

Resistance theory of asymmetric $2 \times n$ circuit network

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Abstract

This article proposes a class of asymmetric $2 \times n$ resistor network model which contains multiple independent resistors, the study of asymmetric resistor network models is a challenge. We conducted in-depth research on this issue using the RT-V theory (Chinese Physics B, 2017, **26**(9): 090503) and achieved new theoretical breakthroughs. This paper derived five original equivalent resistance formulae for this asymmetric $2 \times n$ circuit network, and also discusses the analytical expressions for equivalent resistance in different special cases, and several interesting results have been derived, indicating that the original resistance formula has powerful functionality. Finally, visual graphics of two types of equivalent resistances are provided using MATLAB drawing tools, which reveals the law of equivalent resistance changing with resistance variables. The research theory and technology in this article will provide a new theoretical basis for related scientific, engineering, and simulation research.

Key words: asymmetric network, RT-V method, matrix equation, equivalent resistance

1. Introduction

With the rapid development of electronic technology, circuit systems are becoming increasingly complex. As a basic component of circuits, the complexity and diversity of resistor networks (RNs) have also increased. Many abstract and complex problems need to be solved by establishing intuitive models. The establishment and research of resistor network models (RNMs) not only contribute to a deeper understanding of the working principles of circuits, but also have significant implications for integrated circuit design [1], circuit optimization [2], and fault diagnosis [3]. The research on RNMs are based on the theoretical foundations of multiple disciplines such as circuit theory, mathematical graph theory, matrix theory, and optimization algorithms [4-6]. As a fundamental research method, it has gradually penetrated into many disciplinary fields, promoting their development to a certain extent.

The establishment and research of RNMs have a history of more than 170 years, dating back to 1845 when Kirchhoff, who had just graduated from university, proposed two laws for complex

circuit calculations in his academic paper [7], namely the famous Kirchhoff laws (loop voltage law and node current law). This law establishes the foundation of circuit analysis theory and solves a series of complex scientific problems. With the development of the electrification era, the definition of RNs has begun to generalize, including infinite networks composed of the same resistors or capacitors. At this time, the new theories are needed to study the electrical characteristics of infinite networks. Scholars in references [8-10] have proposed research on the resistance between two points in an infinite grid, as well as some new ideas and methods. In 2000, Professor Cserti from Roland University in Hungary established the Green's function technique (LGF) for calculating the equivalent resistance of infinite rectangular networks [9], which applied the LGF to infinite resistance networks and provided a method for calculating the equivalent resistance of infinite resistance networks. In addition, Asad [11,12] and Owaidat [13-16] used LGF to study complex infinite circuit networks. Their research works have promoted the research and development of infinite networks and made significant contributions. In 2004, Professor Wu FY from Northeastern University proposed the Laplacian matrix method [17] (Wu's LM method) for computing large-scale finite RNs, and used the eigenvalues and eigenvectors of the Laplacian matrix to calculate the resistance between any two nodes in the RN. Wu's LM method has also been well developed and applied in the study of equivalent resistance. For example, references [18-22] applied Wu's LM method to study the equivalent resistance between any two points in a resistance network under special circumstances, but this method is not applicable to networks on any boundary. In 2011, Professor Tan Z-Z from Nantong University in China proposed a new original theory called the recursion-transform (RT) theory [1] to solve RN problems with arbitrary boundaries. By applying Tan's RT method, the equivalent resistance of finite $m \times n$ RNs with different boundaries as shown in Refs. [23-31] can be calculated. This methodology only requires the establishment of matrix equations along one direction, simplifying the expression of the results and suitable for studying infinite and finite RNMs with arbitrary boundaries.

At present, Professor Tan has elevated the problem of RNs to RNMs, opening up a new research field [1]. The establishment of RT theory has promoted the study of RNMs for various complex structures. For example, in reference [32], for the first time, the RT-V method was used to obtain accurate potential formulas for arbitrary $m \times n$ cobweb and Fan networks, and potential formulas for infinite and semi infinite networks were derived. Reference [33] studied an arbitrary $m \times n$ apple surface network with a pair of non-uniform boundary resistances based on RT-V theory. Paper [34] studied the resistance between any node in two n -order periodic resistance networks using the RT-V method. Reference [35] used the RT-V method to establish two new sets of equations to determine the electrical characteristics of an irregular $2 \times n$ hammock network. This paper proposes a class of asymmetric $2 \times n$ RNMs containing multiple independent resistors, as shown in Fig.1. The model is very complex and consists of five independent resistor elements. We conducted in-depth research on this issue by using Professor Tan's RT-V method established in

reference [32]. We derived five original equivalent resistance formulas for asymmetric $2 \times n$ circuit networks and discussed analytical expressions for equivalent resistance in different special cases. Finally, we used MATLAB drawing tools such as those used in references [36, 37] to provide visual representations of two types of equivalent resistance, revealing the variation of equivalent resistance with resistance variables.

Recently, reference [38] studied a class of periodic and asymmetric $2 \times n$ circuit networks embedded with T-shaped structures, and this article intends to study another type of non-periodic and asymmetric $2 \times n$ circuit networks. Although their partial structures are similar, their essential problems are different because their boundary conditions are different (one is periodic structure, the other is non-periodic structure), and their equivalent resistance results are different. People understand that even a slight change in the conditions of the RNM will alter their electrical characteristics. Therefore, it is necessary to study non-periodic and asymmetric $2 \times n$ circuit networks.

2. Main conclusions and formulae

The $2 \times n$ RNM of the embedded inverted T-shaped circuit in Figure 1 has two characteristics : firstly, it is a very complex $2 \times n$ order RNM; Secondly, it can be inferred from its sub model that the model does not have symmetry vertically. Therefore, the n -order RNM embedded in an inverted T-shaped circuit is an asymmetric $2 \times n$ -order RNM. In this RNM, the lower nodes are defined from left to right as A_0 to A_n , the middle nodes corresponding to the lower nodes are defined as B_0 to B_n , and the remaining intermediate nodes are C_1 to C_n in sequence. The top of the model is the horizontal axis D_0 to D_n with the same potential, and D_k in all expressions in the text is represented by O (because they are located on the same wire). Refer to Figure 1, there are the five independent arbitrary resistance elements are arranged in this model. The resistance element r is arranged between any two nodes on the horizontal axis A_0A_n , the r_2 is arranged between any two nodes on the horizontal axis B_0B_n . In the vertical direction, the r_0 is placed between A_k and B_k , r_3 is placed between B_k and D_k , the resistance element r_1 is arranged between C_k and D_k . In this paper, the RT-V method is also used to study and derive the analytical formula for the equivalent resistance between any two nodes A_{x_1} (and B_{x_1}) and $P_{x_2} (= A_{x_2}, B_{x_2}, O)$ in an $2 \times n$ RN.

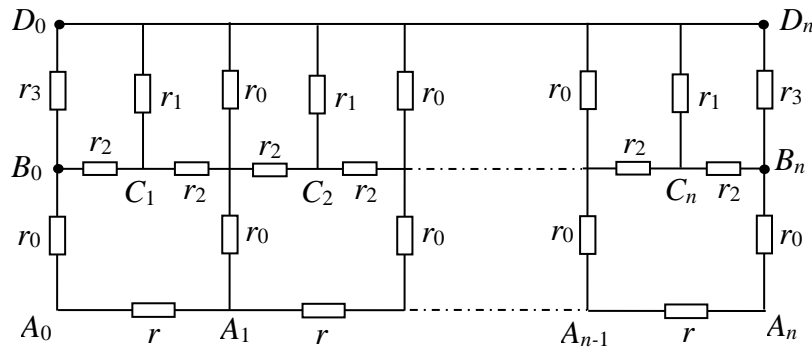


Fig.1. An asymmetric $2 \times n$ resistor network model embedded with T circuit, containing five different resistor elements, and $r_3 = r_0 / (2r_1 + r_2)$.

2.1 Necessary parameter definitions

The RNM studied in this article is very complex. In order to make the expression results more concise, the parameters in the article are defined as follows.

$$\begin{aligned} p_1 &= \frac{1}{2}h_{r0} + (h_{21} + 2)(h_{20} + 1) - \sqrt{\Delta}, \\ p_2 &= \frac{1}{2}h_{r0} + (h_{21} + 2)(h_{20} + 1) + \sqrt{\Delta}, \end{aligned} \quad (1)$$

$$\Delta = \left(\frac{1}{2}h_{r0} + 1 - d\right)^2 + (h_{21} + 2)h_{20}h_{r0}, \quad (2)$$

In the above equation, $d = (h_{20} + 1)(h_{21} + 2) - 1$, and

$$h_{sk} = r_s / r_k, \quad h_{r0} = r / r_0, \quad r_3 = r_0 // (2r_1 + r_2), \quad (3)$$

and define

$$B_k^{(i)} = \frac{\lambda_i^k - \bar{\lambda}_i^k}{\lambda_i - \bar{\lambda}_i}, \quad \Delta B_k^{(i)} = B_{k+1}^{(i)} - B_k^{(i)}, \quad (i=1,2) \quad (4)$$

$$\lambda_i = \frac{1}{2}\left(p_i + \sqrt{p_i^2 - 4}\right), \quad \bar{\lambda}_i = \frac{1}{2}\left(p_i - \sqrt{p_i^2 - 4}\right). \quad (5)$$

$$F_{k,v}^{(i)} = \Delta B_{x_k}^{(i)} \Delta B_{n-x_v}^{(i)}, \quad (6)$$

The above parameters expressions (1)-(6) are applicable to the entire text and facilitate further research in the following sections.

2.2 Five original results

In the RNM shown in Fig. 1, for the horizontal axis A_0A_n , from left to right, the first node on the left side is A_0 , and the k -th node is A_k . alike, assuming that the k -th node on the central axis B_0B_n is B_k . The parameters of each resistor involved in the article are labeled in Fig. 1, in this asymmetric $2 \times n$ RNM, the analytical expressions for the equivalent resistance between any node A_{x_1} and node $P_{x_2} (= A_{x_2}, B_{x_2}, O)$, as well as between node B_{x_1} and node $P'_{x_2} (= B_{x_2}, O)$ are the following five conclusions.

