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RESEARCH ARTICLE

Semi-global stabilization of a class of cascade systems by a separate design approach

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Summary

The semi-global stabilization of a class of cascade systems (e.g., partially linear composite systems) is investigated via partial state feedback. The system comprises a nonlinear subsystem with a cross-term and a linear subsystem in the Byrnes-Isidori normal form. The cross-term that involves any two consecutive states of chains of integrators is incorporated into the nonlinear subsystem. Based on the established lemma for separate design, the semi-global stabilization problem for the entire composite system is reduced to stabilizing its linear subsystem subject to non-peaking constraints on the consecutive states. To address the later problem, a linear low-and-high gain feedback law is developed in the backstepping manner, which can be recognized as partial state feedback for the composite system as it uses only the states of the linear subsystem.

KEYWORDS:
semi-global stabilization, cascade system, partial state feedback, low-gain feedback, peaking phenomenon

1 | INTRODUCTION

In recent times, the focus on under-actuated systems (UASs) has grown significantly. Examples include systems like the under-actuated flexible joint robot[1], the reaction wheel pendulum[2], the ball and beam system[3], the four-DOF crane system[4], the translational oscillator with rotational actuator system[5], the under-actuated autonomous underwater vehicle[6,7]. Some types of UASs can be formulated by the following partially linear composite system[8–10]

$$\begin{cases} \dot{\eta} = f(\eta, x),\ \eta \in \mathbb{R}^l \\ \dot{x} = Ax + Bu,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ y = Cx,\ y \in \mathbb{R}^p, \end{cases} \tag{1}$$

where $f(\cdot,\cdot) : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^l$ is a smooth function with $f(0,0) = 0$. It is well-known that the Byrnes-Isidori normal form[11,12] for a class of affine nonlinear systems shares a similar structure.

There are lot of original stabilization approaches[13–19] for system (1). Under the standard assumption that $\dot{\eta} = f(\eta, 0)$ is globally asymptotically stable (GAS), partial state feedback (using the states of the linear subsystem) is particularly noteworthy, since it not only shows explicit and economical structure, but also acts as a preliminary approach for a class of observer-based output feedback[20–22]. Our work mainly deals with global or semi-global stabilization using partial state feedback. It should be noted that forcing the state x to decay fast arbitrarily may not necessarily stabilize the η subsystem even with $\dot{\eta} = f(\eta, 0)$ being GAS. This is because that the state x incorporated in the η subsystem may suffer from the peaking phenomenon[23–25]. When it concerns partial state feedback, there is a trade-off between peaking states and the growth properties of the nonlinear subsystem[23], and without additional growth conditions on f, global or semi-global stabilization is difficult to achieve (see the reference[26]). In general, there are two kinds of approaches for partial state feedback. One approach imposes restrictions on the nonlinear subsystem while considering little for the linear part, such as input-to-state stability[10] or some particular growth conditions[13,14]. The other approach imposes restrictions on the linear part and tends towards a special structure of the nonlinear subsystem, for example, only the output y enters the η subsystem instead of all states[17,18].

For the latter kind approach, a cascade system[10,11,27,28]

$$\begin{cases} \dot{\eta} = f(\eta, y) \\ \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \tag{2}$$

with $\dot{\eta} = f(\eta, 0)$ being GAS deserves much more attention. Partial state feedback for such a system (2) has been studied heuristically with the development of relaxations on the model assumptions. Firstly, semi-global stabilization for the particular system (2) with the subsystem $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ being chains of integrators and the top integrators of each chain being the output was achieved by a class of high-gain feedback[11]. Clearly, the triple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, namely, chains of integrators, was supposed to be square invertible and have no invariant zeros. Then, a relaxed work for system (2), where the subsystem $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ are chains of integrators too but it allows any integrator of each chain to be the output, was proposed in the reference[27]. Parallelly, for the same cascade system of the work[27], a linear low-and-high gain feedback law was developed for semi-global stabilization[28]. Besides, with the linear subsystem being the chain of integrators, a weaker assumption that the subsystem $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is stabilizable and right invertible with its all invariant zeros lying in the closed left s-plane was manifested in the work[18] to be sufficient for the semi-global stabilization of such a cascade system (2). Moreover, the semi-global stabilization for the cascade system (2) with the linear subsystem being the Byrnes-Isidori normal form was studied in the work[17].

Since then, several works tend to system (1) with its η subsystem containing cross-term. The SISO cascade system (removing the growth condition)

$$\begin{cases} \dot{\eta} = f(\eta, \varphi(\xi_{j_0})\xi_{j_0+1}) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_r = u, \end{cases}$$

was studied by Lin[29], where $u \in \mathbb{R}$, ξ_{j_0}, ξ_{j_0+1} are any two consecutive integrators, and $\varphi(\xi_{j_0})$ is a locally Lipschitz function. Semi-global stabilization for this system is accomplished by a low-and-high gain feedback. Along with this control scheme, a general case that the linear part being a class of normal forms was further investigated in the work[29]. The special SISO system

$$\begin{cases} \dot{\eta} = (-1 + \xi_i \xi_j)\eta^3 \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = u, \end{cases}$$

was studied by Sepulchre et al[17], where $u \in \mathbb{R}$ and ξ_i, ξ_j are any two integrators. In the case of $|i - j| \leq 1$, this cascade can be stabilized semi-globally, while in the case of $|i - j| = 2$, the semi-global stabilization problem deserves further discussions.

Our work preserves similar assumptions in the works[17,18] and further considers a class of cascade systems with cross-term. Since the square-down process is not our focus, we assume for simplicity that the linear part $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is square, and has a nonsingular decoupling matrix, which results in the Byrnes-Isidori normal form[12,30]. Then, after a simple input transformation (depending on the states of the linear part (A, B, C)), the cascade system (1) is formulated as

$$\begin{cases} \dot{\eta} = f(\eta, \varphi(\xi_{j_i}, \xi_{j_i+1})\xi_{j_i+1}) \\ \dot{\xi}_0 = A_0\xi_0 + B_0 y \\ \dot{\xi}_1^i = \xi_2^i \\ \dot{\xi}_2^i = \xi_3^i \\ \quad \vdots \\ \dot{\xi}_{r_i}^i = u_i \\ y = \xi_1, \ i = 1, 2, ..., m, \end{cases} \tag{3}$$

where $\eta \in \mathbb{R}^l$, $\xi_0 \in \mathbb{R}^{n-r}$, $\xi \in \mathbb{R}^r$, $u_i \in \mathbb{R}$, $y \in \mathbb{R}^m$, $\sum_{i=1}^m r_i = r$. Define $\xi_{j_i} = [\, \xi_{j_1}^1 \ \xi_{j_2}^2 \ \cdots \ \xi_{j_m}^m \,]^{\mathrm{T}}$ and $\xi_{j_i+1} = [\, \xi_{j_1+1}^1 \ \xi_{j_2+1}^2 \ \cdots \ \xi_{j_m+1}^m \,]^{\mathrm{T}}$, $j_i \in \{r_1, r_2, ..., r_i - 1\}$, $i = 1, 2, ..., m$ such that the cross-term $\varphi(\xi_{j_i}, \xi_{j_i+1})\xi_{j_i+1}$ is comprised of any two consecutive integrators $\xi_{j_i}^i, \xi_{j_i+1}^i$ in each chain. For a clear interpretation, define $\xi^i = [\, \xi_1^i \ \xi_2^i \ \cdots \ \xi_{r_i}^i \,]^{\mathrm{T}}$, $i = 1, 2, ..., m$, $\xi_j = [\, \xi_j^1 \ \xi_j^2 \ \cdots \ \xi_j^m \,]^{\mathrm{T}}$, $j = 1, 2, ..., r_i$, and $\xi = [\, (\xi^1)^{\mathrm{T}} \ (\xi^2)^{\mathrm{T}} \ \cdots \ (\xi^m)^{\mathrm{T}} \,]^{\mathrm{T}}$. Suppose that the cascade system (3) satisfies

Assumption 1. $f(\cdot, \cdot) : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^l$ is a smooth function with $f(0, 0) = 0$ and $\varphi(\xi_{j_i}, \xi_{j_i+1}) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is a locally Lipschitz function.

Assumption 2. The equilibrium $\eta = 0$ of the nonlinear dynamic $\dot{\eta} = f(\eta, 0)$ is GAS.

Assumption 3. The subsystem (A_0, B_0) is asymptotically null controllable with bounded controls (ANCBC), that is, the pair (A_0, B_0) is stabilizable and all eigenvalues of A_0 lie in the closed left half s-plane.

Remark 1. Assumption 1 is a common assumption in the nonlinear control literature[27–29]. Assumption 2 is the standard assumption to guarantee the equilibrium $\eta = 0$ of the η-subsystem. Assumption 3 is the standard assumption while using the low-gain feedback[31,34] or nested saturation design[27]. In addition, when it concerns the Byrnes-Isidori normal form of the linear system (A, B, C), the invariant-zeros of (A, B, C) are equal to the eigenvalues of A_0[12,30]. Thus our Assumptions 1-3 are also employed as part of assumptions in the reference[18,29]. Besides, Assumptions 1-3 are just the ones used in the work[17] for the cascade system (2) (without the nonlinear cross-term).

