\gamma-1)+1}{N(\gamma-1)+2}} \int_0^t \frac{C_1}{(1+t)^{\frac{N(\gamma-1)+1}{N(\gamma-1)+2}}} dt \\ &> 0 \text{ for } t > t^*, \end{aligned}$$

which contradicts with (3.23)) such that $h(t_0) = 0$, and t_0 can be determined by

$$h(t_0) = a'(t_0) - \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t_0) = 0. \tag{3.25}$$

Thus, after time t_0, (3.22) implies that $h(t) \ge h(t_0)$, namely,

$$a'(t) \ge \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t) > 0, \text{ for } t > t_0, \tag{3.26}$$

where t_0 is determined by (3.25).

If $h(0) \ge 0$, it follows from (3.22) and (3.20) that

$$a'(t) \ge \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t) > 0, \ \text{ for } t > 0, \tag{3.27}$$

so $a(t)$ increases for all time. Thus, it follows from (3.26) and (3.27) that

$$a'(t) > 0 \text{ and } a(t) \ge \underline{a} \text{ is increasing in } (t_0, +\infty). \tag{3.28}$$

Step 3. In this step, we show the asymptotic behaviors of $a(t)$ and $a'(t)$. First, we derive from (3.28) and the monotone bounded principle that the limit $\lim\limits_{t\to+\infty} a(t)$ exists and belongs to $[\underline{a}, +\infty]$. Moreover, we claim that

$$\lim_{t\to+\infty} a(t) = +\infty. \tag{3.29}$$

Otherwise, suppose that there holds

$$\lim_{t\to\infty} a(t) = \bar{r} \in (\underline{a}, +\infty) \text{ and } a(t) \le 2\bar{r} \ \text{ for } t \ge t^*, \tag{3.30}$$

for a suitably large $t^* > 0$, then it follows from (3.11) and (3.30) that

$$\begin{aligned}
a'(t) &= a_1 - \frac{C_2}{N(\gamma-1)+1} a_0^{-[N(\gamma-1)+1]} \\
&\quad + \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t) + \int_0^t \left(\frac{C_1}{a^{N(\gamma-1)+1}(t)} \right) dt \\
&\geq a_1 - \frac{C_2}{N(\gamma-1)+1} a_0^{-[N(\gamma-1)+1]} \\
&\quad + \frac{C_2}{N(\gamma-1)+1} (2\bar{r})^{-[N(\gamma-1)+1]} + \frac{C_1}{(2\bar{r})^{N(\gamma-1)+1}} (t - t^*) \\
&> (2H_0)^{1/2} \text{ for a suitably large } t > 0,
\end{aligned}$$

which contradicts (3.18). So, the supposition (3.30) fails, and (3.29) is true.

Due to (3.29) and (3.18), the following fact holds

$$\lim_{t\to\infty} \frac{a'(t)}{a(t)} = 0,$$

which and (3.8) give that, for a suitably large $t_1 > 0$,

$$\begin{aligned}
a''(t) &= C_1 \frac{1}{a^{N(\gamma-1)+1}(t)} - C_2 \frac{a'(t)}{a^{N(\gamma-1)+2}(t)} \\
&= \frac{C_1}{a^{N(\gamma-1)+1}(t)} \left(1 - \frac{C_2}{C_1} \frac{a'(t)}{a(t)} \right) \\
&\geq \frac{1}{2} \frac{C_1}{a^{N(\gamma-1)+1}(t)} > 0, \text{ for } t > t_1,
\end{aligned} \tag{3.31}$$

which implies that $a(t)$ is convex in $(t_1, +\infty)$. Thus, (3.18), (3.28), (3.31), and the monotone bounded principle yield that

$$\lim_{t\to+\infty} a'(t) = C_0, \ 0 < C_0 \leq (2H_0)^{1/2}, \tag{3.32}$$

and it follows that, for a suitably large $t^* > 0$,

$$a(t) \sim C_0 t + a_0 \text{ for } t > t^*, \tag{3.33}$$

for some positive constant C_0 to be determined later. By (3.33) and (3.9), we know the following integrability:

$$\int_0^t \left(\frac{C_1}{a^{2\gamma-1}(t)} \right) dt \leq \int_0^{+\infty} \left(\frac{C_1}{a^{2\gamma-1}(t)} \right) dt < +\infty. \tag{3.34}$$