2.2.1 Analytical formula of equivalent resistance $R(A_{x_1}, B_{x_2})$

We have derived an accurate equivalent resistance analytical formula $R_n(A_{x_1}, B_{x_2})$ for the study of RNM in Figure 1

$$\frac{R_n(A_{x_1}, B_{x_2})}{r} = (2d - p_1) \frac{F_{1,1}^{(1)} - 2u_1 F_{1,2}^{(1)} + u_1^2 F_{2,2}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - (2d - p_2) \frac{F_{1,1}^{(2)} - 2u_2 F_{1,2}^{(2)} + u_2^2 F_{2,2}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (7)$$

where $d = (h_{20} + 1)(h_{21} + 2) - 1$, and

$$u_i = h_{20}(2 + h_{21}) / (2d - p_i), \quad (8)$$

other parameters $B_k^{(i)}$, h_{sk} , $F_{k,v}^{(i)}$ etc. are defined in equations (1)-(6).

2.2.2 Analytical formula of equivalent resistance $R(A_{x_1}, O)$

We then derived the second equivalent resistance formula $R_n(A_{x_1}, O)$ for the complex $2 \times n$ RN of Fig.1,

$$\frac{R_n(A_{x_1}, O)}{r} = \frac{(2d - p_1)F_{1,1}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - \frac{(2d - p_2)F_{1,1}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}, \quad (9)$$

where $d = (h_{20} + 1)(h_{21} + 2) - 1$, and t_1, t_2 are defined in equation (1).

2.2.3 Analytical formula of equivalent resistance $R(A_{x_1}, A_{x_2})$

Furthermore, we derived the third equivalent resistance formula $R_n(A_{x_1}, A_{x_2})$ for the complex $2 \times n$ RN of Fig.1,

$$\frac{R_n(A_{x_1}, A_{x_2})}{r} = (2d - p_1) \frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - (2d - p_2) \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (10)$$

where all parameters $B_k^{(i)}$, h_{sk} , d , $F_{k,v}^{(i)}$ etc. are defined in equations (1)-(6).

2.2.4 Analytical formula of equivalent resistance $R(B_{x_1}, B_{x_2})$

The fourth equivalent resistance formula $R_n(B_{x_1}, B_{x_2})$ we derived in the complex $2 \times n$ RN of Fig.1 is

$$\frac{R_n(B_{x_1}, B_{x_2})}{r_2} = \frac{2 + h_{21}}{p_2 - p_1} \left(\left(\frac{h_{r0}}{p_1 - 2} - 1 \right) \frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{B_{n+1}^{(1)}} - \left(\frac{h_{r0}}{p_2 - 2} - 1 \right) \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{B_{n+1}^{(2)}} \right) \quad (11)$$

where all parameters $B_k^{(i)}$, h_{sk} , p_1, p_2 , $F_{k,v}^{(i)}$ etc. are defined in equations (1)-(6).

2.2.5 Analytical formula of equivalent resistance $R(B_{x_1}, O)$

The fifth equivalent resistance formula $R_n(B_{x_1}, O)$ we derived in the complex $2 \times n$ RN of Fig.1 is

$$\frac{R_n(B_{x_1}, O)}{r_2} = \frac{2 + h_{21}}{p_2 - p_1} \left(\left(\frac{h_{r0}}{p_1 - 2} - 1 \right) \frac{F_{1,1}^{(1)}}{B_{n+1}^{(1)}} - \left(\frac{h_{r0}}{p_2 - 2} - 1 \right) \frac{F_{1,1}^{(2)}}{B_{n+1}^{(2)}} \right), \quad (12)$$

where all parameters $B_k^{(i)}$, h_{sk} , p_1, p_2 , $F_{k,v}^{(i)}$ etc. are defined in equations (1)-(6).

From the above formulas (7), (9), (10), (11), and (12), we can see that although their expression are succinct, their contents are very abundant, and they are the research conclusions obtained for the first time in this article. The following text will provide the specific calculation process and draw the above conclusions.

3 Basic methods and theories

This article uses the most cutting edge theory, the RT-V theory established by Professor Tan [32], to study the complex circuit network in Figure 1. This theory has five essential research

process. The RT-V theory will be used for calculation and derivation in the following text.

3.1 Construction of Matrix Equations

According to the RT-V theory, the Kirchhoff node current law ($\sum r_i^{-1} V_k = 0$) is first applied to establish a differential equation model for node voltage (including a general equation model and a boundary constrained equation model) to study the complex $2 \times n$ RN in Figure 1. According to the structural characteristics of the RN in Figure 1, assuming that the potential of node O be zero $U_o = 0$, the current I flows into the network from node A_{x_1} and exits at node B_{x_2} (or A_{x_2}, O). In order to establish the equation model, the sub network diagram shown in Figure 2, as well as the resistance and voltage parameters, were provided.

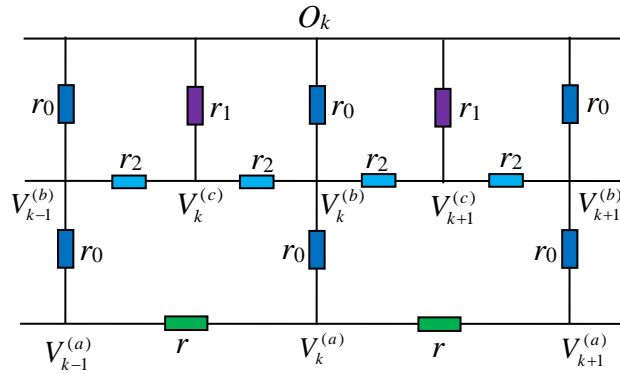


Fig.2. A sub network diagram with resistors and voltage parameters used for analyzing circuits

3.1.1 Voltage equation for any node

When the current is input from node A_{x_1} to node A_{x_2} output, analyze Figure 2 and apply Kirchhoff's node current law ($\sum r_i^{-1} V_k = 0$) to obtain the node voltage equation system as follows

$$(h_{21} + 2)V_k^{(c)} = V_{k-1}^{(b)} + V_k^{(b)}, \quad (13)$$

$$(h_{21} + 2)V_{k+1}^{(c)} = V_{k+1}^{(b)} + V_k^{(b)}. \quad (14)$$

$$V_{k+1}^{(a)} = (h_{r0} + 2)V_k^{(a)} - h_{r0}V_k^{(b)} - V_{k-1}^{(a)} + rI(\delta_{kx_2} - \delta_{kx_1}), \quad (15)$$

$$V_{k+1}^{(c)} + V_k^{(c)} + h_{20}V_k^{(a)} - 2(h_{20} + 1)V_k^{(b)} = 0, \quad (16)$$

In equation (15), $\delta_{k,x}$ is the Dirac function, and $h_{sk} = r_s / r_k$, $h_{r0} = r / r_0$.

To eliminate $V_k^{(c)} + V_{k+1}^{(c)}$, substitute equations (13) and (14) into equation (16) and simplify it

$$V_{k+1}^{(b)} = -(h_{21} + 2)h_{20}V_k^{(a)} + 2[(h_{21} + 2)(h_{20} + 1) - 1]V_k^{(b)} - V_{k-1}^{(b)} = 0. \quad (17)$$

Represent equations (15) and (17) in matrix form

$$\begin{bmatrix} V_{k+1}^{(a)} \\ V_{k+1}^{(b)} \end{bmatrix} = D_{2 \times 2} \begin{bmatrix} V_k^{(a)} \\ V_k^{(b)} \end{bmatrix} - \begin{bmatrix} V_{k-1}^{(a)} \\ V_{k-1}^{(b)} \end{bmatrix} + r(\delta_{kx_2} - \delta_{kx_1}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} I, \quad (18)$$

from the previous text, it is known that $a = (2 + h_{21})(1 + h_{20}) - 1$, so in the above equation

$$D_{2 \times 2} = \begin{pmatrix} (2+h_{r0}) & -h_{r0} \\ -(h_{21}+2)h_{20} & 2d \end{pmatrix}. \quad (19)$$

Equation (18) is the recursive equation of the key matrix required to solve the analytical formula for equivalent resistance.

Consider the boundary condition equation of associated node A_x , when the current I is input from node A_{x_1} , it is obtained from equation (18)

$$\begin{bmatrix} V_{x_1+1}^{(a)} \\ V_{x_1+1}^{(b)} \end{bmatrix} = D_{2 \times 2} \begin{bmatrix} V_{x_1}^{(a)} \\ V_{x_1}^{(b)} \end{bmatrix} - \begin{bmatrix} V_{x_1-1}^{(a)} \\ V_{x_1-1}^{(b)} \end{bmatrix} - r \begin{bmatrix} 1 \\ 0 \end{bmatrix} I, \quad (20)$$

When current I flows into the network from node A_{x_2} , equation (18) yields

$$\begin{bmatrix} V_{x_2+1}^{(a)} \\ V_{x_2+1}^{(b)} \end{bmatrix} = D_{2 \times 2} \begin{bmatrix} V_{x_2}^{(a)} \\ V_{x_2}^{(b)} \end{bmatrix} - \begin{bmatrix} V_{x_2-1}^{(a)} \\ V_{x_2-1}^{(b)} \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \end{bmatrix} I. \quad (21)$$

Moreover, when current I flows into the network from node B_{x_1} on the B_0B_n axis and flows out of the network from node B_{x_2} , based on the above analysis, the matrix equation is obtained using the same method above

$$\begin{bmatrix} V_{k+1}^{(a)} \\ V_{k+1}^{(b)} \end{bmatrix} = D_{2 \times 2} \begin{bmatrix} V_k^{(a)} \\ V_k^{(b)} \end{bmatrix} - \begin{bmatrix} V_{k-1}^{(a)} \\ V_{k-1}^{(b)} \end{bmatrix} + r_2(h_{21}+2)(\delta_{kx_2} - \delta_{kx_1}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} I, \quad (22)$$

the matrix $D_{2 \times 2}$ in the above equation comes from equation (19).

3.1.2 Boundary condition equation

The above equation is not sufficient to solve the node voltage, and it is necessary to establish conditional equations for the left and right boundaries. Establish boundary condition equations using Kirchhoff's law for the circuit shown in Figure 3.

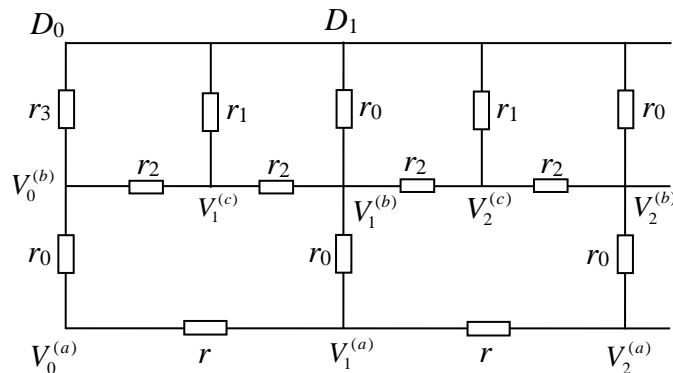


Fig.3. The network model and its parameters for the left boundary, where $r_3=r_0/(2r_1+r_2)$.