We now state the main problem:

Problem 1. Let Assumptions 1-3 be met. Find partial state feedback $u_i = u_i(\xi_0, \xi)$, $i = 1, 2, ..., m$ such that the cascade system (3) is semi-globally asymptotically stabilized, that is, given an arbitrary compact set $\Omega \subset \mathbb{R}^{l+n}$ centered at the origin, the closed-loop system is locally asymptotically stable (LAS) with the region of attraction containing Ω.

We acknowledge that our work is motivated by the pioneer works[17,31]. The motivation and main contribution are outlined as follows.

Motivation: 1) Applicability: Numerous UASs such as the systems[1–5] can be formulated by our concerned system (3) by choosing an appropriate y. 2) Theoretical value: Our concerned system (3) is originated from the partially linear composite system (1). Although system (1) has been studied since the 80s[3,23,27–29], accomplishing global or semi-global partial state feedback control for such a system (1) under weaker model assumptions remains a challenge. 3) Practical value: Our approach requires fewer measurement sensors compared to the full state feedback and is more friendly to users as it exclusively requires the states of the linear subsystem and does not care about the exact form of the nonlinear η-subsystem.

Contribution: Our approach requires fewer assumptions on the nonlinear η subsystem, particularly avoiding global growth conditions or the input-to-state stability condition. Specifically, the concerned cross-term $\varphi(\xi_{j_i}, \xi_{j_i+1})\xi_{j_i+1}$ involves any two consecutive integrators in each chain and allows $\varphi(\xi_{j_i}, \xi_{j_i+1})$ to be a function of $\xi_{j_i}^i, \xi_{j_i+1}^i$, which is not discussed in the works[18,29]. Moreover, the cascade system (3) consists of the concerned cross-term and Byrnes-Isidori normal form, which is not discussed in the work[17].

This paper is structured into five sections. Section 2 introduces the peaking phenomenon, the analysis tool, and the design tool. Section 3 conducts the problem transformation. Section 4 presents the composite controller for the cascade system. Section 5 provides an illustrative example. Section 6 presents a brief concluding remark.

2 | PRELIMINARIES

2.1 | Peaking phenomenon

We begin with two definitions about non-peaking signals in high-gain feedback and low-gain feedback. Consider the parameterized state feedback

$$u = F(k)x,\ k \in [1, \infty)\,,$$

for the pair (A, B), which results in the closed-loop system

$$\dot{x} = (A + BF(k))x,\ h(k,t) = H(k)x.$$

Based on Definition 4.30 in the reference [17], we have the following definition.

Definition 1. The signal $h(k,t)$ is said to be high-gain non-peaking if there exists a constant $k_0 \in [1, \infty)$ such that

$$\sup_{t>0} \|h(k,t)\| \le \gamma \|x(0)\| \,\mathrm{e}^{-k\sigma t} + \epsilon,\ \forall k \in [k_0, \infty),$$

where γ, σ, ϵ are positive constants independent of k.

Similarly, we investigate the low-gain non-peaking phenomenon based on the work [32]. Consider the parameterized state feedback for the linear system (A, B),

$$u = F(\varepsilon)x,\ \varepsilon \in (0, 1],$$

such that the closed-loop system

$$\dot{x} = (A + BF(\varepsilon))x,\ h(\varepsilon,t) = H(\varepsilon)x.$$

Definition 2. The signal $h(\varepsilon,t)$ is said to be low-gain non-peaking if there exists a constant $\varepsilon_0 \in (0, 1]$ such that

$$\sup_{t>0} \|h(\varepsilon,t)\| \le \gamma\varepsilon \|x(0)\| + \varepsilon,\ \forall \varepsilon \in (0, \varepsilon_0],$$

where γ is a positive constant independent of ε.

2.2 | Analysis tool

We now recall the definitions about class $\mathcal{K}$ function $\alpha(\cdot)$ and class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ form the well-known work [33].

Definition 3. A continuous function $\alpha(\cdot): [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$, where a is a nonnegative constant. A continuous function $\beta(\cdot,\cdot): [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the mapping $\beta(s,t)$ belongs to class $\mathcal{K}$ with respect to s and, for each fixed t, the mapping $\beta(s,t)$ is decreasing with respect to t and $\lim_{t\to\infty} \beta(s,t) = 0$, where a, s, t are nonnegative constants.

Motivated by Theorem 4.41 in the work [17], we then introduce a basic analysis tool in our work.

Theorem 1. Consider the nonlinear system

$$\dot{\eta} = f(\eta, u), \tag{4}$$

where $\eta \in \mathbb{R}^l$, $u \in \mathbb{R}^m$ and f is a smooth function with $f(0,0) = 0$. Suppose that

- the equilibrium $\eta = 0$ of the unforced dynamic $\dot{\eta} = f(\eta, 0)$ is GAS;
- $u = u(k,t)$ satisfies

$$\|u(k,t)\| \le \gamma \mathrm{e}^{-kt} + \epsilon,\ \forall k \in [k_0, \infty),\ \forall t \in [0, \infty)\,, \tag{5}$$

$$\lim_{t\to\infty} \|u(k,t)\| = 0, \tag{6}$$

  where $k_0 \in [1, \infty)$, and γ, ϵ are positive constants independent of k.

Then the nonlinear system (4) is semi-globally asymptotically stabilized by such a control input $u = u(k,t)$, that is, given an arbitrary compact set $\Omega_\eta \subset \mathbb{R}^l$ centered at the origin, there exists a constant $k_* \in [k_0, \infty)$ such that, for any $k \in [k_*, \infty)$, the nonlinear system (4) is LAS with the region of attraction containing Ω_η.

Proof. Without loss of generality, we can set $\epsilon = 1$. By applying a decomposition for $f(\eta, u)$ on any convex set, we have

$$\dot{\eta} = f(\eta, 0) + g(\eta, u)u, \tag{7}$$

where $g(\eta, u)$ is a smooth function. Consider an auxiliary subsystem

$$\dot{\chi} = -k\chi + 2k, \ \ \chi_0 = \chi(0) = \gamma + 2, \tag{8}$$

whose solution is given by

$$\chi(t) = \left(\chi_0 - 2\right) \mathrm{e}^{-kt} + 2 = \gamma \mathrm{e}^{-kt} + 2. \tag{9}$$

It follows from (5) that

$$\|u(k,t)\| < \chi(t) \le \chi_0, \ \forall t \ge 0. \tag{10}$$

Since $\dot{\eta} = f(\eta, 0)$ is GAS, there exists a radially unbounded Lyapunov function $W(\eta)$ (see Theorem 4.16 in the reference[33]) such that

$$\frac{\partial W(\eta)}{\partial \eta} f(\eta, 0) \le -\alpha(\|\eta\|), \ \forall \eta \in \mathbb{R}^l, \tag{11}$$

where $\alpha(\cdot)$ is a class $\mathcal{K}$ function defined on $[0, \infty)$. Moreover, for any positive constant c, the level set $\{\eta : W(\eta) \le c\}$ is a compact set. Define a positive definite function $V(\eta, \chi) = W(\eta) + \chi^2$ and a compact level set

$$\Lambda = \left\{(\eta, \chi) : V(\eta, \chi) \le c + \chi_0^2\right\},$$

where $c \ge 1$ is a constant chosen such that $\Omega_\eta \times \{\chi : \chi = \chi(0)\} \subset \Lambda$. Besides, the level surface of Λ is defined as

$$\partial\Lambda = \left\{(\eta, \chi) : V(\eta, \chi) = c + \chi_0^2\right\}.$$

The boundedness of $\eta(t)$ will be shown by a contradiction. Suppose that there exists a finite time t_c such that the trajectory $(\eta(t), \chi(t))$ with $(\eta(0), \chi(0)) \in \Omega_\eta \times \{\chi : \chi = \chi(0)\}$ arrives at the boundary of Λ (namely, $\partial\Lambda$) in the time t_c and leave it thereafter, that is,

$$(\eta(t_c), \chi(t_c)) \in \partial\Lambda \Rightarrow \dot{V}(\eta(t_c), \chi(t_c)) > 0. \tag{12}$$

Along with (10) and χ_0 being independent of k, there exists a positive constant γ_1 independent of k such that

$$\left\|\frac{\partial W(\eta)}{\partial \eta} g(\eta, u)\right\| \le \gamma_1, \ \forall (\eta, \chi) \in \partial\Lambda. \tag{13}$$