Now, letting $t \to +\infty$ in (3.11) and noting (3.34), one has that

$$\begin{aligned}
a'(t) &= a_1 - \frac{C_2}{N(\gamma-1)+1} a_0^{-[N(\gamma-1)+1]} \\
&\quad + \frac{C_2}{N(\gamma-1)+1} a^{-[N(\gamma-1)+1]}(t) + \int_0^t \left(\frac{C_1}{a^{N(\gamma-1)+1}(t)} \right) dt \\
&\to a_1 - \frac{C_2}{N(\gamma-1)+1} a_0^{-[N(\gamma-1)+1]} + \int_0^{+\infty} \left(\frac{C_1}{a^{N(\gamma-1)+1}(t)} \right) dt := C_0,
\end{aligned} \tag{3.35}$$

as $t \to +\infty$. Thus, (2.15) and (2.16) follow from (3.33), (3.35), (3.9), and the the L'Hospital rule, and we finish the proof of Theorem 2.3.

Appendix

Expression for the viscous terms in spherical coordinates in $\mathbb{R}^N$

Noting the definition of viscous stress tensor Ψ given by (1.3), its divergence can be calculated as follows

$$\begin{aligned}\operatorname{div}\Psi &= \operatorname{div}\left(\lambda_1(\rho)\nabla\mathbf{u}\right) + \operatorname{div}\left(\lambda_2(\rho)\nabla\mathbf{u}^T\right) + \nabla\left(\lambda_3(\rho)\operatorname{div}\mathbf{u}\right) &(3.36)\\ &= \left\{\left(\lambda_1'(\rho)+\lambda_2'(\rho)\right)(\nabla\rho\cdot\nabla)\mathbf{u} + \lambda_1(\rho)\triangle\mathbf{u} + \lambda_2(\rho)\operatorname{div}\left(\nabla\mathbf{u}^T\right)\right\}\\ &\quad + \left\{\lambda_3(\rho)\nabla(\operatorname{div}\mathbf{u}) + (\operatorname{div}\mathbf{u})\lambda_3'(\rho)\nabla\rho\right\}. &(3.37)\end{aligned}$$

Now, we calculate the terms in equation (3.37) in the following three cases in the spherical coordinates.

(1) $\nabla\rho$ and $\operatorname{div}\mathbf{u}$

For a scalar functions $f(r,t) = f(\mathbf{x},t)$ with $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $r = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, the chain rule gives us that

$$\frac{\partial}{\partial x_i} f(r,t) = \frac{\partial}{\partial r} f(r,t)\cdot\frac{\partial r}{\partial x_i} = \frac{x_i}{r} f_r,\ \ i = 1, 2, ..., N,$$

then

$$\nabla f(r) = (f_{x_1}, f_{x_2}, \cdots, f_{x_N}) = \left(\frac{x_1}{r}f_r, \frac{x_2}{r}f_r, \cdots, \frac{x_N}{r}f_r\right) = f_r\frac{\mathbf{x}}{r} := f_r\mathbf{e}_r. \tag{3.38}$$

Let the function f in (3.38) to be the density or pressure of the fluid, one will have

$$\nabla\rho = \rho_r\mathbf{e}_r,\ \ \nabla p = p_r\mathbf{e}_r. \tag{3.39}$$

For the vector function velocity field $\mathbf{u}(\mathbf{x},t) = u(r,t)\,\frac{\mathbf{x}}{r}$ (see (2.3)), we can deduce that