According to the circuit on the left boundary shown in Figure 3, a matrix equation can be established using a technique similar to equation (18)

$$\begin{bmatrix} V_1^{(a)} \\ V_1^{(b)} \end{bmatrix} = \begin{pmatrix} (h_{r0} + 1) & -h_{r0} \\ -h_{20}(2 + h_{21}) & [(2 + h_{21})(1 + h_{20} + h_{23}) - 1] \end{pmatrix} \begin{bmatrix} V_0^{(a)} \\ V_0^{(b)} \end{bmatrix}. \quad (23)$$

Research has found that only when $(2 + h_{21})(1 + h_{20} + h_{23}) = 2d = 2[(2 + h_{21})(1 + h_{20}) - 1]$ at that time can the network model have a concise analytical expression, which leads to $h_{23} = h_{20} + h_{21} / (2 + h_{21})$ and further simplifies to $r_3 = r_0 / (2r_1 + r_2)$. This is why we choose this value, otherwise it is impossible to obtain an accurate equivalent resistance analytical expression. So equation (23) can be abbreviated as

$$\begin{bmatrix} V_1^{(a)} \\ V_1^{(b)} \end{bmatrix} = (D_{2 \times 2} - E_{2 \times 2}) \begin{bmatrix} V_0^{(a)} \\ V_0^{(b)} \end{bmatrix}, \quad (24)$$

Where E is the identity matrix.

Similarly, the conditional equation for the right boundary can be written

$$\begin{bmatrix} V_{n-1}^{(a)} \\ V_{n-1}^{(b)} \end{bmatrix} = (D_{2 \times 2} - E_{2 \times 2}) \begin{bmatrix} V_n^{(a)} \\ V_n^{(b)} \end{bmatrix}, \quad (25)$$

The above equations established using the RT-V method are all equations required for calculating the analytical expression of equivalent resistance.

3.2 Transformation of Matrix Equations

3.2.1 Main matrix equation transformation

Use the matrix transformation method in RT theory to indirectly calculate the solution of the above equation. According to the matrix transformation method established in references [39-42], perform matrix transformation on equation (22) (ignoring the input and output of current). Assuming the existence of an undetermined second-order matrix $Q_{2 \times 2}$, multiply both ends of equation (22) by $Q_{2 \times 2}$ simultaneously and perform matrix transformation to obtain

$$Q_{2 \times 2} \begin{bmatrix} V_{k+1}^{(a)} \\ V_{k+1}^{(b)} \end{bmatrix} = Q_{2 \times 2} D_{2 \times 2} \begin{bmatrix} V_k^{(a)} \\ V_k^{(b)} \end{bmatrix} - Q_{2 \times 2} \begin{bmatrix} V_{k-1}^{(a)} \\ V_{k-1}^{(b)} \end{bmatrix}. \quad (26)$$

Find the eigenvalue of $D_{2 \times 2}$ from formula $\det[D_{2 \times 2} - pE] = 0$ below, that is

$$\begin{vmatrix} (2 + h_{r0}) - p & -h_{r0} \\ -(h_{21} + 2)h_{20} & 2d - p \end{vmatrix} = 0, \quad (27)$$

therefore

$$p^2 - (h_{r0} + 2d + 2)p + 2d(2 + h_{r0}) - (h_{21} + 2)h_{20}h_{r0} = 0, \quad (28)$$

the solution of p in the above equation is

$$\begin{aligned} p_1 &= \frac{1}{2} h_{r0} + (h_{21} + 2)(h_{20} + 1) - \sqrt{\Delta}, \\ p_2 &= \frac{1}{2} h_{r0} + (h_{21} + 2)(h_{20} + 1) + \sqrt{\Delta}, \end{aligned} \quad (29)$$

where $\Delta = (\frac{1}{2} h_{r0} + 1 - d)^2 + (h_{21} + 2)h_{20}h_{r0}$, and $d = (h_{20} + 1)(h_{21} + 2) - 1$.

Next, calculate the eigenvector $Q_{2 \times 2}$ through diagonalization matrix transformation

$$Q_{2 \times 2} \begin{pmatrix} (2+h_{r0}) & -h_{r0} \\ -(h_{21}+2)h_{20} & 2d \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} Q_{2 \times 2}, \quad (30)$$

where we assume

$$Q_{2 \times 2} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (31)$$

whereupon, according to the matrix identity, from equations (30) and (31), it can be obtained that

$$Q_{2 \times 2} = \begin{pmatrix} (h_{21}+2)h_{20} & 2+h_{r0}-p_1 \\ (h_{21}+2)h_{20} & 2+h_{r0}-p_2 \end{pmatrix}. \quad (32)$$

Next, use the following transformation matrix to simplify the matrix equation

$$\begin{bmatrix} Y_k^{(1)} \\ Y_k^{(2)} \end{bmatrix} = Q_{2 \times 2} \begin{bmatrix} V_k^{(a)} \\ V_k^{(b)} \end{bmatrix}. \quad (33)$$

Therefore, without considering boundary flow conditions, the main matrix equation (22) is transformed into

$$\begin{bmatrix} Y_{k+1}^{(1)} \\ Y_{k+1}^{(2)} \end{bmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{bmatrix} Y_k^{(1)} \\ Y_k^{(2)} \end{bmatrix} - \begin{bmatrix} Y_{k-1}^{(1)} \\ Y_{k-1}^{(2)} \end{bmatrix}, \quad (34)$$

then, the above equation can be simplified as

$$Y_{k+1}^{(s)} = p_s Y_k^{(s)} - Y_{k-1}^{(s)}. \quad (s=1,2) \quad (35)$$

Through observation, it was found that the main matrix equation (22) is a relatively complex two-dimensional equation, while the transformed equation (35) is a relatively simple one-dimensional equation. It is obvious that the transformed equation is easier to solve.

3.2.2 Transform boundary condition equation

Next, combining the transformation of the boundary condition matrix equation, equations (24) and (25) are transformed using the method of equation (26) to obtain

$$Y_1^{(i)} = (p_i - 1)Y_0^{(i)}. \quad (i=1,2) \quad (36)$$

$$Y_{n-1}^{(i)} = (p_i - 1)Y_n^{(i)}. \quad (i=1,2) \quad (37)$$

3.2.3 Transformation of Current Excitation Matrix Equation

Considering the transformation of boundary condition equations for current flowing in and out of node A_x and out of node B_x in the network, using the above method, when current I flows into the network from node A_{x_1} , equation (20) is transformed into

$$Y_{x_1+1}^{(i)} = p_i Y_{x_1}^{(i)} - Y_{x_1-1}^{(i)} - rI(h_{21}+2)h_{20}. \quad (38)$$

When the current flows out from node A_{x_2} , equation (21) yields

$$Y_{x_2+1}^{(i)} = p_i Y_{x_2}^{(i)} - Y_{x_2-1}^{(i)} + rI(h_{21}+2)h_{20}. \quad (39)$$

Besides, when the current flows in from node B_{x_1} and flows out from node B_{x_2} , by transforming

equation (22), we can obtain

$$Y_{x_1+1}^{(i)} = p_i Y_{x_1}^{(i)} - Y_{x_1-1}^{(i)} - r_2 I(h_{21} + 2)(2 + h_{r0} - p_i). \quad (40)$$

$$Y_{x_2+1}^{(i)} = p_i Y_{x_2}^{(i)} - Y_{x_2-1}^{(i)} + r_2 I(h_{21} + 2)(2 + h_{r0} - p_i). \quad (41)$$

The above equation provides us with preliminary preparation for promoting the main results.

4. Deduction of Main Conclusions

Analysis of the circuit shown in Figure 1 reveals two possible scenarios: $x_2 \geq x_1$ or $x_2 \leq x_1$. Since all the calculation processes of situation $x_2 \leq x_1$ are completely similar to situation $x_2 \geq x_1$, this article only considers the case of $x_2 \geq x_1$.

4.1 Calculation of equivalent resistance $R_n(A_{x_1}, B_{x_2})$

4.1.1 Special solutions of matrix equations under constraint conditions

Firstly, considering the condition that current I is input from node A_{x_1} to node B_{x_2} , the required matrix equation is solved as equation (35), and its corresponding constraint equations are (38) and (41), respectively. Equations (36) and (37) are common boundary conditions.

Equation $\lambda^2 = p_i \lambda - 1$ is the eigenvalue of equation (35), let its two solutions be λ_i and $\bar{\lambda}_i$ respectively, and solve them as follows

$$\lambda_i = \frac{1}{2} \left(p_i + \sqrt{p_i^2 - 4} \right), \quad \bar{\lambda}_i = \frac{1}{2} \left(p_i - \sqrt{p_i^2 - 4} \right). \quad (42)$$

According to the method for solving the difference equation established in references [31-33], equation (35) was solved and its piecewise function solution was obtained as follows

$$Y_k^{(i)} = Y_1^{(i)} B_k^{(i)} - Y_0^{(i)} B_{k-1}^{(i)}, \quad (0 \leq k \leq x_1) \quad (43)$$

$$Y_k^{(i)} = Y_{x_1+1}^{(i)} B_{k-x_1}^{(i)} - Y_{x_1}^{(i)} B_{k-x_1-1}^{(i)}, \quad (x_1 \leq k \leq x_2) \quad (44)$$

$$Y_k^{(i)} = Y_{x_2+1}^{(i)} B_{k-x_2}^{(i)} - Y_{x_2}^{(i)} B_{k-x_2-1}^{(i)}, \quad (x_2 \leq k \leq n) \quad (45)$$

in the above equation $B_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k) / (\lambda_i - \bar{\lambda}_i)$ is defined by equation (4).