The time derivative of $V(\eta, \chi)$ along the trajectories of system (7) and system (8) is given by

$$\begin{aligned} \dot{V}(\eta, \chi) &= \frac{\partial W(\eta)}{\partial \eta} f(\eta, 0) + \frac{\partial W(\eta)}{\partial \eta} g(\eta, u)u + 2\chi\dot{\chi} \\ &\le -\alpha(\|\eta\|) + \gamma_1 \|u\| - 2k\chi^2 + 2k\chi \\ &< -\alpha(\|\eta\|) - 2k\chi^2 + \left(2k + \gamma_1\right)\chi, \ \forall (\eta, \chi) \in \partial\Lambda, \end{aligned} \tag{14}$$

where (10), (11), and (13) are used. In view of (9), we have $\chi(t) = \gamma \mathrm{e}^{-kt} + 2 > 2$. Thus there exists a sufficiently large constant $k_1 \in [1, \infty)$ such that, for all $k \in [k_1, \infty)$,

$$-2k\chi^2 + \left(2k + \gamma_1\right)\chi = -\left(2k\chi - 2k - \gamma_1\right)\chi \le 0. \tag{15}$$

Then substituting (15) into (14) yields, for all $k \in [k_1, \infty)$,

$$\dot{V}(\eta, \chi) < -\alpha(\|\eta\|) \le 0, \ \forall (\eta, \chi) \in \partial\Lambda,$$

which contradicts (12). As a result, the trajectory $(\eta(t), \chi(t))$ with its initial condition $(\eta(0), \chi(0)) \in \Omega_\eta \times \{\chi : \chi = \chi(0)\}$ cannot leave the compact set Λ. This further implies a positively invariant compact set Λ_η (containing Ω_η) for the trajectory $\eta(t)$, that is, for all $k \in [k_1, \infty)$,

$$\eta(0) \in \Lambda_\eta \Rightarrow \eta(t) \in \Lambda_\eta, \ \forall t \ge 0. \tag{16}$$

We next show the convergence when the trajectory $\eta(t)$. The time derivative of $W(\eta)$ along the trajectory of system (7) is

$$\dot{W}(\eta) = \frac{\partial W(\eta)}{\partial \eta} f(\eta, 0) + \frac{\partial W(\eta)}{\partial \eta} g(\eta, u)u.$$

Along with (10), there exists a positive constant γ_2 independent of k such that

$$\left\|\frac{\partial W(\eta)}{\partial \eta} g(\eta, u)\right\| \le \gamma_2, \ \forall \eta \in \Lambda_\eta,$$

substituting of which with (16) and (11) into $\dot{W}(\eta)$ yields

$$\begin{aligned}\eta(0) \in \Lambda_\eta \Rightarrow \dot{W}(\eta) &\leq -\alpha(\|\eta\|) + \gamma_2 \|u\| \\ &= -(1-\theta)\,\alpha(\|\eta\|) - \theta\alpha(\|\eta\|) + \gamma_2 \|u\|,\end{aligned} \tag{17}$$

where $\theta \in (0,1)$ is a constant. It follows from

$$-\theta\alpha(\|\eta\|) + \gamma_2 \|u\| \leq 0,\ \forall \|\eta\| \geq \alpha^{-1}\left(\gamma_2 \|u\|/\theta\right),$$

that $\dot{W}(\eta)$ in (17) can be continued as, for all $k \in [k_1, \infty)$,

$$\eta(0) \in \Lambda_\eta \Rightarrow \dot{W}(\eta) \leq -(1-\theta)\,\alpha(\|\eta\|),\ \forall \|\eta\| \geq \alpha^{-1}\left(\gamma_2 \|u\|/\theta\right).$$

By Theorem 4.18 in the work[33], there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\kappa(\cdot)$ such that, for all $k \in [k_1, \infty)$,

$$\eta(0) \in \Lambda_\eta \Rightarrow \|\eta(t)\| \leq \beta(\|\eta(0)\|, t) + \kappa\left(\sup_{0\leq t\leq\tau} \|u(t)\|\right),$$

where τ is a positive constant. It follows from Exercise 4.58 in the work[33] that, for any $k \in [k_*, \infty)$, $\lim_{t\to\infty} \|\eta(t)\| = 0$ as long as $\lim_{t\to\infty} \|u(k,t)\| = 0$ (namely, (5)). Choose $k_* = \max\{k_0, k_1\}$. It then follows from (5) and $\Omega_\eta \subset \Lambda_\eta$ that, for any $k \in [k_*, \infty)$, the equilibrium $\eta = 0$ of the system (4) is LAS with the region of attraction containing Ω_η. □

Remark 2. It is well-known that a bounded (even vanishing) input u may drive the trajectory of system (4) to infinity (see Exercise 9.13 in[33]) when $\dot{\eta} = f(\eta, 0)$ is limited to be GAS. This underscores the importance of Theorem 1, which identifies a class of inputs capable of semi-globally stabilizing such a nonlinear system (4). Consider the cascade system (2) or its more general case

$$\begin{cases}\dot{\eta} = f(\eta, y) \\ \dot{x} = \Upsilon(\eta, x, u) \\ y = \Pi(x),\end{cases} \tag{18}$$

where the nonlinear dynamic $\dot{\eta} = f(\eta, 0)$ is GAS (namely, Assumption 2 is fulfilled). If the subsystem $\dot{x} = \Upsilon(\eta, x, u)$ is globally (or semi-globally) stabilized by the input $u(\eta, x, k)$ and its output y satisfies (5)-(6), then such a cascade system (18) is semi-globally stabilized. This implies a separate design for semi-global stabilization of cascade systems. In addition, consider the situation involving observer-based output feedback, for instance, η denotes the plant state and x represents the observer error (or observer estimate). If the observer-error subsystem (i.e., x-subsystem) has been proven to be GAS with its observer-error-associated term y satisfying (5)-(6), the stability analysis shall be completed by Theorem 1 directly. Therefore, for the observer-based output feedback, Theorem 1 may serve as an analogy to the separation principle for linear systems.

2.3 | Design tool

Since low-gain feedback serves as one of our design tools, we should introduce some properties of low-gain feedback. Consider the single input linear system $\dot{x} = \Phi x + B_c u$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and

$$\Phi = \left[\begin{array}{c|ccc} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hline * & * & \cdots & * \end{array}\right],\ B_c = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right], \tag{19}$$

with $*$s being constants. Clearly, Φ is a companion matrix whose characteristic polynomial is determined by its last row. Let $\Phi_0 = \Phi$ if the last row of Φ is zero.

For the chain form system (Φ_0, B_c), we recall a result from Lemma 2.2.1 in the reference[31].

Lemma 1. [31] Consider the pair (Φ_0, B_c). Design the low-gain feedback as $u = F(\varepsilon)x$, where $F(\varepsilon) = [\,\varepsilon^n b_1\ \ \varepsilon^{n-1} b_2\ \cdots\ \varepsilon b_n\,]$ and $b_i,\ i = 1, 2, ..., n$ are chosen such that the polynomial $p(s) = s^n + b_n s^{n-1} + \cdots + b_2 s + b_1$ is Hurwitz. Then, for any $\varepsilon \in (0, 1]$, the closed-loop system $\dot{x} = (\Phi_0 + B_c F)x$ is GES and

$$\|F(\varepsilon)\| \leq \beta_1 \varepsilon, \tag{20}$$

$$\left\|e^{(A+BF(\varepsilon))t}\right\| \leq \frac{\beta_2}{\varepsilon^{n-1}} e^{-\varepsilon\sigma t},\ \forall t \geq 0, \tag{21}$$

$$\left\| F(\varepsilon)\mathrm{e}^{(A+BF(\varepsilon))t} \right\| \le \beta_3 \varepsilon \mathrm{e}^{-\varepsilon\sigma t},\ \forall t \ge 0, \tag{22}$$

where β_1, β_2, β_3, σ are positive constants independent of ε.

Proof. As the concerned pair $(\boldsymbol{\Phi}_0, \boldsymbol{B}_c)$ is a particular case of the controllable form in Lemma 2.2.1 of the reference[31], we can perform this proof in another simple way. Consider the coordinate change $\bar{x} = Tx$, $T = \mathrm{diag}(\varepsilon^{n-1}, \varepsilon^{n-2}, \ldots, 1)$ for the closed-loop system. Then, we have $\dot{\bar{x}} = \varepsilon \boldsymbol{\Phi}_b \bar{x}$, where $\boldsymbol{\Phi}_b$ is the companion matrix whose last row consists of the coefficients of the Hurwitz polynomial $p(s)$. It follows that

$$x(t) = T^{-1}\mathrm{e}^{\boldsymbol{\Phi}_b \varepsilon t} T x(0). \tag{23}$$

As $\boldsymbol{\Phi}_b$ is Hurwitz, we have $\|\mathrm{e}^{\boldsymbol{\Phi}_b t}\| \le \beta_2 \mathrm{e}^{-\sigma t}$, where β_2 and σ are positive constants independent of ε. Substituting $\|T^{-1}\| \le 1/\varepsilon^{n-1}$ and $\|T\| \le 1$ into (23) yields

$$\|x(t)\| \le \beta_2 \left\| T^{-1} \right\| \|x(0)\| \mathrm{e}^{-\varepsilon\sigma t} \le \frac{1}{\varepsilon^{n-1}} \beta_2 \|x(0)\| \mathrm{e}^{-\varepsilon\sigma t},$$

which further implies (21). Notice that $F(\varepsilon)T^{-1} = [\, \varepsilon b_1 \ \ \varepsilon b_2 \ \cdots \ \varepsilon b_n \,]$ and $\|T\| \le 1$. Thus, there exists a positive constant β_3 independent of ε such that

$$\|F(\varepsilon)x\| \le \left\| F(\varepsilon)T^{-1} \right\| \|T\| \left\| \mathrm{e}^{\boldsymbol{\Phi}_b \varepsilon t} x(0) \right\| \le \beta_3 \varepsilon \|x(0)\| \mathrm{e}^{-\varepsilon\sigma t},$$

which further implies (22). The proof is finished by noting that (20) holds trivially. □

Remark 3. Similar to the work[32], it follows from Lemma 1 that each integrator x_i, $i = 1, 2, ..., n$ of the closed-loop system peaks slowly with growth rate $1/\varepsilon^{n-i}$, that is, $\sup_{t>0} |x_i| \le \frac{1}{\varepsilon^{n-i}} \beta_2 \|x(0)\| \mathrm{e}^{-\varepsilon\sigma t}$, $i = 1, 2, ..., n$.