$$\begin{aligned}\operatorname{div}\mathbf{u} &= \left(u(r,t)\frac{x_1}{r}\right)_{x_1} + \left(u(r,t)\frac{x_2}{r}\right)_{x_2} + \cdots + \left(u(r,t)\frac{x_N}{r}\right)_{x_N}\\ &= N\frac{u(r,t)}{r} + x_1\left(\frac{u(r,t)}{r}\right)_r\frac{x_1}{r} + x_2\left(\frac{u(r,t)}{r}\right)_r\frac{x_2}{r} + \cdots + x_{_N}\left(\frac{u(r,t)}{r}\right)_r\frac{x_N}{r}\\ &= N\frac{u(r,t)}{r} + r\left(\frac{u(r,t)}{r}\right)_r = u_r + (N-1)\frac{u}{r}. &(3.40)\end{aligned}$$

(2) $\triangle\mathbf{u} = div(\nabla\mathbf{u})$ and $div(\nabla\mathbf{u}^T)$

If we set

$$\begin{aligned}\nabla\mathbf{u} &= \nabla\left(u^1, u^2, ..., u^N\right) = \nabla\downarrow\left(u(r,t)\frac{x_1}{r}, u(r,t)\frac{x_2}{r}, \cdots, u(r,t)\frac{x_N}{r}\right)\\ &= \begin{bmatrix} \left(u(r,t)\frac{x_1}{r}\right)_{x_1} & \left(u(r,t)\frac{x_2}{r}\right)_{x_1} & \cdots & \left(u(r,t)\frac{x_N}{r}\right)_{x_1}\\ \left(u(r,t)\frac{x_1}{r}\right)_{x_2} & \left(u(r,t)\frac{x_2}{r}\right)_{x_2} & \cdots & \left(u(r,t)\frac{x_N}{r}\right)_{x_2}\\ \vdots & \vdots & \vdots & \vdots\\ \left(u(r,t)\frac{x_1}{r}\right)_{x_N} & \left(u(r,t)\frac{x_2}{r}\right)_{x_N} & \cdots & \left(u(r,t)\frac{x_N}{r}\right)_{x_N} \end{bmatrix}\\ &= \begin{bmatrix} \frac{u}{r} + \frac{x_1^2}{r}\left(\frac{u}{r}\right)_r & \frac{x_1x_2}{r}\left(\frac{u}{r}\right)_r & \cdots & \frac{x_1x_N}{r}\left(\frac{u}{r}\right)_r\\ \frac{x_1x_2}{r}\left(\frac{u}{r}\right)_r & \frac{u}{r} + \frac{x_2^2}{r}\left(\frac{u}{r}\right)_r & \cdots & \frac{x_2x_N}{r}\left(\frac{u}{r}\right)_r\\ \vdots & \vdots & \vdots & \vdots\\ \frac{x_1x_N}{r}\left(\frac{u}{r}\right)_r & \frac{x_2x_N}{r}\left(\frac{u}{r}\right)_r & \cdots & \frac{u}{r} + \frac{x_N^2}{r}\left(\frac{u}{r}\right)_r \end{bmatrix}\\ &= \nabla\mathbf{u}^T, &(3.41)\end{aligned}$$

which is a symmetric matrix, and thus

$$\triangle \mathbf{u} = \mathbf{div}(\nabla \mathbf{u}) = \mathbf{div}(\nabla \mathbf{u}^{\mathbf{T}}).$$

Then, it follows that

$$\begin{aligned}
&\Delta u^1 = \operatorname{div} \nabla \left(u^1\right) \\
&= \left(\frac{u}{r} + \frac{x_1^2}{r}\left(\frac{u}{r}\right)_r\right)_{x_1} + \left(\frac{x_1 x_2}{r}\left(\frac{u}{r}\right)_r\right)_{x_2} + \cdots + \left(\frac{x_1 x_N}{r}\left(\frac{u}{r}\right)_r\right)_{x_N} \\
&= \left[\frac{x_1}{r}\left(\frac{u}{r}\right)_r + \frac{2x_1}{r}\left(\frac{u^r}{r}\right)_r + \frac{x_1^3}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] + \left[\frac{x_1}{r}\left(\frac{u}{r}\right)_r + \frac{x_1 x_2^2}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&\quad + \cdots + \left[\frac{x_1}{r}\left(\frac{u}{r}\right)_r + \frac{x_1 x_N^2}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&= \left[\frac{(N+2)\,x_1}{r}\left(\frac{u}{r}\right)_r + x_1 r\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] = \frac{x_1}{r}\left[(N+2)\left(\frac{u}{r}\right)_r + r^2\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&= \frac{x_1}{r}\left[u_r + (N-1)\,\frac{u}{r}\right]_r,
\end{aligned}$$