Let's start calculating the analytical expression for $Y_k^{(i)}$, by (36) and (43) to derive

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)}, \quad (0 \leq k \leq x_1), \quad (46)$$

to solve the equation, we need to eliminate $Y_{x_1+1}^{(i)}$ and $Y_{x_2+1}^{(i)}$ in equations (44) and (45). So, substituting equation (38) into equation (44) together with (46), yielding ($x_1 \leq k \leq x_2$)

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} - I r(h_{21} + 2) h_{20} B_{k-x_1}^{(i)}, \quad (x_1 \leq k \leq x_2) \quad (47)$$

by substituting both equation (41) and equation (47) into equation (45), yielding ($x_2 \leq k \leq n$)

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} + r_2 I(h_{21} + 2)(2 + h_{r0} - p_s) B_{k-x_2}^{(i)} - I r(h_{21} + 2) h_{20} B_{k-x_1}^{(i)}, \quad (48)$$

to find out $Y_0^{(i)}$, take equation (48) with $k = \{n, n-1\}$ into (37), we have

$$Y_0^{(i)} = I(h_{21} + 2) \frac{r h_{20} \Delta B_{n-x_1}^{(i)} - r_2 (2 + h_{r0} - p_i) \Delta B_{n-x_2}^{(i)}}{(p_i - 2) B_{n+1}^{(i)}}. \quad (49)$$

Substitute (49) into (47) and export ($x_1 \leq k \leq x_2$, $i=1,2$)

$$Y_k^{(i)} = I(h_{21} + 2) \frac{rh_{20}\Delta B_{x_1}^{(i)}\Delta F_{n-k}^{(i)} - r_2(h_{r0} + 2 - p_i)\Delta B_k^{(i)}\Delta B_{n-x_2}^{(i)}}{(p_i - 2)B_{n+1}^{(i)}}, \quad (50)$$

The calculation of equivalent resistance $R_n(A_{x_1}, B_{x_2})$ will mainly depend on equation (50).

4.1.2 Calculate the equivalent resistance $R_n(A_{x_1}, B_{x_2})$

Based on the RT-V method for calculating equivalent resistance in reference[32], we can first invert (33) and apply the matrix equation (32) of $Q_{2 \times 2}$ to obtain it

$$V_k^{(a)} = \frac{(2 + h_{r0} - p_1)Y_k^{(2)} - (2 + h_{r0} - p_2)Y_k^{(1)}}{(2 + h_{21})h_{20}(p_2 - p_1)}, \quad (51)$$

$$V_k^{(b)} = \frac{Y_k^{(1)} - Y_k^{(2)}}{p_2 - p_1}. \quad (52)$$

Next, we can derive the following formula from Ohm's law

$$R_n(A_{x_1}, B_{x_2}) = (V_{x_1}^{(a)} - V_{x_2}^{(b)}) / I. \quad (53)$$

Simplify the equations of $V_k^{(a)}$ and $V_k^{(b)}$ by substituting them into the above equation

$$R_n(A_{x_1}, B_{x_2}) = \frac{1}{I} \left(\frac{(h_{r0} + 2 - p_1)Y_{x_1}^{(2)} + (2 + h_{21})h_{20}Y_{x_2}^{(2)}}{(2 + h_{21})h_{20}(p_2 - p_1)} - \frac{(h_{r0} + 2 - p_2)Y_{x_1}^{(1)} + (2 + h_{21})h_{20}Y_{x_2}^{(1)}}{(2 + h_{21})h_{20}(p_2 - p_1)} \right). \quad (54)$$

Because of $p_1 + p_2 = h_{r0} + 2 + 2d$, therefore $h_{r0} + 2 - p_1 = p_2 - 2d$, substituting (50)

with $k = x_1, x_2$ into (54) yields

$$R_n(A_{x_1}, B_{x_2}) = r \left[\frac{(2d - p_1)F_{1,1}^{(1)} - 2h_{20}(2 + h_{21})F_{1,2}^{(1)} - h_{2r}(2 + h_{21})(2d - p_2)F_{2,2}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - \frac{(2d - p_2)F_{1,1}^{(2)} - 2h_{20}(2 + h_{21})F_{1,2}^{(2)} - h_{2r}(2 + h_{21})(2d - p_1)F_{2,2}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}} \right]. \quad (55)$$

Because of

$$(p_1 - 2d)(p_2 - 2d) = (h_{r0} + 2 - p_1)(h_{r0} + 2 - p_2) = -(2 + h_{21})h_{20}h_{r0}, \quad (56)$$

using Eq.(56) to obtain $p_2 - 2d = h_{20}h_{r0}(2 + h_{21})/(2d - p_1)$ and defining $u_i = h_{20}(2 + h_{21})/(2d - p_i)$,

then simplifying using Eq. (55) yields

$$\frac{R_n(A_{x_1}, B_{x_2})}{r} = (2d - p_1) \frac{F_{1,1}^{(1)} - 2u_1F_{1,2}^{(1)} + u_1^2F_{2,2}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - (2d - p_2) \frac{F_{1,1}^{(2)} - 2u_1F_{1,2}^{(2)} + u_1^2F_{2,2}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (57)$$

In summary, the equivalent resistance $R_n(A_{x_1}, B_{x_2})$ between nodes A_{x_1} and B_{x_2} was obtained, and parameter (6) was proved.

4.2 Calculation of equivalent resistance $R_n(A_{x_1}, O)$

4.2.1 Special solutions of matrix equations under constraint conditions

To calculate the equivalent resistance $R_n(A_{x_1}, O)$, the current can be input from node A_{x_1} and

output from node O. At this point, equations (43) - (45) given earlier are applicable to this situation, and equations (46) and (47) are also applicable to this situation. However, the scope of equation (47) needs to be modified to $x_1 \leq k \leq n$, so equation (47) needs to be rewrite.

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(s)} - Ir(2 + h_{21})h_{20}B_{k-x_1}^{(i)}, (x_1 \leq k \leq n) \quad (58)$$

To find out $Y_0^{(s)}$, take equation (58) with $k = \{n, n-1\}$ into (37), we therefore have after algebraic operations

$$Y_0^{(i)} = Ir(2 + h_{21})h_{20} \frac{\Delta B_{n-x_1}^{(i)}}{(t_i - 2)B_{n+1}^{(i)}}. \quad (59)$$

Since equation (46) also applies to this situation, substituting (59) into (46) yields

$$Y_k^{(i)} = Ir(2 + h_{21})h_{20} \frac{\Delta B_k^{(i)} \Delta B_{n-x_1}^{(i)}}{(p_s - 2)B_{n+1}^{(i)}}, (0 \leq k \leq x_1) \quad (60)$$

4.2.2 Calculation of equivalent resistance $R_n(A_{x_1}, O)$

Equation (51) is a universal inverse transformation equation, it also applies to the situation here. Similarly, the analytical formula $R_n(A_{x_1}, O) = (V_{x_1}^{(a)} - 0) / I$ for calculating equivalent resistance through Ohm's law, where $V_{x_1}^{(a)}$ is expressed by equation (51), therefore

$$R_n(A_{x_1}, O) = \frac{1}{I} \left(\frac{(h_{r0} + 2 - p_1)Y_{x_1}^{(2)} - (h_{r0} + 2 - p_2)Y_{x_1}^{(1)}}{(2 + h_{21})h_{20}(p_2 - p_1)} \right). \quad (61)$$

Use equation (60) to eliminate $Y_k^{(i)}$ in the above equation and simplify it to obtain

$$R_n(A_{x_1}, O) = r \frac{(h_{r0} + 2 - p_2)F_{1,1}^{(1)}}{(p_1 - p_2)(p_1 - 2)B_{n+1}^{(1)}} - r \frac{(h_{r0} + 2 - p_1)F_{1,1}^{(2)}}{(p_1 - p_2)(p_2 - 2)B_{n+1}^{(2)}}. \quad (62)$$

Because of $p_1 + p_2 = h_{r0} + 2d + 2$, Substituting $h_{r0} + 2 - p_2 = p_1 - 2d$ into (62) yields conclusion (9). At present, the equivalent resistance $R_n(A_{x_1}, O)$ between nodes A_{x_1} and O is solved, and formula (9) is proven.

4.3 Calculation of equivalent resistance $R_n(A_{x_1}, A_{x_2})$

Consider the special solutions of matrix equations under constraint conditions. Equations (38) and (39) provide the constraint equations for the input current from node A_{x_1} and the output current from node A_{x_2} , while equations (43) - (45) provide the piecewise functional solutions for the main matrix equation (35), respectively. Combining equation (47) ($x_1 \leq k \leq x_2$), solve the system of equations to obtain

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} - Ir(2 + h_{21})h_{20}B_{k-x_1}^{(i)}, (x_1 \leq k \leq x_2) \quad (63)$$

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} + Ir(2 + h_{21})h_{20}(B_{k-x_2}^{(i)} - B_{k-x_1}^{(i)}), (x_2 \leq k \leq n) \quad (64)$$

To find out $Y_0^{(i)}$, take equation (64) with $k = \{n, n-1\}$ into (37), we have

$$Y_0^{(i)} = Ir(2 + h_{21})h_{20} \frac{\Delta B_{n-x_1}^{(i)} - \Delta B_{n-x_2}^{(i)}}{(p_i - 2)B_{n+1}^{(i)}}. \quad (65)$$

Substitute (65) into (63) and export $(x_1 \leq k \leq x_2)$

$$\frac{Y_k^{(i)}}{Ir} = (2 + h_{21})h_{20} \frac{\Delta B_{x_1}^{(i)} \Delta B_{n-k}^{(i)} - \Delta B_k^{(i)} \Delta B_{n-x_2}^{(i)}}{(p_s - 2)B_{n+1}^{(i)}}, (x_1 \leq k \leq x_2) \quad (66)$$

Using a method similar to the calculation process mentioned earlier, and then applying Ohm's law to calculate the analytical formula for equivalent resistance, we obtain

$$R_n(A_{x_1}, A_{x_2}) = (V_{x_1}^{(a)} - V_{x_2}^{(a)}) / I. \quad (67)$$

where $V_{x_1}^{(a)}$ is expressed by equation (51).

Substitute equation (51) into equation (67) to obtain

$$R_n(A_{x_1}, A_{x_2}) = \frac{1}{I} \left(\frac{(p_2 - 2d)(Y_{x_1}^{(b)} - Y_{x_2}^{(b)})}{(2 + h_{21})h_{20}(p_2 - p_1)} - \frac{(p_1 - 2d)(Y_{x_1}^{(a)} - Y_{x_2}^{(a)})}{(2 + h_{21})h_{20}(p_2 - p_1)} \right). \quad (68)$$

Substituting Eq.(66) with $k = \{x_1, x_2\}$ into (68) yields

$$R_n(A_{x_1}, A_{x_2}) = r(2d - p_1) \frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - r(2d - p_2) \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (69)$$

At this point, the equivalent resistance $R_n(A_{x_1}, A_{x_2})$ between nodes A_{x_1} and A_{x_2} is solved, and formula (10) is proven.