For the ANCBC pair (A, B), a low-gain feedback law $u = F(\varepsilon)x$ can be designed as[34]

$$F(\varepsilon) = \Lambda_1 T^{-1}, \tag{24}$$

with

$$\Lambda_1 = \mathrm{diag}(F_1(\varepsilon^{(p_2+1)(p_3+1)\cdots(p_q+1)}), \ldots, F_{q-1}(\varepsilon^{(p_q+1)}), F_q(\varepsilon)),$$

where $F_i(\cdot)$, $i = 1, 2, ..., q$ are gain matrices determined by Lemma 2.2.1 in the reference[31]. The transformation matrix T and positive constants p_i, $i = 1, 2, ..., q$ are determined naturally during searching for the triangular structure of A.

Lemma 2. [34] Consider the ANCBC pair (A, B). Under the low-gain feedback (24), there exists a constant $\varepsilon_0 \in (0, 1]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the closed-loop system $\dot{x} = (A + BF(\varepsilon))x$ is GES and

$$\|F(\varepsilon)\| \le \bar{\beta}_1 \varepsilon, \tag{25}$$

$$\left\| \mathrm{e}^{(A+BF(\varepsilon))t} \right\| \le \frac{\bar{\beta}_2}{\varepsilon^{\nu-1}} \mathrm{e}^{-\varepsilon^{\rho} t},\ \forall t \ge 0, \tag{26}$$

$$\left\| F(\varepsilon)\mathrm{e}^{(A+BF(\varepsilon))t} \right\| \le \bar{\beta}_3 \varepsilon \mathrm{e}^{-\varepsilon^{\rho} t},\ \forall t \ge 0, \tag{27}$$

where $\nu = p_1 \prod_{i=2}^{q}(p_i + 1)$, $\rho = \prod_{i=2}^{q}(p_i + 1)$ and $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\beta}_3$ are positive constants independent of ε.

3 | PROBLEM TRANSFORMATION

Now consider the parameterized state feedback controller

$$u_i = u_i(k, \xi_0, \xi),\ i = 1, 2, ..., m,$$

for the system (3), where $k \in [1, \infty)$ is a parameter to be determined. The closed-loop system is

$$\begin{cases} \dot{\eta} = f(\eta, \varphi(\xi_{j_i}, \xi_{j_i+1})\xi_{j_i+1}) \\ \dot{\xi}_0 = A_0\xi_0 + B_0\xi_1 \\ \dot{\xi}_1^i = \xi_2^i \\ \dot{\xi}_2^i = \xi_3^i \\ \quad \vdots \\ \dot{\xi}_{r_i}^i = u_i(k, \xi_0, \xi),\ i = 1, 2, ..., m. \end{cases}$$

Lemma 3. Suppose that there exists a constant $k_0 \in [1, \infty)$ such that

$$\left\|\xi_{j_i}(k,t)\right\| \le \kappa_1,\ \forall k \in [k_0, \infty),\ \forall t \in [0, \infty), \tag{28}$$

$$\left\|\xi_{j_i+1}(k,t)\right\| \le \kappa_2 \mathrm{e}^{-kt} + \epsilon,\ \forall k \in [k_0, \infty),\ \forall t \in [0, \infty), \tag{29}$$

$$\lim_{t\to\infty} \left\|\xi_{j_i+1}(k,t)\right\| = 0,$$

are satisfied for $i = 1, 2, ..., m$, where $\kappa_1, \kappa_2, \epsilon$ are positive constants independent of k. Then the cross-term $\mu(k,t) \triangleq \varphi(\xi_{j_i}, \xi_{j_i+1})\xi_{j_i+1}$ also satisfies

$$\|\mu(k,t)\| \le \kappa_0 \mathrm{e}^{-kt} + \epsilon,\ \forall k \in [k_0, \infty),\ \forall t \in [0, \infty),$$

$$\lim_{t\to\infty} \|\mu(k,t)\| = 0,$$

where κ_0 is a positive constant independent of k.

Proof. Since the states $\xi_{j_i}(k,t)$ and $\xi_{j_i+1}(k,t)$ are bounded for any $k \in [k_0, \infty)$ and $\varphi(\cdot,\cdot)$ is continuous, we know that $\varphi(\xi_{j_i}, \xi_{j_i+1})$ is bounded. As $\xi_{j_i+1}(k,t)$ satisfies (29) and $\lim_{t\to\infty} \xi_{j_i+1}(k,t) = 0$, we clearly have that $\mu(k,t)$ also satisfies (29) and $\lim_{t\to\infty} \mu(k,t) = 0$. The proof is finished. □

Let conditions in Lemma 3 be met. It then follows from Theorem 1 that the η-subsystem under Assumption 2 can be semi-globally asymptotically stabilized by the cross-term $\mu(k,t)$. Thus the possible stability degradation caused by the η subsystem can be ignored in this case. Since the conditions in Lemma 3 only involve an individual ξ subsystem, it is sufficient to consider the following problem for Problem 1.

Problem 2. Let Assumptions 1-3 be met. Find state feedback $u_i = u_i(k, \xi)$, $i = 1, 2, ..., m$ and a constant $k_0 \in [1, \infty)$ such that, for any $k \in [k_0, \infty)$, the linear part of the cascade system (3) is globally exponentially stabilized and the conditions (28)-(29) are satisfied.

It can be observed from Problem 2 that the stabilization problem for the entire cascade system (3) is reduced to the stabilization problem for the linear part subject to high-gain non-peaking constraint (29) on ξ_{j_i+1} and uniformly bounded (with respect to k) constraint (28) on ξ_{j_i}. As we shall see, Problem 2 can be addressed by a backstepping scheme.

4 | PARTIAL STATE FEEDBACK STABILIZATION

4.1 | SISO case

In the SISO case, the nonlinear cascade system (3) can be rewritten as

$$\begin{cases} \dot{\eta} = f(\eta, \varphi(\xi_{j_0}, \xi_{j_0+1})\xi_{j_0+1}) \\ \dot{\xi}_0 = A_0\xi_0 + B_0\xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_r = u, \end{cases} \tag{30}$$

where $j_0 \in \{1, 2, ..., r-1\}$. We now start the backstepping design.

Step 1: This step will be accomplished by the linear low-gain feedback approach in Lemma 2. Design the low-gain feedback

$$\alpha_1 = F_1(\varepsilon_1)\xi_0$$

as the virtual control input to the ANCBC pair (A_0, B_0), where $F_1(\varepsilon_1)$ is an appropriate gain matrix in the form of (24), with the low-gain parameter ε_1 to be determined. Then, by denoting $\bar{\xi}_j = \xi_j - \alpha_1^{(j-1)}$, $j = 1, 2, ..., r$, the linear part of the SISO cascade system (30) can be formulated as

$$\begin{cases} \dot{\xi}_0 = \left(A_0 + B_0 F_1\right) \xi_0 + B_0 \bar{\xi}_1 \\ \dot{\bar{\xi}}_1 = \bar{\xi}_2 \\ \dot{\bar{\xi}}_2 = \bar{\xi}_3 \\ \quad \vdots \\ \dot{\bar{\xi}}_r = u + \alpha_1^{(r)}. \end{cases} \tag{31}$$