where we have used that

$$\begin{aligned}
r^2\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r &= r^2\left[\frac{u_r}{r^2} - \frac{u}{r^3}\right]_r = r^2\left[\frac{u_{rr}}{r^2} - 2\frac{u_r}{r^3} - \frac{u_r}{r^3} + 3\frac{u}{r^4}\right] \\
&= u_{rr} - 3\left(\frac{u_r}{r} - \frac{u}{r^2}\right) = u_{rr} - 3\left(\frac{u}{r}\right)_r = \left[u_r - 3\frac{u}{r}\right]_r.
\end{aligned}$$

And similarly, one has

$$\begin{aligned}
&\Delta u^2 = \operatorname{div} \nabla \left(u^2\right) \\
&= \left(\frac{x_1 x_2}{r}\left(\frac{u}{r}\right)_r\right)_{x_1} + \left(\frac{u}{r} + \frac{x_2^2}{r}\left(\frac{u}{r}\right)_r\right)_{x_2} + \cdots + \left(\frac{x_2 x_N}{r}\left(\frac{u}{r}\right)_r\right)_{x_N} \\
&= \left[\frac{x_2}{r}\left(\frac{u}{r}\right)_r + \frac{x_1^2 x_2}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] + \left[\frac{x_2}{r}\left(\frac{u}{r}\right)_r + \frac{2x_2}{r}\left(\frac{u}{r}\right)_r + \frac{x_2^3}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&\quad + \cdots + \left[\frac{x_2}{r}\left(\frac{u}{r}\right)_r + \frac{x_2 x_N^2}{r}\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&= \left[\frac{(N+2)\,x_2}{r}\left(\frac{u}{r}\right)_r + x_2 r\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] \\
&= \frac{x_2}{r}\left[(N+2)\left(\frac{u}{r}\right)_r + r^2\left(\frac{1}{r}\left(\frac{u}{r}\right)_r\right)_r\right] = \frac{x_2}{r}\left[u_r + (N-1)\,\frac{u}{r}\right]_r,
\end{aligned}$$

and

$$\Delta\left(u^i\right) = \operatorname{div} \nabla \left(u^i\right) \;\; = \frac{x_i}{r}\left[u_r + (N-1)\,\frac{u}{r}\right]_r, \;\; (i = 3, ..., N)\,.$$

Thus,

$$\triangle \mathbf{u} = \left(\Delta u^1, \Delta u^2, \ldots, \Delta u^N\right) = \left[u_r + (N-1)\,\frac{u}{r}\right]_r \mathbf{e}_r. \tag{3.42}$$

which is exactly the same as that in [25, 37].