4.4 Calculation of equivalent resistance $R_n(B_{x_1}, B_{x_2})$

When the current I is input from node B_{x_1} to output from node B_{x_2} , We obtain constraint equations (40) and (41). Except for the different input and output condition equations of current I , the general solutions (43) - (46) given above still apply to this situation. So the general solution was combined with the constraint equations (40) and (41) to obtain the solution

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} - Ic_i B_{k-x_1}^{(i)}, (x_1 \leq k \leq x_2) \quad (70)$$

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(i)} + Ic_i (B_{k-x_2}^{(i)} - B_{k-x_1}^{(i)}), (x_2 \leq k \leq n) \quad (71)$$

where $c_i = r_2(h_{21} + 2)(h_{r0} + 2 - p_i)$.

To understand $X_0^{(s)}$, take equation (71) with $k = \{n, n-1\}$ into (37), we have

$$\Delta B_{n-1}^{(i)} Y_0^{(i)} + Ic_i (B_{n-1-x_2}^{(i)} - B_{n-1-x_1}^{(i)}) = (p_i - 1) [\Delta B_n^{(i)} Y_0^{(i)} + Ic_i (B_{n-x_2}^{(i)} - B_{n-x_1}^{(i)})]. \quad (72)$$

Simplify to obtain

$$Y_0^{(i)} = Ic_i \frac{\Delta B_{n-x_1}^{(i)} - \Delta B_{n-x_2}^{(i)}}{(p_i - 2)B_{n+1}^{(i)}}. \quad (73)$$

Substitute (73) into (70) and export $(x_1 \leq k \leq x_2)$

$$Y_k^{(i)} = Ic_i \frac{\Delta B_{x_1}^{(i)} \Delta B_{n-k}^{(i)} - \Delta B_k^{(i)} \Delta B_{n-x_2}^{(i)}}{(p_i - 2)B_{n+1}^{(i)}}, \quad (74)$$

the calculation of equivalent resistance $R_n(B_{x_1}, B_{x_2})$ will mainly depend on equation (74).

Thus, Ohm's law can be used to derive

$$R_n(B_{x_1}, B_{x_2}) = (V_{x_1}^{(b)} - V_{x_2}^{(b)}) / I. \quad (75)$$

The expression of $V_k^{(b)}$ is given by equation (52). Substitute equation (52) into equation (75) to derive

$$R_n(B_{x_1}, B_{x_2}) = \frac{1}{I} \left(\frac{Y_{x_1}^{(1)} - Y_{x_2}^{(1)}}{p_2 - p_1} - \frac{Y_{x_1}^{(2)} - Y_{x_2}^{(2)}}{p_2 - p_1} \right). \quad (76)$$

So substituting (74) with $k = \{x_1, x_2\}$ into (76) and simplifies it to obtain

$$R_n(B_{x_1}, B_{x_2}) = \frac{r_2(2 + h_{21})}{p_2 - p_1} \left(\left(\frac{h_{r0}}{p_1 - 2} - 1 \right) \frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{B_{n+1}^{(1)}} - \left(\frac{h_{r0}}{p_2 - 2} - 1 \right) \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{B_{n+1}^{(2)}} \right) \quad (77)$$

Obviously, the equation (11) proposed earlier has been proven.

4.5 Calculation of equivalent resistance $R_n(B_{x_1}, O)$

4.5.1 Special solutions of matrix equations under constraint conditions

To calculate the equivalent resistance $R_n(B_{x_1}, O)$, the current can be input from B_{x_1} and output from node O. At this point, Eqs. (43) - (46) given earlier are applicable to the situation here, Eq. (70) also applies to this situation. But the range of Eq. (70) needs to be modified to $x_1 \leq k \leq n$, so equation (70) needs to be rewritten

$$Y_k^{(i)} = \Delta B_k^{(i)} Y_0^{(s)} - r_2 I (h_{21} + 2)(h_{r0} + 2 - p_i) B_{k-x_1}^{(i)}, \quad (x_1 \leq k \leq n) \quad (78)$$

To find out $Y_0^{(s)}$, take Eq. (78) with $k = \{n, n-1\}$ into (37), we have

$$Y_0^{(s)} = r_2 I (2 + h_{21})(h_{r0} + 2 - p_i) \frac{\Delta B_{n-x_1}^{(i)}}{(p_i - 2) B_{n+1}^{(i)}}. \quad (79)$$

Substitute (79) into (46) to calculate

$$Y_k^{(i)} = r_2 I (2 + h_{21})(h_{r0} + 2 - p_i) \frac{\Delta B_k^{(i)} \Delta B_{n-x_1}^{(i)}}{(p_i - 2) B_{n+1}^{(i)}}, \quad (0 \leq k \leq x_1) \quad (80)$$

4.5.2 Calculation of equivalent resistance $R_n(B_{x_1}, O)$

Equation (52) also applies to this situation. Similarly, using Ohm's law to calculate the analytical formula $R_n(B_{x_1}, O) = (V_{x_1}^{(b)} - 0) / I$ for equivalent resistance, where $V_{x_1}^{(b)}$ is represented by equation (52), therefore

$$R_n(B_{x_1}, O) = \frac{1}{I} \left(\frac{Y_{x_1}^{(1)} - Y_{x_1}^{(2)}}{p_2 - p_1} \right), \quad (81)$$

substituting (80) with $k = x_1$ into (81) yields

$$R_n(B_{x_1}, O) = r_2 \left(\frac{2 + h_{21}}{p_2 - p_1} \right) \left(\left(\frac{h_{r0}}{p_1 - 2} - 1 \right) \frac{\Delta B_{x_1}^{(1)} \Delta B_{n-x_1}^{(1)}}{B_{n+1}^{(1)}} - \left(\frac{h_{r0}}{p_2 - 2} - 1 \right) \frac{\Delta B_{x_1}^{(2)} \Delta B_{n-x_1}^{(2)}}{B_{n+1}^{(2)}} \right), \quad (82)$$

At this point, the equivalent resistance $R_n(B_{x_1}, O)$ between nodes B_{x_1} and O is solved, and formula (12) is proven.

5. Special Situations and Discussions

The asymmetric $2 \times n$ RN has already been calculated in the above text, which including the analytical formulae for the equivalent resistances between five pairs of nodes: (A_{x_1}, B_{x_2}) , (A_{x_1}, O) , (A_{x_1}, A_{x_2}) , (B_{x_1}, B_{x_2}) , (B_{x_1}, O) . Due to the abstract analytical formula for equivalent resistance given in the previous text, in order to better understand the physical meaning of the conclusion and compare and verify its correctness, we will discuss some special cases below.

To facilitate the verification of the equivalent resistance of the RN, we can use the structural transformation between $T \leftrightarrow \nabla$ in the network model of Fig.1 to obtain the $2 \times n$ -order RN of the structure shown in Fig. 4.

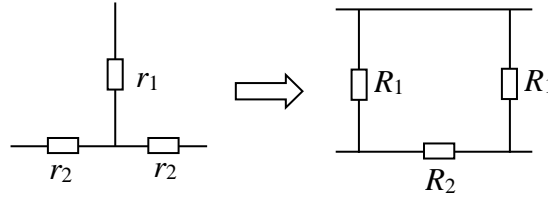


Fig.4. Circuit transformation: $T \leftrightarrow \nabla$

In the transformation shown in Fig. 4, the equivalent resistance can be solved

$$R_1 = 2r_1 + r_2, \quad R_2 = \frac{r_2}{r_1}(2r_1 + r_2). \quad (83)$$

So, we can equate the circuit diagram to the structure of Fig. 5. In Fig. 5, there be

$$r_4 = r_0 \parallel R_1 \parallel R_1 = \frac{r_0(2r_1 + r_2)}{2r_0 + 2r_1 + r_2}. \quad (84)$$

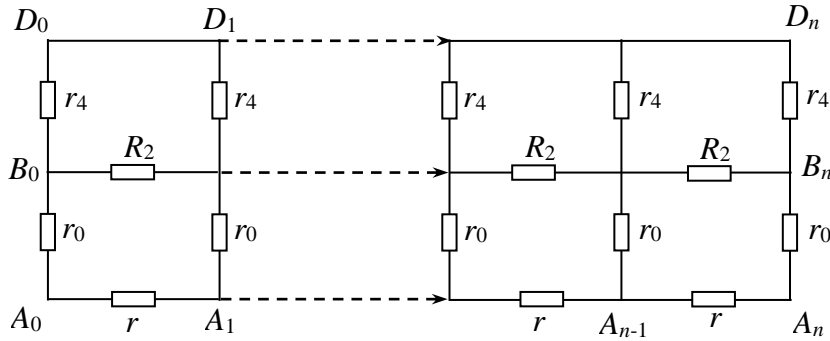


Fig.5. A equivalent circuit network of asymmetric $2 \times n$ resistance network

5.1 The case of $n = 0$

When $n = 0$, the $2 \times n$ -order RNM shown in Fig.1 can be degraded into the 2×0 -order RNM shown in Fig.6 (rotate from vertical to horizontal direction).