Step 2: This step will be accomplished via another linear low-gain feedback in Lemma 1. Divide the chain of integrators into two parts, one part is $\dot{\bar{\xi}}_j = \bar{\xi}_{j+1}$, $j = 1, 2, ..., j_0$, and another is $\dot{\bar{\xi}}_{j_0+j} = \bar{\xi}_{j_0+j+1}$, $j = 1, 2, ..., j_0$. Besides, define $w = [\, w_1 \ \ w_2 \ \cdots \ w_{j_0} \,]^{\mathrm{T}}$, $w_j = \bar{\xi}_j$, $j = 1, 2, ..., j_0$ and $\tilde{r} = r - j_0$. Then, the cascade system (31) becomes

$$\begin{cases} \dot{\xi}_0 = \left(A_0 + B_0 F_1\right) \xi_0 + B_0 w_1 \\ \dot{w} = \Phi_0 w + B_{\mathrm{c}} \bar{\xi}_{j_0+1} \\ \dot{\bar{\xi}}_{j_0+1} = \bar{\xi}_{j_0+2} \\ \dot{\bar{\xi}}_{j_0+2} = \bar{\xi}_{j_0+3} \\ \quad \vdots \\ \dot{\bar{\xi}}_r = u + \alpha_1^{(r)}, \end{cases} \tag{32}$$

where $\Phi_0 \in \mathbb{R}^{j_0 \times j_0}$ and $B_{\mathrm{c}} \in \mathbb{R}^{j_0 \times 1}$ are in the form of (19). Notice that the w subsystem in system (32) is a chain of integrators. Then we can design a low-gain feedback

$$\alpha_2 = F_2(\varepsilon_2) w$$

as the visual control, where $F_2(\varepsilon_2)$ is the gain matrix in Lemma 1, with the low-gain parameter ε_2 to be determined. Subsequently, by defining $e_j = \bar{\xi}_{j_0+j} - \alpha_2^{(j-1)}$, $j = 1, 2, ..., \tilde{r}$, the cascade system (32) becomes

$$\begin{cases} \dot{\xi}_0 = \left(A_0 + B_0 F_1\right) \xi_0 + B_0 w_1 \\ \dot{w} = \left(\Phi_0 + B_{\mathrm{c}} F_2\right) w + B_{\mathrm{c}} e_1 \\ \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \quad \vdots \\ \dot{e}_{\tilde{r}} = u - \alpha_1^{(r)} - \alpha_2^{(\tilde{r})}. \end{cases} \tag{33}$$

Step 3: Finally, the entire composite feedback law can be designed as

$$\begin{aligned} u &= \alpha_1^{(r)} + \alpha_2^{(\tilde{r})} + v, \\ v &= -\sum_{j=1}^{\tilde{r}} k^{\tilde{r}-j+1} c_j e_j, \end{aligned} \tag{34}$$

where k is the high-gain to be determined and c_j, $j = 1, 2, ..., \tilde{r}$, are chosen such that $p_{\mathrm{c}}(s) = s^{\tilde{r}} + c_{\tilde{r}} s^{\tilde{r}-1} + \cdots + c_2 s + c_1$ is Hurwitz. Let $\bar{e}_j = e_j / k^{j-1}$, $j = 1, 2, ..., \tilde{r}$. Then, the cascade system (33) can be formulated as

$$\begin{cases} \dot{\xi}_0 = \left(A_0 + B_0 F_1\right) \xi_0 + B_0 w_1 \\ \dot{w} = \left(\Phi_0 + B_{\mathrm{c}} F_2\right) w + B_{\mathrm{c}} \bar{e}_1 \\ \dot{\bar{e}} = k \Phi_{\mathrm{c}} \bar{e}, \end{cases} \tag{35}$$

where $\Phi_{\mathrm{c}} \in \mathbb{R}^{\tilde{r} \times \tilde{r}}$ is a companion matrix whose last row consists of the coefficients of the polynomial $p_{\mathrm{c}}(s)$.

A selection criterion for the controller parameters $\varepsilon_1, \varepsilon_2, k$ is stated in the following lemma.

Lemma 4. Let Assumption 3 be satisfied and set $\varepsilon_2 = 1/k$ and $\varepsilon_1 = \varepsilon_2^{j_0+1}$. Then, there exists a constant $k_0 \in [1, \infty)$ such that, for any $k \in [k_0, \infty)$, the linear part of the cascade system (30) is globally exponentially stabilized by the linear state feedback law (34) and the states ξ_{j_0} and ξ_{j_0+1} satisfy (28)-(29) respectively.

Proof. Notice that $\boldsymbol{\Phi}_{\mathrm{c}}$, $\boldsymbol{\Phi}_0 + B_{\mathrm{c}}F_2$, and $A_0 + B_0F_1$ are Hurwitz. Then, it remains to show that (28)-(29) holds. Firstly, in view of $\bar{\xi}_j = \xi_j - \alpha_1^{(j-1)}$, $j = 1, 2, \dots, r$, we have the fact that, for all $\varepsilon_1 \in (0, 1]$,

$$\|\bar{\xi}(0)\| \leq \gamma_1, \tag{36}$$

where γ_1 is a positive constant independent of k. With coordinate changes in **Step 2** and **Step 3**, there exist two positive constants γ_2 and γ_3 independent of k such that, for all $k \in [1, \infty)$,

$$\|w(0)\| \leq \gamma_2 \|\bar{\xi}(0)\|, \ \|\bar{e}(0)\| \leq \gamma_3 \|\bar{\xi}(0)\|.$$

In the sequence, the fact that $\bar{\xi}_{j_0+1}$ is high-gain non-peaking will be verified. It follows from the third equation of (35) that $\bar{e}_1 = C_{\mathrm{c}}\mathrm{e}^{k\boldsymbol{\Phi}_{\mathrm{c}}t}\bar{e}(0)$ with $C_{\mathrm{c}} = [\,1 \;\; 0_{1\times(\bar{r}-1)}\,]$, which further implies that, for all $k \in [1, \infty)$,

$$|\bar{e}_1(t)| \leq \gamma_3 \|\bar{\xi}(0)\| \mathrm{e}^{-k\sigma_{\mathrm{c}}t}, \tag{37}$$

where σ_{c} is determined by the eigenvalues of $\boldsymbol{\Phi}_{\mathrm{c}}$. It follows from the second equation of (35) that

$$w(t) = \mathrm{e}^{(\boldsymbol{\Phi}_0 + B_{\mathrm{c}}F_2)t}w(0) + \int_0^t \mathrm{e}^{(\boldsymbol{\Phi}_0 + B_{\mathrm{c}}F_2)(t-\tau)}B_{\mathrm{c}}\bar{e}_1(\tau)\mathrm{d}\tau. \tag{38}$$

Substituting (22) associated with $F_2(\varepsilon_2)$, (37) and (38) into $\alpha_2 = F_2(\varepsilon_2)w$ yields

$$\begin{aligned}
|\alpha_2(t)| &\leq \beta_3\gamma_2\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t} + \beta_3\gamma_3\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t}\int_0^t \mathrm{e}^{-(k\sigma_{\mathrm{c}}-\varepsilon_2\sigma)\tau}\mathrm{d}\tau \\
&\leq \beta_3\gamma_2\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t} + \frac{\beta_3\gamma_3\varepsilon_2\|\bar{\xi}(0)\|}{k\sigma_{\mathrm{c}} - \varepsilon_2\sigma}\mathrm{e}^{-\varepsilon_2\sigma t}\left(-\mathrm{e}^{-(k\sigma_{\mathrm{c}}-\varepsilon_2\sigma)t} + 1\right) \\
&\leq \beta_3\gamma_2\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t} + \frac{\beta_3\gamma_3\varepsilon_2\|\bar{\xi}(0)\|}{k\sigma_{\mathrm{c}} - \varepsilon_2\sigma}\mathrm{e}^{-\varepsilon_2\sigma t}.
\end{aligned}$$

For all $k \in [\sqrt{2\sigma/\sigma_{\mathrm{c}}}, \infty)$, we have $k\sigma_{\mathrm{c}} - \varepsilon_2\sigma \geq k\sigma_{\mathrm{c}}/2$ and thus

$$|\alpha_2(t)| \leq \beta_3\gamma_2\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t} + \frac{2\beta_3\gamma_3\varepsilon_2}{k\sigma_{\mathrm{c}}}\|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2\sigma t}. \tag{39}$$

Since the focus is the existence of k to guarantee that (39) holds, we can denote $\sigma = \sigma_{\mathrm{c}} = 1$ without loss of generality. It then follows from $k - \varepsilon_2 \geq k/2$, $\forall k \in [\sqrt{2}, \infty)$ that, for all $k \in [\sqrt{2}, \infty)$,

$$|\alpha_2(t)| \leq \beta_3\gamma_2\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2 t} + \frac{2\beta_3\gamma_3\varepsilon_2}{k}\|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2 t} = \gamma_4\varepsilon_2 \|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2 t}, \tag{40}$$

with $\gamma_4 = \beta_3\gamma_2 + \sqrt{2}\beta_3\gamma_3$. Substituting (37) and (40) into $\bar{\xi}_{j_0+1} = \bar{e}_1 + \alpha_2$ yields, for all $k \in [\sqrt{2}, \infty)$,

$$\left|\bar{\xi}_{j_0+1}(t)\right| \leq \gamma_3 \|\bar{\xi}(0)\| \mathrm{e}^{-kt} + \frac{\gamma_4}{k}\|\bar{\xi}(0)\| \mathrm{e}^{-t/k}, \tag{41}$$

which implies that the state $\bar{\xi}_{j_0+1}$ is high-gain non-peaking over $k \in [\sqrt{2}, \infty)$, namely,

$$\left|\bar{\xi}_{j_0+1}(t)\right| \leq \gamma_5 \|\bar{\xi}(0)\| \left(\mathrm{e}^{-kt} + \frac{1}{k}\right), \tag{42}$$

where $\gamma_5 = \gamma_3 + \gamma_4$.