(3) $\left(\nabla\rho \cdot \nabla\right)\mathbf{u}$

By direct calculations, one has, for $1 \le i \le N$, that

$$
\begin{aligned}
&\left(\nabla\rho\cdot\nabla\right)\mathbf{u}^i = \left(\rho_{x_1}\partial_{x_1} + \rho_{x_2}\partial_{x_2} + \cdots + \rho_{x_N}\partial_{x_N}\right)\left(\frac{u}{r}x_i\right)\\
&= \rho_{x_1}\left(\frac{u}{r}x_i\right)_{x_1} + \cdots + \rho_{x_{i-1}}\left(\frac{u}{r}x_i\right)_{xi-1} + \rho_{x_i}\left(\frac{u}{r}x_i\right)_{x_i}\\
&\quad + \rho_{x_{i+1}}\left(\frac{u}{r}x_i\right)_{x_{i+1}} + \cdots + \rho_{x_N}\left(\frac{u}{r}x_i\right)_{x_N}\\
&= \rho_r\frac{x_1}{r}\left[x_i\left(\frac{u}{r}\right)_r\frac{x_1}{r}\right] + \cdots + \rho_r\frac{x_{i-1}}{r}\left[x_i\left(\frac{u}{r}\right)_r\frac{x_{i-1}}{r}\right] + \rho_r\frac{x_i}{r}\left[x_i\left(\frac{u}{r}\right)_r\frac{x_i}{r} + \frac{u}{r}\right]\\
&\quad + \rho_r\frac{x_{i+1}}{r}\left[x_i\left(\frac{u}{r}\right)_r\frac{x_{i+1}}{r}\right] + \cdots + \rho_r\frac{x_N}{r}\left[x_i\left(\frac{u}{r}\right)_r\frac{x_N}{r}\right]\\
&= \rho_r\frac{x_i}{r}\left[r\left(\frac{u}{r}\right)_r\right] + \rho_r\frac{x_i}{r}\left[\frac{u}{r}\right] = \rho_r\frac{x_i}{r}\left[r\left(\frac{u}{r}\right)_r + \frac{u}{r}\right] = \frac{x_i}{r}\rho_r u_r,
\end{aligned}
$$

and thus

$$
\left(\nabla\rho\cdot\nabla\right)\mathbf{u} = \rho_r u_r \mathbf{e}_r. \tag{3.43}
$$

Substituting expressions (3.39)-(3.43) into (3.36)-(3.37) produces

$$
\begin{aligned}
&\operatorname{div}\left(\lambda_1(\rho)\nabla u\right) = \lambda_1(\rho)\triangle u + \lambda_1'(\rho)\left(\nabla\rho\cdot\nabla\right)u\\
&= \lambda_1(\rho)\left(\left(u_r + (N-1)\frac{u}{r}\right)_r\mathbf{e}_r\right) + \lambda_1'(\rho)\left(\rho_r u_r \mathbf{e}_r\right)\\
&= \left[\lambda_1(\rho)\left(u_r + (N-1)\frac{u}{r}\right)_r + \left(\lambda_1(\rho)\right)_r u_r\right]\mathbf{e}_r,
\end{aligned} \tag{3.44}
$$

$$
\begin{aligned}
&\operatorname{div}\left(\lambda_2(\rho)\nabla u^T\right) = \operatorname{div}\left(\lambda_2(\rho)\nabla u\right)\\
&= \left[\lambda_2(\rho)\left(u_r + (N-1)\frac{u}{r}\right)_r + \left(\lambda_2(\rho)\right)_r u_r\right]\mathbf{e}_r,
\end{aligned} \tag{3.45}
$$

and

$$
\begin{aligned}
\nabla\left(\lambda_3(\rho)\operatorname{div}u\right) &= \lambda_3(\rho)\nabla\left(\operatorname{div}u\right) + \left(\operatorname{div}u\right)\lambda_3'(\rho)\nabla\rho\\
&= \lambda_3(\rho)\nabla\left(u_r + (N-1)\frac{u}{r}\right) + \left(u_r + (N-1)\frac{u}{r}\right)\lambda_3'(\rho)\nabla\rho\\
&= \lambda_3(\rho)\left(u_r + (N-1)\frac{u}{r}\right)_r\mathbf{e}_r + \lambda_3'(\rho)\rho_r\left(u_r + (N-1)\frac{u}{r}\right)\mathbf{e}_r\\
&= \left[\lambda_3(\rho)\left(u_r + (N-1)\frac{u}{r}\right)_r + \left(\lambda_3(\rho)\right)_r\left(u_r + (N-1)\frac{u}{r}\right)\right]\mathbf{e}_r.
\end{aligned} \tag{3.46}
$$

Finally, inserting (3.44)-(3.46) into (3.36) gives (2.4) directly.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflict of interest

The author declares there is no conflict of interest in relation to this article.

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