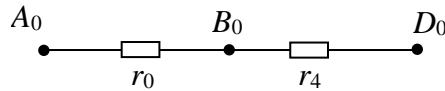


Fig.6. Expressed a 2×0 circuit network with $n=0$ (rotated 90°)

When $n=0$, there be

$$B_0^{(i)} = 0, \quad B_1^{(i)} = 1, \quad \Delta B_0^{(i)} = 1, \quad F_{0,0}^{(i)} = 1. \quad (85)$$

From Eq. (7) or (55), we have

$$R_0(A_0, B_0) = \frac{r(h_{r0} + 2 - p_1) + 2h_{20}r(2 + h_{21}) - r_2(2 + h_{21})(h_{r0} + 2 - p_2)}{(p_2 - p_1)(p_2 - 2)} \\ - \frac{r(h_{r0} + 2 - p_2) + 2h_{20}r(2 + h_{21}) - r_2(2 + h_{21})(h_{r0} + 2 - p_1)}{(p_2 - p_1)(p_1 - 2)}. \quad (86)$$

Simplify (86) then export

$$R_n(A_0, B_0) = r \frac{p_1 + p_2 - 4 - h_{r0} - (2 + h_{21})h_{20}}{(p_2 - 2)(p_1 - 2)}. \quad (87)$$

Because p_1 and p_2 are the two roots of equation (28), therefore

$$p_1 + p_2 = h_{r0} + 2(2 + h_{21})(1 + h_{20}) \\ p_1 p_2 = 2a(h_{r0} + 2) - (2 + h_{21})h_{20}h_{r0} \quad (88)$$

So the molecule of equation (87) can be simplified as

$$(p_1 + p_2 - 4) - h_{r0} - (2 + h_{21})h_{20} = 2h_{21} + (2 + h_{21})h_{20} \quad (89)$$

So, by substituting (88) into the denominator of equation (87), we obtain

$$(p_1 - 2)(p_2 - 2) = p_2 p_1 - 2(p_1 + p_2) + 4 \\ = 2a(h_{r0} + 2) - (2 + h_{21})h_{20}h_{r0} - 2h_{r0} - 4(2 + h_{21})(1 + h_{20}) + 4 \quad (90)$$

Because of $d = (2 + h_{21})(1 + h_{20}) - 1$, Substitute the value of d into equation (90) to calculate

$$(p_1 - 2)(p_2 - 2) = h_{r0}[2h_{21} + (2 + h_{21})h_{20}] \quad (91)$$

Substitute equations (89) and (91) into (87) to calculate

$$R_n(A_0, B_0) = r / h_{r0} = r_0. \quad (92)$$

Equation (92) is identical to the actual circuit result shown in Fig.6, which verifies the correctness of the equation at $n=0$.

Next, verify the situation of $R_0(A_0, 0)$. When $n=0$, substitute equation (85) into equation (9) and export

$$R_0(A_0, 0) = r \frac{(2a - p_1)}{(p_2 - p_1)(p_1 - 2)} - r \frac{(2d - p_2)}{(p_2 - p_1)(p_2 - 2)}, \quad (93)$$

simplify and obtain

$$R_0(A_0, 0) = r \frac{2(d - 1)}{(p_1 - 2)(p_2 - 2)}, \quad (94)$$

because of $d = (h_{20} + 1)(2 + h_{21}) - 1$, therefore $d - 1 = (2 + h_{21})(1 + h_{20}) - 2 = h_{21} + (2 + h_{21})h_{20}$, substituting a and (91) it into equation (94), we have

$$R_0(A_0, 0) = 2r_0 \frac{h_{21} + (2 + h_{21})h_{20}}{2h_{21} + (2 + h_{21})h_{20}} = r_0 \left(1 + \frac{(2 + h_{21})h_{20}}{2h_{21} + (2 + h_{21})h_{20}} \right), \quad (95)$$

simplify by substituting $h_{2k} = r_2 / r_k$ into the above equation

$$R_0(A_0, O) = r_0 \left(1 + \frac{2r_1 + r_2}{2r_0 + 2r_1 + r_2} \right). \quad (96)$$

According to the actual circuit calculation shown in Fig.6, it is obtained that

$$R_0(A_0, O) = r_0 + r_4 = r_0 + r_0 \frac{2r_1 + r_2}{2r_0 + 2r_1 + r_2}. \quad (97)$$

Obviously, the theoretical result (96) is identical to the actual circuit result shown in Fig. 6, which verifies the correctness of $R_n(A_{x_1}, O)$ at $n=0$.

Next, verify the situation of $R_0(B_0, 0)$. When $n=0$, substitute equation (85) into equation (12) and export

$$R_0(B_0, O) = r_2 \left(\frac{2 + h_{21}}{p_2 - p_1} \right) \left(\frac{h_{r0}}{p_1 - 2} - \frac{h_{r0}}{p_2 - 2} \right) = r_2 \frac{h_{r0}(2 + h_{21})}{(p_1 - 2)(p_2 - 2)}. \quad (98)$$

Substituting equation (91) into equation (98) yields

$$R_0(B_0, O) = r_2 \frac{2 + h_{21}}{2h_{21} + (2 + h_{21})h_{20}}, \quad (99)$$

simplify by substituting $h_{2k} = r_2 / r_k$ into the above equation

$$R_0(B_0, O) = r_0 \frac{2r_1 + r_2}{2r_0 + 2r_1 + r_2}. \quad (100)$$

From the actual circuit calculation shown in Fig. 6, $R_0(B_0, O) = r_4$ is obtained. Obviously, the theoretical result (100) is completely identical to the actual circuit result shown in Fig. 6, which verifies the correctness of $R_n(B_{x_1}, O)$ at $n=0$.

From Eqs.(10) and (11), the analytical formula for the equivalent resistance between nodes $A_{x_1} (B_{x_1})$ and $A_{x_2} (B_{x_2})$ in the complex $2 \times n$ RN of Fig.1 is

$$R_0(A_0, A_0) = 0, \quad R_0(B_0, B_0) = 0 \quad (101)$$

Obviously, the equivalent resistance calculated by the theoretical formula when $n=0$ is completely consistent with the actual circuit results, verifying the correctness of the theoretical formula when $n=0$.

In addition, when $n=1$, a similar verification method can be used for verification, and the verification results are completely correct. The calculation process will not be given here.

5.2 The case of $r_1 \rightarrow \infty$ and $r_0 = r, r_2 = 0.5r$

When $r_1 \rightarrow \infty$ and $r_0 = r, r_2 = 0.5r$, the complex $2 \times n$ RNM of Fig.1 is simplified to the RNM shown in Fig. 5 with $R_2=r, r_4=r_0$.

Because $\lim_{r_1 \rightarrow \infty} (h_{21}) = \lim_{r_1 \rightarrow \infty} (r_2 / r_1) = 0$ and $h_{20} = r_2 / r_0 = 1/2$, $h_{r0} = r / r_0 = 1$, $d = (2 + h_{21})(1 + h_{20}) - 1 = 2$, from equations (1), (2), (5), (7) and (8), we can obtain

$$p_1 = 4 - 2\cos\theta_1, \quad p_2 = 4 - 2\cos\theta_2, \quad \theta_i = (2i - 1)\pi/5 \quad (102)$$

$$\lambda_i = 2 - \cos\theta_i + \sqrt{(2 - \cos\theta_i)^2 - 1},$$

$$\bar{\lambda}_i = 2 - \cos \theta_i - \sqrt{(2 - \cos \theta_i)^2 - 1}, \quad (103)$$

Substituting Eqs. (102)-(103) into analytical equations (6), (9) and (10) can be simplified to obtain

$$\frac{R_n(A_{x_1}, B_{x_2})}{r} = \frac{q_2 F_{1,1}^{(1)} + 2F_{1,2}^{(1)} - q_1 F_{2,2}^{(1)}}{(p_1 - p_2)(p_1 - 2)B_{n+1}^{(1)}} - \frac{q_1 F_{1,1}^{(2)} + 2F_{1,2}^{(2)} - q_2 F_{2,2}^{(2)}}{(p_1 - p_2)(p_2 - 2)B_{n+1}^{(2)}}. \quad (104)$$

Inside $q_s = h_{r_0} + 2 - p_s$.

Using (102) to obtain

$$q_i = h_{r_0} + 2 - p_i = 2 \cos \theta_i - 1, \quad \cos \theta_1 = \frac{1 + \sqrt{5}}{4}, \quad \cos \theta_2 = \frac{1 - \sqrt{5}}{4}, \quad (105)$$

so the calculation shows that

$$\frac{q_1}{p_1 - p_2} = \frac{2 \cos \theta_1 - 1}{2 \cos \theta_2 - 2 \cos \theta_1} = -\frac{4}{5} \left(\frac{5 - \sqrt{5}}{8} \right) = -\frac{4}{5} \sin^2(\theta_1) = -\frac{4}{5} \sin^2(2\theta_2) \quad (106)$$

$$\frac{q_2}{p_1 - p_2} = \frac{2 \cos \theta_2 - 1}{2 \cos \theta_2 - 2 \cos \theta_1} = \frac{4}{5} \left(\frac{5 + \sqrt{5}}{8} \right) = \frac{4}{5} \sin^2(\theta_2) = \frac{4}{5} \sin^2(2\theta_1). \quad (107)$$

Substituting equations (106) and (107) into (104) yields

$$R_n(A_{x_1}, B_{x_2}) = \frac{2}{5} r \left[\frac{F_{1,1}^{(1)} \sin^2(2\theta_1) + 2F_{1,2}^{(1)} \sin(\theta_1) \sin(2\theta_1) + F_{2,2}^{(1)} \sin^2(\theta_1)}{(1 - \cos \theta_1)B_{n+1}^{(1)}} + \frac{F_{1,1}^{(2)} \sin^2(2\theta_2) + 2F_{1,2}^{(2)} \sin(\theta_2) \sin(2\theta_2) + F_{2,2}^{(2)} \sin^2(\theta_2)}{(1 - \cos \theta_2)B_{n+1}^{(2)}} \right], \quad (108)$$

the remaining parameters $B_k^{(i)}, \Delta, h_{sk}, F_{k,v}^{(i)}$ etc. are defined in equations (1)-(6).