Subsequently, we show that the state w_{j_0} (also denoted as $\bar{\xi}_{j_0}$) is uniformly bounded with respect to ε_2. Notice from (33) that $\dot{w}_{j_0} = \bar{\xi}_{j_0+1}$. Along with (41) and $k = 1/\varepsilon_2$, integrating $\dot{w}_{j_0}(t) = \bar{\xi}_{j_0+1}(t)$ yields

$$\begin{aligned}
\left|w_{j_0}(t)\right| &\leq \left|w_{j_0}(0)\right| + \int_0^t \left|\bar{\xi}_{j_0+1}(\tau)\right| \mathrm{d}\tau \\
&\leq \left|w_{j_0}(0)\right| + \|\bar{\xi}(0)\| \int_0^t \left(\gamma_3\mathrm{e}^{-k\tau} + \frac{\gamma_4}{k}\mathrm{e}^{-\tau/k}\right)\mathrm{d}\tau \\
&\leq \left|w_{j_0}(0)\right| + \gamma_3\varepsilon_2 \|\bar{\xi}(0)\| + \gamma_4 \|\bar{\xi}(0)\|,
\end{aligned} \tag{43}$$

where $w_{j_0}(0)$ is the initial value independent of ε_2. Parallel to the procedure of (38)-(40), substituting (21) associated with $F_2(\varepsilon_2)$ and (37) into (38), we have, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\|w(t)\| \leq \frac{\gamma_4}{\varepsilon_2^{j_0-1}}\|\bar{\xi}(0)\| \mathrm{e}^{-\varepsilon_2 t}, \tag{44}$$

which together with (43) implies that, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| w_{j_0}(t) \right| \le \gamma_4 \left\| \bar{\xi}(0) \right\| \mathrm{e}^{-\varepsilon_2 t}. \tag{45}$$

Then, in view of Remark 3, we have, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| w_j(t) \right| \le \frac{\gamma_4}{\varepsilon_2^{j_0 - j}} \left\| \bar{\xi}(0) \right\| \mathrm{e}^{-\varepsilon_2 t},\ j = 1, 2, \dots, j_0. \tag{46}$$

We now show the low-gain non-peaking property of $\alpha_1^{(j)}$, $j = 1, 2, \dots, j_0$. It follows from (35) that the jth order derivative of $\alpha_1 = F_1(\varepsilon_1)\xi_0$ is

$$\alpha_1^{(j)}(t) = F_1 \left(A_0 + B_0 F_1 \right)^j \xi_0(t) + \sum_{l=1}^{j} F_1 \left(A_0 + B_0 F_1 \right)^{j-l} B_0 w_l(t). \tag{47}$$

Define

$$\Delta_1(t) = F_1 (A_0 + B_0 F_1)^j \xi_0(t),\ \Delta_2(t) = \sum\nolimits_{l=1}^{j} F_1 (A_0 + B_0 F_1)^{j-l} B_0 w_l(t).$$

Solving $\xi_0(t)$ from (35) yields

$$\xi_0(t) = \mathrm{e}^{\left(A_0 + B_0 F_1\right)t} \xi_0(0) + \int_0^t \mathrm{e}^{\left(A_0 + B_0 F_1\right)(t-\tau)} B_0 w_1(\tau) \mathrm{d}\tau. \tag{48}$$

Substituting $|w_1(t)|$ in (46) and (48) into $\Delta_1(t)$ yields, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\begin{aligned} \left| \Delta_1(t) \right| \le {} & \frac{\gamma_4}{\varepsilon_2^{j_0 - 1}} \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \int_0^t \left\| F_1 \left(A_0 + B_0 F_1 \right)^j \mathrm{e}^{\left(A_0 + B_0 F_1\right)(t-\tau)} \right\| \mathrm{e}^{-\varepsilon_2 \tau} \mathrm{d}\tau \\ & + \left\| F_1 \left(A_0 + B_0 F_1 \right)^j \mathrm{e}^{\left(A_0 + B_0 F_1\right)t} \right\| \left\| \xi_0(0) \right\|. \end{aligned} \tag{49}$$

By using

$$\left(A_0 + B_0 F_1 \right) \mathrm{e}^{\left(A_0 + B_0 F_1\right)t} = \mathrm{e}^{\left(A_0 + B_0 F_1\right)t} \left(A_0 + B_0 F_1 \right),$$

and (27) associated with $F_1(\varepsilon_1)$, we have

$$\left\| F_1 \left(A_0 + B_0 F_1 \right)^j \mathrm{e}^{\left(A_0 + B_0 F_1\right)t} \right\| \le \gamma_6 \bar{\beta}_3 \varepsilon_1 \mathrm{e}^{-\varepsilon_1^\rho t}, \tag{50}$$

where $\gamma_6 = \sup_{\varepsilon_1 \in (0,1]} \{ \|(A_0 + B_0 F_1(\varepsilon_1))^j\|,\ j = 1, 2, \dots, j_0 \}$ is independent of ε_1 by checking (25) associated with $F_1(\varepsilon_1)$. Substituting (50) into (49) yields, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| \Delta_1(t) \right| \le \gamma_4 \gamma_6 \bar{\beta}_3 \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \mathrm{e}^{-\varepsilon_1^\rho t} \frac{\varepsilon_1}{\varepsilon_2^{j_0 - 1}} \int_0^t \mathrm{e}^{-(\varepsilon_2 - \varepsilon_1^\rho)\tau} \mathrm{d}\tau + \gamma_6 \bar{\beta}_3 \varepsilon_1 \left\| \xi_0(0) \right\| \mathrm{e}^{-\varepsilon_1^\rho t}. \tag{51}$$

Notice that

$$\frac{\varepsilon_1}{\varepsilon_2^{j_0 - 1}} \int_0^t \mathrm{e}^{-(\varepsilon_2 - \varepsilon_1^\rho)\tau} \mathrm{d}\tau = \frac{-\varepsilon_1}{\varepsilon_2^{j_0 - 1} \left(\varepsilon_2 - \varepsilon_1^\rho \right)} \left(\mathrm{e}^{-(\varepsilon_2 - \varepsilon_1^\rho)t} - 1 \right) \le \frac{\varepsilon_1}{\varepsilon_2^{j_0 - 1} \left(\varepsilon_2 - \varepsilon_1^\rho \right)} \le \frac{\varepsilon_2}{1 - \varepsilon_2^{j_0}},$$

where $\rho \ge 1$ in Lemma 2 and $\varepsilon_1 = \varepsilon_2^{j_0 + 1}$ are used in the last inequality. Then, (51) can be continued as, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\begin{aligned} \left| \Delta_1(t) \right| & \le \gamma_4 \gamma_6 \bar{\beta}_3 \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \frac{\varepsilon_2}{1 - \varepsilon_2^{j_0}} \mathrm{e}^{-\varepsilon_1^\rho t} + \gamma_6 \bar{\beta}_3 \varepsilon_1 \left\| \xi_0(0) \right\| \mathrm{e}^{-\varepsilon_1^\rho t} \\ & \le \gamma_4 \gamma_6 \bar{\beta}_3 \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \frac{\varepsilon_2}{1 - \left(\sqrt{2}/2 \right)^{j_0}} \mathrm{e}^{-\varepsilon_1^\rho t} + \gamma_6 \bar{\beta}_3 \varepsilon_1 \left\| \xi_0(0) \right\| \mathrm{e}^{-\varepsilon_1^\rho t}. \end{aligned} \tag{52}$$

Let us focus on $\Delta_2(t)$. Substituting $|w_1(t)|$ in (46) and (50) into $\Delta_2(t)$ yields, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| \Delta_2(t) \right| \le \frac{j \gamma_4}{\varepsilon_2^{j_0 - 1}} \left\| F_1(\varepsilon_1) \right\| \left\| \left(A_0 + B_0 F_1 \right)^j \right\| \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \mathrm{e}^{-\varepsilon_2 t}.$$

By using $\|F_1(\varepsilon_1)\| \le \bar{\beta}_1 \varepsilon_1$ in Lemma 2, $\varepsilon_1 = \varepsilon_2^{j_0 + 1}$ and $\gamma_6 = \sup_{\varepsilon_1 \in (0,1]} \{ \|(A_0 + B_0 F_1(\varepsilon_1))^j\|,\ j = 1, 2, \dots, j_0 \}$, we have, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| \Delta_2(t) \right| \le j \gamma_6 \bar{\beta}_1 \gamma_4 \varepsilon_2^2 \left\| B_0 \right\| \left\| \bar{\xi}(0) \right\| \mathrm{e}^{-\varepsilon_2 t}. \tag{53}$$

Until now, we can investigate $\alpha_1^{(j)}(t)$. Substituting (52) and (53) into (47) yields, for all $\varepsilon_2 \in (0, \sqrt{2}/2]$,

$$\left| \alpha_1^{(j)}(t) \right| \le \left| \Delta_1(t) \right| + \left| \Delta_2(t) \right|$$

$$
\begin{aligned}
&\leq \gamma_4\gamma_6\bar{\beta}_3 \left\|B_0\right\| \left\|\bar{\xi}(0)\right\| \frac{\varepsilon_2}{1-\left(\sqrt{2}/2\right)^{j_0}} \mathrm{e}^{-\varepsilon_1^{\rho}t} + \gamma_6\bar{\beta}_3\varepsilon_1 \left\|\xi_0(0)\right\| \mathrm{e}^{-\varepsilon_1^{\rho}t} + j\gamma_6\bar{\beta}_1\gamma_4\varepsilon_2 \left\|B_0\right\| \left\|\bar{\xi}(0)\right\| \mathrm{e}^{-\varepsilon_2 t} \\
&= \gamma_7\varepsilon_2 \left\|\bar{\xi}(0)\right\| \mathrm{e}^{-\varepsilon_1^{\rho}t},\ j = 1,2,\ldots,j_0,
\end{aligned} \tag{54}
$$

with γ_7 being a positive constant independent of ε_2 and ε_1.

Finally, substituting (36), (42), (45) and (54) into $\xi_{j_0} = w_{j_0} + \alpha_1^{(j_0-1)}$ and $\xi_{j_0+1} = \bar{\xi}_{j_0+1} + \alpha_1^{(j_0)}$ respectively, we have, for all $k \in [\sqrt{2}, \infty)$,

$$
\begin{aligned}
\left|\xi_{j_0}(t)\right| &\leq \gamma_4 \left\|\bar{\xi}(0)\right\| + \gamma_7\varepsilon_2 \left\|\bar{\xi}(0)\right\| \leq \left(\gamma_4+\gamma_7\right)\gamma_1, \\
\left|\xi_{j_0+1}(t)\right| &\leq \left(\gamma_5+\gamma_7\right) \left\|\bar{\xi}(0)\right\| \left(\mathrm{e}^{-kt}+\frac{1}{k}\right) \leq \left(\gamma_5+\gamma_7\right)\gamma_1\left(\mathrm{e}^{-kt}+1\right),
\end{aligned}
$$

which, together with $\gamma_1, \gamma_4, \gamma_5, \gamma_7$ being independent of k, implies that ξ_{j_0} and ξ_{j_0+1} satisfy (28)-(29). This proof is completed. □

With Lemma 4, we can state the following result for the cascade system (30).

Theorem 2. Let Assumptions 1-3 be met. Given an arbitrary compact set $\Omega \subset \mathbb{R}^{l+n}$ centered at the origin, there exists a constant $k_* \in \left[k_0, \infty\right)$ such that, for any $k \in [k_*, \infty)$, the cascade system (30) under the partial state feedback (34) is LAS with the region of attraction containing Ω.

Proof. This proof is finished by using Lemma 4, Lemma 3, and Theorem 1 in turn. □

4.2 | MIMO case

For the MIMO case, it shares the same backstepping scheme as the SISO case.

Step 1: This step will be accomplished via linear low-gain feedback in Lemma 2. Design the low-gain feedback

$$\alpha_1 = F_1(\varepsilon_1)\xi_0$$

to the ANCBC pair (A_0, B_0), where $F_1(\varepsilon_1)$ is the gain matrix in the form of (24) with the low-gain parameter ε_1 to be determined. By defining $\bar{\xi}_1 = \xi_1 - \alpha_1$, $\bar{\xi}_j^i = \xi_j^i - (\alpha_1^i)^{(j-1)}$, $\bar{\xi}_1 = [\,\bar{\xi}_1^1\ \bar{\xi}_1^2\ \cdots\ \bar{\xi}_1^m\,]^{\mathrm{T}}$, $\alpha_1 = [\,\alpha_1^i\ \alpha_1^2\ \cdots\ \alpha_1^m\,]^{\mathrm{T}}$, $j = 1,2,...,r_i$, $i = 1,2,...,m$, the linear part of the cascade system (3) can be formulated as

$$
\begin{cases}
\dot{\xi}_0 = \left(A_0 + B_0F_1\right)\xi_0 + B_0\bar{\xi}_1 \\
\dot{\bar{\xi}}_1^i = \bar{\xi}_2^i \\
\dot{\bar{\xi}}_2^i = \bar{\xi}_3^i \\
\quad\vdots \\
\dot{\bar{\xi}}_r^i = u_i + \left(\alpha_1^i\right)^{(r_i)},\ i = 1,2,...,m.
\end{cases} \tag{55}
$$

Step 2: This step will be accomplished via a set of linear low-gain feedback in Lemma 1. We take the ith chain of integrators as an example. Divide the chain of integrators into two parts, one part is $\dot{\bar{\xi}}_j^i = \bar{\xi}_{j+1}^i$, $j = 1,2,...,j_i$, and another is $\dot{\bar{\xi}}_{j_i+j}^i = \bar{\xi}_{j_i+j+1}^i$, $j = 1,2,...,j_i$. Besides, define $w_1 = \bar{\xi}_1$, $w_j^i = \bar{\xi}_j^i$, $w^i = [\,w_1^i\ w_2^i\ \cdots\ w_{j_i}^i\,]^{\mathrm{T}}$, $j = 1,2,...,j_i$, $i = 1,2,...,m$ and $\tilde{r}_i = r_i - j_i$, $i = 1,2,...,m$. Design the low-gain feedback

$$\alpha_2^i = F_2^i(\varepsilon_2)w^i,\ i = 1,2,...,m$$

as the visual control to the w^i subsystem, where $F_2^i(\varepsilon_2)$ is the gain matrix in Lemma 1, with the low-gain parameter ε_2 to be determined. By defining $e_j^i = \bar{\xi}_{j_i+j}^i - (\alpha_2^i)^{(j-1)}$, $j = 1,2,...,\tilde{r}_i$, $i = 1,2,...,m$, the cascade system (55) becomes

$$
\begin{cases}
\dot{\xi}_0 = \left(A_0 + B_0F_1\right)\xi_0 + B_0w_1 \\
\dot{w}^i = \left(\Phi_0 + B_{\mathrm{c}}^i\right)w + B_{\mathrm{c}}^ie_1^i \\
\dot{e}_1^i = \dot{e}_2^i \\
\dot{e}_2^i = \dot{e}_3^i \\
\quad\vdots \\
\dot{e}_{r_i}^i = u_i - \left(\alpha_1^i\right)^{(r_i)} - \left(\alpha_2^i\right)^{(\tilde{r}_i)},\ i = 1,2,...,m,
\end{cases} \tag{56}
$$

where $\Phi_0^i \in \mathbb{R}^{j_i \times j_i}$ and $B_{\mathrm{c}}^i \in \mathbb{R}^{j_i \times 1}$, $i = 1, 2, ..., m$ share the same structures as (19).

Step 3: Finally, the entire composite feedback law can be designed as

$$u_i = \left(\alpha_1^i\right)^{(r_i)} + \left(\alpha_2^i\right)^{(\tilde{r}_i)} + v_i,\ i = 1, 2, ..., m, \tag{57}$$
$$v_i = -\sum_{j=1}^{\tilde{r}_i} k^{\tilde{r}_i - j + 1} c_j^i e_j^i,\ i = 1, 2, ..., m,$$

where k is the high-gain to be determined, and c_j^i, $j = 1, 2, ..., \tilde{r}_i$, $i = 1, 2, ..., m$ are chosen such that $p_{\mathrm{c}}^i(s) = s^{\tilde{r}_i} + c_{\tilde{r}_i}^i s^{\tilde{r}_i - 1} + \cdots + c_2^i s + c_1^i$ are Hurwitz. Let $\bar{e}_j^i = e_j^i / k^{j-1}$, $j = 1, 2, ..., \tilde{r}_i$, $i = 1, 2, ..., m$ and denote $\bar{e}^i = [\,\bar{e}_1^i\ \bar{e}_2^i\ \cdots\ \bar{e}_{\tilde{r}_i}^i\,]^{\mathrm{T}}$, $i = 1, 2, ..., m$. Then the closed-loop system of the linear part of (3) can be eventually formulated as

$$\begin{cases} \dot{\xi}_0 = \left(A_0 + B_0 F_1\right) \xi_0 + B_0 w_1 \\ \dot{w}^i = \left(\Phi_0^i + B_{\mathrm{c}}^i F_2^i\right) w^i + B_{\mathrm{c}}^i \bar{e}_1^i \\ \dot{\bar{e}}^i = k \Phi_c^i \bar{e}^i,\ i = 1, 2, ..., m, \end{cases} \tag{58}$$

where $\Phi_c^i \in \mathbb{R}^{\tilde{r}_i \times \tilde{r}_i}$, $i = 1, 2, ..., m$ are the companion matrices whose last rows consist of the coefficients of the Hurwitz polynomial $p_{\mathrm{c}}^i(s)$. Parallel to the SISO case, we have the following lemma.