The analytical formula for the equivalent resistance between nodes A_{x_1} and O in the $2 \times n$ RN of Fig.1 is

$$R_n(A_{x_1}, O) = \frac{2r}{5} \left(\frac{F_{1,1}^{(1)} \sin^2(2\theta_1)}{(1 - \cos \theta_1)B_{n+1}^{(1)}} + \frac{F_{1,1}^{(2)} \sin^2(2\theta_2)}{(1 - \cos \theta_2)B_{n+1}^{(2)}} \right), \quad (109)$$

Because using (105) to derive $\frac{2}{5} \left(\frac{\sin^2(2\theta_i)}{1 - \cos \theta_i} \right) = \cot^2(\theta_i)$, equation (109) can be rewritten as

$$\frac{R_n(A_{x_1}, O)}{r} = \frac{\Delta B_{x_1}^{(1)} \Delta B_{n-x_1}^{(1)}}{B_{n+1}^{(1)}} \cot^2(\theta_1) + \frac{\Delta B_{x_1}^{(2)} \Delta B_{n-x_1}^{(2)}}{B_{n+1}^{(2)}} \cot^2(\theta_2), \quad (110)$$

The analytical formula for the equivalent resistance between nodes B_{x_1} and O in the $2 \times n$ RN of Fig.1 is

$$R_n(B_{x_1}, O) = \frac{2r}{5} \left(\frac{F_{1,1}^{(1)} \sin^2(\theta_1)}{(1 - \cos \theta_1)B_{n+1}^{(1)}} + \frac{F_{1,1}^{(2)} \sin^2(\theta_2)}{(1 - \cos \theta_2)B_{n+1}^{(2)}} \right), \quad (111)$$

because $\sin^2(\theta_i) = 4 \sin^2(\frac{1}{2} \theta_i) \cos^2(\frac{1}{2} \theta_i) = 2(1 - \cos \theta_i) \cos^2(\frac{1}{2} \theta_i)$, equation (111) can be rewritten as

$$R_n(B_{x_1}, O) = \frac{4r_0}{5} \left(\frac{F_{1,1}^{(1)}}{B_{n+1}^{(1)}} \cos^2(\frac{1}{2} \theta_1) + \frac{F_{1,1}^{(2)}}{B_{n+1}^{(2)}} \cos^2(\frac{1}{2} \theta_2) \right), \quad (112)$$

the analytical formula for the equivalent resistance between nodes A_{x_1} and A_{x_2} in the $2 \times n$ RN of Fig.1 is

$$\frac{R_n(A_{x_1}, A_{x_2})}{r} = \frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{B_{n+1}^{(1)}} \cot^2(\theta_1) + \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{B_{n+1}^{(2)}} \cot^2(\theta_2). \quad (113)$$

The analytical formula for the equivalent resistance between nodes B_{x_1} and B_{x_2} in the $2 \times n$ RN of Fig.1 is

$$\frac{R_n(B_{x_1}, B_{x_2})}{r} = \frac{4}{5} \left(\frac{F_{1,1}^{(1)} - 2F_{1,2}^{(1)} + F_{2,2}^{(1)}}{B_{n+1}^{(1)}} \cos^2\left(\frac{1}{2}\theta_1\right) + \frac{F_{1,1}^{(2)} - 2F_{1,2}^{(2)} + F_{2,2}^{(2)}}{B_{n+1}^{(2)}} \cos^2\left(\frac{1}{2}\theta_2\right) \right). \quad (114)$$

The equivalent resistance $R_n(A_{x_1}, O)$ of the $2 \times n$ fan network model was once a research topic in Refs. [41, 42], and Ref. [39] investigated more general issues. Comparing these conclusions, it was found that the conclusions drawn in this paper (108) - (114) are consistent with those given in references [41,42], which indirectly proves the correctness of the calculation results in this paper.

5.3 The case of $x_2 = x_1$

Here we mainly discuss the problem of equation (7). When $x_2 = x_1$, there be $F_{1,2}^{(i)} = F_{2,2}^{(i)} = F_{1,1}^{(i)}$, equation (7) can be simplified,

$$\frac{R_n(A_{x_1}, B_{x_1})}{r} = (2d - p_1) \frac{(1 - u_1)^2 \Delta B_{x_1}^{(1)} \Delta B_{n-x_1}^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - (2d - p_2) \frac{(1 - u_2)^2 \Delta B_{x_1}^{(2)} \Delta B_{n-x_1}^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (115)$$

Among them $F_{1,1}^{(i)} = \Delta B_{x_1}^{(i)} \Delta B_{n-x_1}^{(i)}$, $u_i = h_{20}(2 + h_{21})/(2d - p_i)$.

Specifically, when $x_2 = x_1 = 0$ and $F_{1,1}^{(i)} = \Delta B_0^{(i)} \Delta B_{n-0}^{(i)} = \Delta B_n^{(i)}$, it is derived from (115)

$$\frac{R_n(A_0, B_0)}{r} = (2d - p_1) \frac{(1 - u_1)^2 \Delta B_n^{(1)}}{(p_2 - p_1)(p_1 - 2)B_{n+1}^{(1)}} - (2d - p_2) \frac{(1 - u_2)^2 \Delta B_n^{(2)}}{(p_2 - p_1)(p_2 - 2)B_{n+1}^{(2)}}. \quad (116)$$

Obviously, when $x_2 = x_1$, the analytical formula for equivalent resistance degenerates into a relatively simple result, and they are all original results derived from this article.

5.4 Visualize resistance relationship

In order to further reveal the relationship between equivalent resistance with r_1 and node position, we have used the formulae given in the previous text to draw their visualization graphs, which can clearly show the relationship between equivalent resistance with r_1 and node position. Here we consider the visualization of equivalent resistances $R_n(A_0, B_x)$ and $R_n(A_x, 0)$. The parameter design in the graph is $r_0 = r_2 = r = 1$ (b Ω), where unit b is any value, such as b=1, b=1000, and so on. Let r_1 be a variable, A_x be a movable point on the A_0A_n axis, and B_x be a movable point on the B_0B_n axis.

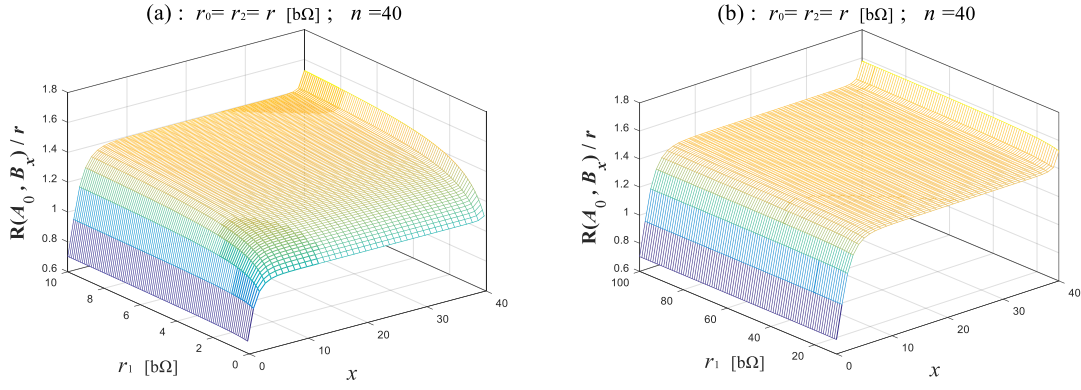


Fig. 6. Visual characteristic of equivalent resistance $R_n(A_0, B_x)/r$ varying with r_1 .

In Fig. (a), $r_1 \in [0, 10]$, and in Fig. (b), $r_1 \in [10, 100]$

5.4.1 Visual representation of equivalent resistance $R_n(A_0, B_x)$

Draw a graph of the change in equivalent resistance between A_0 and B_x . Since $R_n(A_0, B_x)/r$ is a dimensionless value, regardless of the unit $[\text{b}\Omega]$ of resistance, the unit $[\text{b}\Omega]$ does not affect the structure of the graph. We take the maximum order of the network as $n=40$. In order to facilitate the investigation of the variation characteristics of equivalent resistance $R_n(A_0, B_x)/r$ with r_1 , we considered two segmented intervals $r_1 \in [0, 10]$ and $r_1 \in [10, 100]$.

From Fig. 6 (a), it can be seen that when x is determined (e.g. $x=20$, etc.), the equivalent resistance $R_n(A_0, B_x)/r$ for $r_1 \in [0, 10]$ gradually increases with the increase of r_1 ; When r_1 is determined (e.g. $r_1=4$, etc.), $R_n(A_0, B_x)/r$ within the range of $x \in [0, 10]$ significantly increases with the increase of x , and $R_n(A_0, B_x)/r$ within the range of $x \in [10, 35]$ hardly changes with the increase of x , but $R_n(A_0, B_x)/r$ within the range of $x \in [35, 40]$ significantly increases with the increase of x . From Fig. 6 (b), it can be seen that when x is determined (such as $x=20$, etc.), the equivalent resistance $R_n(A_0, B_x)/r$ of $r_1 \in [10, 100]$ hardly changes with the increase of r_1 , indicating that the magnitude of the equivalent resistance $R_n(A_0, B_x)/r$ when $r_1 > 10$ is not significantly affected by r_1 .

5.4.2 Visual representation of equivalent resistance $R_n(A_x, O)$

Draw a graph of the change in equivalent resistance between A_x and O . Since $R_n(A_x, O)/r$ is a dimensionless value, regardless of the unit $(\text{b}\Omega)$ of resistance, it does not affect the structure of the graph. We take the maximum order of the network as $n=40$. In order to facilitate the investigation of the variation characteristics of equivalent resistance $R_n(A_x, O)/r$ with respect to r_1 , we consider two segmented intervals $r_1 \in [0, 10]$ and $r_1 \in [10, 100]$.

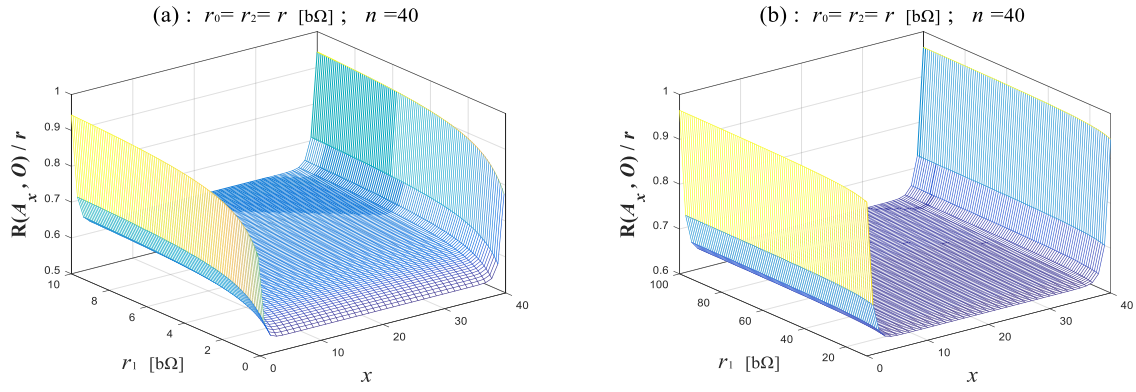


Fig.7. Characteristics of equivalent resistance $R_n(A_x, O)/r$ as a function of r_1 , where $r_1 \in [0, 10]$ in Fig. (a), $r_1 \in [10, 100]$ in Fig. (b)

From Fig.7 (a), it can be seen that when x is determined (e.g. $x=10$, etc.), and $r_1 \in [0, 10]$, the equivalent resistance $R_n(A_x, O)/r$ gradually increases with the increase of r_1 ; When r_1 is determined (such as $r_1=4$, etc.), the equivalent resistance $R_n(A_x, O)/r$ within the range of $x \in [0, 40]$ shows a U-shaped change with the increase of x , showing a significant increase in the equivalent resistance at both ends. The equivalent resistance $R_n(A_x, O)/r$ within the range of $x \in [5, 35]$ exhibits a relatively small resistance value. From Fig. 7 (b), it can be seen that when x is determined (such as $x=20$, etc.), when $r_1 \in [10, 100]$, the equivalent resistance $R_n(A_x, O)/r$ remains almost unchanged as r_1 increases, this indicates that the magnitude of the equivalent resistance $R_n(A_x, O)/r$ when $r_1 > 10$ is not significantly affected by r_1 .