Lemma 5. Let Assumption 3 be satisfied and set $\varepsilon_2 = 1/k$ and $\varepsilon_1 = \varepsilon_2^{j_{\mathrm{c}}+1}$, $j_{\mathrm{c}} = \max\{j_1, j_2, ..., j_m\}$, $j_i \in \{r_1, r_2, ..., r_i - 1\}$, $i = 1, 2, ..., m$. Then there exists a constant $k_0 \in [1, \infty)$ such that, for any $k \in [k_0, \infty)$, the linear part of the cascade system (3) is globally exponentially stabilized by the linear state feedback law (57) and the states ξ_{j_0} and ξ_{j_0+1} satisfy (28)-(29) respectively.

Proof. This proof is performed in a decoupled way. Following the same line of the proof to Theorem 2, we choose $j_{\mathrm{c}} = \max\{j_1, j_2, ..., j_m\}$ to guarantee that each state w_1^i, $i = 1, 2, ..., m$ is bounded in the form of (46), and the other procedures deserve a similar operation as in the proof to Theorem 2. □

With the development of Lemma 5, we can state a solution to Problem 1 as follows.

Theorem 3. Let Assumptions 1-3 be met. Given an arbitrary compact set $\Omega \subset \mathbb{R}^{l+n}$ centered at the origin, there exists a constant $k_* \in \left[k_0, \infty\right)$ such that, for any $k \in [k_*, \infty)$, the cascade system (3) under the partial state feedback (57) is LAS with the region of attraction containing Ω.

Proof. This proof is finished by using Lemma 5, Lemma 3, and Theorem 1 in turn. □

5 | AN ILLUSTRATIVE EXAMPLE

Consider the following cascade systems

$$\Sigma_1 : \begin{cases} \dot{\eta} = -0.5(1 + \xi_2 \xi_3)\eta^3 \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_1 + \xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = u, \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{\eta} = -0.5(1 + \xi_2 \xi_3 + \xi_3^2)\eta^3 \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_1 + \xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = u. \end{cases}$$

It can be observed that the η subsystems in Σ_1 and Σ_2 are consistent with the cascade system (3) and the η subsystem in Σ_2 includes a nonlinear cross-term $(\xi_2 + \xi_3)\xi_3$, which seems not treated in the work[29]. A particular example of Σ_1 is

$$\begin{cases} \dot{\eta} = -0.5(1 + \xi_3)\eta^3 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = u. \end{cases}$$

Although the dynamic $\dot{\eta} = -0.5\eta^3$ is GAS, this cascade system cannot be globally stabilized or not always be semi-globally stabilized by the linear high gain-feedback $u(\xi_2, \xi_3)$[27]. This is also the case for the systems Σ_1 and Σ_2. Besides of this fact, with

ξ_3 being the output, the invariant-zeros of the linear part are j, −j, 0, 0 and the mode 0 is unstable. Thus, semi-global stabilization for the cascade systems Σ_1 and Σ_2 using partial state feedback $u(z, \xi)$ is not trivial as it seems.

By using Theorem 2, we can construct a partial state feedback law to Σ_1 and Σ_2 as

$$
\begin{aligned}
u &= \dddot{\alpha}_1 + \dot{\alpha}_2 + v, \\
\dddot{\alpha}_1 &= -\varepsilon_1^2 \left(\xi_2 - z_2\right) - 2\varepsilon_1 \left(\xi_3 - \xi_1\right), \\
\dot{\alpha}_2 &= -\frac{b_1}{\varepsilon_2^2} \left[\xi_2 + \varepsilon_1^2 z_2 + 2\varepsilon_1 \left(\xi_1 - z_1\right)\right] - \frac{b_2}{\varepsilon_2} \left[\xi_3 + \varepsilon_1^2 \left(\xi_1 - z_1\right) + 2\varepsilon_1 \left(\xi_2 - z_2\right)\right], \\
v &= -kc \left[\xi_3 + \varepsilon_1^2 \left(\xi_1 - z_1\right) + 2\varepsilon_1 \left(\xi_2 - z_2\right)\right] - kc\frac{b_1}{\varepsilon_2^2} \left[\xi_1 + \varepsilon_1^2 z_1 + 2\varepsilon_1 z_2 + \xi_2 + \varepsilon_1^2 z_2 + 2\varepsilon_1 \left(\xi_1 - z_1\right)\right].
\end{aligned}
$$

Choose $b_1 = 1$, $b_2 = 2$, $c = 1$, $k = 3$, $\varepsilon_2 = 1/k$, $\varepsilon_1 = 1/k^3$ and set the initial conditions as $\eta(0) = -1$, $z_1(0) = -2$, $z_2(0) = 2$, $\xi_1(0) = 1$, $\xi_2(0) = -2$, $\xi_3(0) = 2$. Since both Σ_1 and Σ_2 share the same linear part, the simulation results of the linear part, as show in Figure 1, are the same. It can also be observed from Figure 1 that, for both Σ_1 and Σ_2, the convergence of $\eta(t)$ is achieved, which reveals that the semi-global stabilization has been accomplished via such a partial state feedback. As observed in the above two simulations, regardless of the specific form of the η-subsystem $\dot{\eta} = f(\eta, \varphi(\xi_{j_0}, \xi_{j_0+1})\xi_{j_0+1})$, our partial state feedback $u(z, \xi)$ works effectively for any practical initial conditions. This result demonstrates the practical value and advantage of our separate design.

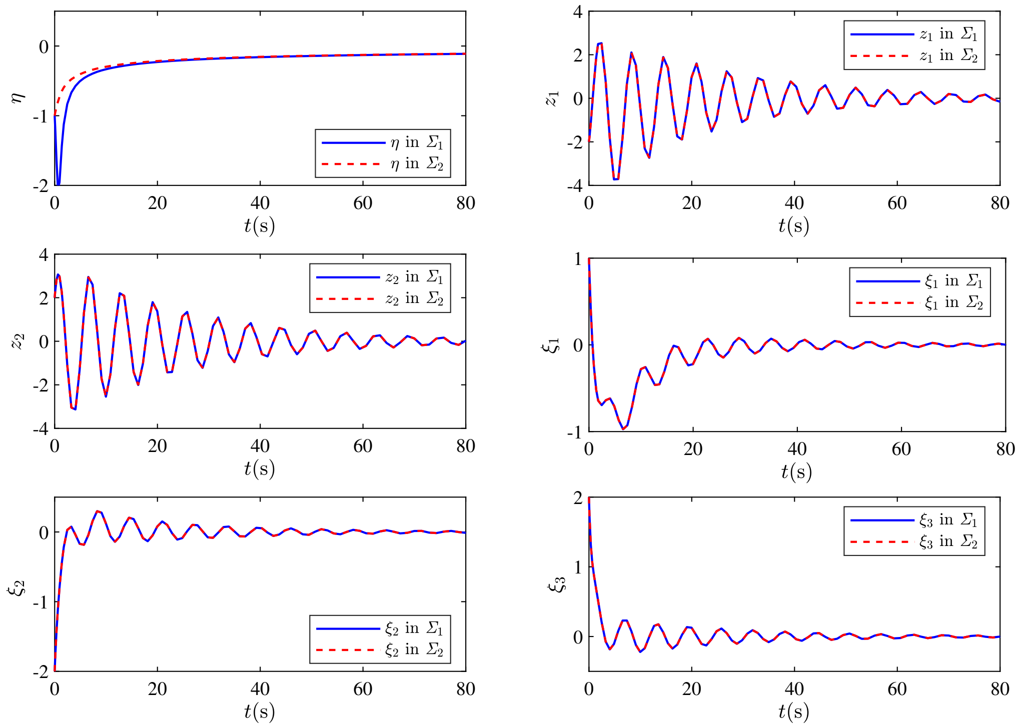

Figure 1 State responses of the closed-loop systems.

6 | CONCLUSIONS

The semi-global stabilization problem of a class of cascade systems (e.g., partially linear composite systems) is solved by partial state feedback. This system comprises a nonlinear subsystem with a cross-term and a linear subsystem in the Byrnes-Isidori normal form. The cross-term that involves any two consecutive states of chains of integrators is incorporated into the nonlinear subsystem. Our work considers little assumptions on the nonlinear subsystem. By taking into account the peaking phenomenon, a lemma that identifies the types of inputs capable of semi-globally stabilizing the affine nonlinear system, with its unforced system being GAS, is established, which is crucial for preparing separate designs. Then the semi-global stabilization of the cascade system is reduced to stabilizing its linear subsystem subject to non-peaking constraints on the consecutive states. Subsequently, a class of linear low-and-high gain feedback is developed for the remaining problem, thereby ensuring that the objective states do not exhibit peaking behavior. For the whole cascade system, this low-and-high gain feedback is partial state feedback using only the states of the linear subsystem.

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Conflict of interest

The authors declare no potential conflict of interests.

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□