6 Summary and comment

This article proposes an asymmetric $2 \times n$ RNM that is a challenging issue. The research here indicates that the RT-V method established in reference [32] has broad application value and can solve complex circuit networks. The five original analytical formulas for equivalent resistance proposed in the article are all derived using the RT-V method. Due to the presence of five independent resistive elements in the model, such as r_0, r, r_1, r_2, r_3 all of which are random values, this model has generality and can generate a series of special network models. The article validates and compares the results of this article when discussing special cases, indicating that formulas (7) - (12) hold for everything, especially when discussing $r_1 \rightarrow \infty$ and $r_0=r, r_2=0.5r$, equivalent resistance analytical equations (108) - (114) expressed in trigonometric functions are derived.

Solving the analytical formula for equivalent resistance is a fundamental problem, as once the equivalent resistance formula is obtained, many complex circuits can be solved through variable substitution techniques. For example, taking the circuit network in this article as an example, consider a class of RLC complex impedance circuits as shown in Fig.1, one can assume $r_0 = -j/\omega C$, $r = j\omega L$, $r_2 = j\omega L_2$, $r_1 = -j/\omega C_1$, by substituting these relationships into the equivalent resistance formulae, the equivalent complex impedance analytical formula can be

derived. For example, in a fractional order complex impedance circuit, let $r_0 = \frac{1}{\omega^{\lambda_0} C_0} e^{j(-\lambda_0 \pi/2)}$,
 $r_1 = \frac{1}{\omega^{\lambda_1} C_1} e^{j(-\lambda_1 \pi/2)}$, $r = \omega^{\bar{\lambda}_1} L e^{j(\bar{\lambda}_1 \pi/2)}$, $r_2 = \omega^{\bar{\lambda}_2} L_2 e^{j(\bar{\lambda}_2 \pi/2)}$, where $0 \leq \lambda_i \leq 1$, $0 \leq \bar{\lambda}_i \leq 1$. By substituting these relationships into the equivalent resistance formulae, the equivalent complex impedance analytical formula for fractional order circuits can be derived. The research work in this article can promote the research and development of complex circuit network models; The analytical formula for equivalent resistance we have derived can provide a theoretical basis for related research and simulation studies in engineering.

REFERENCES

- [1] Tan Z Z. Resistance network model. China, Xi'an: Xidian Univ. Press (in Chinese), 2011.
- [2] Yamagata R, Iokibe K, Toyota Y. Optimization of Resistances in RL Snubbers for Power Distribution Network of Integrated Circuits[J]. IEICE Proceeding Series, 2013, **30**:226-229.
- [3] Guerrero JM, Daniel S-J, Mahtani K, and Platero, CA, A Ground Fault Location Method for DC Systems Through Multiple Grounding Connections. IEEE Transactions on Industry Applications, 2022, **58**(6): 7022-7033.
- [4] Hadad Yakir, Jason C. Soric, Alexander B. Khanikaev, et al. Self-induced topological protection in nonlinear circuit arrays[J]. Nat. Electron, 2018, **1**(3):178–182.
- [5] Lin X Y, Tan Z Z, Unified construction of two n-order circuit networks with diodes. Front Inform Technol Electron Eng. 2023, **24**(2): 289-298.
- [6] Stefan Imhof, Christian Berger, Florian Bayer, et al. Topoelectrical-circuit realization of topological corner modes[J]. Nat. Phys, 2018, **14**(9):925–929.
- [7] Kirchhoff G. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. Annals of Physical Chemistry, 1847, **148**(12):497-508.
- [8] D. Atkinson and F. J. van Steenwijk. Infinite resistive lattices. Am. J. Phys, 1999, **67**(6):486-492.
- [9] Cserti J. Application of the lattice Green's function for calculating the resistance of an infinite network of resistors. American Journal of Physics, 2000, **68**(10):896–906.
- [10] Giordano S . Disordered lattice networks: General theory and simulations. International Journal of Circuit Theory and Applications, 2005, **33**(6):519-540.
- [11] Asad JH. Exact Evaluation of the Resistance in an Infinite Face-Centered Cubic Network. J Stat Phys 2013, **150**:1177–1182.
- [12] Asad JH, Diab AA, Hijjawi RS, Khalifeh JM. Infinite face-centered-cubic network of identical resistors: Application to lattice Green's function. Eur Phys J Plus 2013, **128**:13002.
- [13] Owaidat MQ, Hijjawi RS, Khalifeh JM. Perturbation theory of uniform tiling of space with resistors. Eur Phys J Plus 2014, **129**(29): 2190–5444.
- [14] Owaidat MQ, Asad JH, Tan Z-Z. On the perturbation of a uniform tiling with resistors. Int J Mod Phys B 2016, **30**(24):1650166.
- [15] Owaidat MQ, Al-Badawi AA, Asad JH, Al-Twiessi M. Two-point resistance on the centered-triangular lattice. Chin Phys Lett 2018, **35**(2):020502.
- [16] Owaidat MQ, Asad JH. Resistance calculation of three-dimensional triangular and hexagonal prism lattices. Eur Phys J Plus 2016, **131**:309.
- [17] Wu F Y. Theory of resistor networks: the two-point resistance. Journal of Physics A: Mathematical and General, 2004, **37**(26):6653–6673.

- [18] Tzeng W J, Wu F Y. Theory of impedance networks: the two-point impedance and LC resonances[J]. Journal of Physics A: Mathematical and General, 2006, **39**: 8579-8591.
- [19] Izmailian N S, Huang M C. Asymptotic expansion for the resistance between two maximum separated nodes on an $M \times N$ resistor network[J]. Physical Review E, 2010, **82**: 011125.
- [20] Izmailian N S, Kenna R. A generalised formulation of the Laplacian approach to resistor networks[J]. Journal of Statistical Mechanics Theory and Experiment, 2014, **9**(9): P09016.
- [21] Izmailian N S, Kenna R, Wu F Y. The two-point resistance of a resistor network: a new formulation and application to the cobweb network[J]. Journal of Physics A: Mathematical and Theoretical, 2014, **47**(3): 035003.
- [22] Essam J W, Izmailyan N S, Kenna R, et al. Comparison of methods to determine point-to-point resistance in nearly rectangular networks with application to a ‘hammock’ network[J]. Royal Society Open Science, 2015, **2**(4): 2054-5703.
- [23] Tan Z Z, Zhou L, Yang J H. The equivalent resistance of a $3 \times n$ cobweb network and its conjecture of an $m \times n$ cobweb network[J]. Journal of Physics A: Mathematical and Theoretical, 2013, **46**(19): 195202.
- [24] Tan Z Z, Zhou L, Luo D F. Resistance and capacitance of $4 \times n$ cobweb network and two conjectures[J]. International Journal of Circuit Theory and Applications, 2015, **43**: 329-341.
- [25] Tan Z Z. Theory on resistance of $m \times n$ cobweb network and its application[J]. International Journal of Circuit Theory and Applications, 2015, **43**: 1687-1702.
- [26] Tan Z Z, Fang J H. Two-point resistance of a cobweb network with a $2r$ boundary[J]. Communications in Theoretical Physics, 2015, **63**(1): 36-44.
- [27] Tan Z Z, Essam J W, Wu F Y. Two-point resistance of a resistor network embedded on a globe[J]. Physical Review E, 2014, **90**(1): 012130.
- [28] Essam J W, Tan Z Z, Wu F Y. Resistance between two nodes in general position on an $m \times n$ fan network[J]. Physical Review E, 2014, **90**(3): 032130.
- [29] Tan Z Z, Zhang, Q H. Formulae of resistance between two corner nodes on a common edge of the $m \times n$ rectangular network[J]. International Journal of Circuit Theory and Applications, 2015, **43**: 944-958.
- [30] Tan Z Z. Two-point resistance of an $m \times n$ resistor network with an arbitrary boundary and its application in RLC network[J]. Chinese Physics B, 2016, **25** (5): 050504.
- [31] Tan Z Z, Tan Z. Electrical properties of an arbitrary $m \times n$ rectangular network. Acta Physica Sinica, 2020, **69**(2): 020502.
- [32] Tan Z-Z. Recursion-transform method and potential formulae of the $m \times n$ cobweb and fan networks. Chinese Physics B, 2017, **26**(9): 090503.
- [33] Tan, Z Z. Theory of an $m \times n$ apple surface network with special boundary. Communications in Theoretical Physics, 2022, **75**(6):065701.
- [34] Tan, Z Z. Resistance theory for two classes of n -periodic networks. The European Physical Journal Plus, 2022, **137**(5):546.
- [35] Chen J X, Ji W X, Tan Z Z. Electrical properties of a $2 \times n$ non-regular hammock network. Indian J Phys, 2024, **98**(8):2851–2860.
- [36] Luo X L and Tan Z Z. Fractional circuit network theory with n -V-structure. Physica Scripta, 2023, **98**: 045224.
- [37] Tan Z Z, Wang X. Electrical characteristics of a fractional order $3 \times n$ Fan network. Communications in Theoretical Physics, 2022, **75**(6): 065701.
- [38] Xin-Yu Fang , Zhi-Li Zhang , Tan Z-Z, Equivalent resistance of a periodic and asymmetric $2 \times n$ resistor network, Results in Physics, 2024, **60**:107683.
- [39] Tan Z Z. Recursion–transform approach to compute the resistance of a resistor network with an arbitrary boundary. Chinese Physics B, 2015, **24**(2):020503.
- [40] Tan Z Z. Recursion-transform method for computing resistance of the complex resistor network with three arbitrary boundaries. Phys Rev E, 2015, **91**(5):052122.

- [41] Tan Zhen, Tan Z-Z, Chen J-X, Potential formula of the nonregular $m \times n$ fan network and its application, Scientific Reports, 2018, **8**: 5798
- [42] Wu Jin-Yi , Wang Lu, *et al.* , Electrical properties of an arbitrary $2 \times n$ order fan network. Phys. Scr. 2019, **94**: 105818